

QUOTIENTS OF L -FUNCTIONS

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Abstract. In this paper a certain type of Dirichlet series, attached to a pair of Jacobi forms and Siegel modular forms is studied. It is shown that this series can be analyzed by a new variant of the Rankin-Selberg method. We prove that for eigenforms the Dirichlet series have an Euler product and we calculate all the local L -factors. Globally this Euler product is essentially the quotient of the standard L -functions of the involved Jacobi- and Siegel modular form.

Introduction

Let $F, G \in S_2^k$ be two Siegel cusp forms of weight k and degree 2. It had been discovered by Kohnen and Skoruppa [K-S89] that the Dirichlet series

$$(1) \quad D_{F,G}^{\text{KS}}(s) = \zeta(2s - 2k + 4) \sum_{N=1}^{\infty} \langle \Phi_N^F, \Phi_N^G \rangle_{\mathcal{J}} N^{-s},$$

where Φ_N^F, Φ_N^G are the N^{th} coefficients of the Fourier-Jacobi expansion of F and G , respectively and $\langle \cdot, \cdot \rangle_{\mathcal{J}}$ denotes the Petersson scalar product on Jacobi cusp forms, can be studied by the Rankin-Selberg method. Moreover they proved, that if F is a Hecke eigenform and G in the Maass space, then $D_{F,G}^{\text{KS}}(s)$ is proportional to the spinor zeta function of F , i.e.,

$$(2) \quad D_{F,G}^{\text{KS}}(s) = \langle \Phi_1^F, \Phi_1^G \rangle_{\mathcal{J}} Z_F(s).$$

In this paper we study Dirichlet series $D_{\Phi,F}^{\circ}(s)$ attached to Jacobi cusp forms Φ on $\mathbb{H} \times \mathbb{C}$ and Siegel cusp forms $F \in S_2^k$ of degree 2 and even weight k of formally similar type, but of surprisingly different properties. Let U_{λ} be the operator $\Phi(\tau, z) \mapsto \Phi(\tau, \lambda z)$. Then by definition

$$(3) \quad D_{\Phi,F}^{\circ}(s) = \sum_{\lambda=1}^{\infty} \langle \Phi|U_{\lambda}, \Phi_{t\lambda^2}^F \rangle_{\mathcal{J}} \lambda^{-(2s+2k-4)},$$

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where $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$, i.e., Φ is a Jacobi cusp form of weight k and index t . We employ the method introduced in [He97], to obtain a Rankin type integral representation of $D_{\Phi,F}^{\diamond}(s)$. Thus analytic and arithmetic properties can be deduced from a certain kind of Jacobi Eisenstein series. In contrast to other generalizations of Kohnen and Skoruppa’s work e.g. Yamazaki [Ya90] which do not lead to an Euler product, we can prove the following: Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ a Hecke-Jacobi newform (cf. Section 5.1), then $D_{\Phi,F}^{\diamond}(s)$ has an Euler product. More precisely, let $D_F(s)$ and $L(s, \Phi)$ be the standard L -functions attached to F and Φ . Then

THEOREM. *Let $k, t \in \mathbb{N}$ and let k be even. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform. Then*

$$D_{\Phi,F}^{\diamond}(s) = \langle \Phi, \Phi_t^F \rangle_{\mathcal{J}} \zeta(4s + 2k - 4)^{-1} \times D_F(2s + k - 2)L(2s + 2k - 3, \Phi)^{-1}.$$

Hence in contrast to Kohnen and Skoruppa’s Dirichlet series we do not need the existence of a Maass space to get an Euler product, which gives some hope for generalization to higher degrees. Moreover the Euler product in the theorem involves L -functions of Φ and F . This is not the case for $D_{F,G}^{\text{KS}}(s)$. Let the index of the Jacobi form be one, then our results have some direct relation to the work of Murase and Sugano [M-S91]. Let $X = \{s \in \mathbb{C} \mid 2 \operatorname{Re}(s) + k > 5\}$ and $\mathcal{H}(X)$ the vector space of all holomorphic functions on X . Then the construction of the Dirichlet series $D_{\Phi,F}^{\diamond}(s)$ can be interpreted as a bilinear map

$$\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k \longrightarrow \mathcal{H}(X),$$

which can be continued to $\mathcal{J}_k^{\text{cusp}} = \bigoplus_{t=1}^{\infty} \mathcal{J}_{k,t}^{\text{cusp}}$, the *Jacobi-Siegel pairing*. This pairing can be used to study either standard L -functions of Siegel modular forms of degree 2 or Jacobi forms of arbitrary index. It follows from the work of [He98], that every analytic Klingen-Jacobi Eisenstein series attached to a Hecke-Jacobi eigenform $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ has a meromorphic continuation on the whole complex plane. Thus the image of $\Phi \times S_2^k$ has a meromorphic continuation. This means for example, that the image of $\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k$ for t square free has the same property.

Finally we would like to mention, that we believe that our results can be generalized to Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{n,1}$ and Siegel modular forms of degree n . But for this more knowledge on the involved Hecke-Jacobi theory has to be obtained.

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Table of contents.

- 1 Automorphic forms.
- 1.1 Review of Siegel modular forms.
- 1.2 Jacobi forms.
- 2 Jacobi-Siegel pairing.
- 3 Hecke-Jacobi theory.
- 4 Euler products.
- 5 Quotients of L -functions.
- 5.1 Jacobi newforms.
- 5.2 Local L -factors for newforms.
- 5.3 Main results.

