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PARTITIONS OF $\mathbb{Z}_m$ WITH IDENTICAL REPRESENTATION FUNCTION

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Abstract

For a given set $S \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, let $R_S(\overline{n})$ denote the number of solutions of the equation $\overline{n} = \overline{s} + \overline{s'}$ with ordered pairs $(\overline{s}, \overline{s'}) \in S^2$. We determine the structure of $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m - 2$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, where m is an even integer.

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1. Introduction

For a fixed integer $m \geq 2$, let $\overline{x}$ be the residue class x modulo m, the set of all integers y such that $y \equiv x \pmod{m}$, and let $\mathbb{Z}_m$ be the set of all residue classes mod m. For a given set $S \subseteq \mathbb{Z}_m$ and $\overline{n} \in \mathbb{Z}_m$, the representation function $R_S(\overline{n})$ is defined as the number of solutions of $\overline{n} = \overline{s} + \overline{s'}$ with $\overline{s}, \overline{s'} \in S$. For $\overline{a} \in \mathbb{Z}_m$ and $X \subseteq \mathbb{Z}_m$, define

$$\overline{a} + X = \{\overline{a} + \overline{x} : \overline{x} \in X\}.$$

In 2012, Yang and Chen [10] studied the analogue of Sárközy's problem in $\mathbb{Z}_m$. They determined the structure of the sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.

THEOREM A ([10], Theorem 2). *The equality $R_A(\overline{n}) = R_{\mathbb{Z}_m \setminus A}(\overline{n})$ holds for all $\overline{n} \in \mathbb{Z}_m$ if and only if m is even and $|A| = m/2$.*

In 2017, Yang and Tang [11] considered the case of $|(A \cup B) \setminus (A \cap B)| = m - 1$ and proved the following result.

THEOREM B ([11], Theorem 3.1). *Let $m \geq 3$ be an odd integer. Then there do not exist sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m - 1$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.*

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Related results about partitions of $\mathbb{N}$ with the same representation functions can be found in [1–9].

In this paper we determine the structure of $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m - 2$ such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$. If $R_A(\overline{n}) = R_B(\overline{n})$ holds for all $\overline{n} \in \mathbb{Z}_m$, then $|A| = |B|$, and so $|(A \cup B) \setminus (A \cap B)| = m - 2$ is even. Hence we always assume that m is an even integer.

If $|(A \cup B) \setminus (A \cap B)| = m - 2$, then there are three cases to consider:

(1) $A \cup B = \mathbb{Z}_m$, $|A \cap B| = 2$;
(2) $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}\}$, $|A \cap B| = 1$;
(3) $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}, \overline{r_2}\}$, $|A \cap B| = 0$.

So we have the following three theorems.

THEOREM 1.1. *Let m be an even integer and let A, B be sets with $A \cup B = \mathbb{Z}_m$, $|A \cap B| = 2$. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = A + \overline{m/2}$.*

THEOREM 1.2. *Let m be an even integer and let A, B be sets with $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}\}$, $|A \cap B| = 1$.*

(i) *If $m \equiv 0 \pmod 4$, then there do not exist two sets A, B such that $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.*
(ii) *If $m \equiv 2 \pmod 4$, then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if there exist two sets C, D with $|C| = |D| = (m/2 - 1)/2$, $C \cup D = [\overline{0}, \overline{m/2 - 1}] \setminus \{\overline{r_1}, \overline{r_1 + m/2}\}$, and $C \cap D = \emptyset$ such that*

$$A = C \cup (C + \overline{m/2}) \cup \{\overline{r_1 + m/2}\}, \quad B = D \cup (D + \overline{m/2}) \cup \{\overline{r_1 + m/2}\}.$$

THEOREM 1.3. *Let m be an even integer and let A, B be sets with $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}, \overline{r_2}\}$, $|A \cap B| = 0$. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $B = A + \overline{m/2}$.*

Throughout this paper, the characteristic function of the set $A \subseteq \mathbb{Z}_m$ is denoted by

$$\chi_A(t) = \begin{cases} 0 & \bar{t} \notin \text{ A}, \\ 1 & \bar{t} \in \text{ A}. \end{cases}$$

For any sets $A, B \subseteq \mathbb{Z}_m$, $R_{A,B}(\overline{n})$ is defined as the number of solutions of the equation $\overline{a} + \overline{b} = \overline{n}$ with $\overline{a} \in A$ and $\overline{b} \in B$. For a property P, define $\theta(P) = 1$ if P is true, otherwise $\theta(P) = 0$. Without loss of generality, for every residue class $\overline{n} \in \mathbb{Z}_m$, we may assume that $0 \le n \le m - 1$.

