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Surfaces of Rotation with Constant Mean Curvature in the Direction of a Unitary Normal Vector Field in a Randers Space

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Abstract. We consider the Randers space (V^n, F_b) obtained by perturbing the Euclidean metric by a translation, $F_b = \alpha + \beta$, where α is the Euclidean metric and β is a 1-form with norm b, $0 \leq b < 1$. We introduce the concept of a hypersurface with constant mean curvature in the direction of a unitary normal vector field. We obtain the ordinary differential equation that characterizes the rotational surfaces (V^3, F_b) of constant mean curvature (cmc) in the direction of a unitary normal vector field. These equations reduce to the classical equation of the rotational cmc surfaces in Euclidean space, when $b = 0$. It also reduces to the equation that characterizes the minimal rotational surfaces in (V^3, F_b) when $H = 0$, obtained by M. Souza and K. Tenenblat. Although the differential equation depends on the choice of the normal direction, we show that both equations determine the same rotational surface, up to a reflection. We also show that the round cylinders are cmc surfaces in the direction of the unitary normal field. They are generated by the constant solution of the differential equation. By considering the equation as a nonlinear dynamical system, we provide a qualitative analysis, for $0 < b < \frac{\sqrt{3}}{3}$. Using the concept of stability and considering the linearization around the single equilibrium point (the constant solution), we verify that the solutions are locally asymptotically stable spirals. This is proved by constructing a Lyapunov function for the dynamical system and by determining the basin of stability of the equilibrium point. The surfaces of rotation generated by such solutions tend asymptotically to one end of the cylinder.

Introduction

The concept of mean curvature for immersions in Finsler spaces was introduced by Z. Shen [S1] in 1998. This concept differs from the Riemannian case, since in Finsler spaces the metric depends on the point of the manifold and on the direction in the tangent space. Shen proved that the mean curvature form $\mathcal{H}_\varphi(p, W)$ for an immersion $\varphi\colon M^n \to (\widetilde{M}^m, \widetilde{F})$, of a manifold in a Finsler space $(\widetilde{M}, \widetilde{F})$ always vanishes when (p, W) is an element of the tangent bundle TM. In Riemannian geometry besides the minimal hypersurfaces, one also studies nonzero constant mean curvature hypersurfaces. But, in Finsler geometry, due to the fact that the mean curvature form always vanishes on tangent directions, one cannot have nonzero constant mean curvature form in all directions. However, the mean curvature form is linear (see [S1]). Therefore, if N is a unitary vector field normal to the submanifold, then the curvature vector in any direction W is determined by its normal component W^N and the mean curvature on the normal direction, *i.e.*, $H_\varphi(p, W) = W^N H_\varphi(p, N)$. Hence, once the

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mean curvature form is determined in the direction of a unitary normal vector field, it is determined in any other direction W. Due to these observations, we decided to study surfaces with *constant mean curvature in the direction of a unitary normal vector field*. The purpose of this paper is to study such surfaces in the Randers space obtained by perturbing the three-dimensional Euclidian space by a translation.

Namely we consider the Randers metric $F = \alpha + \beta$, where α is the euclidian metric and β is a 1-form with constant coefficients. This is a Minkowski metric, *i.e.*, it does not depend on the point. Three-dimensional spaces equipped with this metric are Randers spaces and they will be denoted by (V^3, F_b). We will consider $x \in V^3$ with coordinates x^1, x^2, x^3, so that

$$F_b(x, y) = \sqrt{\textstyle\sum_{i=1}^{3}(y^i)^2} + by^3,$$

$0 < b < 1$ and $y = \sum_{i=1}^{3} y^i \frac{\partial}{\partial x^i} \in T_x V^3$.

In this Randers space, the minimal surfaces of rotation ($\mathcal{H}_\varphi \equiv 0$) and a Bernstein type theorem for $0 \le b < \frac{\sqrt{3}}{3}$ were already studied by M. Souza and K. Tenenblat in [ST] and by Souza, Spruck, and Tenenblat in [SST], see also [Y]. Therefore, we concentrate our studies in the case $\mathcal{H}_\varphi \not\equiv 0$. The main objective of this paper is to characterize the surfaces of rotation around the axis Ox_3, with constant mean curvature in the direction of the unitary normal vector fields in Randers spaces (V^3, F_b). Such immersions are characterized by two ordinary differential equations, one for each normal vector field (due to the non-reversibility of the norm in Finsler spaces, in general, the normal vectors are not necessarily parallel). We show that, although each normal vector field gives origin to a distinct equation, they both generate the same rotational surfaces in V_3. This result is important in the sense that we can choose a single differential equation (and a single normal vector field) to accomplish the study of the possible solutions for the curves generating the rotational surfaces. We show that the round cylinders are cmc surfaces in the direction of a unitary normal vector field. They are generated by the constant solutions of the differential equation. We consider the equation as a nonlinear dynamical system and we provide its qualitative analysis, for $0 \le b < \frac{\sqrt{3}}{3}$, by introducing what we call the *surface method*. It consists of using the implicit function theorem to study the behavior of the solutions of the differential equation. It uses the relationship between the surface associated to the equation and the curves that describe the behavior of the critical points and the points of inflection of the solutions. Using the concept of stability and considering the linearization around the single equilibrium point (the constant solution), we verify that the solutions are locally asymptotically stable spirals. This is proved by constructing a Lyapunov function for the dynamical system.

1 Preliminaries

We follow the notation and terminology of [ST], and we will make use of the following conventions: we will use Greek letters γ, ϵ, η, τ for indices running from 1 to n, and Latin letters i, j, k, l for indices running from 1 to $n+1$. We will also use the Einstein convention for repeated indices.

Let M^n be a C^∞ n-dimensional manifold and let (x, y) be a point of the tangent bundle TM, $x \in M$, $y \in T_xM$. We consider local coordinates $(x^l, \dots, x^n)$ on an open subset U of M. As usual, $\frac{\partial}{\partial x^i}$ and dx^i are the induced coordinate basis for T_xM and T_x^*M and (x^i, y^i) are local coordinates on TM, where $y = y^i \frac{\partial}{\partial x^i}$. A function $F\colon TM \to [0, \infty)$ is called a *Finsler metric* on M if F has the following properties:

(i) (Regularity) $F \in C^\infty$ in $TM\backslash\{0\}$;

(ii) (Positive Homogeneity) $F(x, ty) = tF(x, y), \forall t > 0, (x, y) \in TM$;

(iii) (Strong Convexity) $g = \big(g_{ij}(x, y)\big) = \big(\frac{1}{2}[F^2(x, y)]_{y_i y_j}\big)$ is positive definite at each point of $TM\backslash\{0\}$.

The pair (M, F) is called a *Finsler space*. The strong convexity property is equivalent to saying that the symmetric bilinear form g_y on M is positive definite $\forall y \in TM\backslash\{0\}$, where

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y + su + tv)]|_{s=t=0}. \tag{1.1}$$

Denote by V^n the standard n-dimensional real vector space. A *Minkowski space* is a vector space V^n equipped with a Minkowski norm, *i.e.*, a Finsler metric $F(x, y)$ that depends only on $y \in T_xV^n$. Minkowski spaces are the simplest Finsler manifolds. A *Randers metric* on M is a Finsler structure F on TM given by

$$F(x, y) = \alpha(x, y) + \beta(x, y), \tag{1.2}$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$, a_{ij} are the components of the Riemannian metric, a^{ij} denotes the inverse matrix of a_{ij} and $\beta := z_k dx^k$ is the 1-form whose norm $b(x) = \sqrt{a^{ij}(x)z_i z_j}$ satisfies $0 \le b < 1$.

An interesting property of a Minkowski norm is that in general $F(y) \neq F(-y)$. When $F(y) = F(-y)$ we say that the norm is *reversible*. It follows from expression (1.1) and from property (iii) that

$$\begin{gathered} g_y(y, u) = \frac{1}{2}\frac{\partial}{\partial s}[F^2(y + su)]\Big|_{s=0}, \\ g_y(y, y) = F^2(y). \end{gathered} \tag{1.3}$$

One can prove (see [S2]) that if V^{n+1} is a vector space and $F\colon TV \to \mathbb{R}$ is a Randers metric given by $F = \alpha + \beta$, then there exists $\{e_i\}$, $1 \le i \le n + 1$, an orthonormal frame in the metric α, such that F has the following normal form

$$F(x, y) = \sqrt{\textstyle\sum_{i=1}^{n+1}(y^i)^2} + b(x)y^{n+1}, \quad \forall x \in V_{n+1}, \quad \text{and} \quad \forall y = y^i e_i \in T_xV^{n+1},$$

where $b(x)$ is the norm of the 1-form β given in (1.2).

If (M^n, F) is a Finsler space, then F induces a smooth volume form defined by

$$d\mu_F := \sigma_F dx^1 \wedge \cdots \wedge dx^n,$$

where

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}\{(y) \in T_xM; F(x, y) \le 1\}},$$

$\mathbb{B}^n$ is a unit ball in $\mathbb{R}^n$ and Vol is the Euclidean volume. $d\mu_F$ is the *volume element of Busemann–Hausdorff*, which owes its name to the fact that Busemann [B] proved that if F is reversible, then the volume element is the Hausdorff measure of the metric induced by F.

Let $(\widetilde{M}^m, \widetilde{F})$ be a Finsler space with local coordinates $(\tilde{x}^1, \dots, \tilde{x}^m)$ and let $\varphi\colon M^n \to \widetilde{M}^{n+1}$ be an immersion. Then there is an *induced Finsler metric* on M, defined by

$$F(x, y) = (\varphi * \widetilde{F})(x, y) = \widetilde{F}\big(\varphi(x), \varphi_*(y)\big), \quad \forall (x, y) \in TM.$$

The concept of mean curvature in Finsler spaces was introduced by Z. Shen in [S1] and it is obtained by considering a variational volume problem as follows. Let $\varphi\colon M^n \to (\widetilde{M}^n, \widetilde{F})$ be an immersion in a Finsler space and let $\varphi_t\colon M^n \to (\widetilde{M}^m, \widetilde{F})$, $t \in (-\epsilon, \epsilon)$ be a variation such that for all t, φ_t is an immersion, $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ outside a compact set $\Omega \subset M$. Then φ_t induces a family of Finsler metrics $F_t := \varphi_t^* \widetilde{F}$ and $\widetilde{X} := \frac{\partial \varphi_t}{\partial t}|_{t=0}$ is the variational vector field along φ. Let $V(t) := \int_\Omega d\mu_{F_t}$, then

$$V'(0) = \int_M \mathcal{H}_\varphi(\widetilde{X})\, d\mu_F,$$

where $\mathcal{H}_\varphi$ denotes the *mean curvature* of the immersion φ.

In [S1], Z. Shen showed that $\mathcal{H}_\varphi(v)$ depends linearly on v and $\mathcal{H}_\varphi$ vanishes on $\varphi_*(TM)$. Therefore, it follows that in Finsler spaces one cannot determine a mean curvature form that is constant and it does not vanish for all $(x, y) \in TM$. However, a minimal immersion is defined as usual, *i.e.*, the immersion φ is said to be *minimal* when $\mathcal{H}_\varphi \equiv 0$.

The proof of the following result can be found in [S2].

Proposition 1.1 ([S2]) *Let (V, F) be a Minkowski space. Given a hyperplane $\Upsilon \subset V$ there exists, in each half space determined by Υ, a single unitary vector $N \in V$ such that*

$$\Upsilon = \{w \in V : g_N(N, w) = 0\}.$$

N is called a normal vector to the hyperplane Υ.

From now on, we will consider immersed hypersurfaces $\varphi\colon M^n \to (\widetilde{V}^{n+1}, \widetilde{F}_b)$, in a special Randers space, where $\widetilde{V}$ is an $(n+1)$-dimensional real vector space, $\widetilde{F}_b = \alpha + \beta$, where α is the Euclidian metric, and β is a 1-form with norm $b \in \mathbb{R}$, $0 \leq b < 1$. Without loss of generality we will consider $\beta = b dx^{n+1}$. Let $x = (x^\epsilon)$, $\epsilon = 1, \dots, n$, be local coordinates of M^n and $\varphi(x) = \big(\varphi^i(x^\epsilon)\big) \in \widetilde{V}$, $i = 1, \dots, n+1$.

In local coordinates, the mean curvature form $\mathcal{H}_\varphi$ is given by (see [S1]),

$$\mathcal{H}_\varphi(\widetilde{X}) = \frac{1}{\mathcal{F}} \Big\{ \frac{\partial^2 \mathcal{F}}{\partial z^i_\epsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} + \frac{\partial^2 \mathcal{F}}{\partial \tilde{x}^j \partial z^i_\epsilon} \frac{\partial \varphi^j}{\partial x^\epsilon} - \frac{\partial \mathcal{F}}{\partial \tilde{x}^i} \Big\} \widetilde{X}^i, \tag{1.4}$$

where

$$\mathcal{F}(x, z) = (1 - b^2 A^{\tau\gamma} z^{n+1}_\tau z^{n+1}_\gamma)^{\frac{n+1}{2}} \sqrt{\det A}, \tag{1.5}$$

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A is the matrix whose components are given by

$$(A_{\gamma\tau}) = \Big(\sum_{i=1}^{n+1} z^i_\tau z^i_\gamma\Big), \quad z^i_\gamma = \frac{\partial \varphi^i}{\partial x^\gamma}, \tag{1.6}$$

and $(A^{\tau\gamma})$ is the inverse of A.

Observe that whenever $(\tilde{V}, \tilde{F})$ is a Minkowski space, $\widetilde{F}_b$ does not depend on x, consequently $\mathcal{F}$ also is independent of x. Therefore, the expression of the mean curvature (1.4) reduces to

$$\mathcal{H}_\varphi(W) = \frac{1}{\mathcal{F}}\Big\{ \frac{\partial^2 \mathcal{F}}{\partial z^i_\epsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} \Big\} W^i. \tag{1.7}$$

Since $\mathcal{H}_\varphi$ vanishes when applied to tangent vector fields, one cannot have constant nonzero mean curvature. Due to this restriction we introduce the concept of *constant mean curvature (cmc) form in the direction of a unitary normal vector field*. For such a hypersurface $H_\varphi(v) = v^n c$, where $c = H_\varphi(N)$ is constant and v^n is in the normal component of v. As we saw in Proposition 1.1, at each point $x \in M$, we have two normal vectors, not necessarily parallel, one in each half space determined by the hyperplane $T_x M$.

Intuition leads us to suppose that if we have two distinct normal vectors, we should obtain two distinct differential equations associated to the immersion. In fact, as we will see later, considering rotational surfaces, for each normal vector field we have a distinct differential equation. However, an important result will show that although these differential equations are distinct, they determine the same rotational surface, up to a reflection.