Notation: For an associative ring R with identity element, we denote by R^\times the group of all invertible elements of R . If M is a matrix, M^t , $\det(M)$, and $\text{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R) = R^{n,n}$, $Gl_n(R) = M_n(R)^\times$. The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the symplectic group of degree n is defined by $Sp_n(R) = \{M \in Gl_{2n}(R) \mid M^t J_n M = J_n\}$.

$$P_{n,r}(R) = \left\{ \begin{pmatrix} \alpha & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\},$$

$$C_{n,r}(R) = \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & \alpha \end{pmatrix} \in Sp_n(R) \right\}.$$

Let $P_{n,r}^J, C_{n,r}^J$ be the subgroups of $P_{n,r}$ and $C_{n,r}$ respectively, where $\alpha = 1_{n-r}$. Let R be a subring of \mathbb{R} . Then $R^+ = \{r \in R \mid r > 0\}$ and

$$G^+ Sp_n(R) = \{M \in Gl_{2n}(R) \mid M^t J_n M = \mu(M) J_n \text{ with } \mu(M) \in R^+\}.$$

For real symmetric matrices A and B , we put $A[B] = B^t A B$ if A, B are suitable. If A_1, A_2, \dots, A_n are square matrices, $[A_1, A_2, \dots, A_n]$ denotes the matrix with A_1, A_2, \dots, A_n in the diagonal blocks and 0 in all other blocks. Let $Z \in \mathbb{C}^{n,n}$, then we put $e\{Z\} = e^{2\pi i \text{tr}(Z)}$ and $\text{Re}(Z), \text{Im}(Z)$ for the real and imaginary part of Z . Further let $\delta(Z) = \det(\text{Im}(Z))$.

§1. Automorphic forms

1.1. Review of Siegel modular forms

The group of positive symplectic similitudes $G^+Sp_n(\mathbb{R})$ acts on Siegel’s half space $\mathbb{H}_n = \{Z = Z^t \in M_n(\mathbb{C}) \mid \text{Im}(Z) > 0\}$ of degree n as a group of biholomorphic automorphisms by

$$(M, Z) \mapsto M(Z) = (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_n(\mathbb{R})$. We denote the factor of automorphy by $j(M, Z) = \det(CZ + D)$. For $F : \mathbb{H}_n \rightarrow \mathbb{C}$ and $k \in \mathbb{Z}$ we define the Petersson operator

$$(4) \quad (F|_k M)(Z) := \mu(M)^{nk-n(n+1)/2} j(M, Z)^{-k} F(M(Z)).$$

Let us denote by M_n^k the space of Siegel modular forms and by S_n^k its subspace of cuspforms of degree n and weight k for $\Gamma_n = Sp_n(\mathbb{Z})$. Let $\langle \cdot, \cdot \rangle$ denote the Petersson scalar product, i.e., for arbitrary complex valued functions F and G on \mathbb{H}_n , which satisfy the same transformation law as modular forms the Petersson integral, convergence assumed, it is given by

$$(5) \quad \langle F, G \rangle = \int_{\Gamma_n \backslash \mathbb{H}_n} F(Z) \overline{G(Z)} \det(\text{Im } Z)^k d^*Z.$$

Here $d^*Z = \det(Y)^{-(n+1)} dXdY$ denotes the symplectic volume element. For more details the reader is referred to Klingen [KI90].

Let $F \in S_n^k$ be a Hecke eigenform with Satake parameter $(\alpha_{0,p}; \alpha_{1,p}; \dots; \alpha_{n,p})$. Then the standard zeta function $D_F^n(s)$ of F is given by

$$(6) \quad D_F^n(s) = \prod_p \{D_{p,F}(p^{-s})\}^{-1},$$

where the Rankin polynomial is

$$(7) \quad D_{p,F}(X) = (1 - X) \prod_{j=1}^n (1 - \alpha_{j,p}X)(1 - \alpha_{j,p}^{-1}X).$$

1.2. Jacobi forms

Let $k, n, t \in \mathbb{N}$. Then we denote by $\mathcal{J}_{k,t}^n$ and $\mathcal{J}_{k,t}^{n,\text{cusp}}$ the space of Jacobi forms and Jacobi cusp forms, respectively, on $\mathcal{D}_{n,1} = \mathbb{H}_n \times \mathbb{C}^{1,n}$. We shall write $\langle \cdot, \cdot \rangle_{\mathcal{J}}$ for the Petersson scalar product of Jacobi forms. Let

$((\lambda, \mu, \rho), M)$ be the parameterization of $g \in P_{n+1,n}^J(\mathbb{Z})$ as given in [A-H98, Section 1.1]. Then $\Phi \in \mathcal{J}_{k,t}^n$ satisfies

$$(8) \quad \Phi(\tau, z) = j_{k,t}(g, (\tau, z))^{-1} \Phi(g(\tau, z)).$$

Here $g(\tau, z) = (M(\tau), zJ(M, \tau)^{-1} + \lambda M(\tau) + \mu)$, where $J(M, \tau) = c\tau + d$ and

$$(9) \quad j_{k,t}(g, (\tau, z)) = j(M, \tau)^k e\{-t\rho\} \times e\{-t[\lambda]M(\tau) - 2\lambda^t t z J(M, \tau)^{-1} + t[z]J(M, \tau)^{-1}c\}.$$

At the same time we could also consider Φ as a Jacobi form with respect to $C_{n+1,n}^J(\mathbb{Z})$. Moreover let us introduce the projection map

$$(10) \quad *_{J,r} : \begin{cases} \mathcal{D}_{n,1} & \longrightarrow & \mathcal{D}_{r,1} \\ (\tau, z) & \longmapsto & \left(\tau \begin{bmatrix} 0 \\ 1_r \end{bmatrix}, z \begin{bmatrix} 0 \\ 1_r \end{bmatrix} \right), \end{cases}$$

which is a generalization of the projection $\mathbb{H}_n \rightarrow \mathbb{H}_r$, where $\tau \mapsto \tau_* = \tau \begin{bmatrix} 0 \\ 1_r \end{bmatrix}$. The groups

$$(11) \quad G_{n,1,r}^J(\mathbb{R}) = \left\{ ((0\lambda_2, \mu, \rho), M) \mid \begin{array}{l} \lambda_2 \in \mathbb{R}^{1,r}, \mu \in \mathbb{R}^{1,n}, \rho \in \mathbb{R}^{1,1} \\ \text{and } M \in P_{n,r}(\mathbb{R}). \end{array} \right\},$$

for $r \leq n$, are involved in the definition of Eisenstein series.