2. A preliminary lemma

LEMMA 2.1. *Let $S \subseteq \mathbb{Z}_m$. Then $R_{\mathbb{Z}_m \setminus S}(\overline{n}) = m - 2|S| + R_S(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$.*

PROOF. For all $\overline{n} \in \mathbb{Z}_m$, it is clear that

$$\begin{aligned} R_{\mathbb{Z}_m \setminus S}(\overline{n}) &= |\{(s, s') : \overline{s}, \overline{s'} \in \mathbb{Z}_m \setminus S, 0 \le s, s' \le m - 1, s + s' = n \text{ or } s + s' = n + m\}| \\ &= \sum_{0 \le i \le n} (1 - \chi_S(i))(1 - \chi_S(n - i)) + \sum_{n+1 \le i \le m-1} (1 - \chi_S(i))(1 - \chi_S(n + m - i)) \end{aligned}$$

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$$= m - 2|S| + \sum_{0 \le i \le n} \chi_S(i)\chi_S(n-i) + \sum_{n+1 \le i \le m-1} \chi_S(i)\chi_S(n+m-i)$$
$$= m - 2|S| + R_S(\overline{n}).$$

This completes the proof of Lemma 2.1. □

3. Proof of Theorem 1.1

The sufficiency of the condition in Theorem 1.1 is obvious. We prove the necessity. Assume that

$$A \cup B = \mathbb{Z}_m, \quad A \cap B = \{\overline{r_1}, \overline{r_2}\}, \quad R_A(\overline{n}) = R_B(\overline{n}) \quad \text{for all } \overline{n} \in \mathbb{Z}_m.$$

Since $B = (\mathbb{Z}_m \setminus A) \cup \{\overline{r_1}, \overline{r_2}\}$, it follows from Lemma 2.1 that

$$\begin{aligned} R_B(\overline{n}) &= R_{\mathbb{Z}_m \setminus A}(\overline{n}) + 2R_{\mathbb{Z}_m \setminus A, \{\overline{r_1}, \overline{r_2}\}}(\overline{n}) + 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) + \theta(\overline{n} = 2\overline{r_1}) + \theta(\overline{n} = 2\overline{r_2}) \\ &= R_{\mathbb{Z}_m \setminus A}(\overline{n}) + 2\sum_{i=1}^{2}(1 - \chi_A(n - r_i)) + 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) + \theta(\overline{n} = 2\overline{r_1}) + \theta(\overline{n} = 2\overline{r_2}) \\ &= m - 2|A| + R_A(\overline{n}) + 2\sum_{i=1}^{2}(1 - \chi_A(n - r_i)) \\ &\quad + 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) + \theta(\overline{n} = 2\overline{r_1}) + \theta(\overline{n} = 2\overline{r_2}) \end{aligned} \tag{3.1}$$

for all $\overline{n} \in \mathbb{Z}_m$. Since $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, it follows that $|A| = |B| = m/2 + 1$. Then by (3.1),

$$2(\chi_A(n - r_1) + \chi_A(n - r_2)) = 2 + 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) + \theta(\overline{n} = 2\overline{r_1}) + \theta(\overline{n} = 2\overline{r_2}) \tag{3.2}$$

for all $\overline{n} \in \mathbb{Z}_m$.

If $2\overline{r_1} \ne 2\overline{r_2}$, then $2(\chi_A(r_1) + \chi_A(2r_1 - r_2)) = 3$ by taking $\overline{n} = 2\overline{r_1}$ in (3.2), a contradiction. Hence $2\overline{r_1} = 2\overline{r_2}$, that is, $\overline{r_2} = \overline{r_1} + \overline{m/2}$. It follows from (3.2) that

$$\chi_A(n - r_1) + \chi_A\Big(n - \Big(r_1 + \frac{m}{2}\Big)\Big) = 1 + \theta\Big(\overline{n} = 2\overline{r_1} + \overline{\frac{m}{2}}\Big) + \theta(\overline{n} = 2\overline{r_1}) \tag{3.3}$$