2 The Differential Equation for a Surface cmc in the Direction of a Unitary Normal Vector Field in (V^{n+1}, F_b)

In this section we will deduce the differential equation for immersions in the space (V^{n+1}, F_b) with constant mean curvature in a normal direction.

Let $\{e_i\}$ be an orthonormal basis of V^{n+1} in the Euclidean metric, and let F_b be given by

$$F(x,y) = \sqrt{\textstyle\sum_{i=1}^{n+1} (y^i)^2} + b y^{n+1}, \quad 0 \le b < 1, \quad y = y^i \tfrac{\partial}{\partial x^i}.$$

Theorem 2.1 *Let $\varphi\colon M^n \to (V^{n+1}, F_b)$ be an immersion. Let $\big(\varphi^i(x^\epsilon)\big)$ be coordinate functions of φ. Then the immersion is cmc in the direction of a normal unitary vector field N_ξ if and only if the following differential equation is satisfied*

(2.1)

$$\frac{1}{(1-B)^2 C}\Big\{ \frac{(n^2-1)}{4} \frac{\partial B}{\partial z^i_\epsilon} \frac{\partial B}{\partial z^j_\eta} C - \frac{(n+1)}{2}(1-B)\Big[\frac{\partial^2 B}{\partial z^i_\epsilon \partial z^j_\eta} C + \frac{\partial B}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\epsilon} + \frac{\partial B}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta}\Big]$$
$$+ (1-B)^2 \frac{\partial^2 B}{\partial z^i_\epsilon \partial z^j_\eta}\Big\} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} N^i_\xi - H = 0, \quad H \in \mathbb{R}, \quad \xi = \pm 1,$$

where $0 \leq b < 1$, $N = N^i e_i$, *A is given by* (1.6), *and*

$$C = \sqrt{\det A}, \quad B = b^2 A^{\epsilon\eta} z_\epsilon^{n+1} z_\eta^{n+1}, \quad z_\epsilon^i = \frac{\partial \varphi^i}{\partial x^\epsilon}. \tag{2.2}$$

Proof From (1.5) we have that, $\mathcal{F}(x,z) = (1 - b^2 A^{\tau\gamma} z_\tau^{n+1} z_\gamma^{n+1})^{\frac{n+1}{2}} \sqrt{\det A}$. Using the notation introduced in (2.2), we have

$$\mathcal{F}(z) = (1-B)^{\frac{n+1}{2}} C. \tag{2.3}$$

Taking the derivatives of $\mathcal{F}$ with respect to z_η^j, we obtain

$$\frac{\partial \mathcal{F}}{\partial z_\eta^j} = -\frac{(n+1)}{2}(1-B)^{\frac{n-1}{2}} \frac{\partial B}{\partial z_\eta^j} C + (1-B)^{\frac{n+1}{2}} \frac{\partial C}{\partial z_\eta^j}. \tag{2.4}$$

Taking now the derivatives of (2.4) with respect to z_ϵ^i, we have

$$\begin{aligned} \frac{\partial^2 \mathcal{F}}{\partial z_\epsilon^i \partial z_\eta^j} = {} & \frac{(n^2-1)}{4}(1-B)^{\frac{n-3}{2}} \frac{\partial B}{\partial z_\epsilon^i} \frac{\partial B}{\partial z_\eta^j} C \\ & - \frac{(n+1)}{2}(1-B)^{\frac{n-1}{2}} \Big[\frac{\partial^2 B}{\partial z_\epsilon^i \partial z_\eta^j} C + \frac{\partial B}{\partial z_\eta^j} \frac{\partial C}{\partial z_\epsilon^i} + \frac{\partial B}{\partial z_\epsilon^i} \frac{\partial C}{\partial z_\eta^j} \Big] \\ & + (1-B)^{\frac{n+1}{2}} \frac{\partial^2 C}{\partial z_\epsilon^i \partial z_\eta^j}. \end{aligned} \tag{2.5}$$

It follows from the expression of $\mathcal{H}$ given by (1.7) that

$$\frac{1}{\mathcal{F}} \frac{\partial^2 \mathcal{F}}{\partial z_\epsilon^i \partial z_\eta^j} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} N_\xi^i - H = 0.$$

Therefore, using (2.3) and (2.5), we obtain the differential equation (2.1). ∎

From now on, we will consider the dimension $n = 2$.

Theorem 2.2 *Let $\varphi\colon M^2 \to (V^3, F_b)$ be an immersion in a Randers space under the conditions of Theorem* 2.1. *Then, φ has cmc H in the direction of a normal unitary field N_ξ if and only if the following differential equation is satisfied*

(2.6)

$$\begin{aligned} \frac{1}{(C^2-E)C} \Big[& \Big(\frac{12E^2 - (2E+C^2)^2}{C(C^2-E)} \Big) \frac{\partial C}{\partial z_\epsilon^i} \frac{\partial C}{\partial z_\eta^j} - \frac{3C}{2} \frac{\partial^2 E}{\partial z_\eta^j \partial z_\epsilon^i} \\ & - \frac{3}{2} \Big(\frac{2E - C^2}{C^2 - E} \Big) \Big(\frac{\partial C}{\partial z_\epsilon^i} \frac{\partial E}{\partial z_\eta^j} + \frac{\partial C}{\partial z_\eta^j} \frac{\partial E}{\partial z_\epsilon^i} \Big) + \frac{3C}{4(C^2-E)} \frac{\partial E}{\partial z_\epsilon^i} \frac{\partial E}{\partial z_\eta^j} \\ & + \Big(\frac{2E + C^2}{2C} \Big) \frac{\partial^2 \det A}{\partial z_\eta^j \partial z_\epsilon^i} \Big] \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} N_\xi^i - H = 0, \quad \forall N = N^i e_i, \quad H \in \mathbb{R}, \end{aligned}$$

where

$$E = b^2 \sum_{k=1}^{3} [(z_2^k)^2 (z_3^1)^2 - 2 z_1^k z_2^k z_1^3 z_2^3 + (z_1^k)^2 (z_2^3)^2], \tag{2.7}$$

C is given by (2.2), *$A = A_{\epsilon\eta}(z)$, and $z = (z_\epsilon^i)$ are given by* (1.6).

Proof Considering $n = 2$ in Theorem 2.1, equation (2.1) reduces to

$$\frac{1}{(1-B)^2 C} \mathcal{G}_{ij}^{\epsilon\eta} \frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta} N_\xi^i - H = 0, \tag{2.8}$$

where

(2.9)

$$\mathcal{G}_{ij}^{\epsilon\eta} = \frac{3}{4} \frac{\partial B}{\partial z_\epsilon^i} \frac{\partial B}{\partial z_\eta^j} C - \frac{3}{2}(1-B)\Big(\frac{\partial^2 B}{\partial z_\epsilon^i \partial z_\eta^j} C + \frac{\partial B}{\partial z_\eta^j} \frac{\partial C}{\partial z_\epsilon^i} + \frac{\partial B}{\partial z_\epsilon^i} \frac{\partial C}{\partial z_\eta^j} \Big) + (1-B)^2 \frac{\partial^2 C}{\partial z_\epsilon^i \partial z_\eta^j}.$$

Moreover, the inverse matrix of $A_{\epsilon\eta}$ is given by

$$(A^{\epsilon\eta}) = \frac{1}{\det A} \begin{pmatrix} \sum_{i=1}^3 z_2^i z_2^i & -\sum_{i=1}^3 z_1^i z_2^i \\ -\sum_{i=1}^3 z_1^i z_2^i & \sum_{i=1}^3 z_1^i z_1^i \end{pmatrix}. \tag{2.10}$$

It follows from (2.2), (2.7) and (2.10) that

$$B = \frac{1}{C^2} E. \tag{2.11}$$

We also have from (2.3) that $\mathcal{F} = (1-B)^{\frac{3}{2}} C$.

We will now compute separately the components of the expression (2.9). Taking both first and second order derivatives of B with respect to z_ϵ^i, and using (2.11) we obtain

$$\frac{\partial B}{\partial z_\epsilon^i} = \frac{1}{C^2} \frac{\partial E}{\partial z_\epsilon^i} - \frac{2}{C} \frac{\partial C}{\partial z_\epsilon^i} B \tag{2.12}$$

and

(2.13)

$$\frac{\partial^2 B}{\partial z_\epsilon^i \partial z_\eta^j} = \frac{1}{C^2} \Big[-\frac{2E}{C} \frac{\partial^2 C}{\partial z_\eta^j \partial z_\epsilon^i} + \frac{6E}{C^2} \frac{\partial C}{\partial z_\eta^j} \frac{\partial C}{\partial z_\epsilon^i} - \frac{2}{C} \Big(\frac{\partial C}{\partial z_\epsilon^i} \frac{\partial E}{\partial z_\eta^j} + \frac{\partial C}{\partial z_\eta^j} \frac{\partial E}{\partial z_\epsilon^i} \Big) + \frac{\partial^2 E}{\partial z_\eta^j \partial z_\epsilon^i} \Big].$$

The derivative of C with respect to z_ϵ^i is given by

$$\frac{\partial C}{\partial z_\epsilon^i} = \frac{1}{2C} \frac{\partial C^2}{\partial z_\epsilon^i} = \frac{1}{2C} \frac{\partial \det A}{\partial z_\epsilon^i}, \tag{2.14}$$

where we used (2.2) and (2.11). Therefore, from (2.14), we have that

$$\frac{\partial^2 C}{\partial z_\eta^j z_\epsilon^i} = \frac{1}{C} \Big[\frac{1}{2} \frac{\partial^2 \det A}{\partial z_\eta^j \partial z_\epsilon^i} - \frac{\partial C}{\partial z_\eta^j} \frac{\partial C}{\partial z_\epsilon^i} \Big]. \tag{2.15}$$

Substituting (2.12) and (2.13) into (2.9) and observing from (2.11) that $(1-B)/C = (C^2-E)/C^3$, we have

$$\begin{aligned}\mathcal{G}_{ij}^{\epsilon\eta} = \Big(\frac{C^2-E}{C^4}\Big)\Big[\frac{3E}{C}\Big(\frac{-C^2+2E}{C^2-E}\Big)\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial C}{\partial z_\eta^j} \\ -\frac{3}{2}\Big(\frac{-C^2+2E}{C^2-E}\Big)\Big(\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j}+\frac{\partial C}{\partial z_\eta^j}\frac{\partial E}{\partial z_\epsilon^i}\Big) \\ -\frac{3C}{2}\frac{\partial^2 E}{\partial z_\eta^j\partial z_\epsilon^i}+\frac{3C}{4(C^2-E)}\frac{\partial E}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j}+(C^2+2E)\frac{\partial^2 C}{\partial z_\epsilon^i\partial z_\eta^j}\Big].\end{aligned}$$

It follows from (2.15) that

$$\begin{aligned}\mathcal{G}_{ij}^{\epsilon\eta} = \Big(\frac{C^2-E}{C^4}\Big)\Big\{\frac{12E^2-(2E+C^2)^2}{C(C^2-E)}\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial C}{\partial z_\eta^j}-\frac{3C}{2}\frac{\partial^2 E}{\partial z_\eta^j\partial z_\epsilon^i} \\ -\frac{3}{2}\Big(\frac{2E-C^2}{C^2-E}\Big)\Big(\frac{\partial C}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j}+\frac{\partial C}{\partial z_\eta^j}\frac{\partial E}{\partial z_\epsilon^i}\Big) \\ +\frac{3C}{4(C^2-E)}\frac{\partial E}{\partial z_\epsilon^i}\frac{\partial E}{\partial z_\eta^j}+\frac{(2E+C^2)}{2C}\frac{\partial^2 \det A}{\partial z_\eta^j\partial z_\epsilon^i}\Big\}.\end{aligned} \tag{2.16}$$

Substituting this expression into (2.8), we obtain (2.6). This concludes the proof of the theorem. ■

Next we will compute the derivatives of C and E in terms of the z_η^j's for subsequent use. We have from (2.2) and (1.6) that

$$\det A = \sum_{k\neq l}(z_1^k)^2(z_2^l)^2 - \sum_{k\neq l} z_1^k z_2^k z_1^l z_2^l \quad \text{and} \quad C = \sqrt{\det A}.$$

We can write the derivative of $\det A$ with respect to z_ϵ^i as follows:

$$\begin{aligned}\frac{\partial \det A}{\partial z_\epsilon^i} = \frac{1}{2C}\Big[\sum_{k\neq l}\big(\delta_{ik}\delta_{\epsilon 1}2z_1^k(z_2^l)^2+\delta_{il}\delta_{\epsilon 2}2(z_1^k)^2z_2^l-\delta_{ik}(\delta_{\epsilon 1}z_2^kz_1^lz_2^l+\delta_{\epsilon 2}z_1^kz_1^lz_2^l)\big) \\ -\sum_{k\neq l}\delta_{il}(\delta_{\epsilon 1}z_1^kz_2^kz_2^l+\delta_{\epsilon 2}z_1^kz_2^kz_1^l)\Big].\end{aligned}$$

Hence the derivative of C with respect to z_ϵ^i reduces to

$$\frac{\partial C}{\partial z_\epsilon^i} = \frac{1}{C}\sum_{l\neq i}(z_1^iz_2^l - z_2^iz_1^l)(\delta_{\epsilon 1}z_2^l - \delta_{\epsilon 2}z_1^l). \tag{2.17}$$

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Therefore,

(2.18)
$$\frac{\partial^2 \det A}{\partial z^i_\epsilon \partial z^j_\eta} = \sum_{l\neq i}\Big\{ \delta_{\epsilon 1}\big(\delta_{\eta 1}\delta_{ji}(z^l_2)^2 + 2z^i_1\delta_{\eta 2}\delta_{jl}z^l_2\big) + \delta_{\epsilon 2}\big(\delta_{\eta 2}\delta_{ji}(z^l_1)^2 + 2z^i_2\delta_{\eta 1}\delta_{jl}z^l_1\big)$$
$$- \delta_{\epsilon 1}[\delta_{\eta 2}\delta_{ij}z^l_1z^l_2 + z^i_2(\delta_{\eta 1}\delta_{jl}z^l_2 + \delta_{\eta 2}\delta_{jl}z^l_1)]$$
$$- \delta_{\epsilon 2}[\delta_{\eta 1}\delta_{ji}z^l_1z^l_2 + z^i_1(z^l_2\delta_{\eta 1}\delta_{jl} + z^l_1\delta_{\eta 2}\delta_{jl})]\Big\}.$$

Moreover, it follows from (2.7) that

$$\frac{\partial E}{\partial z^i_\epsilon} = 2b^2\Big[z^3_{\tilde\epsilon}(-1)^{\tilde\epsilon+\tau}z^i_{\tilde\tau}z^3_\tau + \delta_{i3}\sum_{k=1}^{3}(-1)^{\epsilon+\tau}z^3_\tau z^k_{\tilde\tau}z^k_{\tilde\epsilon}\Big],$$

where $\tilde\epsilon = 1 + \delta_{\epsilon 1}$, $\epsilon = 1, 2$. Taking the derivative with respect to z^j_η, we have

(2.19)
$$\frac{\partial^2 E}{\partial z^j_\eta \partial z^i_\epsilon} = 2b^2\Big[\delta_{j3}\delta_{\eta\tilde\epsilon}(-1)^{\tilde\epsilon+\tau}z^i_{\tilde\tau}z^3_\tau + z^3_{\tilde\epsilon}\big((-1)^{\tilde\epsilon+\tilde\eta}\delta_{ji}z^3_{\tilde\eta} + (-1)^{\tilde\epsilon+\eta}\delta_{j3}z^i_{\tilde\eta}\big)$$
$$+ \delta_{i3}\sum_{k=1}^{3}(-1)^{\epsilon+\eta}\delta_{j3}z^k_{\tilde\eta}z^k_{\tilde\epsilon} + \delta_{i3}(-1)^{\epsilon+\tilde\eta}z^3_{\tilde\eta}z^j_{\tilde\epsilon} + \delta_{i3}(-1)^{\epsilon+\tau}\delta_{\eta\tilde\epsilon}z^3_\tau z^j_{\tilde\tau}\Big].$$

These expressions will be used in next section.