To simplify our notation we put $\mathcal{J}_{k,t}^{\text{cusp}} = \mathcal{J}_{k,t}^{1,\text{cusp}}$, $\mathbb{H} = \mathbb{H}_1$, $\Gamma_{n,1,r}^J = G_{n,1,r}^J(\mathbb{Z})$ and $\Gamma_{n,1}^J = G_{n,1,n}^J(\mathbb{Z})$. Jacobi cusp forms $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ of index t on $\mathbb{H} \times \mathbb{C}$ can be completed to functions $\widehat{\Phi}$ on \mathbb{H}_2 via

$$(12) \quad \Phi(\tau, z) \longmapsto \widehat{\Phi} \begin{pmatrix} \tau' & z \\ z & \tau \end{pmatrix} = \Phi(\tau, z) e\{t\tau'\}.$$

Then $\widehat{\Phi}|_k g = \widehat{\Phi}$ for $g \in P_{2,1}^J(\mathbb{Z})$. Further if we would put $\widehat{\Phi} \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \Phi(\tau, z) e\{t\tau'\}$, then $\widehat{\Phi}|_k g = \widehat{\Phi}$ for $g \in C_{2,1}^J(\mathbb{Z})$.

We denote the space of completed Jacobi cusp forms of index t and weight k by $\widehat{\mathcal{J}}_{k,t}^{\text{cusp}}$. On this space the Petersson scalar product is simulated by

$$(13) \quad \langle \widehat{\Phi}, \widehat{\Psi} \rangle_{\mathcal{A}} = \int_{P_{2,1}^J(\mathbb{Z}) \backslash \mathbb{H}_2} \widehat{\Phi}(Z) \overline{\widehat{\Psi}(Z)} \det(\text{Im } Z)^k d^* Z,$$

where $\Phi, \Psi \in \mathcal{J}_{k,t}^{\text{cusp}}$. We have $\langle \Phi, \Psi \rangle_{\mathcal{J}} = \beta_k t^{k-2} \langle \widehat{\Phi}, \widehat{\Psi} \rangle_{\mathcal{A}}$. Here $\beta_k = (4\pi)^{k-2} \Gamma(k-2)^{-1}$. It is convenient to alternate frequently between the two notations Φ and $\widehat{\Phi}$ of a Jacobi form.

Next we state the definition of an analytic Jacobi Eisenstein series.

DEFINITION 1.1. Let $k, t, n \in \mathbb{N}$ with k even, $0 \leq r \leq n$. To $\Phi \in \mathcal{J}_{k,t}^{r,\text{cusp}}$ we attach an analytic Jacobi Eisenstein series of Klingen type on $\mathcal{D}_{n,1} = \mathbb{H}_n \times \mathbb{C}^{1,n}$ defined by:

$$(14) \quad E_{n,r}^{k,t}((\tau, z), \Phi; s) = \sum_{\gamma \in \Gamma_{n,1,r}^J \backslash \Gamma_{n,1}^J} \Phi(\gamma(\tau, z)_{*J}) j_{k,t}(\gamma, (\tau, z))^{-1} \left(\frac{\delta(M(\tau))}{\delta(M(\tau)_*)} \right)^s,$$

here $\gamma = (h, M)$ and $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large. If $r = 0$ we denote by $E_n^{k,t}((\tau, z); s) = E_{n,0}^{k,t}((\tau, z), 1; s)$ and $E_n^{k,t}((\tau, z)) = E_n^{k,t}((\tau, z); 0)$ the (analytic) Siegel-Jacobi Eisenstein series.

The Eisenstein series is absolutely convergent for $k + 2 \text{Re}(s) > n + r + 2$. We have proven in [He97], that under certain conditions the Klingen Jacobi Eisenstein series has a meromorphic continuation on the whole complex plane. For example, the conditions are satisfied for Hecke-Jacobi eigenforms $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$. This has been proven recently in [He98]. If $k > n + r + 2$, then $E_{n,r}^{k,t}((\tau, z), \Phi) = E_{n,r}^{k,t}((\tau, z), \Phi; 0) \in \mathcal{J}_{k,t}^n$.

§2. Jacobi-Siegel pairing

We introduce a bilinear map from $\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k$ to the space $\mathcal{H}(X)$ of holomorphic complex-valued functions on $\{s \in \mathbb{C} \mid 2 \text{Re}(s) + k > 5\}$. It will turn out that these functions will have a meromorphic continuation on the whole complex plane, if we restrict ourselves to the subspace generated by Hecke-Jacobi eigenforms. Moreover these complex-valued functions can be described essentially as the quotient of L -series.

Let $\Phi, \Psi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $\tau' \in \mathbb{H}$, then we put $\widehat{\Phi}(Z) = \widehat{\Phi}\left(\begin{smallmatrix} \tau & z \\ z' & \tau' \end{smallmatrix}\right) = \Phi(\tau', z) e\{t\tau\}$. In (13) and [He99, Definition 3.10], we have introduced a Petersson scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ for the so called P-forms $\widehat{\Phi}$ on \mathbb{H}_2 . We have

$$\langle \Phi, \Psi \rangle_J = \beta_k t^{k-2} \langle \widehat{\Phi}, \widehat{\Psi} \rangle_{\mathcal{A}},$$

where $\beta_k = (4\pi)^{k-2} \Gamma(k-2)^{-1}$ (see also Section 1.2). Moreover if A is a linear operator on the graded algebra of Jacobi cusp forms, then we denote

by \tilde{A} the corresponding operator on the graded space of P-forms. Let us denote the Fourier-Jacobi expansion of F by

$$F \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \sum_{m=1}^{\infty} \Phi_m^F(\tau', z) e\{m\tau\}.$$

Moreover let U_λ be the operator $\Phi(\tau, z) \mapsto \Phi(\tau, \lambda z)$ and \tilde{U}_λ the corresponding one on the space of P-forms.