for all $\overline{n} \in \mathbb{Z}_m$. Taking $n = k + (r_1 + m/2)$ in (3.3),

$$\chi_A(k) + \chi_A\Big(k + \frac{m}{2}\Big) = 1 + \theta(\overline{k} = \overline{r_1}) + \theta\Big(\overline{k} = \overline{r_1} + \overline{\frac{m}{2}}\Big)$$

for all $\overline{k} \in \mathbb{Z}_m$, which implies that $B = A + \overline{m/2}$.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Assume that $A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}\}$, $A \cap B = \{\overline{r_2}\}$. Since $B = (\mathbb{Z}_m \setminus (A \cup \{\overline{r_1}\})) \cup \{\overline{r_2}\}$, it follows from Lemma 2.1 that

$$\begin{aligned} R_B(\overline{n}) &= R_{\mathbb{Z}_m \setminus (A \cup \{\overline{r_1}\})}(\overline{n}) + 2R_{\mathbb{Z}_m \setminus (A \cup \{\overline{r_1}\}), \{\overline{r_2}\}}(\overline{n}) + \theta(\overline{n} = 2\overline{r_2}) \\ &= R_{\mathbb{Z}_m \setminus (A \cup \{\overline{r_1}\})}(\overline{n}) + 2(1 - \chi_{A \cup \{\overline{r_1}\}}(n - r_2)) + \theta(\overline{n} = 2\overline{r_2}) \end{aligned}$$

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$$
\begin{aligned}
&= m - 2|A \cup \{\overline{r_1}\}| + R_{A\cup\{\overline{r_1}\}}(\overline{n}) + 2(1 - \chi_{A\cup\{\overline{r_1}\}}(n - r_2)) + \theta(\overline{n} = 2\overline{r_2})\\
&= m - 2|A \cup \{\overline{r_1}\}| + R_A(\overline{n}) + 2\chi_A(n - r_1) + \theta(\overline{n} = 2\overline{r_1})\\
&\quad + 2 - 2\chi_A(n - r_2) - 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) + \theta(\overline{n} = 2\overline{r_2}) \qquad (4.1)
\end{aligned}
$$

for all $\overline{n} \in \mathbb{Z}_m$. Then $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$ if and only if $|A| = |B| = m/2$ and

$$2\chi_A(n - r_1) + \theta(\overline{n} = 2\overline{r_1}) + \theta(\overline{n} = 2\overline{r_2}) = 2\chi_A(n - r_2) + 2\theta(\overline{n} = \overline{r_1} + \overline{r_2}) \qquad (4.2)$$

for all $\overline{n} \in \mathbb{Z}_m$.

If $2\overline{r_1} \neq 2\overline{r_2}$, then $2\chi_A(2r_2 - r_1) + 1 = 2$ by taking $\overline{n} = 2\overline{r_2}$ in (4.2), a contradiction. Hence $2\overline{r_1} = 2\overline{r_2}$, that is, $\overline{r_2} = \overline{r_1} + \overline{m/2}$. Then (4.2) is equivalent to $\overline{r_2} = \overline{r_1} + \overline{m/2}$ and

$$\chi_A(n - r_1) + \theta(\overline{n} = 2\overline{r_1}) = \chi_A\Big(n - \Big(r_1 + \frac{m}{2}\Big)\Big) + \theta\Big(\overline{n} = 2\overline{r_1} + \overline{\frac{m}{2}}\Big) \qquad (4.3)$$

for all $\overline{n} \in \mathbb{Z}_m$. Taking $n = k + (r_1 + m/2)$ in (4.3),

$$\chi_A\Big(k + \frac{m}{2}\Big) + \theta\Big(\overline{k} = \overline{r_1} + \overline{\frac{m}{2}}\Big) = \chi_A(k) + \theta(\overline{k} = \overline{r_1}) \qquad (4.4)$$

for all $\overline{k} \in \mathbb{Z}_m$.

(i) If $m \equiv 0 \pmod 4$ and $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n} \in \mathbb{Z}_m$, then $|A| = m/2$ is even and (4.4) holds for all $\overline{k} \in \mathbb{Z}_m$. However, for any $\overline{k} \neq \overline{r_1}$, $\overline{r_1} + \overline{m/2}$ with $\overline{k} \in A$, it follows from (4.4) that $\overline{k}, \overline{k} + \overline{m/2} \in A$, which implies that $|A|$ is odd, a contradiction.