3 Rotational Surfaces with cmc in the Direction of a Unitary Normal Vector Field in (V^3, F_b)

In this section we will determine the unitary vector fields, normal to an immersion in the Randers space (V^3, F_b) and obtain two ordinary differential equations, which describe rotational surfaces with constant mean curvature H in the direction of these vector fields. It is well known from the classical theory of surfaces (corresponding to $b = 0$) that such equations reduce to the ordinary differential equation that describes the Delaunay surfaces (cylinder, sphere, onduloids and nodoids).

Let (V^3, F_b) be the Randers space with $F_b = \alpha + \beta$, where α is the Euclidian metric perturbed by a translation β of norm b. Let $\varphi\colon M^2 \to (V^3, F_b)$ be given by

$$\varphi(t,\theta) = \big(g(t)\cos\theta, g(t)\sin\theta, t\big), \quad t \in \mathbb{R}, \theta \in [0, 2\pi], \tag{3.1}$$

where $g(t)$ is a nonvanishing differentiable function. It follows from Proposition 1.1, that there exists a unique unitary normal vector N in each half space determined by a plane tangent to M^2, *i.e.*,

$$g_N(N, w) = 0, \quad \forall w \in TM.$$

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In order to obtain the normal vectors, it is sufficient to find the solutions $N = (N^1, N^2, N^3)$ of the following system,

$$\begin{cases} g_N(N, \varphi_t) = 0, \\ g_N(N, \varphi_\theta) = 0, \\ g_N(N, N) = 1, \end{cases} \tag{3.2}$$

where φ_t and φ_θ denote the derivatives of the immersion $\varphi(t, \theta)$ with respect to t and θ respectively.

From (1.3), we have

$$g_N(N, v) = \frac{\partial}{\partial s} F(N + sv)|_{s=0}, \quad \forall v \in V^3.$$

Hence,

$$g_N(N, v) = \frac{\sum_{i=1}^3 N^i v^i}{\sqrt{\sum_{i=1}^3 (N^i)^2}} + bv^3. \tag{3.3}$$

Since $F(N) = 1$, it follows from (3.1) and (3.3) that

$$g_N(N, \varphi_t) = \frac{N^1 g' \cos\theta + N^2 g' \sin\theta + N^3}{\sqrt{(N^1)^2 + (N^2)^2 + (N^3)^2}} + b \tag{3.4}$$

and

$$g_N(N, \varphi_\theta) = \frac{N^2 g \cos\theta - N^1 g \sin\theta}{\sqrt{(N^1)^2 + (N^2)^2 + (N^3)^2}}. \tag{3.5}$$

From (1.3), we have

$$g_N(N, N) = F^2(N) = 1. \tag{3.6}$$

Proposition 3.1 *The unit vectors normal to the tangent planes of M, in the Randers metric in (V^3, F_b), are given by*

$$N_\xi = \frac{1}{\sqrt{1 - b^2 + (g')^2}} \left(-\xi \cos\theta, -\xi \sin\theta, \frac{-b\sqrt{1 - b^2 + (g')^2} + \xi g'}{1 - b^2} \right), \tag{3.7}$$

where $\xi = \pm 1$.

Proof We will solve the two first equations of (3.2). Using (3.4), (3.5) and the fact that $g(t) \neq 0$, it follows that

$$\begin{cases} N^2 \cos\theta - N^1 \sin\theta = 0, \\ N^1 g' \cos\theta + N^2 g' \sin\theta + N^3(1 - b^2) + b = 0. \end{cases} \tag{3.8}$$

If $g'(t) \neq 0$ we have that

$$N^1 = -\frac{\cos\theta}{g'}\big(N^3(1-b^2)+b\big) \quad \text{and} \quad N^2 = -\frac{\sin\theta}{g'}\big(N^3(1-b^2)+b\big). \tag{3.9}$$

It follows from (3.6) and (3.9) that

$$N^3 = \frac{-b\sqrt{1-b^2+(g')^2}+\xi g'}{(1-b^2)\sqrt{1-b^2+(g')^2}}, \quad \xi = \pm 1. \tag{3.10}$$

Therefore N^1 and N^2 are determined by (3.9) and hence,

$$N^1 = -\frac{\xi\cos\theta}{\sqrt{1-b^2+(g')^2}} \quad \text{and} \quad N^2 = -\frac{\xi\sin\theta}{\sqrt{1-b^2+(g')^2}}, \quad \xi = \pm 1. \tag{3.11}$$

When $g' = 0$, it follows from (3.8) that

$$N^1 = \frac{\chi\cos\theta}{\sqrt{1-b^2}}, \quad N^2 = \frac{\chi\sin\theta}{\sqrt{1-b^2}}, \quad N^3 = -\frac{b}{1-b^2}, \quad \chi = \pm 1. \tag{3.12}$$

Observe that we can choose the sign of N^1 and N^2 in (3.12). Since we need differentiable vector fields, we choose $\chi = -\xi$. This provides us the unit normal vector fields given in (3.7). ■

From now on, we will consider rotational surfaces with cmc H in the direction of a unitary normal vector field N_ξ in (V^3, F_b). We will prove that there exist two ordinary differential equations, whose solutions characterize these surfaces.

Theorem 3.2 *Consider the Randers space (V^3, F_b). Let $\varphi\colon M^2 \to V^3$ be an immersion given by $\varphi(t,\theta) = \big(g(t)\cos\theta, g(t)\sin\theta, t\big)$ such that $\forall t$, $g(t) \neq 0$. Then φ has constant mean curvature H in the direction of the unitary normal vector field N_ξ, if and only if g satisfies the following ordinary differential equation:*

$$\begin{aligned}\Big\{ -gg''\big[w_b\big(1+2b^2+(1-3b^2)(g')^2\big) + 3b^4(g')^2\big] \\ + w_0 w_b\big(1-b^2+(1-3b^2)(g')^2\big)\Big\}(-\xi w_b - bg'\sqrt{w_b}) \\ - Hg(1-b^2)w_0^2(w_b)^{\frac{5}{2}} = 0, \quad H \in \mathbb{R},\end{aligned} \tag{3.13}$$

where

$$\xi = \pm 1, \quad w_0 = 1+(g')^2 \quad \text{and} \quad w_b = 1-b^2+(g')^2. \tag{3.14}$$

In order to prove Theorem 3.2, we will need the following lemma.

Lemma 3.3 *Let $\varphi\colon M^2 \to (V^3, F_b)$ be an immersion given by*

$$\varphi(t,\theta) = \big(g(t)\cos\theta, g(t)\sin\theta, t\big), \quad g(t) \neq 0\ \forall t.$$

Then the following equalities hold:

$$\frac{\partial C}{\partial z^i_\epsilon} N^i_\xi = \frac{\lambda\xi\delta_{\epsilon 1} g}{\sqrt{w_0}\sqrt{w_b}}[-\xi g' + Z], \tag{3.15}$$

$$\frac{\partial C}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon \partial x^\eta} = \frac{\lambda\xi g'}{\sqrt{w_0}}\delta_{\epsilon 1}(gg'' + w_0), \tag{3.16}$$

$$\frac{\partial E}{\partial z^i_\epsilon} N^i_\xi = \frac{2b^2g^2}{\sqrt{w_b}}\delta_{\epsilon 1} Z, \tag{3.17}$$

$$\frac{\partial E}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon \partial x^\eta} = 2b^2 g g'\delta_{\epsilon 1}, \tag{3.18}$$

$$\frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon \partial x^\eta} = 2b^2 g(\delta_{i3}2g' - \delta_{i1}\cos\theta - \delta_{i2}\sin\theta), \tag{3.19}$$

where A, C and E are given by (1.6), (2.2) *and* (2.7), *respectively, N^i, $i = 1,2,3$, are given by* (3.10) *and* (3.11), $\lambda = \operatorname{sgn}(g)$ *and Z is given by*

$$Z = \frac{-b\sqrt{w_b} + \xi g'}{1-b^2}. \tag{3.20}$$

Proof By considering $x_1 = t$ and $x_2 = \theta$ in (2.2), we have

(3.21)
$$\begin{aligned} z^i_\epsilon &= \delta_{\epsilon 1}(\delta_{i1}z^1_1 + \delta_{i2}z^2_1 + \delta_{i3}z^3_1) + \delta_{\epsilon 2}(\delta_{i1}z^1_2 + \delta_{i2}z^2_2 + \delta_{i3}z^3_2) \\ &= \delta_{\epsilon 1}\big(\delta_{i1}g'(t)\cos\theta + \delta_{i2}g'(t)\sin\theta + \delta_{i3}\big) + \delta_{\epsilon 2}\big(-\delta_{i1}g(t)\sin\theta + \delta_{i2}g(t)\cos\theta\big). \end{aligned}$$

Using the same notation, we will write the second order derivative of the immersion as

$$\begin{aligned} \frac{\partial^2\varphi^j}{\partial x^\epsilon \partial x^\eta} &= (\delta_{\epsilon 1}\delta_{\eta 2} + \delta_{\epsilon 2}\delta_{\eta 1})g'(t)(-\delta_{j1}\sin\theta + \delta_{j2}\cos\theta) \\ &\quad + \big(\delta_{\epsilon 1}\delta_{\eta 1}g''(t) - \delta_{\epsilon 2}\delta_{\eta 2}g(t)\big)(\delta_{j1}\cos\theta + \delta_{j2}\sin\theta). \end{aligned} \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$z^3_\epsilon = \delta_{\epsilon 1} \quad\text{and}\quad \frac{\partial^2\varphi^3}{\partial x^\epsilon \partial x^\eta} = 0, \quad \forall \epsilon, \eta. \tag{3.23}$$

From now on we will omit the parameter t whenever there is no possibility of ambiguity. It follows from (1.6), (2.2), (2.7), (3.21) and (3.23), that

$$A = \begin{pmatrix} 1 + (g')^2 & 0 \\ 0 & g^2 \end{pmatrix}, \quad C = |g|\sqrt{1 + (g')^2}, \quad E = b^2g^2. \tag{3.24}$$

Using (2.17), it follows from (3.21) that $\forall i$, $1 \le i \le 3$, and $\epsilon = 1, 2$, we have

$$\frac{\partial C}{\partial z^i_\epsilon} = \frac{\lambda}{\sqrt{1+(g')^2}}\Big\{\delta_{\epsilon 1} g[g'(\delta_{i1}\cos\theta + \delta_{i2}\sin\theta) + \delta_{i3}] + \delta_{\epsilon 2}\big(1+(g')^2\big)(-\delta_{i1}\sin\theta + \delta_{i2}\cos\theta)\Big\}, \tag{3.25}$$

where $\lambda = \mathrm{sgn}(g)$.

We will rewrite the components of the normal vector field N_ξ given by (3.7) in the following way,

$$N^i_\xi = \frac{1}{\sqrt{w_b}}(-\delta_{i1}\xi\cos\theta - \delta_{i2}\xi\sin\theta + \delta_{i3}Z), \tag{3.26}$$

where w_b and Z are given by (3.14) and (3.20), respectively. Multiplying (3.25) by (3.26) and adding in i, we conclude, using (3.14), that (3.15) is verified. Equation (3.16) is obtained from (3.22) and (3.25), by adding in j and η. It follows from (2.7) that the derivatives of E are given by

$$\frac{\partial E}{\partial z^i_\epsilon} = 2b^2 g(\delta_{\epsilon 1}\delta_{i3} g - \delta_{\epsilon 2}\delta_{i1}\sin\theta + \delta_{\epsilon 2}\delta_{i2}\cos\theta). \tag{3.27}$$

Multiplying (3.27) by N^i_ξ and adding in i, we obtain equation (3.17). Substituting i by j and ϵ by η in the equation (3.27), multiplying by (3.22) and adding in j and η, we obtain equation (3.18).

We observe that (2.19) gives us the second order derivatives of E for any immersion. Therefore, considering the immersion $\varphi(t,\theta)$ and using (3.21), we obtain

$$\begin{aligned}\frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta} &= 2b^2\Big[\delta_{j3}\delta_{\eta\widetilde{\epsilon}}(-1)^{1+\widetilde{\epsilon}} g(-\delta_{i1}\sin\theta + \delta_{i2}\cos\theta)\\ &\quad + \delta_{\widetilde{\epsilon}1}\big((-1)^{1+\widetilde{\eta}}\delta_{ji}\delta_{\widetilde{\eta}1} + (-1)^{1+\eta}\delta_{j3} z^i_{\widetilde{\eta}}\big)\\ &\quad + \delta_{i3}\Big(\sum_{k=1}^{3}(-1)^{\eta+\epsilon}\delta_{j3} z^k_{\widetilde{\eta}} z^k_{\widetilde{\epsilon}}\\ &\qquad + (-1)^{\epsilon+\widetilde{\eta}}\delta_{\widetilde{\eta}1}[\delta_{j1}(\delta_{\widetilde{\epsilon}1} g'\cos\theta - \delta_{\widetilde{\epsilon}2} g\sin\theta)\\ &\qquad + \delta_{j2}(\delta_{\widetilde{\epsilon}1} g'\sin\theta + \delta_{\widetilde{\epsilon}2} g\cos\theta)]\\ &\qquad + (-1)^{\epsilon+1} g\delta_{\eta\widetilde{\epsilon}}(-\delta_{j1}\sin\theta + \delta_{j2}\cos\theta)\Big)\Big],\end{aligned} \tag{3.28}$$

where $\widetilde{\epsilon} = 1 + \delta_{\epsilon 1}$, $\epsilon = 1, 2$.

