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Zero dissipation limit and stability of boundary layers for the heat conductive Boussinesq equations in a bounded domain

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In this paper, we study the zero dissipation limit of the initial boundary-value problem of the multi-dimensional Boussinesq equations with viscosity and heat conductivity. Such equations are used as models for the motion of multi-dimensional incompressible fluids in atmospheric and oceanographic turbulence. In particular, they describe the thermal convection of an incompressible flow, and constitute the relations between the velocity field, the pressure and the local temperature. Under the Navier slip boundary condition in the velocity field and the thermal isolation boundary condition for the temperature, we prove the existence of weak amplitude characteristic boundary layers. Then, by a standard energy method, we prove the L^2 convergence of the solutions when both the viscosity and the heat conductivity coefficients tend to 0.

1. Introduction and main results

In theoretical hydrodynamics, the inviscid Euler equations are used to describe the motion of an ideal fluid but, except for some special cases, such equations cannot describe the motion of actual fluids. Great difficulties of a mathematical nature may arise in connection with this. In fact, for fluids with small viscosity, only a very thin region adjacent to the solid boundary is affected by the viscosity, and thus the Navier–Stokes equations describing viscous flows were introduced along with a small parameter as a coefficient of the highest order derivatives (see [15, 22]). This thin region is called the boundary layer. Most of the initial boundary-value problems for fluid dynamics systems arise in various domains and the boundary conditions for these problems are chosen according to the physical properties of each situation. Compared with the viscous models, there is in general a loss of boundary conditions when the viscosity ε goes to 0 (and hence a boundary layer appears). Thus, it is commonly believed that solutions for the viscous parabolic equations cannot be uniformly close to those for the inviscid hyperbolic equations. There is an immense

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literature on this aspect of theory; see [20, 25, 29] and references therein. The in-flow and out-flow boundary conditions cause the boundary to be non-characteristic and, in this case, there are boundary layers of size ε that are stable when the amplitude is small (see [18, 23, 25, 27]). The no-slip boundary condition states that at a solid boundary the fluid will have zero velocity, which means that particles close to a surface do not move along with a flow. For the Navier–Stokes equations, such a boundary condition always makes the boundary be characteristic. It is pointed out in Prandtl's theory that such a boundary condition will cause the so-called characteristic boundary layers, of thickness $\mathcal{O}(\sqrt{\varepsilon})$, to develop, and many nonlinear phenomena may occur in the layers. In this case, the leading boundary layer functions appear in the $O(1)$-term of the asymptotic approximate solution and they satisfy a set of nonlinear Prandtl-type equations (for details, see [19, 28]). For the case of analytic data and linearized problems, the reader is referred to [20, 21, 24, 26, 29]. However, as with most engineering approximations, the no-slip condition does not always hold in reality. A common approximation for fluid slip is given by the slip boundary conditions (1.4)–(1.5), which were first proposed by Navier. The slip boundary conditions allow the fluid to slip at the boundary and have important applications in aerodynamics, weather forecasts and haemodynamics (see [1]). The inviscid limit of the Leray solutions of the incompressible Navier–Stokes equations with such boundary conditions was studied in [8], where they carried out a descriptive method to describe the error in two and three dimensions and then proved that the boundary layer has a linear behaviour and that its thickness is of order $\mathcal{O}(\sqrt{\varepsilon})$, as in Prandtl's theory of no-slip boundary conditions.

From a thermodynamical point of view, heat will be generated in any motion, and thus there must be transformations among the temperature, the velocity and the pressure. The following n-dimensional Boussinesq model with heat conductivity is a simplified approximation for the motion of incompressible viscous and heat-conductive fluids [3, 9]:

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \varepsilon \Delta u^\varepsilon + \theta^\varepsilon e_n; \tag{1.1}$$

$$\operatorname{div} u^\varepsilon = 0; \tag{1.2}$$

$$\partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon = \kappa \Delta \theta^\varepsilon, \tag{1.3}$$

where $u^\varepsilon = (u_1^\varepsilon, \dots, u_n^\varepsilon)^{\mathrm{T}}$ denotes the velocity vector field, p is the scalar pressure, θ^ε is the absolute temperature, $\varepsilon, \kappa > 0$ are the viscosity and the diffusivity coefficients, respectively, and $e_n = (0, \dots, 1)^{\mathrm{T}}$. This model plays an important role in atmospheric and oceanographic sciences (see [13, 16]). Furthermore, because of its close connection to the incompressible Euler and Navier–Stokes equations, it has received significant attention in the mathematical fluid dynamics community (see [2, 3, 9, 10, 12, 17]). As is stated in [14], problems related to the vanishing viscosity limit ($\varepsilon \to 0$ and $\kappa > 0$), vanishing diffusivity limit ($\kappa \to 0$ and $\varepsilon > 0$) or zero dissipation limit ($\varepsilon, \kappa \to 0$) are important and challenging for (1.1)–(1.3). For the Cauchy problem, the vanishing viscosity limit and the vanishing diffusivity limit in the two-dimensional case are established in [3]. For the initial boundary-value problem, the vanishing diffusivity limit for (1.1)–(1.3) in a half plane is investigated in [10], where the existence of a boundary layer for the temperature is proved.

