THE USE OF LIE SERIES IN THE CONSTRUCTION OF A PERTURBATION THEORY AND SOME RECENT RESULTS IN THE THEORY OF THE MOTION OF HYPERION

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Abstract. This paper begins with a brief review of a form of the Lie series transformation, and then reports some new results in the study, using Lie series methods, of the orbit of Saturn's satellite Hyperion. In particular, improved expressions are given for the long-period perturbations of the orbital elements which describe the motion in the orbit plane, and also first results for expressions for the short-period perturbations in the apse longitude, derived from the Lie series generating function.

1. Lie Series Transformations

At this Colloquium, whose topic is the impact of modern dynamics on astronomy, it is appropriate to mark the advances in the study of the motions of celestial bodies which have been made possible by the use of Lie series transformations, with a brief review of which we will begin below. Many applications of Lie series transformations have been in the context of near-commensurability of orbital motions, including the series of investigations into the motion of Saturn's satellite Hyperion, some new results in which are reported in this paper. Let us begin, then, by setting out the main features of the Lie series transformation, in the form in which it is used in this work on the orbit of Hyperion. We suppose that we have a dynamical system, of n degrees of freedom, with co-ordinates $q = (q_1, q_2, ..., q_n)$, whose conjugate momenta are $p = (p_1, p_2, ..., p_n)$, respectively, and with Hamiltonian function $\mathcal{H}(q,p)$. So the equations of motion are Hamilton's equations:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \text{and,} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i},, \quad \text{for,} \quad i = 1, 2, ..., n. \tag{1}$$

Suppose we have chosen a function $\mathcal{W}(q,p)$, the *generating function*, then let us define the operator $\mathcal{L}_{\mathcal{W}}$ so that, for any function $f(q,p)$,

$$\begin{aligned} \mathcal{L}_{\mathcal{W}}(f) &= \{f, \mathcal{W}\} \\ &= \sum_{i=1}^{n} \left\{ \frac{\partial f}{\partial q_i}\frac{\partial \mathcal{W}}{\partial p_i} - \frac{\partial \mathcal{W}}{\partial q_i}\frac{\partial f}{\partial p_i} \right\}, \end{aligned} \tag{2}$$

which is the Poisson bracket of f and $\mathcal{W}$. In turn define $\mathcal{L}_{\mathcal{W}}^{(2)}(f) = \mathcal{L}_{\mathcal{W}}\{\mathcal{L}_{\mathcal{W}}(f)\}$, and $\mathcal{L}_{\mathcal{W}}^{(3)}(f) = \mathcal{L}_{\mathcal{W}}\{\mathcal{L}_{\mathcal{W}}^{(2)}(f)\}$, and, in general, $\mathcal{L}_{\mathcal{W}}^{(n)}(f) = \mathcal{L}_{\mathcal{W}}\{\mathcal{L}_{\mathcal{W}}^{(n-1)}(f)\}$, for $n = 2, 3, 4, ...$, (understanding $\mathcal{L}^{(1)}$ as $\mathcal{L}$). Then the Lie series transformation $(Q,P) \longmapsto (q,p)$ is defined by

$$q_i = Q_i + \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{L}_{\mathcal{W}}^{(k)}(Q_i), \text{and} p_i \;=\; P_i + \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{L}_{\mathcal{W}}^{(k)}(P_i), \tag{3}$$

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for $i = 1, 2, ..., n$. Now in fact $\mathcal{L}_{\mathcal{W}}(Q_i) = \frac{\partial \mathcal{W}}{\partial P_i}$, and $\mathcal{L}_{\mathcal{W}}(P_i) = -\frac{\partial \mathcal{W}}{\partial Q_i}$, and so

$$q_i = Q_i + \frac{\partial \mathcal{W}}{\partial P_i} + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \mathcal{L}_{\mathcal{W}}^{(k)} \left(\frac{\partial \mathcal{W}}{\partial P_i} \right), \tag{4}$$

$$\text{and} \quad p_i = P_i - \frac{\partial \mathcal{W}}{\partial P_i} - \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \mathcal{L}_{\mathcal{W}}^{(k)} \left(\frac{\partial \mathcal{W}}{\partial P_i} \right), \tag{5}$$

for $i = 1, 2, ..., n$. In fact, for any function $f(q, p)$,

$$f(q,p) = f(Q,P) + \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{L}_{\mathcal{W}}^{(k)} (f(Q,P)). \tag{6}$$