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Multiplying (3.28) by (3.22) and adding in ϵ, η and j, we obtain

$$\begin{aligned}
&\frac{\partial^2 E}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta}\\
&\quad = 2b^2\Big\{\,\delta_{\epsilon 2}(-1)^{1+\widetilde{\eta}}\delta_{ij}\delta_{\eta 2} + \delta_{i3}(-1)^{\epsilon+\widetilde{\eta}}\big[\,\delta_{\eta 2}\big(\,\delta_{j1}(\delta_{\epsilon 2}g'\cos\theta + \delta_{\epsilon 1}g\sin\theta)\\
&\qquad\quad + \delta_{j2}(\delta_{\epsilon 2}g'\sin\theta + \delta_{\epsilon 1}g\cos\theta)\big) + (-1)^{1+\epsilon}g\delta_{\widetilde{\epsilon}\eta}(-\delta_{j1}\sin\theta\\
&\qquad\quad + \delta_{j2}\cos\theta)\big]\,\Big\}\,\Big\{\,(\delta_{\epsilon 1}\delta_{\eta 2} + \delta_{\epsilon 2}\delta_{\eta 1})g'(t)(-\delta_{j1}\sin\theta + \delta_{j2}\cos\theta)\\
&\qquad\quad + \big(\,\delta_{\epsilon 1}\delta_{\eta 1}g''(t) - \delta_{\epsilon 2}\delta_{\eta 2}g(t)\big)\,(\delta_{j1}\cos\theta + \delta_{j2}\sin\theta)\Big\}\\
&\quad = \delta_{i3}4b^2gg' - \delta_{i1}2b^2g\cos\theta - \delta_{i2}2b^2g\sin\theta\\
&\quad = 2b^2g(\delta_{i3}2g' - \delta_{i1}\cos\theta - \delta_{i2}\sin\theta).
\end{aligned}$$

This proves (3.19) and concludes the proof of Lemma 3.3. ∎

Lemma 3.4 *Let $\varphi\colon M^2 \to (V^3, F_b)$ be an immersion as in Lemma 3.3. Then*

$$\text{(3.29)}\quad \frac{\partial^2 \det A}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta}N^i_\xi = \frac{2g}{\sqrt{w_b}(1-b^2)}[\xi(1-b^2)(1-gg'') + \xi(g')^2(1+b^2) - 2bg'\sqrt{w_b}],$$

where A and N^i_ξ are given by (3.24) and (3.26) respectively.

Proof Equation (2.18) gives us the second order derivatives of $\det A$ with respect to z^i_ϵ. We will compute separately each term of this expression. Before that, we note that we do not need to compute the terms involving δ_{j3}, since they will be multiplied by zero when we consider the product $\frac{\partial^2 \det A}{\partial z^i_\epsilon \partial z^j_\eta}\frac{\partial^2 \varphi^j}{\partial x^\epsilon \partial x^\eta}$ (see (3.23)). In the next expressions $R(\delta_{j3})$ will denote a term involving δ_{j3}.

It follows from the expression of $z^i_\epsilon = \frac{\partial \varphi^i}{\partial x^\epsilon}$ given by (3.21) that $\forall i, j$, $1 \le i, j \le 3$,

$$\text{(3.30)}\qquad \sum_{l\neq i}\delta_{ij}(z^l_2)^2 = g^2(\delta_{i1}\delta_{j1}\cos^2\theta + \delta_{i2}\delta_{j2}\sin^2\theta) + R_1(\delta_{j3}).$$

Moreover,

$$\text{(3.31)}\qquad \sum_{l\neq i}\delta_{ij}(z^l_1)^2 = (g')^2(\delta_{i1}\delta_{j1}\sin^2\theta + \delta_{i2}\delta_{j2}\cos^2\theta) + (\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + R_2(\delta_{j3}),$$

$$\text{(3.32)}\quad \sum_{l\neq i}\delta_{jl}z^i_1z^l_2 = g[g'(\delta_{i1}\delta_{j2}\cos^2\theta - \delta_{i2}\delta_{j1}\sin^2\theta) + \delta_{i3}(-\delta_{j1}\sin\theta + \delta_{j2}\cos\theta)] + R_3(\delta_{j3}).$$

Similarly,

$$-\sum_{l\neq i}\delta_{jl}z_2^i z_1^l = -gg'(-\delta_{i1}\delta_{j2}\sin^2\theta+\delta_{i2}\delta_{j1}\cos^2\theta)+R_4(\delta_{j3}), \tag{3.33}$$

$$-\sum_{l\neq i}\delta_{ij}z_1^l z_2^l = -gg'\sin\theta\cos\theta(\delta_{i1}\delta_{j1}-\delta_{i2}\delta_{j2})+R_5(\delta_{j3}). \tag{3.34}$$

Moreover,

$$\sum_{l\neq i}\delta_{jl}z_1^i z_1^l = (g')^2\sin\theta\cos\theta(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})+g'\delta_{i3}(\delta_{j1}\cos\theta+\delta_{j2}\sin\theta)+R_6(\delta_{j3}). \tag{3.35}$$

Finally,

$$-\sum_{l\neq i}\delta_{jl}z_2^i z_2^l = g^2[(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})\sin\theta\cos\theta]+R_7(\delta_{j3}). \tag{3.36}$$

Therefore, it follows from equations (3.30)–(3.36) that,

$$\begin{aligned}
&\frac{\partial^2\det A}{\partial z_\epsilon^i\partial z_\eta^j}\\
&=\delta_{\epsilon1}\Big\{\delta_{\eta1}[g^2(\delta_{i1}\delta_{j1}\cos^2\theta+\delta_{i2}\delta_{j2}\sin^2\theta)+(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})g^2\sin\theta\cos\theta]\\
&\qquad+\delta_{\eta2}\big[2g\big(g'(\delta_{i1}\delta_{j2}\cos^2\theta-\delta_{i2}\delta_{j1}\sin^2\theta)+\delta_{i3}(-\delta_{j1}\sin\theta+\delta_{j2}\cos\theta)\big)\\
&\qquad\qquad-gg'\sin\theta\cos\theta(\delta_{i1}\delta_{j1}-\delta_{i2}\delta_{j2})\\
&\qquad\qquad-gg'(-\delta_{i1}\delta_{j2}\sin^2\theta+\delta_{i2}\delta_{j1}\cos^2\theta)\big]\Big\}\\
&\qquad+\delta_{\epsilon2}\Big\{\delta_{\eta1}\Big[-gg'\sin\theta\cos\theta(\delta_{i1}\delta_{j1}-\delta_{i2}\delta_{j2})\\
&\qquad\qquad\quad+2gg'(-\delta_{i1}\delta_{j2}\sin^2\theta+\delta_{i2}\delta_{j1}\cos^2\theta)\\
&\qquad\qquad\quad-g[g'(\delta_{i1}\delta_{j2}\cos^2\theta-\delta_{i2}\delta_{j1}\sin^2\theta)\\
&\qquad\qquad\quad+\delta_{i3}(-\delta_{j1}\sin\theta+\delta_{j2}\cos\theta)]\Big]\\
&\qquad+\delta_{\eta2}\Big[-g'\sin\theta\cos\theta[g'(\delta_{j1}\delta_{i1}-\delta_{i2}\delta_{j1})+\delta_{i3}(\delta_{j1}\cos\theta+\delta_{j2}\sin\theta)]\\
&\qquad\quad+\Big(\delta_{i1}\delta_{j1}\big(1+(g')^2\sin^2\theta\big)+\delta_{i2}\delta_{j2}\big(1+(g')^2\cos^2\theta\big)\Big)\Big]\Big\}+R(\delta_{j3}).
\end{aligned} \tag{3.37}$$

Multiplying equation (3.37) by $\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}$ and N_ξ^i, given by (3.22) and (3.26) respectively, and adding in all the indices, we obtain (3.29). ∎

We will now use Lemmas 3.3 and 3.4 to prove Theorem 3.2.

Proof of Theorem 3.2 For the proof, in view of equation (2.6), we will obtain the following expressions:

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(i) $\dfrac{\partial C}{\partial z^i_\epsilon}\dfrac{\partial C}{\partial z^j_\eta}\dfrac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}N^i_\xi = \dfrac{gg'b}{(1-b^2)w_0\sqrt{w_b}}[(gg''+w_0)(\xi bg'-\sqrt{w_b})]$;

(ii) $\dfrac{\partial E}{\partial z^i_\epsilon}\dfrac{\partial E}{\partial z^j_\eta}\dfrac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}N^i_\xi = \dfrac{4b^4g^3g'}{\sqrt{w_b}(1-b^2)}(-b\sqrt{w_b}+\xi g')$;

(iii) $\Big(\dfrac{\partial C}{\partial z^i_\epsilon}\dfrac{\partial E}{\partial z^j_\eta}+\dfrac{\partial C}{\partial z^j_\eta}\dfrac{\partial E}{\partial z^i_\epsilon}\Big)\dfrac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}N^i_\xi$

$$= \lambda\frac{2b^2g^2g'}{(1-b^2)\sqrt{w_b}\sqrt{w_0}}[g'(w_0+gg''+b^2)-b\xi\sqrt{w_b}(1+g''+w_0)];$$

(iv) $\dfrac{\partial^2 E}{\partial z^j_\eta\partial z^i_\epsilon}\dfrac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}N^i_\xi = \dfrac{2b^2g}{\sqrt{w_b}(1-b^2)}\big[\xi\big(1+2(g')^2-b^2\big)-2bg'\sqrt{w_b}\big]$.

In fact, from (3.15) and (3.16), we have that

$$\frac{\partial C}{\partial z^i_\epsilon}\frac{\partial C}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial x^\epsilon\partial x^\eta}N^i_\xi = \Big[\frac{\lambda\xi\delta_{\epsilon1}g}{\sqrt{w_0}\sqrt{w_b}}(-\xi g'+Z)\Big]\Big[\frac{\lambda\xi g'}{\sqrt{w_0}}\delta_{\epsilon1}(gg''+w_0)\Big]. \tag{3.38}$$

Therefore, a simple computation, using (3.20), gives us (i) (observe that $\lambda^2 = 1$). (ii) follows directly from the product of (3.17) by (3.18), using (3.20). (iii) follows directly from the product of the equations (3.15), (3.18), (3.16) and (3.17). And finally, to prove (iv) we multiply equation (3.19) by N^i_ξ, given by (3.26).

A straightforward computation shows that the coefficients that appear in equation (3.13) are given by the following equalities:

$$\frac{12E^2-(2E+C^2)^2}{C(C^2-E)} = \frac{\lambda g(8b^4-4b^2w_0-w_0^2)}{\sqrt{w_0}w_b}, \tag{3.39}$$

$$\frac{3C}{2} = \frac{3|g|}{2}\sqrt{w_0}, \tag{3.40}$$

$$-\frac{3}{2}\Big(\frac{2E-C^2}{C^2-E}\Big) = -\frac{3}{2}\Big(\frac{2b^2-w_0}{w_b}\Big), \tag{3.41}$$

$$\frac{3C}{4(C^2-E)} = \lambda\frac{3}{4}\frac{\sqrt{w_0}}{g\sqrt{w_b}}, \tag{3.42}$$

$$\frac{2E+C^2}{2C} = \lambda g\frac{(2b^2+w_0)}{2\sqrt{w_0}}, \tag{3.43}$$

$$\frac{1}{C(C^2-E)} = \frac{1}{g^3\sqrt{w_0}w_b}, \tag{3.44}$$

where $\lambda = \frac{g}{|g|} = \mathrm{sgn}(g)$, C and E are given by (3.24).

Substituting equalities (i)–(iv), equation (3.29) and equations (3.39)–(3.44) into

(2.6), we obtain

$$
\begin{aligned}
H = {} & \frac{1}{|g|g^2\sqrt{w_0}w_b} \\
& \cdot \Big\{ \frac{\lambda g(8b^4 - 4b^2 w_0 - w_0^2)}{\sqrt{w_0}w^b} \Big[\frac{gg'b}{(1-b^2)w_0\sqrt{w_b}} (gg'' + w_0)(\xi b g' - \sqrt{w_b}) \Big] \\
& \quad - \frac{3|g|}{2}\sqrt{w_0}\frac{2b^2 g}{\sqrt{w_b}(1-b^2)} \big[\xi\big(1 + 2(g')^2 - b^2\big) - 2bg'\sqrt{w_b} \big] \\
& \quad - \frac{3}{2}\Big(\frac{2b^2 - w_0}{w_b}\Big)\lambda \frac{2b^2 g^2 g'}{(1-b^2)\sqrt{w_b}\sqrt{w_0}} \\
& \quad \cdot [g'(w_0 + gg'' + b^2) - b\xi\sqrt{w_b}(1 + g'' + w_0)] \\
& \quad + \lambda \frac{3}{4}\frac{\sqrt{w_0}}{g\sqrt{w_b}} \Big[\frac{4b^4 g^3 g'}{\sqrt{w_b}(1-b^2)}(-b\sqrt{w_b} + \xi g') \Big] \\
& \quad + \Big(\frac{\lambda g(1 + 2b^2 + (g')^2)}{2\sqrt{w_0}} \Big) \frac{2g}{\sqrt{w_b}(1-b^2)} \\
& \quad \cdot [\xi(1-b^2)(1-gg'') + \xi(g')^2(1+b^2) - 2bg'\sqrt{w_b}] \Big\}.
\end{aligned}
$$

Simplifying this expression, we obtain (3.13). This concludes the proof of Theorem 3.2. ∎

It follows from Theorem 3.2 that we have two distinct differential equations, depending on choice of ξ in equation (3.13), *i.e.*, depending on the normal vector field. This is in contrast to the Euclidian case, $b = 0$, when we have just one equation.

4 Characterization of Rotational Surfaces with cmc in the Direction of the Unitary Normal Vector Fields in (V^3, F_b)

In this section we show that, in spite of finding two distinct differential equations that characterize the surfaces of rotation with cmc in the direction of the unitary normal vector fields in (V^3, F_b), both equations determine the same rotational surface. This allows us to choose only one equation to be analyzed. We also show, that the cylinder is a cmc surface in the direction of a unitary normal vector field and that there are no nonconstant linear solutions of the differential equation, for any b, $0 \le b < 1$.

A particular case of equation (3.13) is obtained by considering $H = 0$. This case represents the minimal rotational surfaces in Randers spaces (V^3, F_b), and it was already studied by Souza and Tenenblat in [ST]. In [SST], Souza, Spruck, and Tenenblat concluded that the equation that describes the minimal surfaces as a graph of a function in (V^3, F_b) is elliptic for $b < \frac{\sqrt{3}}{3}$. This same value of b appears also in our work. Therefore, we will restrict the analysis of equation (3.13) to the interval $0 < b < \frac{\sqrt{3}}{3}$. The Euclidean case is obtained taking $b = 0$ in equation (3.13). In this case, the solutions are curves that generate the Delaunay surfaces (see [D]).