In this paper, we consider the Boussinesq model (1.1)–(1.3) in $\Omega \times [0,T]$, where $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) is a bounded domain with smooth boundary $\partial\Omega$. To specify the boundary condition, we introduce (as in [8]) a smooth distance function $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ for x in a neighbourhood $\mathcal{V}$ of $\partial\Omega$. One has that $\Omega \cap \mathcal{V} = \{\varphi > 0\} \cap \mathcal{V}$, $\Omega^c \cap \mathcal{V} = \{\varphi < 0\} \cap \mathcal{V}$ and $\partial\Omega := \{\varphi = 0\} \cap \mathcal{V}$. Furthermore, we normalize it such that $|\nabla\varphi(x)| = 1$ for all $x \in \mathcal{V}$. We define a smooth extension of the normal unit vector n inside Ω by taking $n := \nabla\varphi(x)$. For a vector field $\tilde{u}$ defined on Ω, we define the tangential part of $\tilde{u}$ to be $\tilde{u}_{\tan}(x) = \chi(x)[\tilde{u} - (\tilde{u} \cdot n)n]$, where $\chi(x)$ is a cut-off function such that $\operatorname{supp}\chi \subset \mathcal{V}$ and $\chi = 1$ in a neighbourhood of the boundary $\partial\Omega$. Now, for the Boussinesq equations (1.1)–(1.2), we add the following Navier slip boundary conditions for the velocity, and the Neumann boundary condition for the temperature on $\partial\Omega$:

$$u^\varepsilon \cdot n = 0; \tag{1.4}$$

$$[D(u^\varepsilon) \cdot n + \alpha u^\varepsilon]_{\tan} = 0; \tag{1.5}$$

$$\frac{\partial\theta^\varepsilon}{\partial n} = 0, \tag{1.6}$$

where $\varepsilon > 0$ is the coefficient of kinematic viscosity, n stands for the outward unit normal to Ω, α is a scalar friction function of class C^2 and $D(u)$ is the rate-of-strain tensor defined by $D_{ij}u = (\partial_i u_j + \partial_j u_i)/2$.

The initial conditions are taken as

$$u^\varepsilon(0,\cdot) = u_0(x) \quad \text{in } \Omega, \tag{1.7}$$

$$\theta^\varepsilon(0,\cdot) = \theta_0(x) \quad \text{in } \Omega, \tag{1.8}$$

which satisfy the compatibility conditions

$$u_0 \cdot n|_{\partial\Omega} = 0, \tag{1.9}$$

$$\operatorname{div} u_0 = 0, \tag{1.10}$$

$$\left.\frac{\partial\theta_0}{\partial n}\right|_{\partial\Omega} = 0. \tag{1.11}$$

It has been shown that the two-dimensional (2D) Cauchy problem of (1.1)–(1.3) has a unique global solution in various function spaces (see [2,3]; the initial boundary-value problem of (1.1)–(1.3) in the 2D case is investigated in [30]). The local existence of three-dimensional (3D) smooth solutions, the blow-up criteria to the Cauchy problem of (1.1)–(1.3) and the initial boundary-value problem are studied in [6,12,17]. To isolate the effect of the boundaries, we consider the solutions before the development of singularities. By retracing a similar argument to that found in [6], we prove the following proposition.

Proposition 1.1. *If* $(u_0(x), \theta_0(x)) \in H^3(\Omega)$ *satisfies the compatibility condition* (1.9)–(1.11)*, then there exist* $T_1 > 0$ *and* $\varepsilon_0, \kappa_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$ *and* $\kappa \in (0, \kappa_0]$*, the initial boundary-value problem* (1.1)–(1.8) *admits a unique weak solution*

$$(u^\varepsilon, \theta^\varepsilon) \in C(0, T_1; H^3(\Omega)) \cap L^2(0, T_1; H^1(\Omega)). \tag{1.12}$$

For the limiting case in which $\varepsilon = 0$ and $\kappa = 0$, we have the following inviscid equations:

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = \theta^0 e_n; \tag{1.13}$$

$$\operatorname{div} u^0 = 0; \tag{1.14}$$

$$\partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = 0. \tag{1.15}$$

We impose the same initial conditions (1.7)–(1.8). Since the inviscid system is of first order, only the normal component of the velocity

$$u^0 \cdot n = 0 \tag{1.16}$$

survives on the boundary. The local existence of the solutions to the inviscid problem is guaranteed by the following proposition.