Note that this transformation represents the progression over unit time of a notional dynamical system whose Hamiltonian function is $\mathcal{W}$. Therefore it is a contact transformation, and so preserves the Hamiltonian form of the equations of motion. If $\tilde{\mathcal{H}}(Q,P)$ is the Hamiltonian function, in the actual motion, which gives the equations of motion for the (Q,P), then, since the transformation is autonomous, *i.e.* does not involve the time explicitly, we have

$$\tilde{\mathcal{H}}(Q,P) = \mathcal{H}(q,p) \tag{7}$$

$$= \mathcal{H}(Q,P) + \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{L}_{\mathcal{W}}^{(k)} \{\mathcal{H}(Q,P)\}, \tag{8}$$

where equation (6) has been used for $\mathcal{H}$. Now in a large class of problems encountered in celestial mechanics, we find that the Hamiltonian function may be expressed in the form $\mathcal{H} = \mathcal{H}_0 - \varepsilon \mathcal{R}$, where $\mathcal{H}_0$ is a function only of the momenta p_i, and ε is a small parameter. The case $\varepsilon = 0$ is spoken of as the "unperturbed" motion, and in this case the p_i are the action variables, and the q_i are their conjugate angle variables. Then $\mathcal{R}$ is the *disturbing function*, and it may usually be expanded as a multiple Fourier series:

$$\mathcal{R} = \sum_{\nu} K_\nu \cos N_\nu, \tag{9}$$

where $\nu = (\nu_1, \nu_2, \nu_3, ..., \nu_n)$, and $N_\nu = \sum_{j=1}^{n} \nu_j q_j$, and the summation over ν is over all sets of integers ν_j with ν_1 non-negative and with $\sum_{j=1}^{n} \nu_j = 0$. Very often the transformation is to be chosen so that the motion of the transformed variables, Q_i and P_i, contains none of the short-period features of the motion, so that these features will be encompassed within the generating function $(\mathcal{W})$, leaving the long-period features to be dealt with in isolation in the transformed system. Usually in celestial mechanics the co-ordinates fall into two sets, the *fast-moving* co-ordinates (e.g. the mean longitudes) and the *slow-moving* (e.g. the apse and node longitudes). In the class of problems where the unperturbed motion is in fact Keplerian elliptic motion, the co-ordinates of the latter set are constant in the unperturbed motion. In the case of motion near to a small-integer commensurability of the mean motions, so that there is a linear combination of the mean longitudes which changes slowly in the unperturbed motion, choose co-ordinates so that some of them (the *critical arguments*) contain the mean longitudes in this particular combination, and so will be classed as *slow-moving* for our purposes. In any case we suppose that we are able to identify which terms of $\mathcal{R}$ are of short-period, and which of long-period. The choice of the transformation is made by choosing the function $\mathcal{W}$. It will generally be necessary to choose it, stage-by-stage, as an expansion in powers of the perturbation parameter ε, of the form: $\mathcal{W} = \sum_{i=1}^{\infty} \varepsilon^i \mathcal{W}_i$, and likewise for the Hamiltonian function for the transformed problem: $\tilde{\mathcal{H}} = \sum_{i=0}^{\infty} \varepsilon^i \tilde{\mathcal{H}}_i$. Then, equating terms independent of ε in equation (8) gives

$$\tilde{\mathcal{H}}_0(Q,P) = \mathcal{H}_0(Q,P), \tag{10}$$

so that the unperturbed motion in the transformed system is the same as that in the untransformed system. Equating terms of the first order in ε gives

$$\tilde{\mathcal{H}}_1(Q,P) = -\mathcal{R}(Q,P) + \{\mathcal{H}_0(Q,P), \mathcal{W}_1\}, \tag{11}$$

and equating terms of the second order in ε gives

$$\tilde{\mathcal{H}}_2(Q,P) = \{\mathcal{H}_0(Q,P), \mathcal{W}_2\} - \{(\mathcal{R}(Q,P) - \frac{1}{2}\{\mathcal{H}_0, \mathcal{W}_1\}), \mathcal{W}_1\}. \tag{12}$$