One knows that in general, nonlinear ordinary differential equations cannot be analytically solved. Equation (3.13) is highly nonlinear. This entails enormous difficulties in the analysis of the possible solutions and mainly in the determination of the interval of existence and uniqueness of the solutions. In view of these difficulties we will use methods of qualitative analysis to determine the behavior of the solutions for all $0 < b < \frac{\sqrt{3}}{3}$ and $H \neq 0$. We will see that we only need to choose $H > 0$.

We can rewrite equation (3.13) as follows,

$$\{-gg''Q + P\}A^{\pm} - HgR = 0, \tag{4.1}$$

where we used the notation

$$P := P_b(g') = w_0 w_b[1 - b^2 + (1 - 3b^2)(g')^2], \tag{4.2}$$

$$Q := Q_b(g') = w_b[1 + 2b^2 + (1 - 3b^2)(g')^2] + 3b^4(g')^2, \tag{4.3}$$

$$R := R_b(g') = (1 - b^2)w_0^2 w_b^{\frac{5}{2}}, \tag{4.4}$$

$$A^+ := A_b^+(g') = w_b - bg'\sqrt{w_b}, \tag{4.5}$$

$$A^- := A_b^-(g') = -w_b - bg'\sqrt{w_b}, \tag{4.6}$$

with w_0 and w_b given by (3.14).

Remark 4.1 It follows from the fact that w_0 and w_b are both positive $\forall b, 0 \leq b \leq \frac{\sqrt{3}}{3}$ (see (3.14)), that:

(i) $P_b(g')$ vanishes for $\frac{\sqrt{3}}{3} < b < 1$, when g' assumes the values

$$r_1(b) = \pm\sqrt{\frac{1 - b^2}{3b^2 - 1}}. \tag{4.7}$$

(ii) $P_b(g') > 0$ (resp. < 0) if and only if $|g'| < |r_1(b)|$ (resp. $|g'| > |r_1(b)|$).

(iii) $P_b(g') > 0, \forall b, 0 \leq b \leq \frac{\sqrt{3}}{3}$ and $\forall g'$.

(iv) The polynomial $Q_b(g')$ vanishes if $\frac{\sqrt{3}}{3} < b < 1$, when g' assumes the values

$$r_2(b) = \pm\sqrt{\frac{1 - b^2 + 3b^4 + \sqrt{12b^4 - 12b^6 + 9b^8}}{3b^2 - 1}}.$$

(v) $Q_b(g') > 0$ (resp. < 0) if and only if $|g'| < |r_2(b)|$ (resp. $|g'| > |r_2(b)|$).

(vi) $Q_b(g') > 0, \forall b, 0 \leq b \leq \frac{\sqrt{3}}{3}$ and $\forall g'$.

Remark 4.2 $R_b(g') > 0, \forall b, 0 \leq b < 1$ and $\forall g'$.

Proposition 4.3 $A_b^-(g') < 0$ *(resp.* $A_b^+(g') > 0$*),* $\forall b, 0 \leq b < 1$.

Proof If $g' > 0$, it follows from (3.14) and (4.6) that $A^- < 0$. If $g' \leq 0$, the proof follows from the fact that $(1 - b^2)(1 + g'^2) > 0, \forall b, 0 \leq b < 1$. For A^+ the proof is analogous. ∎

For a fixed $H \neq 0$, consider the following differential equations given by (4.1)

$$\{-gg''Q(g') + P(g')\}A^{-}(g') - gHR(g') = 0, \tag{4.8}$$

$$\{-gg''Q(g') + P(g')\}A^{+}(g') - gHR(g') = 0. \tag{4.9}$$

The following proposition, whose proof is immediate, shows that both equations define the same surface of rotation.

Proposition 4.4 *A differentiable function $h(t) = -g(t)$ is a solution of* (4.8) *if and only if $-h(t)$ is a solution of* (4.9).

The next proposition shows us that we can restrict the study of the equations (4.1) to the case $H > 0$. In fact, consider the two differential equations

$$\{-gg''Q(g') + P(g')\}A^{-}(g') + gHR(g') = 0, \tag{4.10}$$

$$\{-gg''Q(g') + P(g')\}A^{+}(g') + gHR(g') = 0. \tag{4.11}$$

Then one can easily see that the following result holds.

Proposition 4.5 *A differentiable function, $h(t)$, $t \in I \subset \mathbb{R}$, is a solution of* (4.8) *(resp.* (4.9)*) if and only if $-h(-t)$ is a solution of* (4.11) *(resp.* (4.10)*).*

In short, together Propositions 4.4 and 4.5 tell us that, although we have two equations describing the surfaces of rotation, in the space (V^3, F_b), with cmc H in the direction of the normal unitary vector fields, both describe the same surface, independently of the choice of the normal field and also of the sign of H. This result is important for the fact that it allows us to choose just one equation to analyze. It is also important for the fact that it would not be desirable, in the development of our theory, that the choice of the normal field would affect the surface of rotation.

Due to this fact, from now on we will work only with equation (4.9) (it corresponds to taking A^{+} in equation (4.1)). Moreover we will consider $g(t) > 0$, $\forall t \in \mathbb{R}$.

Lemma 4.6 *For a fixed constant $H > 0$,*

$$g(t) = \frac{1}{H\sqrt{1-b^2}} \quad (\textit{respectively } g(t) = -\frac{1}{H\sqrt{1-b^2}}), \quad \forall t \in \mathbb{R}, \tag{4.12}$$

is a solution of (4.9) *(respectively* (4.8)*). Moreover, this is the unique constant solution of* (4.9) *(respectively* (4.8)*).*

The proof of Lemma 4.6 is trivial and shows that (4.12) is the unique constant solution. We observe that this constant solution generates the cylinder of radius $\frac{1}{H\sqrt{1-b^2}}$.

Remark 4.7 If $g(t)$ is a solution of (4.9), where H is a nonvanishing constant and $g(t_0) \neq \frac{1}{H\sqrt{1-b^2}}$, for some $t_0 \in \mathbb{R}$, then $g'(t_0)$ and $g''(t_0)$ do not vanish simultaneously.

The next lemma excludes any nonconstant linear solution for the equation (4.9), whenever H is a nonvanishing constant. In the case $H = 0$, the linear solutions generate minimal cones and they were obtained in [ST].

Lemma 4.8 *Let $d \neq 0$ be a constant.*

(i) *$g(t) = r_1(b)t + d$ is a solution of* (4.9) *with H constant if and only if $H = 0$, where $r_1(b)$ is given by* (4.7) *and $\frac{\sqrt{3}}{3} < b < 1$.*

(ii) *Let H be a nonvanishing constant. If $g(t) = at + d$ is a solution of* (4.9)*, for all $t \in \mathbb{R}$, then $a = 0$.*

Proof If $H = 0$, it is enough to substitute g and its derivatives into equation (4.9) to verify that g is a solution of (4.9). Similarly, if $g(t) = \pm r_1(b)t + a$ is a solution, then it follows from (4.9), that $H = 0$. This proves (i). In order to prove (ii), we conclude from equation (4.9) that if $H \neq 0$, then $a = 0$. ■

In short, we proved that the cylinder has cmc in the direction of the normal vector field N in (V^3, F_b) and it is generated by the constant solution. Moreover, we excluded all nonconstant linear solutions. Although we do not have an explicit nonconstant solution of equation (4.1), we can describe qualitative information on their solutions. In the next section, we present a qualitative analysis of equation (4.9).

5 Qualitative Analysis

One knows that, for smooth functions $f(x)$, the solution of the initial value problem

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad x(0) = x_0,$$

is defined at least in some neighborhood $t \in (-\epsilon, \epsilon)$ of $t = 0$. This, in the nonlinear case, means that a local flux $\varphi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\varphi_t(x_0) = x(t, x_0)$ in a similar way to the linear case. One knows also that, it is not always possible to determine the interval $(-\epsilon, \epsilon)$ of existence and uniqueness of the solutions.

In this section, we consider equation (4.9) as a nonlinear dynamical system. We use the concept of stability and through the linearization around the single equilibrium point (the constant solution), we verify that the solutions are, locally, asymptotically stable spirals.

Making the change of variables $x(t) = g(t)$ and $y(t) = \dot{x}(t)$, we have that $\dot{\overline{X}} = \mathcal{G}(t, \overline{X}, b)$, where $\overline{X} = (x, y) \in \mathbb{R}^2$ and $\mathcal{G}$ is of class C^r. We can rewrite equation (4.9) in the form of an autonomous system of two equations of first order as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \phi_b(x, y), \end{cases} \tag{5.1}$$

where $\mathcal{G}(t, \overline{X}, b) = \big(y, \phi_b(x, y)\big)$ and ϕ_b is given by

$$\phi_b(x, y) = \frac{P(y)A(y) - xR(y)H}{xQ(y)A(y)}. \tag{5.2}$$

Moreover, the expressions P, Q, R and A, given by equations (4.2)–(4.5), are written now in the following form:

$$P(y) = w_0 w_b[1 - b^2 + (1 - 3b^2)y^2], \tag{5.3}$$

$$Q(y) = w_b[1 + 2b^2 + (1 - 3b^2)y^2] + 3b^4 y^2, \tag{5.4}$$

$$R(y) = (1 - b^2)w_0^2 w_b^{\frac{5}{2}}, \tag{5.5}$$

$$A(y) = w_b - by\sqrt{w_b}, \tag{5.6}$$

$w_0 = 1 + y^2$ and $w_b = 1 - b^2 + y^2$.

Observe that (5.1) defines an autonomous nonlinear system of differential equations, whose equilibrium points (or fixed points) are determined by the solutions of the system $\dot{\overline{X}} = 0$, where $\dot{\overline{X}} = (\dot{x}, \dot{y})$. That is, the equilibrium points are all the points (x, y) of the form

$$\overline{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{H\sqrt{1-b^2}} \\ 0 \end{pmatrix}. \tag{5.7}$$

Observe that the equilibrium solution is exactly the generatrix of the cylinder defined in Lemma 4.6. From now on, we will restrict the study of the solutions of (5.1) to the interval $0 < b < \frac{\sqrt{3}}{3}$ and we will consider only $x > 0$ and $H > 0$.

Hartman–Grobman's theorem (see [H] and [WS]) tells us that, in a neighborhood of a hyperbolic equilibrium point, the linearized system behaves as the linear part of the linearization. In the case of the nonhyperbolic equilibrium points, the eigenvalues of the linearization matrix are purely imaginary, and consequently the trajectories are closed orbits around the equilibrium points and small perturbations can transform a closed orbit in a spiral, stable or not.

Lemma 5.1 *The partial derivatives of ϕ_b with respect to x and y, at the point $\overline{X}_0 = (\frac{1}{\sqrt{1-b^2}H}, 0)$, are given respectively by*

$$[\phi_b(x, y)]_x|_{\overline{X}_0} = -\frac{(1 - b^2)^2 H^2}{1 + 2b^2} \quad \textit{and} \quad [\phi_b(x, y)]_y|_{\overline{X}_0} = -\frac{Hb(1 - b^2)}{1 + 2b^2}, \tag{5.8}$$

where $\phi_b(x, y)$ is given by (5.2).

Proof For the proof of this lemma, we take the derivatives of $\phi_b(x, y)$, with respect to x and then with respect to y at the point $\overline{X}_0$, and we use the fact that $P_{y|_{y=0}} = R_{y|_{y=0}} = Q_{y|_{y=0}} = 0$. ∎

It follows from (5.8) that the Jacobian matrix $\mathcal{A}$ associated to the linearization of the system (5.1) at the point $\overline{X}_0 = (\frac{1}{(\sqrt{1-b^2}H)}, 0)$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\frac{(1-b^2)^2 H^2}{1+2b^2} & -\frac{Hb(1-b^2)}{1+2b^2} \end{pmatrix}.$$

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Therefore, the eigenvalues of $\mathcal{A}$ are given by

$$\lambda\pm = \frac{(1-b^2)H(-b \pm i\sqrt{7b^2+4})}{2(1+2b^2)}.$$

For $0 < b < \frac{\sqrt{3}}{3}$, the eigenvalues of $\mathcal{A}$ are distinct complex numbers whose real parts are nonvanishing and negative. Thus, the following remark follows from Hartman–Grobman theorem.

Remark 5.2 For all $b, 0 < b < \frac{\sqrt{3}}{3}$ the unique equilibrium point $\overline{X}_0$ of the system (5.1) is *hyperbolic* and this equilibrium is locally asymptotically stable in the sense that the orbits are spirals inward the equilibrium point. In this case, $\overline{X}_0$ is said be a *stable focus* for $0 < b < \frac{\sqrt{3}}{3}$. Therefore, for these values of b, all the solutions tend to the equilibrium point, when the parameter t tends to infinity. For $b = 0$, the eigenvalues of $\mathcal{A}$ are complex numbers with vanishing real part, therefore the equilibrium point is *nonhyperbolic* and it is denominated a *center*. In this case, the orbits are closed (ellipses centered at the equilibrium point).

5.1 Surface Method

Due to the great difficulty in obtaining any information on the behavior of the solutions of the nonlinear equation (4.9), in order to accomplish the study of the fields of directions, we introduce what we call *surface method*. This method consists in using the implicit function theorem to study the behavior of the solutions of the differential equation. It uses the existent relationship between the surface associated to the equation and the curves that describe the behavior of the critical points and the points of inflection of the solutions. With this information, we determine the behavior of the fields of direction in the phase plane. We remind once again that we are considering $0 < b < \frac{\sqrt{3}}{3}$.

In coordinates x, y and z, equation (4.9) takes the following form,

$$\{-xzQ + P\}A - HxR = 0, \tag{5.9}$$

where H is a nonvanishing constant and the expressions of P, Q, R and A are given by equations (5.3)–(5.6) respectively, by replacing g by x, g' by y and g'' by z. Equation (5.9) defines implicitly a surface $G(x, y, z) = 0$ (see Figure 1), where

$$G = [-xzQ(y) + P(y)]A(y) - HxR(y). \tag{5.10}$$

On the left side of Figure 1, the surface $G(x, y, z) = 0$ is visualized with its two connected components. On the right side of Figure 1 we can see its connected component given by $x > 0$.