(ii) If $m \equiv 2 \pmod 4$, then $|A| = |B| = m/2$, $\overline{r_2} = \overline{r_1} + \overline{m/2}$ and (4.4) holds for all $\overline{k} \in \mathbb{Z}_m$ if and only if there exist two sets C, D satisfying $|C| = |D| = (m/2 - 1)/2$, $C \cup D = [\overline{0}, \overline{m/2 - 1}] \setminus \{\overline{r_1}, \overline{r_1 + m/2}\}$ and $C \cap D = \emptyset$ such that

$$A = C \cup (C + \overline{m/2}) \cup \{\overline{r_1 + m/2}\}, \quad B = D \cup (D + \overline{m/2}) \cup \{\overline{r_1 + m/2}\},$$

which implies Theorem 1.2.

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

The sufficiency of the condition in Theorem 1.3 is obvious. We prove the necessity. Assume that

$$A \cup B = \mathbb{Z}_m \setminus \{\overline{r_1}, \overline{r_2}\}, \quad A \cap B = \emptyset, \quad R_A(\overline{n}) = R_B(\overline{n}) \quad \text{for all } \overline{n} \in \mathbb{Z}_m.$$

Since $B = \mathbb{Z}_m \setminus (A \cup \{\overline{r_1}, \overline{r_2}\})$, it follows from Lemma 2.1 that

$$\begin{aligned}R_B(\overline{n}) &= R_{\mathbb{Z}_m\setminus(A\cup\{\overline{r_1},\overline{r_2}\})}(\overline{n})\\ &= m - 2|A\cup\{\overline{r_1},\overline{r_2}\}| + R_{A\cup\{\overline{r_1},\overline{r_2}\}}(\overline{n})\\ &= m - 2|A\cup\{\overline{r_1},\overline{r_2}\}| + R_A(\overline{n}) + 2\sum_{i=1}^{2}\chi_A(n-r_i)\\ &\quad + 2\theta(\overline{n}=\overline{r_1}+\overline{r_2}) + \theta(\overline{n}=2\overline{r_1}) + \theta(\overline{n}=2\overline{r_2}) \qquad (5.1)\end{aligned}$$

for all $\overline{n}\in\mathbb{Z}_m$. Since $R_A(\overline{n}) = R_B(\overline{n})$ for all $\overline{n}\in\mathbb{Z}_m$, it follows that $|A| = |B| = m/2-1$. Then by (5.1),

$$2\chi_A(n-r_1) + 2\chi_A(n-r_2) + 2\theta(\overline{n}=\overline{r_1}+\overline{r_2}) + \theta(\overline{n}=2\overline{r_1}) + \theta(\overline{n}=2\overline{r_2}) = 2 \qquad (5.2)$$

for all $\overline{n}\in\mathbb{Z}_m$.

If $2\overline{r_1}\neq 2\overline{r_2}$, then $2\chi_A(2r_2-r_1)+1 = 2$ by taking $\overline{n} = 2\overline{r_2}$ in (5.2), a contradiction. Hence $2\overline{r_1} = 2\overline{r_2}$, that is, $\overline{r_2} = \overline{r_1} + \overline{m/2}$. It follows from (5.2) that

$$\chi_A(n-r_1) + \chi_A\Big(n-\Big(r_1+\frac{m}{2}\Big)\Big) + \theta\Big(\overline{n} = 2\overline{r_1} + \overline{\frac{m}{2}}\Big) + \theta(\overline{n} = 2\overline{r_1}) = 1 \qquad (5.3)$$

for all $\overline{n}\in\mathbb{Z}_m$. Taking $n = k + (r_1 + m/2)$ in (5.3),

$$\chi_A(k) + \chi_A\Big(k+\frac{m}{2}\Big) + \theta(\overline{k} = \overline{r_1}) + \theta\Big(\overline{k} = \overline{r_1} + \overline{\frac{m}{2}}\Big) = 1$$

for all $\overline{k}\in\mathbb{Z}_m$, which implies that $B = A + \overline{m/2}$.

This completes the proof of Theorem 1.3.

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