THEOREM 2.1. *Let $k, t \in \mathbb{N}$ and $k > 5$ be even and $2 \operatorname{Re}(s) + k > 5$. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $F \in S_2^k$. Then*

$$\begin{aligned} JS(\Phi, F; s) &:= \langle E_{2,1}^{k,t}((*, 0), \Phi; s), F(*) \rangle \\ &= (4\pi)^{-(s+k-2)} \Gamma(s+k-2) \sum_{\lambda=1}^{\infty} \langle \Phi | U_\lambda, \Phi_{t\lambda^2}^F \rangle_{\mathcal{J}} (t\lambda^2)^{-(s+k-2)}. \end{aligned}$$

Proof. Before we start to compute $JS(\Phi, F; s)$ explicitly, it is convenient to simplify the (restricted) Eisenstein series

$$E_{2,1}^{k,t}((Z, 0), \Phi; s) = \sum_{\gamma \in \Gamma_{2,1,1}^J \setminus \Gamma_{2,1}^J} \Phi(\gamma(Z, 0)_{*J}) j_{k,t}(\gamma, (Z, 0))^{-1} \left(\frac{\delta(M(Z))}{\delta(M(Z)_*)} \right)^s.$$

Here $\gamma = (h, M)$. We choose the complete representative system

$$\bigcup_{\lambda \in \mathbb{Z}} (\lambda 0, 00; 0) P_{2,1} \setminus \Gamma_2$$

of $\Gamma_{2,1,1}^J \setminus \Gamma_{2,1}^J$. Let $h = (\lambda 0, 00; 0)$ be in the Heisenberg group, then it is easy to see, that

$$\begin{aligned} \Phi(h(Z, 0)_{*J}) &= \Phi(\tau', \lambda z), \\ j_{k,t}(h, (Z, 0))^{-1} &= e\{\lambda^2 t\tau\}. \end{aligned}$$

Now we are ready to compute $JS(\Phi, F; s)$. After *unwinding* we reach to

$$JS(\Phi, F; s) = \sum_{\lambda=-\infty}^{\infty} \int_{P_{2,1} \setminus \mathbb{H}_2} \Phi(\tau', \lambda z) e\{\lambda^2 t\tau\} \overline{F(Z)} \delta(Z)^{s+k-3} \delta(\tau')^{-s} dZ$$

To use the formalism and reduction theory given in Heim [He99, §3.4], it is convenient to exchange $P_{2,1}$ by $C_{2,1}$. Moreover let $w \in \mathbb{C}$, then x_w and y_w denote the real and imaginary part of w . This leads to

$$\begin{aligned}
 JS(\Phi, F; s) &= 2 \sum_{\lambda=1}^{\infty} \int_{C_{2,1} \setminus \mathbb{H}_2} \Phi(\tau, \lambda z) e\{\lambda^2 t \tau'\} \overline{\Phi_{t\lambda^2}^F(\tau, z) e\{t\lambda^2 \tau'\}} \\
 &\quad \times \delta \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}^{k+s-3} \delta(\tau)^{-s} d \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \\
 &= 2 \sum_{\lambda=1}^{\infty} \int_{\mathcal{B}_{1,1}} d\tau dz \Phi(\tau, \lambda z) \overline{\Phi_{t\lambda^2}^F(\tau, z)} \\
 &\quad \times \int_{y_{\tau'} > y_{\tau}^{-1}[y_z]} dy_{\tau'} \int_0^1 dx_{\tau'} e^{-4\pi t \lambda^2 y_{\tau'} y_{\tau}^{-s}} \delta \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}^{k+s-3}.
 \end{aligned}$$

Let $\mathcal{B}_{1,1}$ be a fundamental domain of the action of $(C_{2,1}^J(\mathbb{Z})/\text{center})$ on $\mathbb{H} \times \mathbb{C}$. Then

$$(15) \quad \mathcal{Q}_{1,1} = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2 \mid (\tau, z) \in \mathcal{B}_{1,1} \text{ and } |x_{\tau'}| \leq 1/2 \right\}$$

is a fundamental domain of the action of $C_{2,1}^J(\mathbb{Z})$ on \mathbb{H}_2 .

Next we substitute $y_{\tau'}$ by $y + y_{\tau}^{-1}[y_z]$. Hence we get

$$\begin{aligned}
 JS(\Phi, F; s) &= 2 \sum_{\lambda=1}^{\infty} \int_{\mathcal{B}_{1,1}} d\tau dz \Phi(\tau, \lambda z) \overline{\Phi_{t\lambda^2}^F(\tau, z)} \\
 &\quad \times y_{\tau}^{k-3} e^{-4\pi t \lambda^2 y_{\tau}^{-1}[y_z]} \int_0^{\infty} \frac{dy}{y} y^{s+k-2} e^{-4\pi t \lambda^2 y}.
 \end{aligned}$$

After some obvious simplifications we get the desired result. □

Putting everything together shows that the Jacobi-Siegel pairing

$$\begin{aligned}
 \mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k &\longrightarrow \mathcal{H}(X) \\
 (\Phi, F) &\longmapsto JS(\Phi, F; s)
 \end{aligned}$$

leads to a Dirichlet series (which has a meromorphic continuation on the whole complex plane, if we assume t to be square free or if we restrict ourselves to $\mathcal{J}_{k,t}^{\text{cusp,new}}$).

§3. Hecke-Jacobi theory

Let $\mathcal{R} = \mathbb{Z}, \mathbb{Z}[\frac{1}{p}]$ or \mathbb{Q} . We put $G_{\mathcal{R}}^J = \{(\begin{smallmatrix} * & & & \\ & 0 & 0 & \beta \end{smallmatrix}) \in C_{2,1}(\mathcal{R}) \mid \beta > 0\}$, $H_{\mathcal{R}} = \{(X, \kappa) \mid X \in \mathcal{R} \times \mathcal{R}, \kappa \in \mathcal{R}\}$, $\Gamma_{\mathcal{R}}^J = G^+Sp_1(\mathcal{R}) \times \mathcal{R}^+$ and $\Gamma^J = C_{2,1}^J(\mathbb{Z})$. Then the exact sequence

$$(16) \quad 1 \longrightarrow H_{\mathcal{R}} \xrightarrow{\varphi} G_{\mathcal{R}}^J \xrightarrow{p} \Gamma_{\mathcal{R}}^J \longrightarrow 1$$