In the same way we can define the intersection of the planes $y = 0$ or $z = 0$ with the surface $G = 0$. That is, the curves $G(x, 0, z) = 0$ and $G(x, y, 0) = 0$. We denote

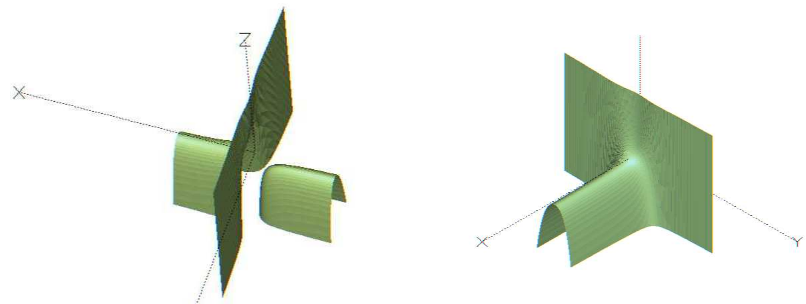

Figure 1: Surface $G(x, y, z) = 0$, with $b = 0.3$ and $H = 1$. Figure constructed using ACOGEO.

$F_1(x, z) := G(x, 0, z)$ and $F_2(x, y) := G(x, y, 0)$. Thus,

$$F_1(x,z) = \{-xz[(1-b^2)(1+2b^2)] + (1-b^2)^2\}(1-b^2) - Hx(1-b^2)^{\frac{7}{2}}, \tag{5.11}$$

$$F_2(x,y) = \left[w_0 w_b\left(1-b^2+(1-3b^2)y^2\right)\right](w_b - by\sqrt{w_b}) - Hx(1-b^2)w_0^2(w_b)^{\frac{5}{2}}. \tag{5.12}$$

Proposition 5.3 *$G(x, y, z) = 0$ is a regular surface for all $x > 0$ (resp. $x < 0$).*

Proof It follows from (5.10), that $\frac{\partial G}{\partial z} = -xQA$ and, from equations (5.4) and (5.6), that $QA > 0\ \forall x > 0$. We also have that

$$\frac{\partial G}{\partial y} = -zx\Big[\frac{\partial Q}{\partial y}A + Q\frac{\partial A}{\partial y}\Big] + \Big[\frac{\partial P}{\partial y}A + P\frac{\partial A}{\partial y}\Big] - Hx\frac{\partial R}{\partial y} \quad \text{and} \quad \frac{\partial G}{\partial x} = zQA + RH.$$

Therefore $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial y}$ and $\frac{\partial G}{\partial z}$ do not vanish simultaneously for all $x > 0$. It follows that $G(x, y, z) = 0$ is the inverse image of zero, that is a regular value of G for all $x > 0$ and hence G is a regular surface for all $x > 0$. The proof for $x < 0$ is analogous. ∎

We observe that $G(x, y, z) = 0$ is not defined for $x = 0$, because in this case we would have $PA = 0$ and we know from (5.3) and (5.6) that P and A are positive for all $0 < b < \frac{\sqrt{3}}{3}$. From this fact and from Proposition 5.3, we conclude that the surface G has two connected components. In what follows, $G(x, y, z) = 0$ will designate the connected component where $x > 0$, which is the one we are interested in.

Let $G\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ be the function defined by (5.10) and let $p = (x_0, y_0, z_0) = (\frac{1}{\sqrt{1-b^2}H}, 0, 0) \in U$. We have that G, $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial y}$, are continuous $\forall b, 0 < b < \frac{\sqrt{3}}{3}$ and $\frac{\partial G}{\partial x}|_p = -H(1-b^2)^{\frac{7}{2}} \neq 0$. Moreover $G(p) = 0$. Therefore, applying the Implicit Function Theorem, there is an interval I and an open ball $B \subset \mathbb{R}^2$, centered at $(0, 0)$, such that $G^{-1}(0) \cap (B \times I)$ is the graph of a function $W\colon B \to I$ of class C^2. This allows us to express x as a function of (y, z) in U, as follows:

$$x = W(y,z) = \frac{PA}{zQA + HR}, \tag{5.13}$$

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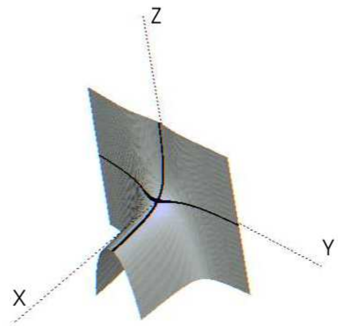

Figure 2: In this figure we can see the intersection between curves $h(y)$ and $f(z)$ on the surface $G(x, y, z)$.

where $P = P(y), Q = Q(y), R = R(y), A = A(y)$, are given by equations (5.3)–(5.6), respectively. This expression will be useful in the next section.

Proposition 5.4 *Consider* $p_1 = (x_0, z_0) = (\frac{1}{\sqrt{1-b^2}H}, 0)$ *and* $p_2 = (x_0, y_0) = (\frac{1}{\sqrt{1-b^2}H}, 0)$. *Then*

$$\frac{\partial F_1}{\partial x}\Big|_{p_1} \neq 0, \quad \frac{\partial F_1}{\partial z}\Big|_{p_1} \neq 0, \quad \frac{\partial F_2}{\partial y}\Big|_{p_2} \neq 0, \quad \frac{\partial F_2}{\partial x}\Big|_{p_2} \neq 0,$$

$\forall x \neq 0$ *and* $\forall b, 0 < b < 1$.

Proof The proof is a straightforward computation that follows from equations (5.11) and (5.12). ■

Let $F_1 \colon U \subset \mathbb{R}^2 \to \mathbb{R}$ be the function defined by (5.11) and let $p_1 = (x_0, z_0) = (\frac{1}{\sqrt{1-b^2}H}, 0) \in U$. The functions F_1, $\frac{\partial F_1}{\partial x,}$, $\frac{\partial F_1}{\partial z}$, are continuous $\forall b, 0 < b < \frac{\sqrt{3}}{3}$ and $F_1(p_1) = 0$. From Proposition 5.4, it follows that $\frac{\partial F_1}{\partial x}|_{p_1} \neq 0$. Therefore, applying the Implicit Function Theorem, there is a rectangle $I \times J$ centered at (x_0, z_0) such that $x = f(z)$, for a unique function $f \colon J \to I$ of class C^1 that satisfies $F_1\big(f(z), z\big) = 0$. That is, $F_1^{-1}(0) \cap (J \times I)$ is the graph of the function f.

Similarly, considering $F_2 \colon V \subset \mathbb{R}^2 \to \mathbb{R}$ defined by (5.12) and $p_2 = (x_0, y_0) = (\frac{1}{\sqrt{1-b^2}H}, 0) \in V$ one can apply the Implicit Function Theorem to ensure the existence of a function $h \colon K \to I$ of class C^1 satisfying $F_2\big(h(y), y\big) = 0$.

Observe that the curves f and h intersect at the point $p_1 = p_2$ (see Figure 2), which is the equilibrium point of the system (5.1) obtained in (5.7).

The curves $f(z)$ and $h(y)$ can be obtained by considering $F_1(x, z) = 0$ and $F_2(x, y) = 0$. Thus we get

$$f(z) = \frac{(1 - b^2)}{z(1 + 2b^2) + (1 - b^2)^{\frac{3}{2}} H} \tag{5.14}$$

and

$$h(y) = \frac{PA}{RH} = \frac{w_b - 3b^2y^2}{H(w_b + by\sqrt{w_b})(w_b)^{\frac{1}{2}}}, \tag{5.15}$$

where $w_b = 1 - b^2 + y^2$.

Remark 5.5 Considering $g(t) = x(t)$ in Remark 4.7, we have that, for any given initial condition, the solution $x(t)$ of equation (5.9) does not have inflexion points such that $\dot{x}(t) = y(t) = 0$. Therefore, we can say that, in the phase plane, f represents a curve where the tangent fields to the trajectories are not defined (*i.e.*, the tangent fields are parallel to the axis Ox).

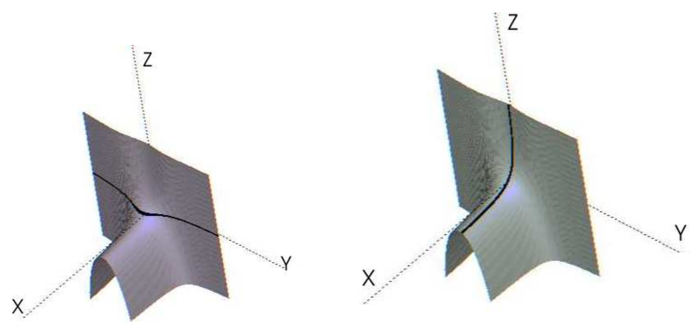

Figure 3: The figure on the left side shows the curve $h(y)$ and the figure on the right side shows the curve $f(z)$.

The curves $f(z)$ and $h(y)$ are represented in Figure 3, with $b = 0.3$ and $H = 1$. With respect to the curve h, we have the following properties.

Proposition 5.6 *Let $h(y)$ be a function defined by* (5.15).
(i) *There exists a unique y_0 such that $h'(y_0) = 0$ and $y_0 < 0$.*
(ii) $\lim_{y\to\pm\infty} h(y) = 0$.
(iii) *y_0 is the point of global maximum of h.*

Proof (i) Taking the derivative of h with respect to y and equating to zero, we obtain

$$y(w_b)^{\frac{1}{2}}[9b^2y^2 - (1 + 6b^2)(w_b)] = b[(w_b)^2 + 3b^2y^2(w_b) - 6b^2y^4], \tag{5.16}$$

where $w_b = 1-b^2+y^2$. Taking squares on both sides of (5.16) we obtain the following polynomial of degree 8:

$$P(y) = a_0 + a_2y^2 + a_4y^4 + a_6y^6 + a_8y^8, \tag{5.17}$$

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where

$$a_0 = -b^2(1-b^2)^4, \quad a_2 = (1-b^2)^3(30b^4 + 8b^2 + 1),$$
$$a_4 = 3(1-b^2)^3(1 + 5b^2 + 3b^4), \quad a_6 = (1-b^2)(3b^2-1)(6b^4 - 5b^2 - 3),$$
$$a_8 = (1-b^2)(1-3b^2)^2.$$

Since $1 - b^2 > 0$, it follows that, $a_0 < 0$, $a_2 > 0$, $a_4 > 0$, $\forall b, 0 < b < 1$ and $a_6 > 0$. Moreover, $a_8 > 0, \forall b, 0 < b < \frac{\sqrt{3}}{3}$.

Observe that $P(y) = P(-y)$ and in both cases we have just one sign change among the coefficients, thus, by the Descartes sign rules for polynomials, we have only one positive root and only one negative root. We can show that the positive one is inadequate. In fact, assuming that we have a positive root r and using the fact that $r > 0$, $(3b^2 - 1) < 0$ and $(-1 + b^2) < 0$, we show that $\forall b, 0 < b < \frac{\sqrt{3}}{3}$, the left-hand side of (5.16) is negative and the right-hand side is positive. This is a contradiction. It follows then, that the polynomial (5.17) has a unique negative root. This concludes the proof of (i).

For the proof of (ii), we apply L'Hospital's rule twice and verify that $\lim\limits_{y\to\pm\infty} \frac{PA}{RH} = 0$.

In order to prove (iii), we observe that it follows from (5.5) that, for both b and y real, the denominator of (5.15) does not vanish and consequently the curve (5.15) is continuous for any real value. Moreover, as a consequence of (i) and (ii), there is a unique critical point of h which satisfies $y_0 < 0$ and h tends to zero when $y \to \pm\infty$. Therefore y_0 may be either a maximum or a minimum global point of h. From the fact that $h(y) > 0, \forall b, y \in \mathbb{R}$, we conclude that $h(y_0)$ is a global maximum point of $h(y)$. This concludes the proof of (iii). ■

For $\phi_b(y, x) \neq 0$, the system (5.1) defines a vector field in the phase plane, whose directions are tangent to the trajectories, pointing to the direction of time increasing. Its directions are determined by $\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{y}{\phi_b(y,x)}$, where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$.

It follows from (5.2) that, $\forall b, 0 < b < \frac{\sqrt{3}}{3}$. $\phi_b(y, x) > 0$ if and only if $x < \frac{PA}{RH}$ and $\phi_b(y, x) < 0$ if and only if $x > \frac{PA}{RH}$. Thus, taking into account these results, we will analyze the behavior of the fields dx/dy in the phase plane.

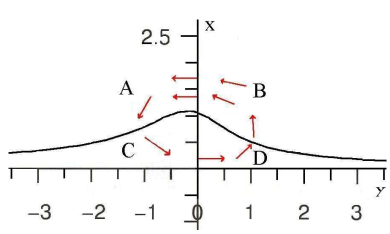

Figure 4: Behavior of the directions fields in the phase space and the graph of the function $h(y)$.

One can observe the following cases (see Figure 4):

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(i) $\frac{dx}{dy} = 0$ if and only if $y = 0$ (tangent fields parallel to the y axis);

(ii) for $\frac{dx}{dy} > 0$ we have either $y > 0$ and $x < \frac{PA}{RH}$ (region D) or $y < 0$ and $x > \frac{PA}{RH}$ (region A);

(iii) for $\frac{dx}{dy} < 0$ we have either $y < 0$ and $x < \frac{PA}{RH}$ (region C) or $y > 0$ and $x > \frac{PA}{RH}$ (region B);

(iv) $\frac{dx}{dy}$ is not defined for $x = \frac{PA}{RH}$ (see Remark 5.5).

Based on this information and knowing that in the phase plane the trajectories behave asymptotically as stable spirals, we sketched, using the resources of Maple, the *phase portrait* of the system (5.1), as we can see in Figure 5. We used the initial conditions $x(0) = 1$, $y(0) = 0$ with $b = 0.3$ and $H = 1$.

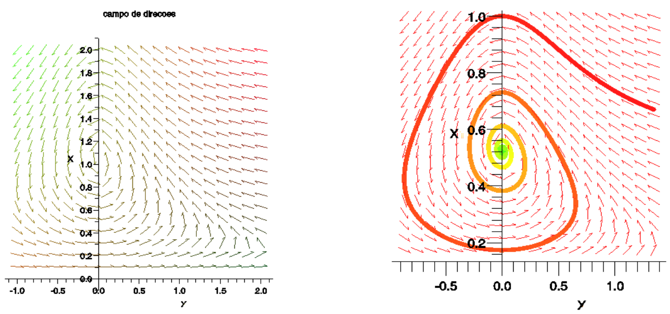

Figure 5: The figure on the left side shows the direction fields in the phase space and the figure on the right side shows a path in the phase space.

Observe that we are considering just the interval $0 < b < \frac{\sqrt{3}}{3}$, where the polynomial Q does not have real roots. Since equation (4.9) has continuous partial derivatives, it follows from the Picard–Lindelöf theorem, that given the initial conditions, there exists a unique solution in the neighborhood of $t = 0$. The difficult problem is to determine the extension of this neighborhood.