Proposition 1.2. *If $(u_0(x), \theta_0(x)) \in H^3(\Omega)$ and $u_0(x)$ satisfies the divergence-free condition, then there exists $T_2 > 0$ such that the inviscid problem (1.13)–(1.16) admits a unique solution*

$$(u^0, \theta^0) \in C(0, T_2; H^3(\Omega)) \cap C^1(0, T_2; H^2(\Omega)). \tag{1.17}$$

The proof of proposition 1.2 is due to the argument in [4] and the hyperbolic theory [11]. The aim of this paper is to study the asymptotic equivalence between (1.1)–(1.6) and (1.13)–(1.16) with the same initial data. Noticing that condition (1.16) causes the boundary to be characteristic and comparing with the boundary conditions (1.4)–(1.6), there is a reduction in the number of boundary conditions, and such a reduction leads to the formation of characteristic boundary layers in the limiting process when $\varepsilon \to 0$ and $\kappa \to 0$. Assume that $\kappa = h\varepsilon$ with uniform constant $h > 0$. We introduce a fast variable $z = \varphi(x)/\sqrt{\varepsilon}$ and, for $m, p \in \mathbb{N}$, the anisotropic Sobolev space

$$H^{m,p} = \{\psi(x,z) \in L^2(\Omega \times \mathbb{R}_1^+) \mid \partial_x^\alpha \partial_z^\beta \psi \in L^2(\Omega \times \mathbb{R}_1^+)\ \forall |\alpha| \leqslant m,\ 0 \leqslant \beta \leqslant p\}$$

with β being an integer. We now give a precise statement of our main theorem.

Theorem 1.3. *Let $T = \min(T_1, T_2) > 0$. There then exists a unique solution $(u^\varepsilon, \theta^\varepsilon)(t,x)$ of (1.1)–(1.3) such that*

$$\sup_{0 \leqslant t \leqslant T} \|u^\varepsilon - \bar{u}^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \int_0^T \|u^\varepsilon - \bar{u}^\varepsilon\|_{H^1(\Omega)}^2 \,\mathrm{d}t \leqslant C\varepsilon^2 \tag{1.18}$$

and

$$\sup_{0 \leqslant t \leqslant T} \|\theta^\varepsilon - \bar{\theta}^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \int_0^T \|\theta^\varepsilon - \bar{\theta}^\varepsilon\|_{H^1(\Omega)}^2 \,\mathrm{d}t \leqslant C\varepsilon^2, \tag{1.19}$$

where $\bar{u}^\varepsilon = u^0(t,x) + \sqrt{\varepsilon} u_b^1(t,x,z)$, $\bar{\theta}^\varepsilon = \theta^0(t,x) + \sqrt{\varepsilon}\theta_b^1(t,x,z)$ with u_b^1, θ_b^1 being the boundary layer profiles such that

$$(u_b^1, \theta_b^1)(t,x,z) \in L^\infty(0,T; H^{2,0}(\Omega \times \mathbb{R}_1^+)) \cap L^2(0,T; H^{2,1}(\Omega \times \mathbb{R}_1^+)) \tag{1.20}$$

and

$$(\partial_z u_b^1, \partial_z \theta_b^1)(t,x,z) \in L^\infty([0,T]\times\Omega\times\mathbb{R}_1^+). \tag{1.21}$$

In particular, $u^\varepsilon - u^0$ *and* $\theta^\varepsilon - \theta^0$ *tend to 0 in* $L^\infty(0,T;L^2(\Omega))$ *as* ε *tends to 0.*

REMARK 1.4. Theorem 1.3 tells us that in the inviscid limit the solution $(u^\varepsilon,\theta^\varepsilon)$ can be seen as the sum of the inviscid solution (u^0,θ^0) and a boundary layer of width $\mathcal{O}(\sqrt{\varepsilon})$, which is of the same thickness as Prandtl's boundary layer in the setting of the no-slip boundary conditions. The underlying reason for this is the Navier slip boundary condition (1.5) and the Neumann boundary condition (1.6), which will be clear in our analysis.