Now, as noted above, we suppose that $\mathcal{H}_0$ is a function of the momenta p_i only, and we also suppose that we are able to separate the expansion (9) of $\mathcal{R}$ into its long-period part $[\mathcal{R}]_{lp}$, say, and its short-period part $[\mathcal{R}]_{sp}$, say, so that $\mathcal{R} = [\mathcal{R}]_{lp} + [\mathcal{R}]_{sp}$. Then we choose $\mathcal{H}_1(Q,P)$ to be equal to $-[\mathcal{R}]_{lp}(Q,P)$, so that it contains only long-period terms, and so equation (11) becomes

$$\{\mathcal{H}_0, \mathcal{W}_1\} = [\mathcal{R}]_{sp}. \tag{13}$$

If we write $\frac{\partial \mathcal{H}_0}{\partial P_j} = n_j$, for $j = 1, 2, ..., n$, then this becomes

$$\sum_{j=1}^{n} n_j \frac{\partial \mathcal{W}_1}{\partial Q_j} = \sum_{\nu, sp} K_\nu \cos N_\nu, \tag{14}$$

the summation being over the short-period terms, and the solution of it which we take is

$$\mathcal{W}_1 = \sum_{\nu, sp} \frac{K_\nu}{\rho_\nu} \sin N_\nu, \tag{15}$$

where $\rho_\nu = \sum_{j=1}^{n} \nu_j n_j$. Then equation (12), using equation (13), is

$$\tilde{\mathcal{H}}_2(Q,P) = \{\mathcal{H}_0(Q,P), \mathcal{W}_2\} - \{(\mathcal{R} - \frac{1}{2}[\mathcal{R}]_{sp}), \mathcal{W}_1\}, \tag{16}$$

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and so we choose $\tilde{\mathcal{H}}_2(Q,P)$ to be the long-period part of $-\frac{1}{2}\{[\mathcal{R}]_{sp}(Q,P),\mathcal{W}_1\}$, which leaves $\{\mathcal{H}_0(Q,P),\mathcal{W}_2\}$ equal to the short-period part of $\{([\mathcal{R}]_{lp}(Q,P)+\frac{1}{2}[\mathcal{R}]_{sp}(Q,P)),\mathcal{W}_1\}$, which we may write as $\sum_{\nu,sp} L_\nu \cos N_\nu$, say, leading to the solution $\mathcal{W}_2 = \sum_{\nu,sp} \frac{L_\nu}{\rho_\nu}\sin N_\nu$. In a similar manner may be found successively $\tilde{\mathcal{H}}_3(Q,P)$, $\mathcal{W}_3$, $\tilde{\mathcal{H}}_4(Q,P), \mathcal{W}_4$, &ce.

2. The application of Lie Series Transformations to the Theory of the Motion of Hyperion. Some Recent Results.

Two earlier papers (*Message 1989*) and (*Message 1993*) set out the method being used to study the long-period features of the motion of Saturn's satellite Hyperion, using a Lie series transformation to separate the long-period aspects of the motion. We recall that, very shortly after the discovery of this satellite at Harvard (and, independantly, in Liverpool) in 1848, unusual aspects of its motion drawing attention to it included the retrograde motion of the apse, and the large eccentricity of the orbit. It was first shown by Newcomb (*1891*) that these arose because of the very close near-commensurability of the orbital period of Hyperion with that of Titan, which is by far the most massive satellite of the system. The critical argument $\theta = 4\lambda_H - 3\lambda_T - \varpi_H$ in fact librates about the mean value 180 degrees, the main term having amplitude of about 36 degrees, and period about 21 months. (Here λ_H and λ_T are, respectively, the mean longitudes of Hyperion and Titan, and ϖ_H is the apse longitude of Hyperion. Another important argument is $\sigma = \varpi_H - \varpi_T$, the difference between the apse longitudes of the two satellites.