In what follows, we will prove some results that will provide information about the behavior of the solutions of the differential equation (4.9). When convenient, we will look at the solutions of our differential equation as a graph and, we will refer to equation (4.9) in the variables g, g', and g''. Alternatively we will look at the same solutions as a trajectory in the phase plane. We will then follow the notation introduced in the previous section, considering $g = x$, $g' = y$, and $g'' = z$, and refer to equation (5.9).

Remark 5.7 From equation (5.9) we have that $x(zQA + HR) = PA$. Since $PA > 0$, $\forall b, 0 < b < \frac{\sqrt{3}}{3}$, we have that $zQA + HR \neq 0$, $\forall x > 0$. Thus, the solution of (5.9) is not defined for

$$z = -\frac{HR}{QA}.$$

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In particular, at the points where y vanishes, this expression is given by

$$z = -\frac{(1-b^2)^{\frac{3}{2}}H}{1+2b^2}.$$

In the previous section, we saw that the intersection of the surface $G(x, y, z) = 0$ with the plane $y = 0$ is a curve in the coordinate plane $z0x$, denoted by $f(z)$ and given by (5.14). Observe that $y(t) = g'(t)$, thus $f(z)$ defines a curve on the surface $G(x, y, z) = 0$ that contains all the critical points of any solution $g(t)$ of equation (4.9).

Lemma 5.8 *For any nonconstant solution $g(t)$ of the differential equation* (4.9)*, there do not exist $t_1, t_2 \in \mathbb{R}$ such that $g(t_1) = g(t_2)$ and $g'(t_1) = g'(t_2) = 0$, $\forall b, 0 < b < \frac{\sqrt{3}}{3}$.*

Proof Suppose that for b, $0 < b < \frac{\sqrt{3}}{3}$, there exists $t_1 \neq t_2 \in \mathbb{R}$ such that $g(t_1) = g(t_2)$ and $g'(t_1) = g'(t_2) = 0$. This implies the existence of a closed orbit in a neighborhood of the equilibrium point (or equivalently that the equilibrium point is a center), but this contradicts the fact that the equilibrium point is a focus (Remark 5.2). ∎

Observe that when $b = 0$, the equilibrium point is a center. This means that the orbits are all closed in a neighborhood of the equilibrium point. In this case, the solutions g are the curves whose rotation around the t axis generate the Delaunay surfaces [D].

The next two lemmas assure that, when t grows, the norm of the second derivative decreases at the points where the first derivative vanishes (minimum and maximum points). At these points, the solution g approaches the value $\frac{1}{\sqrt{1-b^2}H}$, $\forall b$, $0 < b < \frac{\sqrt{3}}{3}$. In other words, we will prove that the solution g behaves as in Figure 6, where we used the initial conditions $g(0) = 1$, $g'(0) = 0$, $b = 0.3$ and $H = 1$.

Before starting the next lemma we will rewrite equation (5.13) in the variables g' and g'', *i.e.*,

$$g = W(g', g'') = \frac{PA}{g''QA + HR}, \tag{5.18}$$

where $P = P(g')$, $Q = Q(g')$, $R = R(g')$ and $A = A(g')$ are given by equations (4.2)–(4.5), respectively.

Lemma 5.9 *Let $g(t)$ be a solution of equation* (4.9) *with $b \neq 0$. The minimum and maximum values of g, are respectively below and above the constant solution $g = \frac{1}{\sqrt{1-b^2}H}$.*

The proof follows directly from equation (4.9), by considering $g'(t_0) = 0$.

Lemma 5.10 *Let t_0 and t_1 be such that $g'(t_0) = g'(t_1) = 0$ and $g''(t_0) > 0$, $g''(t_1) > 0$ (resp. $g''(t_0) < 0$, $g''(t_1) < 0$),* i.e.*, t_0 and t_1 are points of local minimum (resp. maximum). Then $g(t_0) < g(t_1)$ (resp. $g(t_0) > g(t_1)$) if and only if $g''(t_0) > g''(t_1)$ (resp. $g''(t_0) < g''(t_1)$). Moreover $g(t_0) = g(t_1) = 0$ if and only if $b = 0$.*

Proof Observe that, at the points where the derivative g' vanishes, equation (5.18) is given by

$$g = f(g'') = \frac{(1-b^2)}{g''(1+2b^2) + (1-b^2)^{\frac{3}{2}} H}. \tag{5.19}$$

It follows from Lemma 5.8 that $g(t_0) \neq g(t_1)$, $\forall b, 0 < b < \frac{\sqrt{3}}{3}$. Thus, assuming that $g''(t_0) > 0$ and $g''(t_1) > 0$, it follows from (5.19) that $g(t_0) > g(t_1)$ (resp. $g(t_0) < g(t_1)$) if and only if $g''(t_0) < g''(t_1)$ (resp. $g''(t_0) > g''(t_1)$). Similarly, one proves the case in which $g''(t_0) < 0$ and $g''(t_1) < 0$. Moreover, it follows from Lemma 5.8 that equality occurs if and only if $b = 0$. ∎

Lemmas 5.9 and 5.10 mean that, when t tends to infinity, local minimum and maximum values of g approximate the equilibrium solution. Hence, any solution approaches the equilibrium solution (see Figure 6).

Lemma 5.11 *Let* $a = -\frac{(1-b^2)^{\frac{3}{2}} H}{1+2b^2}$ *and consider the function* $f(z)$ *given by* (5.14), *with* $0 \leq b < 1$. *Then*

(i) f *is decreasing for all* $z > a$;
(ii) $\lim_{z\to 0} f(z) = \frac{1}{\sqrt{1-b^2}H}$;
(iii) $\lim_{z\to a^\pm} f(z) = \pm\infty$;
(iv) $\lim_{z\to \pm\infty} f(z) = 0$.

Proof We will only prove (i); the other items are trivially verified. Taking the derivative of (5.14) with respect to z we obtain

$$f'(z) = \frac{(b^2-1)(1+2b^2)}{[z(1+2b^2) + (1-b^2)^{\frac{3}{2}} H]^2}.$$

It follows from the fact that the denominator of f' is always positive and the numerator is negative that $f' < 0$, $\forall b, 0 \leq b < 1$. Hence, f is a decreasing function. ∎

Rotating the generatrix curve (Figure 6) around the $z = t$ axis, we obtain the corresponding surface of rotation, which is represented in Figure 7, where we used $b = 0.3$, $H = 1$ and the initial conditions $g(0) = 1$, $g'(0) = 0$. Figures 6 and 7 were obtained by numerical methods.

6 Basin of Attraction

Next we will determine the existence of a region containing the equilibrium point of the system (5.1) in whose interior, a function of Liapunov to be defined, has strictly negative derivative. This region is denominated the *basin of asymptotic stability* of the equilibrium point.

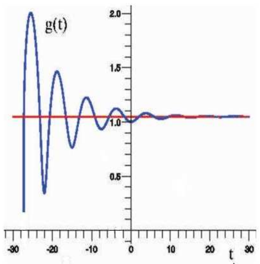

Figure 6: A solution $\big(t, g(t)\big)$ of the equation (4.1) and the equilibrium solution.

Figure 7: Surface of rotation cmc in the direction of the unitary normal vector field.

Let

$$S(y) = \frac{P(y)}{Q(y)}, \quad B(y) = \frac{R(y)H}{Q(y)A(y)}, \tag{6.1}$$

where P, Q, R and A are given by equations (5.3)–(5.6) respectively. In this condition, (5.1) can be rewriten in the following form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{S(y)}{x} - B(y). \end{aligned} \tag{6.2}$$

After the translation $X = x - x_0$, $Y = y$, the above system is given in the new coordinates by

$$\dot{X} = Y, \quad \text{and} \quad \dot{Y} = \frac{S(Y)}{X + x_0} - B(Y). \tag{6.3}$$

Remark 6.1 Consider the function

$$V(X, Y) = S(0) \ln\Big(\frac{x_0}{X + x_0}\Big) + \frac{S(0)}{x_0} X + \frac{Y^2}{2}. \tag{6.4}$$

Then

$$V(0,0) = 0, \quad V_X = -\frac{S(0)}{X + x_0} + \frac{S(0)}{x_0}, \quad V_Y = Y,$$
$$V_{XX} = \frac{S(0)}{(X + x_0)^2}, \quad V_{YY} = 1, \quad V_{XY} = 0.$$

Thus, the Hessian matrix of $V(X, Y)$ at $(0, 0)$ is given by

$$\operatorname{Hess} V(0,0) = \begin{pmatrix} \frac{S(0)}{(X+x_0)^2} & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, $\det \operatorname{Hess} V(0,0) = \frac{S(0)}{(X+x_0)^2} > 0$ and $V(X, Y) \geq 0$.

Taking the derivative of (6.4) with respect to t, using (6.3) and the fact that $\frac{S(0)}{x_0} = \frac{(1-b^2)^{\frac{3}{2}}}{1+2b^2}H = B(0)$, we obtain

$$\dot{V}(X,Y) = \Big(-\frac{S(0)}{X+x_0} + \frac{S(0)}{x_0}\Big)Y + \Big(\frac{S(Y)}{X+x_0} - B(Y)\Big)Y \tag{6.5}$$
$$= \frac{Y}{X+x_0}\{(X+x_0)[B(0)-B(Y)] + [S(Y)-S(0)]\}.$$

Next, we will prove a series of properties, which will be useful for us to conclude the central objective of this section, that is to prove that there exists a region where the function $\dot{V}(X,Y)$ is negative definite.

Lemma 6.2 $\lim\limits_{Y\to\pm\infty} B(Y) = +\infty$, *where* $B(Y)$ *is given by* (6.1).

Proof In fact, simplifying the expression of $B(Y)$, we obtain

$$B(Y) = \frac{(1-b^2)H(1+Y^2)^2(1-b^2+Y^2)^2}{[(1-b^2+Y^2)^{\frac{1}{2}} - bY][(1+Y^2)^2 + b^4(-2+6Y^2) + b^2(1-2Y^2-3Y^4)]}. \tag{6.6}$$

It follows from Remark 4.1 and from Proposition 4.3 that $Q(y) > 0$ and $A(y) > 0$, $\forall b, 0 < b < \frac{\sqrt{3}}{3}$. Therefore, we complete the proof observing that the denominator of (6.6) is positive and the numerator is a polynomial in Y of degree 16, while in the denominator Y appears with powers of, at most degree 5. ∎

Lemma 6.3 *The function*

$$S(Y) = \frac{P(Y)}{Q(Y)} = \frac{(1+Y^2)(1-b^2+Y^2)[1-b^2+(1-3b^2)Y^2]}{(1-b^2+Y^2)[1+2b^2+(1-3b^2)Y^2] + 3b^4Y^2} \tag{6.7}$$

is positive $\forall b, 0 < b < \frac{\sqrt{3}}{3}$. *Moreover,* $Y = 0$ *is the only critical point of* S *and it is a global minimum.*

Proof Proving that the function has a unique critical point is equivalent to showing that the first derivative has a unique real root. We can rewrite (6.7) in the following form:

$$S(Y) = \frac{e_1Y^6 + e_2Y^4 + e_3Y^2 + e_4}{\bar{e}_1Y^4 + \bar{e}_2Y^2 + \bar{e}_3}, \tag{6.8}$$

where $e_1 = 1-3b^2$, $e_2 = 3b^4 - 8b^2 + 3$, $e_3 = 4b^4 - 7b^2 + 3$, $e_4 = (1-b^2)^2$, $\bar{e}_1 = e_1$, $\bar{e}_2 = 6b^4 - 2b^2 + 2$ and $\bar{e}_3 = (1-b^2)(1+2b^2)$.

Taking the derivative of (6.8) with respect to Y, we obtain

$$S'(Y) = \frac{6e_1Y^5 + 4e_2Y^3 + 2e_3Y}{\bar{e}_1Y^4 + \bar{e}_2Y^2 + \bar{e}_3} \tag{6.9}$$
$$- \frac{(e_1Y^6 + e_2Y^4 + e_3Y^2 + e_4)(4\bar{e}_1Y^3 + 2\bar{e}_2Y)}{(\bar{e}_1Y^4 + \bar{e}_2Y^2 + \bar{e}_3)^2}.$$

Therefore, one can see that $S'(Y) = 0$ if and only if

(6.10)
$$2e_1^2Y^9+4e_1\bar{e}_2Y^7+[6e_1\bar{e}_3+2(e_2\bar{e}_2-e_1e_3)]Y^5+4(e_2\bar{e}_3-e_1e_4)Y^3+[2(e_3\bar{e}_3-\bar{e}_2e_4)]Y = 0.$$

It follows from Remark 4.1 that the polynomial Q does not vanish for $0 < b < \frac{\sqrt{3}}{3}$. Moreover, for such a b all the coefficients of Y in the expression (6.10) are positive. It follows that $S'(Y) = 0$ if and only if $Y = 0$. Consequently $Y = 0$ is the unique critical point of $S(Y)$. In order to show that $Y = 0$ is a minimum point, it is sufficient to observe that, the coefficients are all positive and Y appears with odd power, hence $S(Y)$ is decreasing for negative values of Y and increasing for positive values of Y. Therefore, $Y = 0$ is a global minimum for $S(Y)$ and consequently, $S(Y) > 0, \forall Y \in \mathbb{R}$. ■

Consider the derivative $B'(Y)$ of the function $B(Y)$ given by (6.6). We have that

$$B'(Y) = \frac{R'(Y)HA(Y)Q(Y) - R(Y)H[A'(Y)Q(Y) + Q'(Y)A(Y)]}{[A(Y)Q(Y)]^2}. \tag{6.11}$$

Observe that $B' = 0$ if and only if $QAR'H = (QA' + AQ')RH$, *i.e.*,

$$QAY[4w_b + 5w_0] = (QA' + AQ')w_0w_b, \tag{6.12}$$

where prime indicates the derivative with respect to Y, $w_0 = 1+Y^2$, $w_b = 1-b^2+Y^2$ and we used R given by (5.5). Using the notation

$$Z = 1 + 2b^2 + (1 - 3b^2)Y^2,$$

we have that Q and A, given by (5.4) and (5.6) respectively, reduce to $Q = w_bZ+3b^4Y^2$ and $A = w_b - bY(w_b)^{\frac{1}{2}}$. Thus,

$$QAY = (w_bZ + 3b^4)AY[4w_b + 5w_0]. \tag{6.13}$$

Moreover, we have

(6.14)
$$\begin{aligned} QA' + AQ' = {}& 4Yw_bZ - 3bY^2(w_b)^{\frac{1}{2}}Z - 9b^5Y^2(w_b)^{\frac{1}{2}} - b(w_b)^{\frac{3}{2}}Z - 3b^5Y^4(w_b)^{-\frac{1}{2}} \\ & + 2Y(1 - 3b^2)(w_b)^2 + 6b^4Yw_b - 2bY^2(1 - 3b^2)(w_b)^{\frac{3}{2}} + 6b^4Y^3. \end{aligned}$$