The proof of our theorem has two parts. First, we use the method of multi-scale analysis to construct an approximate solution to the initial boundary-value problem (1.1)–(1.8). This approximate solution is close to the inviscid solution away from the boundary and possesses a boundary layer profile of width $\mathcal{O}(\sqrt{\varepsilon})$. The construction of the approximate solution is carried out rigorously in the next section. Due to the transformations between the velocity, the temperature and the pressure, we define the following approximate solutions, which are different from those in [8]:

$$u_a^\varepsilon(t,x) = u^0(t,x) + u_b^0(t,x,z) + \sqrt{\varepsilon}u_b^1(t,x,z) + \varepsilon\omega(t,x,z); \tag{1.22}$$

$$\theta_a^\varepsilon(t,x) = \theta^0(t,x) + \theta_b^0(t,x,z) + \sqrt{\varepsilon}\theta_b^1(t,x,z); \tag{1.23}$$

$$p_a^\varepsilon(t,x) = p^0(t,x) + p_b^0(t,x,z) + \sqrt{\varepsilon}p_b^1(t,x,z) + \varepsilon q(t,x,z), \tag{1.24}$$

where (u^0,θ^0,p^0) satisfies the inviscid problem (1.13)–(1.16) and $z=\varphi(x)/\sqrt{\varepsilon}$ is a fast variable. Thus, we have to determine the first order boundary layer profiles u_b^0 and θ_b^0, which are shown to satisfy the nonlinear Prandtl equations (2.9) and (2.12). In contrast to the case of the no-slip boundary, the boundary layers are much weaker for the Navier slip condition. It will be clear from our analysis that the $O(1)$-term boundary layer functions u_b^0 and θ_b^0 are identically 0 by the boundary conditions and the orthogonality property of u_b^1. Consequently, u_b^1 as a leading boundary layer profile at order $\sqrt{\varepsilon}$ satisfies a linear equation, which is in sharp contrast to Prandtl's boundary layer equations in no-slip boundary conditions. In §§ 2.3 and 2.4 we carry out an H_x^m, $m=0,1,2$, estimate similar to that in [8] for the system of u_b^1 and θ_b^1 to verify (1.20). In comparison with [8], here we additionally need the L^∞ bound of $\partial_z\theta_b^1$, which is carried out by using the structure of the boundary layer equations in § 2.5.

The main part of the proof is an energy estimate of the error equations given in § 3, where we have to estimate both R^ε and S^ε. Since the approximate solution u^a does not satisfy the divergence free condition exactly and it is not tangent to the boundary, the error term R^ε has the same properties. So it is difficult to carry out the standard energy estimate directly on the error equations because there is no further information for the pressure term. Here we adapt the strategy of the L^2 estimate of $\mathbb{P}R^\varepsilon$ in [8], where $\mathbb{P}$ denotes the Leray projector [8], that is, the L^2 orthogonal projection on the space of divergence free vector fields tangent to the boundary. Then, together with the basic L^2 estimate of the error term S^ε, we can finally verify the regularity results of the error terms, and we thus verify the L^2 equivalence between the viscous solutions and the inviscid solutions.

2. Construction of the approximate solutions

In this section, we discuss how we can obtain the approximate solutions given by (1.22)–(1.24) through different scaling and asymptotic expansions. The approximate solutions to the Boussinesq equations $(u_a^\varepsilon, \theta_a^\varepsilon)$ are expected to approximate the inviscid solution (u^0, θ^0) away from the boundary and possess a sharp change near the boundary. The introduction of a multi-scale method, typical in perturbation theory [7], is formally necessary to describe different regions of the flow: the inviscid region and the viscous region. The inner functions u^0, θ^0 and p^0 in (1.22)–(1.24) are exactly the solutions to the inviscid equations (1.13)–(1.15) with the boundary condition (1.16) and the initial data (1.7)–(1.8). In the following, we will determine the boundary layer functions term by term.