Methods were described in the earlier papers to express derivatives of $[\mathcal{R}]_{lp}$, the long-period part of the disturbing function for the action of Titan, and other quantities required for the equations of motion for the long-period motion, as double Fourier series in σ, and ω, the latter being related to θ by $\theta = \pi(1+0.3\sin\omega)$. The short-period argument is $\phi = \lambda_H - \lambda_T$, and we find that the equation (13), which we need to calculate the first-order part of the generating function, $\mathcal{W}_1$, takes the form

$$(n_H - n_T)\frac{\partial \mathcal{W}_1}{\partial \phi} + (4n_H - 3n_T)\frac{\partial \mathcal{W}_1}{\partial \theta} = [\mathcal{R}]_{sp}, \tag{17}$$

where n_H and n_T are the mean motions of Hyperion and Titan, respectively. To use this, in the course of carrying out integrations over ϕ, we need to use an expansion in powers of the small quantity ϵ, which is defined by $(4n_H - 3n_T) = \epsilon(n_H - n_T)$, and which has a value near to $\frac{1}{120}$. In terms of this, we put

$$\mathcal{W}_1 = \mathcal{W}_{10} + \epsilon\mathcal{W}_{11} + \epsilon^2\mathcal{W}_{12} + \ldots\ldots \tag{18}$$

Then, equating terms of each power of ϵ in equation (17), gives in succession

$$(n_H - n_T)\frac{\partial \mathcal{W}_{10}}{\partial \phi} = [\mathcal{R}]_{sp}, \tag{19}$$

$$\frac{\partial \mathcal{W}_{11}}{\partial \phi} = \frac{\partial \mathcal{W}_{10}}{\partial \theta}, \tag{20}$$

$$\frac{\partial \mathcal{W}_{12}}{\partial \phi} = \frac{\partial \mathcal{W}_{11}}{\partial \theta}, \tag{21}$$

$$\frac{\partial \mathcal{W}_{13}}{\partial \phi} = \frac{\partial \mathcal{W}_{12}}{\partial \theta}, \text{\&ce.} \tag{22}$$

Then the solution of equation (19) is

$$\mathcal{W}_{10}(\phi,\theta) = \mathcal{W}_{10}(\phi_0,\theta) + \frac{1}{(n_H - n_T)}\int_{\phi_0}^{\phi}[\mathcal{R}]_{sp}(\phi',\theta)\mathrm{d}\phi', \tag{23}$$

the initial value having been found by use of

$$\mathcal{W}_{10}(\phi_0,\theta) = \frac{1}{(n_H - n_T)}\int_{\phi_0}^{\phi_0+2\pi}\phi'.[\mathcal{R}]_{sp}(\phi',\theta)\mathrm{d}\phi'. \tag{24}$$

2.1. **Improved solutions for the parameters of the motion in the orbit plane.**

The expressions for the long-period terms in the orbital elements describing the motion of Hyperion in its orbit plane, corresponding to the very-close commensurability type of libration are (*Message 1993*)

$$\theta = 180^\circ + \sum \theta_{i,j}\sin(i\tau + j\zeta), \tag{25}$$

$$\sigma = \zeta + \sum \varpi_{i,j}\sin(i\tau + j\zeta), \tag{26}$$

$$a = a_{0,0} + \sum a_{i,j}\cos(i\tau + j\zeta), \tag{27}$$

$$e = e_{0,0} + \sum e_{i,j}\cos(i\tau + j\zeta), \tag{28}$$

where a is the major-semi axis, e is the eccentricity, of Hyperion's orbit, $\tau = \nu t + \tau_0$ is the argument of the free libration, of period about 21 months, and $\zeta = \chi t + \zeta_0$ is the linear part of the difference between the apse longitudes ($\sigma = \varpi_H - \varpi_T$), which has period about 18 $\frac{3}{4}$ years. (The summations are over all integer pairs (i,j) with i non-negative.) The earlier papers (*Message, 1989, 1993*) described how a least-squares fit was carried out, beginning with a set of estimates of some of the co-efficients $\theta_{i,j}, \varpi_{i,j}, a_{i,j}, e_{i,j}$, and of the rate of change, ν, of the libration argument, and of the mean motion, $\hat{n}$, and finding a set consistent with the dynamical equations. The work reported in the previous paper (*Message, 1993*), has since been extended, solving for an enlarged set of co-efficients. This was carried out twice, using different sets of data. First this was done using data arising from the main sequences of observations made between 1875 and 1922, which were reduced by Woltjer (*1928*), who gave opposition mean values of the orbital elements, analysis