Substituting (6.13) and (6.14) into (6.12), we obtain that $B' = 0$ if and only if

$$Y(w_b)^{\frac{1}{2}}[D(Y)] = b[J(Y)], \tag{6.15}$$

where

$$\begin{aligned} D(Y) = {}& (9w_b^2 + 5b^2w_b - 4w_0w_b)Z + 27b^4Y^2 \\ & + 15b^6Y^2 - 2(1 - 3b^2)w_0w_b^2 - 6b^4w_0w_b - 6b^4Y^2w_0, \end{aligned} \tag{6.16}$$

(6.17)
$$\begin{aligned} J(Y) = {}& -1 + 3b^4 - 2b^6 + (15b^2 - 39b^4 + 24b^6)Y^2 \\ & + (6 + 21b^2 - 15b^4 - 6b^6)Y^4 + (8 - 3b^2 + 27b^4)Y^6 + (3 - 9b^2)Y^8. \end{aligned}$$

Lemma 6.4 *The function $K(Y) = Y(w_b)^{\frac{1}{2}}D(Y)$ is increasing for all $Y \in \mathbb{R}$ and $0 < b < \frac{\sqrt{3}}{3}$. Moreover, $K(0) = 0$ if and only if $Y = 0$.*

Proof The expression of D given by (6.16) can be written as

$$D(Y) = d_6Y^6 + d_4Y^4 + d_2Y^2 + d_0, \tag{6.18}$$

where $d_6 = 3(1 - 3b^2)$, $d_4 = 30b^4 - 7b^2 + 9$, $d_2 = (1 - b^2)(15b^4 + 11b^2 + 4) + (5 - 4b^2)(1 + 2b^2) + 3b^6$, $d_0 = (1 - b^2)(3 + 14b^2 - 20b^4)$. It is easy to verify that all d_i, $i = 0, 2, 4, 6$ are positive, for $0 < b < \frac{\sqrt{3}}{3}$. Taking the derivative of $K(Y)$ with respect to Y, we obtain

$$K'(Y) = (w_b)^{\frac{1}{2}}D(Y) + \frac{Y^2D(Y)}{(w_b)^{\frac{1}{2}}} + (w_b)^{\frac{1}{2}}[6d_6Y^6 + 4d_4Y^4 + 2d_2Y^2]. \tag{6.19}$$

Since the coefficients in (6.18) are all positive for $0 < b < \frac{\sqrt{3}}{3}$ and the powers of Y in (6.18) are all even, we have from (6.19) that $K'(Y) > 0$. We also have from (6.15) and from the positivity of $D(Y)$ that $K(Y) = 0$ if and only if $Y = 0$. This tells us that $K(Y)$ is increasing for all $Y \in \mathbb{R}\ \forall b, 0 < b < \frac{\sqrt{3}}{3}$. ∎

Lemma 6.5 *$Y = 0$ is the unique critical point of the function $J(Y)$ given by (6.17), where $0 < b < \frac{\sqrt{3}}{3}$.*

Proof In fact, taking the derivative of (6.17) with respect to Y, we get the following polynomial of degree 7 in Y,

$$J'(Y) = 6(a_1 + 8b^6)Y + 6(a_2 + a_3)Y^3 + 6(a_4 + 27b^4)Y^5 + 6a_5Y^7, \tag{6.20}$$

where $a_1 = 5b^2 - 13b^4$, $a_2 = 4 - 10b^4$, $a_3 = 14b^2 - 4b^6$, $a_4 = 8 - 3b^2$ and $a_5 = 4 - 12b^2$.

It is easy to prove that a_i, $i = 1, \dots, 5$ are all positive for $0 < b < \frac{\sqrt{3}}{3}$. Since the powers of Y in (6.20) are all odd, we have $J'(Y) > 0$ for $Y > 0$ and $J'(Y) < 0$ for $Y < 0$. Moreover, since the coefficients do not vanish simultaneously, it is clear that $J'(Y) = 0$ if and only if $Y = 0$. Therefore, $J(Y)$ is decreasing for $Y < 0$ and increasing for $Y > 0$ and has $Y = 0$ as a critical point. Moreover, from the fact that $J(0) = -(1 - b^2)^2(1 + 2b^2) < 0$, we have that $J(Y)$ has exactly two real roots. ∎

Lemma 6.6 *The function $B(Y)$ given in (6.1) has a single point of minimum $Y_1 < 0$.*

Proof Again we will prove that the derivative of the function $B'(Y)$ has only one real root, which is negative. From (6.15) we have that $B'(Y) = 0$ if and only if $Y(w_b)^{\frac{1}{2}}[D(Y)] = b[J(Y)]$. From Lemmas 6.4 and 6.5 we infer that the graphs of $K(Y)$ and $J(Y)$ are either as in Figure 8 or as in Figure 9. Suppose that $K(Y)$ and $J(Y)$ have two distinct points $Y_1 < 0$ and $Y_2 > 0$ (as in Figure 8) such that $K(Y_1) = J(Y_1)$ and $K(Y_2) = J(Y_2)$. Then $B'(Y_1) = B'(Y_2)$. We will show that, actually, there exists only one point Y_1 satisfying $K(Y_1) = J(Y_1)$ (as in Figure 9).

It follows from (6.15) that:

(i) $B'(Y) < 0$ when $K(Y) < bJ(Y)$, and this can only occur for $Y < Y_1$.
(ii) $B'(Y) > 0$ when $K(Y) > bJ(Y)$, and this can only occur for $Y_1 < Y < Y_2$.
(iii) $B'(Y) < 0$ when $K(Y) < bJ(Y)$, and this can only occur for $Y > Y_2$.

Thus, under these conditions the graph of $B(Y)$ has the following behavior: $B(Y)$ is decreasing for $Y < Y_1$, $B(Y)$ is increasing for $Y_1 < Y < Y_2$ and $B(Y)$ is decreasing for $Y > Y_2$. It is a contradiction, because from Lemma 6.2 we have that

$$\lim_{Y\to\pm\infty} B(Y) = \infty.$$

Hence the function $B(Y)$ is concave upward and has a unique real root given by Y_1, which is negative (see Figure 9). This completes the proof of this lemma. ∎

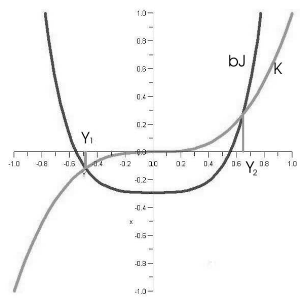

Figure 8: False intersections between the curves K and bJ.

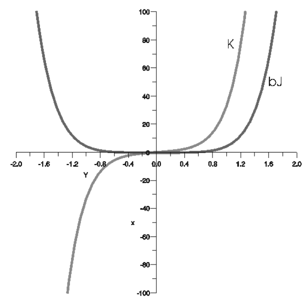

Figure 9: True intersection between the curves K and bJ.

Lemma 6.7 *There exists a region* $\Delta := \{(X,Y) \mid X + x_0 > 0, \overline{Y} < Y < 0\}$ *in which* $\dot{V}(X,Y) < 0$, *where* $x_0 = \frac{1}{\sqrt{1-b^2}H}$ *and* $\overline{Y}$ *is the unique point of the function* $B(Y)$ *such that* $B(Y) = B(0)$.

Proof Consider the expression given by (6.5),

$$\dot{V}(X,Y) = \frac{Y}{X+x_0}\{(X+x_0)[B(0)-B(Y)] + S(Y) - S(0)\}. \tag{6.21}$$

We have that $\frac{1}{X+x_0} > 0$, when $X + x_0 > 0$. In Lemma 6.3, we proved that $S(Y)$ is positive and $Y = 0$ is the global point of minimum of $S(Y)$, *i.e.*, $S(Y)$ is a parabola with vertex in $(0, \frac{1-b^2}{1+2b^2})$. Furthermore, we have from Lemma 6.6 that $B(Y)$ has a global minimum point, which is negative. We also have from Proposition 6.2 that $\lim_{Y\to\pm\infty} B(Y) = +\infty$. Therefore the function $B(Y)$ is a parabola with concavity facing upward and vertex $\big(Y_1, B(Y_1)\big)$ (remember that $B(Y)$ is positive for all $Y \in \mathbb{R}$), then there is a single point $\overline{Y} \in \mathbb{R}$ such that $B(\overline{Y}) = \frac{(1-b^2)^{\frac{3}{2}}H}{1+2b^2} = B(0)$. It follows that, if $\overline{Y} < Y < 0$, then $B(0) - B(Y) > 0$. Hence, for $\overline{Y} < Y < 0$ the expression (6.21) is negative, *i.e.*, $\dot{V}(X,Y) < 0$. ∎

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Lemma 6.8 *There exists a region* $\overline{\Delta} := \{(X,Y) \mid X + x_0 > \frac{S'(Y)}{B'(Y)}, 0 < Y < \infty\}$ *in which* $\dot{V}(X,Y) < 0$.

Proof In order to prove this lemma, we will prove that, if $X + x_0 > \frac{S'(Y)}{B'(Y)}$, $\forall Y > 0$, then $\dot{V}(X,Y) < 0$. Expression (6.21) tells us that if $Y > 0$, then

$$\dot{V}(X,Y) < 0 \Leftrightarrow X + x_0 > \frac{S(Y) - S(0)}{B(Y) - B(0)}. \tag{6.22}$$

Let $Y > 0$ be arbitrary. Observe that the derivatives $S'(Y)$ and $B'(Y)$ are continuous in $(0,\infty)$ and $B'(Y) \neq 0$, for $0 < b < \frac{\sqrt{3}}{3}$. It follows, from the mean value theorem, that there is a number $0 < \overline{\epsilon} < Y$ such that

$$\frac{S'(\overline{\epsilon})}{B'(\overline{\epsilon})} = \frac{S(Y) - S(0)}{B(Y) - B(0)}. \tag{6.23}$$

By hypothesis we have that $X + x_0 > \frac{S'(Y)}{B'(Y)} \forall Y > 0$. In particular, for $0 < \overline{\epsilon} < Y$, we have that $X + x_0 > \frac{S'(\overline{\epsilon})}{B'(\overline{\epsilon})}$. Hence, from (6.23) we conclude that $X + x_0 > \frac{S(Y)-S(0)}{B(Y)-B(0)}$. This is the condition given in (6.22) to have $\dot{V}(X,Y) < 0$. This concludes the proof. ■

Note that for purposes of calculations, we considered the translation $X = x - x_0$, $Y = y$. In what follows we will go back to the original coordinates (x, y). Thus, $\Delta := \{(x,y) \mid x > 0, \overline{y} < y < 0\}$ and $\overline{\Delta} := \{(x,y) \mid x > \frac{S'(y)}{B'(y)}, 0 < y < \infty\}$.

Using Lemma 6.3 and equations (6.9) and (6.11), we can infer the following information on the function $\frac{S'(y)}{B'(y)}$, where the numerator and the denominator are polynomials in y of degree 11 and 12 respectively.

$$\frac{S'(0)}{B'(0)} = 0, \quad \frac{S'(y)}{B'(y)} > 0, \quad \forall y > 0, \quad \lim_{y\to\infty} \frac{S'(y)}{B'(y)} = 0.$$

It follows, from Lemmas 6.7 and 6.8 and from equation (6.21), that $\dot{V}(x,y) \leq 0$ (equality follows from the fact that $\dot{V}(x,0) = 0$) for all $\overline{y} < y < \infty$, where $\overline{y} = \overline{Y}$. Hence, it follows from the Liapunov theorem [LSL] that the equilibrium point $(x_0, 0)$ is globally stable within the region

$$\begin{aligned} \overline{G} &= \{(x,y) \in \mathbb{R} : 0 \leq x \leq c, \overline{y} \leq y \leq 0\} \\ &\cup \left\{ (x,y) \in \mathbb{R} : \frac{S'(y)}{B'(y)} \leq x \leq c, 0 < y \leq y_2 \right\}, \end{aligned}$$

where c and y_2 are arbitrary positive real numbers (this region is visualized in Figure 10). In what follows we will show that, indeed, in this region it is globally asymptotically stable.

Theorem 6.9 *The equilibrium point* $(x_0, 0)$ *of the system* (6.2) *is globally asymptotically stable within the region* $\overline{G}$.

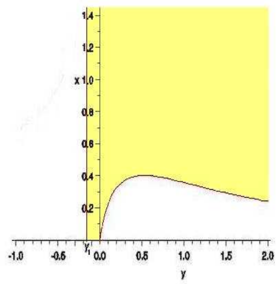

Figure 10: In this figure, we visualize the region G, that is the basin of attraction of the equilibrium point $(\frac{1}{\sqrt{1-b^2H}}, 0)$, where we used $b = 0.3$ and $H = 1$.

Proof Let G be the set consisting of the interior points of $\overline{G}$. Thus G contains the solutions of the system (6.2). Let $E = \{(x, y) \in G : \dot{V}(x, y) = 0\}$. Note that we can rewrite the set E in the following form, $E = \{(x, 0) \in G : x > 0\}$. Thus, $(x_0, 0) \in E$. This means that the trajectories contained in E are such that $y(t) = 0$, $\forall t \in \mathbb{R}$. It follows from Lemma 4.6 that the unique solution satisfying $y(t) = 0$, $\forall t \in \mathbb{R}$ is the constant solution $x(t) = \frac{1}{\sqrt{1-b^2H}} = x_0$. By the definition of invariant set (see [W]), we have that $\{(x_0, 0)\}$ is the bigger (because it is the unique) invariant set contained in E. Therefore, by the Liapunov–La Salle Theorem (see [LSL]), all solutions $\big(x(t), y(t)\big)$ in G tend to $\{(x_0, 0)\}$ when t tends to infinity, *i.e.*, $(x_0, 0)$ is globally asymptotically stable in G. This proves the theorem. ∎

We finally observe that in the set

$$G = \{(x, y) \in \mathbb{R} : 0 < x < c, \overline{y} < y \leq 0\} \\ \cup \left\{ (x, y) \in \mathbb{R} : \frac{S'(y)}{B'(y)} < x \leq c, 0 < y < y_2 \right\},$$

one has that $\dot{V} \leq 0$. Moreover, $\dot{V}$ vanishes only when $y = 0$, because we are excluding the other set of points where $\dot{V}$ vanishes, which belong to the boundary of G. This set is given by $\overline{\Delta} = \{(x, y) : x = \frac{S'(y)}{B'(y)}, 0 < y < \infty\}$. Thus, the equality $E = \{(x, 0) \in G : \dot{V}(x, y) = 0\} = \{(x, 0) \in G : x > 0\}$ holds.

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