with

$$\varphi(\lambda, \mu, \kappa) = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } p \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & \alpha & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & \beta \end{pmatrix} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta \right)$$

splits. Hence $G_{\mathcal{R}}^J$ can be viewed as the semi-direct product of $H_{\mathcal{R}}$ with $\Gamma_{\mathcal{R}}^J$. We consider $H_{\mathcal{R}}, \Gamma_{\mathcal{R}}^J$ as subgroups of $G_{\mathcal{R}}^J$. Further we denote by $\mathcal{H}^n, \mathcal{H}^J, \mathcal{H}_p^J$ the Hecke algebras of the Hecke pairs $(\Gamma_n, G^+Sp_n(\mathbb{Q}))$, $(\Gamma^J, G_{\mathbb{Q}}^J)$ and $(\Gamma^J, G_{\mathbb{Z}[\frac{1}{p}]}^J)$. It is known that \mathcal{H}^n is commutative and has no zero divisors in contrast to \mathcal{H}^J . Several maps $*$, j_-, j_+ will be used to study \mathcal{H}^J . We start with the important map $*$. It is an anti-automorphism of \mathcal{H}^J given by

$$(17) \quad \Gamma^J(h; (M, \beta))\Gamma^J \longmapsto (\Gamma^J(h; (M, \beta))\Gamma^J)^* = \Gamma^J \mu(M)(h; (M, \beta))^{-1}\Gamma^J,$$

where $(h; (M, \beta))$ is the parametrization of $g \in G_{\mathbb{Q}}^J$ via the splitting of (16). This map somehow simulates the rule how to construct the adjoint operator of a Hecke operator with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. Let us put $\Gamma = \Gamma_1$. We have two algebra monomorphism

$$(18) \quad j_- : \begin{cases} \mathcal{H}^1 & \longrightarrow & \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \longmapsto & \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, 1 \right) \Gamma^J \end{cases},$$

$$(19) \quad j_+ : \begin{cases} \mathcal{H}^1 & \longrightarrow & \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \longmapsto & \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, ad \right) \Gamma^J \end{cases}.$$

We have the relation $j_+(X) = j_-(X)^*$ for $X \in \mathcal{H}^1$. This means that $j_+(X)$ is the adjoint operator of $j_-(X)$.

Let $T(n, n) = \Gamma[n, n]\Gamma$. We introduce some elements of \mathcal{H}^J . Let $r \in \mathbb{Q}$ and $n \in \mathbb{N}$:

$$\begin{aligned} T_-^g(n) &:= j_-(T(n)), & T_+^g(n) &:= j_+(T(n)), \\ \Lambda_-^g(n) &:= j_-(T(n, n)), & \Lambda_+^g(n) &:= j_+(T(n, n)), \\ \nabla(r) &:= \Gamma^J((0; r); (1_2, 1))\Gamma^J, & \Delta_n &:= \Gamma^J[n, n, n, n]\Gamma^J, \\ \nabla_n^g &:= \Delta_n \sum_{b \bmod n} \nabla\left(\frac{b}{n}\right), & \Xi_n^g &:= \Delta_n \sum_{\lambda, \mu, \kappa \bmod n} \Gamma^J\left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\kappa}{n}\right), \\ T^{J,g}(n) &:= \Gamma^J[1, n, n^2, n]\Gamma^J. \end{aligned}$$

We also put $\nabla_n^r = (\Delta_n)^{-1}\nabla_n^g$, $\Xi_n^r = (\Delta_n)^{-1}\Xi_n^g$, $\Lambda_\pm^r(n) = (\Delta_n)^{-1}\Lambda_\pm^g(n)$ and $T^{J,r}(n) = (\Delta_n)^{-1}T^{J,g}(n)$ (as a rule, we do this only when it is convenient).

Proposition 3.4 in [He99] states, that the elements T_-^g , T_+^g , $T^{J,g}$, ∇^g , Ξ^g , Λ_-^g , Λ_+^g and Δ in \mathcal{H}^J commute with each other when we only allow co-prime arguments. Moreover the functions $\Lambda_-^g(n)$, $\Lambda_+^g(n)$ and Δ_n are strong multiplicative.

The subalgebra $\tilde{\mathcal{H}}^J$ of \mathcal{H}^J generated by T_-^g , T_+^g , $T^{J,g}$, ∇^g , Ξ^g , Λ_-^g , Λ_+^g , Δ is called Hecke-Jacobi algebra. The local Hecke-Jacobi algebra is given by $\tilde{\mathcal{H}}_p^J = \mathcal{H}_p^J \cap \tilde{\mathcal{H}}^J$. We have

$$(20) \quad \tilde{\mathcal{H}}^J = \bigotimes_p \tilde{\mathcal{H}}_p^J.$$

Moreover let $\tilde{\mathcal{H}}_0^J$ be the subalgebra generated by $T^{J,g}$, ∇^g , Ξ^g and Δ_n and $\tilde{\mathcal{H}}_{0,p}^J = \tilde{\mathcal{H}}_0^J \cap \tilde{\mathcal{H}}_p^J$.

Remark 3.1. The Hecke-Jacobi algebra $\tilde{\mathcal{H}}^J$ is not commutative and has zero divisors, because $\Lambda_-^g(p) \cdot (\nabla_p^r - p) = 0$ and $\Lambda_-^g(p)T_+^g(p) \neq T_+^g(p)\Lambda_-^g(p)$.

The heart of our considerations is the following result proven in [He99, Section 3.3].