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of which gave the following estimates of the co-efficients of the long-period terms, and of ν and of $\hat{n}$:

$$e_{0,0} = +0.10419 \pm 0.00027, \quad e_{0,1} = +0.02414 \pm 0.00044,$$

$$e_{0,2} = -0.00183 \pm 0.00040, \quad e_{1,0} = -0.00401 \pm 0.00034,$$

$$\varpi_{0,1} = -13.905^\circ \pm 0.273^\circ, \quad \varpi_{0,2} = +0.754^\circ \pm 0.249^\circ,$$

$$\varpi_{1,0} = -0.314^\circ \pm 0.262^\circ, \quad \varpi_{2,0} = -0.795^\circ \pm 0.304^\circ,$$

$$\lambda_{0,1} = -0.054^\circ \pm 0.019^\circ, \quad \lambda_{0,2} = +0.007^\circ \pm 0.018^\circ,$$

$$\lambda_{1,0} = +9.112^\circ \pm 0.018^\circ, \quad \lambda_{2,0} = +0.039^\circ \pm 0.018^\circ,$$

$$\hat{n} = 16.9199890^\circ \pm 0.0000027^\circ \text{ per day,}$$

and

$$\nu = 0.562025^\circ \pm 0.000025^\circ \text{ per day.}$$

Using these to provide equations of condition, and solving, as described in (*Message 1993*), but with an extended set of co-efficients in the scheme of solution, gives the following set of estimates of independent parameters:

$$\hat{n} = 16.9199888^\circ \pm 0.0000066^\circ \text{ per day,} \tag{29}$$

$$\theta_{1,0} = 36.877^\circ \pm 0.180^\circ, \tag{30}$$

$$\text{and } m' = 0.0002364220 \pm 0.0000000075, \tag{31}$$

where m' is the mass of Titan in terms of that of Saturn. From these are derived (as described in *Message 1993*) the following dynamically consistent set of expressions:

$$\nu = 0.562024124^\circ \text{ per day,}$$

$$e = 0.1046696 + 0.024515\cos\zeta - 0.001427\cos2\zeta + 0.000175\cos3\zeta$$

$$-0.000025\cos4\zeta + 0.000003\cos5\zeta - 0.003888\cos\tau - 0.000049\cos2\tau$$

$$+0.000019\cos3\tau + 0.000182\cos(\tau - \zeta) - 0.000140\cos(\tau + \zeta)$$

$$-0.000041\cos(\tau - 2\zeta) - 0.000024\cos(\tau + 2\zeta) + 0.000007\cos(2\tau - \zeta)$$

$$-0.000007\cos(2\tau + \zeta),$$

$$a = a_{0,0}\{1 - 0.003227\cos\tau - 0.000004\cos2\tau + 0.000082\cos(\tau - \zeta)$$

$$-0.000054\cos(\tau+\zeta)\},$$

$$\varpi = \varpi_T + \zeta - 13.5880°\sin\zeta + 1.6233°\sin 2\zeta - 0.2605°\sin 3\zeta$$

$$+0.0468°\sin 4\zeta - 0.0092°\sin 5\zeta - 0.4302°\sin\tau - 0.0177°\sin 2\tau$$

$$-0.0004°\sin 3\tau + 0.0003°\sin 4\tau + 0.3606°\sin(\tau-\zeta)$$

$$-0.2715°\sin(\tau+\zeta) - 0.0904°\sin(\tau-2\zeta) + 0.0569°\sin(\tau+2\zeta)$$

$$+0.0216°\sin(\tau-3\zeta) - 0.0133°\sin(\tau+3\zeta) + 0.0133°\sin(2\tau-\zeta)$$

$$-0.0066°\sin(2\tau+\zeta),$$

and

$$\lambda = \hat{n}t + \lambda_0 + 9.11258°\sin\tau + 0.00300°\sin 2\tau - 0.01698°\sin 3\tau$$