THEOREM 3.2. *The Rankin polynomial $d_p^2(X)$ has the following factorization in $\tilde{\mathcal{H}}_p^J[X]$:*

$$(21) \quad d_p^2(X) = (1 - X)(1 - p^{-2}\Lambda_-^r(p)X)S^{(2)}(X)(1 - p^{-2}\Lambda_+^r(p)X)$$

with $S^{(2)}(X) = \sum_{j=0}^3 (-1)^j S_j^{(2)} X^j$. Here

$$S_0^{(2)} = 1,$$

$$\begin{aligned}
 S_1^{(2)} &= p^{-2}(T^{J,r}(p) + \nabla_p^r - p^2), \\
 S_2^{(2)} &= p^{-3}(T^{J,r}(p)(\nabla_p^r - p) + \Xi_p^r - p\nabla_p^r + p^2), \\
 S_3^{(2)} &= p^{-2}(\nabla_p^r - p).
 \end{aligned}$$

§4. Euler products

We call $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ a weak Hecke-Jacobi eigenform, if Φ is an eigenform for all $T^{J,r}(p)$, where $(p, t) = 1$. It is known that $\mathcal{J}_{k,t}^{\text{cusp}}$ has a basis of weak Hecke-Jacobi eigenforms. At this point we are satisfied with this definition, but later on we have to assume stronger conditions on Φ .

In this section we show that the Dirichlet series obtained in Theorem 2.1

$$(22) \quad D_{\Phi,F}(s) = \sum_{\lambda=1}^{\infty} \langle \widehat{\Phi}|_k \widetilde{U}_\lambda, \widehat{\Phi}_{t\lambda^2}^F \rangle_{\mathcal{A}} (t\lambda^2)^{-s}$$

can be written essentially as an Euler product times a function which only depends on the ‘ramified part of $D_{\Phi,F}(s)$ ’, if F is a Hecke eigenform and Φ a weak Hecke-Jacobi eigenform. The proof depends on the Hecke-Jacobi theory developed in [He99, §3]. Moreover, the following results can also be considered as an extension and application of the Hecke-Jacobi theory summarized in the last section.

We start with a representation of the Hecke-Jacobi algebra. Let $F : \mathbb{H}_2 \rightarrow \mathbb{C}$ be invariant under the $|_k$ -action of Γ^J , then for $X = \sum a_j \Gamma^J g_j \in \widetilde{\mathcal{H}}^J$ we put

$$(23) \quad F|_k \mathcal{D}(X) := \sum_j a_j (F|_k g_j).$$

DEFINITION 4.1. Let $p \nmid t$ and let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform. Let $\widehat{\Phi}|_k \mathcal{D}(T^{J,r}(p)) = \lambda \widehat{\Phi}$ and $\lambda^{\text{EZ}} := p^{k-3} \lambda$. We define

$$\widetilde{L}_p^{\text{EZ}}(s, \Phi) = 1 - \lambda^{\text{EZ}} p^{-s} + p^{2k-3} p^{-2s}.$$

The following observations will turn out to provide a transparent interpretation of how these L -factors occur in a natural way.

Remark 4.2. In [He99, Sections 3.3 and 3.4], certain operators $S^{(2)}(X)^{\text{factor}}$ and $S^{(2)}(X)^{\text{n.prim}}$ have been introduced. They are closely related to $S^{(2)}(X)$. For instance we have

$$S^{(2)}(X)^{\text{factor}} = 1 - p^{-2} T^{J,r}(p) X + p^{-1} X^2.$$

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $(p, t) = 1$. Then the action of $S^{(2)}(X)$ is given by

$$(24) \quad \widehat{\Phi}|_k \mathcal{D}(S^{(2)}(p^{-s+k-1})) = (1 + p^{-s+k-1}) \widehat{\Phi}|_k \mathcal{D}(S^{(2)}(p^{-s+k-1})_{\text{factor}}) \\ = (1 + p^{-s+k-1}) \widetilde{L}_p^{\text{EZ}}(s, \Phi) \widehat{\Phi}.$$

Let $\Phi \in \mathcal{J}_{k,1}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform and let $\varphi \in S_1^{2k-2}$ be the corresponding elliptic cusp form by the Saito-Kurokawa lift, then

$$(25) \quad \widehat{\Phi}|_k \mathcal{D}(S^{(2)}(p^{-2s-k+2})) = (1 + p^{-2s-k+2}) L_p(2s + 2k - 3, \varphi) \widehat{\Phi},$$

where $L_p(s, \varphi)$ denotes the local L -factor of the Hecke L -series of φ .

THEOREM 4.3. *Let $k, t \in \mathbb{N}$ and $k > 5$ and $2\text{Re}(s) + k > 5$. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform. Let $t = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_d^{\alpha_d}$ be the prime number decomposition of t , then*

$$(26) \quad D_{\Phi,F}(s) = \zeta(2s + k - 2)^{-1} D_F(2s + k - 2) \\ \times \left(\prod_{p \nmid t} (1 + p^{-2s-k+2}) \widetilde{L}_p^{\text{EZ}}(2s + 2k - 3, \Phi) \right) \\ \times \sum_{\delta_1, \dots, \delta_d=0}^{\infty} \left\langle \widehat{\Phi}, \widehat{\Phi}_{tp_1^{2\delta_1} \dots p_d^{2\delta_d}}^F |_k \mathcal{D}(\Lambda_+^g(p_1^{\delta_1} \cdots p_d^{\delta_d})) \right\rangle_{\mathcal{A}} \\ \times (p_1^{\delta_1} \cdots p_d^{\delta_d})^{-2s+6-3k}.$$

COROLLARY 4.4. *Let $k \in \mathbb{N}$ be even. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,1}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform. Then*

$$(27) \quad D_{\Phi,F}(s) = \langle \widehat{\Phi}, \widehat{\Phi}_1^F \rangle_{\mathcal{A}} \zeta(4s + 2k - 4)^{-1} D_F(2s + k - 2) \\ \times L(2s + 2k - 3, \Phi)^{-1}.$$

Here $L(s, \Phi) = \prod_p \widetilde{L}_p^{\text{EZ}}(s, \Phi)^{-1}$. Let $\varphi \in S_1^{2k-2}$ correspond to Φ with respect to the Saito-Kurokawa lift, then the Hecke L -function of φ is equal to $L(s, \Phi)$, i.e.,

$$L(s, \varphi) = L(s, \Phi).$$

Proof. We would like to apply [He99, Proposition 3.13], to analyze the Dirichlet series $D_{\Phi,F}(s)$. Hence we have to introduce the adjoint operator of \widetilde{U}_λ . Therefore it is advisable to use the rescaling rule

$$\widetilde{U}_\lambda = \lambda^{3(2-k)} \lambda^{2k-6} \Lambda_-^r(\lambda)$$

and the fact that the adjoint operator of $\Lambda_-^r(\lambda)$ is $\Lambda_+^r(\lambda)$, see [He99, Section 3.4], for more details. Let $(\tilde{U}_\lambda)^{\text{ad}}$ be the adjoint operator of \tilde{U}_λ . Then putting $X = p^{-2s-k+2}$ gives

$$(28) \quad \widehat{\Phi}_{tp^{2\delta}}^F|_k(\tilde{U}_{p^\delta})^{\text{ad}} p^{-2\delta s} = \widehat{\Phi}_{tp^{2\delta}}^F|_k \mathcal{D}(\Lambda_+^r(p^\delta))(p^{-2}X)^\delta.$$

In [He99, Proposition 3.13], we have proven the following: Let $F \in S_2^k$ be a Hecke eigenform and $l \in \mathbb{N}$, then

$$(29) \quad D_{p,F}^2(\overline{X}) \sum_{\delta=0}^\infty \widehat{\Phi}_{tp^{2\delta}}^F|_k \mathcal{D}(\Lambda_+^r(p^\delta))(p^{-2}\overline{X})^\delta \\ = (1 - \overline{X}) \left(\widehat{\Phi}_t^F|_k \mathcal{D}(S^{(2)}(\overline{X})) - \widehat{\Phi}_{t/p^2}^F|_k \mathcal{D}(\Lambda_-^r(p)S^{(2)}(\overline{X})p^{-2}\overline{X}) \right).$$

Here $S^{(2)}(X)$ is defined in Theorem 3.2, a polynomial of degree 3 in $\tilde{\mathcal{H}}_{0,p}^J[X]$. At this point we would like to mention that our argument for getting an Euler product only works because of [He99, Proposition 3.4], which gives a sufficient condition for the commutativity of certain Hecke-Jacobi operators.

Now we assume that p does not divide the index of Φ . Then $S^{(2)}(X)$ simplifies to $(1 + X)S^{(2)}(X)^{\text{factor}}$, i.e.,

$$\begin{aligned} \langle \widehat{\Phi}, \widehat{\Phi}_t^F | \mathcal{D}(S^{(2)}(\overline{X})) \rangle_{\mathcal{A}} &= (1 + X) \langle \widehat{\Phi} | \mathcal{D}(S^{(2)}(X)^{\text{factor}}), \widehat{\Phi}_t^F \rangle_{\mathcal{A}} \\ &= \langle \widehat{\Phi}, \widehat{\Phi}_t^F \rangle_{\mathcal{A}} (1 + X)(1 - p^{-2}\lambda X + p^{-1}X^2) \\ &= \langle \widehat{\Phi}, \widehat{\Phi}_t^F \rangle_{\mathcal{A}} (1 + p^{-2s-k+2}) \tilde{L}_p^{\text{EZ}}(2s + 2k - 3, \Phi). \end{aligned}$$

Finally we would like to remark that $\overline{D_F(\overline{s})} = D_F(s)$. Hence the theorem is proven. □

§5. Quotients of L -functions

We begin with the concept of (strong) Hecke-Jacobi eigenforms and newforms in the context of Jacobi forms. Then we compute the local factors of $D_{\Phi,F}(s)$ at the bad primes and define a global L -function attached to a Jacobi form. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$. Then Φ is a Hecke-Jacobi eigenform, if Φ is eigenform with respect to $T^{J,r}(n)$ for all $n \in \mathbb{N}$. We have seen in [He98], that this implies also, that Φ is automatically an eigenform with respect to Ξ_p^r .

5.1. Jacobi newforms

We would like to recall the definition of newforms in the setting of Jacobi forms introduced by Skoruppa and Zagier [S-Z88]. The space of Jacobi newforms is defined as always as the orthogonal complement of oldforms and is equal to

$$(30) \quad \widehat{\mathcal{J}}_{k,t}^{\text{cusp,new}} = \bigcap_{p|t} \text{Ker} \left(\widehat{\mathcal{J}}_{k,t}^{\text{cusp}} |_{k\mathcal{D}(T_+^r(p))} \right) \cap \text{Ker} \left(\widehat{\mathcal{J}}_{k,t}^{\text{cusp}} |_{k\mathcal{D}(\Lambda_+^r(p))} \right).$$

The space $\mathcal{J}_{k,t}^{\text{cusp,new}}$ has a basis of Hecke Jacobi eigenforms and

$$(31) \quad \widehat{\mathcal{J}}_{k,t}^{\text{cusp}} = \widehat{\mathcal{J}}_{k,t}^{\text{cusp,new}} \oplus \bigoplus_{\substack{l,d>0 \\ ld^2|t, ld^2>1}} \widehat{\mathcal{J}}_{k,t/(ld^2)}^{\text{cusp,new}} |_{k\mathcal{D}(\Lambda_-^r(d)T_-^r(l))}.$$

Hecke-Jacobi eigenforms which are newforms are called Hecke-Jacobi newforms. (see [Gr95, page 80], and [He98] for more details).

Remark 5.1. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform and $F \in S_2^k$ be a Hecke eigenform. Then Φ is an eigenform with respect to the operator $S^{(2)}(X)$. Let us denote the eigenvalue by $S_\Phi^{(2)}(X)$. In particular we have $\widehat{\Phi}|_{k\mathcal{D}(\Lambda_+^r(p))} = 0$ and

$$\begin{aligned} & \overline{D_{p,F}^2(\overline{X})} \left\langle \widehat{\Phi}, \sum_{\delta=0}^{\infty} \widehat{\Phi}_{tp^{2\delta}}^F |_{k\mathcal{D}(\Lambda_+^r(p^\delta)(p^{-2}\overline{X})^\delta)} \right\rangle_{\mathcal{A}} \\ &= (1 - X) \langle \widehat{\Phi}|_{k\mathcal{D}(S^{(2)}(X))}, \widehat{\Phi}_t^F \rangle_{\mathcal{A}} \\ &= (1 - X) S_\Phi^{(2)}(X) \langle \widehat{\Phi}, \widehat{\Phi}_t^F \rangle_{\mathcal{A}}. \end{aligned}$$

5.2. Local L-factors for newforms

In this section we restrict our attention on the computation of the local L -factors of $D_{\Phi,F}(s)$ at the bad primes. We give a complete solution for Hecke-Jacobi newforms $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ of weight k and arbitrary index t .