$$+0.00006°\sin 4\tau - 0.07132°\sin\zeta - 0.00069°\sin 2\zeta + 0.00273°\sin 3\zeta$$

$$-0.00067°\sin 4\zeta - 0.00036°\sin 5\zeta - 0.21473°\sin(\tau-\zeta) + 0.18253°\sin(\tau+\zeta)$$

$$+0.00125°\sin(\tau-2\zeta) - 0.00107°\sin(\tau+2\zeta)$$

$$+0.00022°\sin(2\tau-\zeta) - 0.00052°\sin(2\tau+\zeta).$$

The contributions of the influence of the Sun, the figure of Saturn, and of the other satellites to the secular motions of the apse and of the mean longitude have been included. These influences will of course also give rise to very small periodic terms, of periods different from those of the terms given here.

The second solution made use of data from the main sequences of observations made between 1967 and 1983, which were fitted to a numerical integration of the equations of motion by Taylor (*1992*), who derived from it estimates of the co-efficients of the long-period terms, and of ν and $\hat{n}$. (These values were given in *Message 1993*, section 5, though notice that $e_{0,0} = +0.104550 \pm 0.000028$.) A solution for estimates of the independent parameters, carried out in the same way as for the previous (1875 to 1922) set of observational data was carried out also with this set of values (except that the value used for $\hat{n}$ was that from the previous series of observations, which covers a longer time-span). The set of co-efficients in the scheme of solution was extended from that in the solution reported in (*Message 1993*). This gave the following set of estimates of the independent parameters:

$$\hat{n} = 16.9199464° \pm 0.0000396° \text{ per day}, \quad (32)$$

$$\theta_{1,0} = 36.955° \pm 0.096°, \quad (33)$$

$$\text{and } m' = 0.000236398 \pm 0.00000007, \quad (34)$$

The dynamically consistent set of expressions derived from these, proceeding again as described in (*Message 1993*), is:

$$\nu = 0.56220934^\circ \text{ per day,}$$

$$e = 0.104506 + 0.024330\cos\zeta - 0.001417\cos2\zeta + 0.000172\cos3\zeta$$
$$-0.000025\cos4\zeta + 0.000003\cos5\zeta - 0.003904\cos\tau - 0.000049\cos2\tau$$
$$+0.000019\cos3\tau + 0.000182\cos(\tau - \zeta) - 0.000144\cos(\tau + \zeta)$$
$$-0.000041\cos(\tau - 2\zeta) - 0.000026\cos(\tau + 2\zeta) + 0.000007\cos(2\tau - \zeta)$$
$$-0.000008\cos(2\tau + \zeta),$$

$$a = a_{0,0}\{1 - 0.003234\cos\tau - 0.000004\cos2\tau - 0.000061\cos\zeta$$
$$+0.000083\cos(\tau - \zeta) - 0.000058\cos(\tau + \zeta)\},$$

$$\varpi = \varpi_T + \zeta - 13.5006^\circ\sin\zeta + 1.6109^\circ\sin2\zeta - 0.2558^\circ\sin3\zeta$$
$$+0.0461^\circ\sin4\zeta - 0.0088^\circ\sin5\zeta - 0.4333^\circ\sin\tau - 0.0178^\circ\sin2\tau$$
$$-0.0004^\circ\sin3\tau + 0.0003^\circ\sin4\tau + 0.3603^\circ\sin(\tau - \zeta)$$
$$-0.2716^\circ\sin(\tau + \zeta) - 0.0900^\circ\sin(\tau - 2\zeta) + 0.0558^\circ\sin(\tau + 2\zeta)$$
$$+0.0213^\circ\sin(\tau - 3\zeta) - 0.0136^\circ\sin(\tau + 3\zeta) + 0.0133^\circ\sin(2\tau - \zeta)$$
$$-0.0067^\circ\sin(2\tau + \zeta), -0.0045^\circ\sin(2\tau - 2\zeta) + 0.0017^\circ\sin(2\tau + 2\zeta),$$
$$-0.0011^\circ\sin(3\tau - \zeta) + 0.0015^\circ\sin(3\tau + \zeta),$$