Let us first fix some notation. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi eigenform. We denote the eigenvalues of Φ with respect to $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$ by λ and ε , respectively. The operator $p^{-2}\Xi_p^r$ has some connection with the well known involution W_p in the theory of Jacobi forms, when $p|t$, i.e., $p|t$ and $p^2 \nmid t$, see [He98, Remark 3.2]. Let $S^{(2)}(X)$ be as in Section 3, Theorem 3.2. Let $X = p^{-2s-k+2}$. For $p|t$ we have the following simplification

$$(32) \quad \widehat{\Phi}|_{k\mathcal{D}(S^{(2)}(X))} = \widehat{\Phi}|_{k\mathcal{D}(1 - (p^{-2}T^{J,r}(p) + p^{-1} - 1)X + p^{-3}\Xi_p^r X^2)}.$$

From now on, we assume Φ to be a Hecke-Jacobi newform. Thus Φ lies in the kernel of $T_+^r(p)$. This can be used to find a certain hidden relation between the operators $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$ on the space $\mathcal{J}_{k,t}^{\text{cusp,new}}$. We have (cf. [He98, Section 3.2]): $0 = \widehat{\Phi}|_k\mathcal{D}(pT^{J,r}(p) + p^2 + \Xi_p^r)$, which allows us to identify $T^{J,r}(p)$ with $-(p + p^{-1}\Xi_p^r)$. Thus we have

$$\begin{aligned} \widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) &= (1 + X)\widehat{\Phi}|_k\mathcal{D}(1 + p^{-3}\Xi_p^r X) \\ &= (1 + X)(1 + p^{-1}\varepsilon X)\widehat{\Phi}. \end{aligned}$$

We know that $\varepsilon = \pm 1$ if $p|t$ with $p^2 \nmid t$.

Let $p^2|t$. Then similar as in [Gr95] and [He98], we obtain $\widehat{\Phi}|_k\mathcal{D}(\Xi_p^r) = 0$. In other words let $p^2|t$ and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$, then we have

$$\widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) = (1 + X)\widehat{\Phi}.$$

Actually in this case we have to calculate

$$(33) \quad \left\langle \widehat{\Phi}, \left(\widehat{\Phi}_t^F|_k\mathcal{D}(S^{(2)}(\overline{X})) - \widehat{\Phi}_{t/p^2}^F|_k\mathcal{D}(\Lambda_-(p)S^{(2)}(\overline{X})p^{-2}\overline{X}) \right) \right\rangle_{\mathcal{A}}.$$

But this leads after some simplifications essentially to the computation of the two expressions $\langle \widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)), \widehat{\Phi}_t^F \rangle_{\mathcal{A}}$ and $\langle \widehat{\Phi}|_k\mathcal{D}(\Lambda_+(p)), \widehat{\Phi}_{t/p^2}^F \rangle_{\mathcal{A}}$. Hence it is obvious that the term related to $\mathcal{D}(\Lambda_-(p))$ does not contribute to our formula.

The standard L -function $L(s, \Phi)$ attached to a Hecke-Jacobi newform $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ is defined in the following way: Let $L_p(s, \Phi) = \widetilde{L}^{\text{EZ}}(s, \Phi)$ if $p \nmid t$ and $L_p(s, \Phi) = 1 + \varepsilon_p p^{k-2} p^{-s}$ otherwise. Here ε_p is the eigenvalue of the operator $p^{-2}\Xi_p^r$. We know that $\varepsilon = \pm 1$ in the case $p|t$ with $p^2 \nmid t$ and 0 if $p^2|t$. Then

$$L(s, \Phi) = \prod_p L_p(s, \Phi)^{-1}.$$

The definition of $L(s, \Phi)$ is compatible with the one given in Corollary 4.4. Let $X = p^{-2s-k+2}$ and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$ be a Hecke-Jacobi newform. For convenience we put $\zeta_p(s) = 1 - p^{-s}$. Then we have

$$(34) \quad (1 - X)\widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) = \zeta_p(4s + 2k - 4)L_p(2s + 2k - 3, \Phi)\widehat{\Phi}.$$

5.3. Main results

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $F \in S_2^k$, where $k, t \in \mathbb{N}$ and k be even. In Section 2 we obtained an integral representation of the Dirichlet series

$$(35) \quad D_{\Phi,F}(s) = \sum_{\lambda=1}^{\infty} \langle \widehat{\Phi} |_k \widetilde{U}_\lambda, \widehat{\Phi}_t^F \rangle_{\mathcal{A}} (t\lambda^2)^{-s}.$$

More precisely we proved that

$$\langle E_{2,1}^{k,t}((*, 0), \Phi; s), F(*) \rangle = \beta_k (4\pi)^{-(s+k-2)} \Gamma(s+k-2) D_{\Phi,F}(s).$$

Let F be a Hecke eigenform and Φ be a Hecke-Jacobi newform, then we showed that $D_{\Phi,F}(s)$ has an Euler product. We obtained the following result

THEOREM 5.2. *Let $k, t \in \mathbb{N}$ and let k be even. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform. Let $s \in \mathbb{C}$. Then we have*

$$(36) \quad D_{\Phi,F}(s) = t^{-s} \langle \widehat{\Phi}, \widehat{\Phi}_t^F \rangle_{\mathcal{A}} \zeta(4s+2k-4)^{-1} \times D_F(2s+k-2) L(2s+2k-3, \Phi)^{-1}.$$

The formula presented in the theorem is well-defined, because $D_{\Phi,F}(s)$ possesses a meromorphic continuation on the whole complex plane. This follows from the integral representation (35). The reader familiar with the methods and results of our recent paper [He98] should be able to formulate Theorem 5.2 without the assumption newform, when we only assume t to be square-free.

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