and

$$\lambda = \hat{n}t + \lambda_0 + 9.1311^\circ\sin\tau + 0.0028^\circ\sin2\tau - 0.0169^\circ\sin3\tau$$
$$+0.0001^\circ\sin4\tau + 0.0001^\circ\sin5\tau - 0.0700^\circ\sin\zeta - 0.0024^\circ\sin2\zeta$$
$$+0.0002^\circ\sin3\zeta - 0.0005^\circ\sin4\zeta - 0.0004^\circ\sin5\zeta - 0.2167^\circ\sin(\tau - \zeta)$$
$$+0.1909^\circ\sin(\tau + \zeta) - 0.0008^\circ\sin(\tau - 2\zeta) + 0.0020^\circ\sin(\tau + 2\zeta)$$
$$+0.0012^\circ\sin(3\tau - \zeta) - 0.0012^\circ\sin(3\tau + \zeta).$$

The degree of agreement between these two sets of expressions, which are derived using two quite separate sets of observational data, is an indicator of the reliability of the results obtained so far in this work. (The methods, with analytical basis, being used here, are of course quite different from those of Duriez and Vienne *(1997)*, based on numerical integration. Comparison of the results of the two approaches must take account of the use of different parameters.)

2.2. Preliminary results for short-period terms in the apse longitude.

As was indicated earlier, the generating function, $\mathcal{W}$, of the Lie series transformation must contain all the information needed to construct expressions for the short-period perturbations. Thus the first equation of (3), with ϖ for q_i, gives the short-period perturbations of the apse longitude, and the first-order part is

$$-\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial \mathcal{W}}{\partial e}$$

Preliminary calculations give, for the largest few terms of this:

$$+1.0228^\circ\sin\phi + 0.4248^\circ\sin2\phi + 0.2942^\circ\sin3\phi + 0.2813^\circ\sin4\phi$$

$$+0.0445^\circ\sin5\phi + 0.0307^\circ\sin6\phi + 0.0299^\circ\sin7\phi + 0.0217^\circ\sin8\phi$$

$$-0.1540^\circ\sin(\phi+\zeta) - 0.0979^\circ\sin(\phi-\zeta) - 0.0874^\circ\sin(\phi+\tau)$$

$$+0.0941^\circ\sin(\phi-\tau) + 0.0364^\circ\sin(\phi+2\zeta) + 0.0231^\circ\sin(\phi-2\zeta)$$

$$-0.0297^\circ\sin(\phi+\tau-\zeta) + 0.0269^\circ\sin(\phi+\tau+\zeta) - 0.0463^\circ\sin(\phi-\tau+\zeta)$$

$$-0.0508^\circ\sin(2\phi+\zeta) - 0.0369^\circ\sin(2\phi-\zeta) + 0.0208^\circ\sin(2\phi+\tau)$$

$$-0.0381^\circ\sin(3\phi+\zeta) - 0.0299^\circ\sin(3\phi-\zeta) + 0.0216^\circ\sin(3\phi-\tau)$$

$$-0.0448^\circ\sin(4\phi+\zeta) - 0.0288^\circ\sin(4\phi+\tau) + 0.0459^\circ\sin(4\phi-\tau)$$

$$+0.0201^\circ\sin(5\phi+\tau).$$

For comparison, those terms derived by Taylor (*1992*) from Fourier analysis of the results of his numerical integration, are:

$$+1.0391^\circ\sin\phi + 0.4209^\circ\sin2\phi + 0.3115^\circ\sin3\phi + 0.2795^\circ\sin4\phi$$

$$-0.1674^\circ\sin(\phi+\zeta) - 0.0972^\circ\sin(\phi-\zeta) + 0.0833^\circ\sin(\phi+\tau)$$

$$+0.0946^\circ\sin(2\phi+\tau) - 0.0568^\circ\sin(2\phi+\zeta) + 0.0731^\circ\sin(3\phi+\tau)$$

$$-0.0482^\circ\sin(3\phi+\zeta) + 0.0636^\circ\sin(4\phi+\tau) - 0.0511^\circ\sin(4\phi+\zeta).$$

For comparison with the results of Vienne and Duriez (*1991*), note that 0.1° in ϖ corresponds to about 266 km. along the orbit of Hyperion.

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