2.1. Boundary layer functions u_b^0 and θ_b^0

Substituting (1.22) into the divergence free condition (1.2) yields

$$\mathcal{O}(1/\sqrt{\varepsilon}): \quad n \cdot \partial_z u_b^0 = 0, \tag{2.1}$$

$$\mathcal{O}(1): \quad \operatorname{div}_x u_b^0 + n \cdot \partial_z u_b^1 = 0, \tag{2.2}$$

$$\mathcal{O}(\sqrt{\varepsilon}): \quad \operatorname{div}_x u_b^1 + n \cdot \partial_z \omega = 0. \tag{2.3}$$

The boundary condition (1.4) gives that, for any $t \in [0, T]$ and $x \in \Omega$, the boundary layer functions satisfy

$$\mathcal{O}(1): \quad (u_b^0 \cdot n)|_{z=0} = 0, \tag{2.4}$$

$$\mathcal{O}(\sqrt{\varepsilon}): \quad (u_b^1 \cdot n)|_{z=0} = 0. \tag{2.5}$$

It follows from (2.1) and (2.4) that

$$u_b^0 \cdot n = 0 \quad \forall (t, x, z) \in [0, T] \times \Omega \times \mathbb{R}_1^+. \tag{2.6}$$

Putting (1.22)–(1.24) into (1.1), using (2.6) and comparing the terms of order $\mathcal{O}(1/\sqrt{\varepsilon})$ give that

$$n \cdot \partial_z p_b^0 = 0. \tag{2.7}$$

Notice that $p_b^0 \to 0$ as $z \to +\infty$ for any $t \in [0, T]$ and $x \in \Omega$. We then have

$$p_b^0(t, x, z) \equiv 0 \quad \forall (t, x, z) \in [0, T] \times \Omega \times \mathbb{R}_1^+. \tag{2.8}$$

Thus, the $\mathcal{O}(1)$ terms of (1.1) can be reduced to

$$\partial_t u_b^0 + b(t, x) z \partial_z u_b^0 + u^0 \nabla_x u_b^0 + u_b^0 \nabla u^0 + u_b^0 \nabla_x u_b^0 - \theta_b^0 \cdot e_n + n \cdot \partial_z p_b^1 = \partial_z^2 u_b^0 \tag{2.9}$$

with

$$(\partial_z u_b^0)|_{z=0} = 0, \qquad u_b^0(t, x, +\infty) = 0 \tag{2.10}$$

and

$$u_b^0(0, x, z) = 0,$$

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where the boundary condition on $z = 0$ in (2.10) is derived from (1.4) and (2.6). Furthermore, by (1.17) and [8, lemma 4], we have

$$b(t,x) \in C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)), \quad b(t,x) = \frac{u^0 \cdot n}{\varphi(x)}. \tag{2.11}$$

Putting (1.22)–(1.24) into (1.3) and using the boundary condition (1.16), we have the equation of the boundary layer function θ_b^0,

$$\partial_t \theta_b^0 + u^0 \nabla_x \theta_b^0 + b(t,x) z \partial_z \theta_b^0 + u_b^0 \nabla \theta^0 + u_b^0 \nabla_x \theta_b^0 = \partial_z^2 \theta_b^0 \tag{2.12}$$

with the boundary condition

$$\partial_z \theta_b^0(t,x,0) = 0, \qquad \theta_b^0(t,x,+\infty) = 0 \tag{2.13}$$

and initial data $\theta_b^0(0,t,x,z) = 0$. Next, we carry out the L^2 estimate of the above nonlinear problem to prove

$$u_b^0(t,x,z) = \theta_b^0(t,x,z) = 0. \tag{2.14}$$

In view of the orthogonality property $u_b^1 \cdot n = 0$, which will be proved in the next subsection, it follows from (2.3) that

$$\operatorname{div}_x u_b^0 = 0. \tag{2.15}$$

Then, together with the divergence free condition (1.14) and the fact that u^0 and u_b^0 are both tangent to the boundary, we have

$$\iint_{\Omega \times \mathbb{R}_1^+} u^0 \nabla_x u_b^0 \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z = \iint_{\Omega \times \mathbb{R}_1^+} u_b^0 \nabla_x u_b^0 \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z = 0,$$

and

$$\iint_{\Omega \times \mathbb{R}_1^+} n \cdot \partial_z p_b^1 \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z = 0.$$

Then, multiplying (2.9) by u_b^0, integrating over $\Omega \times \mathbb{R}_1^+$ and using the boundary condition (2.10) yield

$$\begin{aligned}
\tfrac{1}{2}\partial_t \|u_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)} &+ \iint_{\Omega \times \mathbb{R}_1^+} |\partial_z u_b^0|^2 \, \mathrm{d}x \, \mathrm{d}z \\
&= -\frac{1}{2} \iint_{\Omega \times \mathbb{R}_1^+} b(t,x) z \partial_z u_b^0 \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z - \iint_{\Omega \times \mathbb{R}_1^+} u_b^0 \nabla u^0 \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z \\
&\quad + \iint_{\Omega \times \mathbb{R}_1^+} \theta_b^0 \cdot e_n \cdot u_b^0 \, \mathrm{d}x \, \mathrm{d}z \\
&\leqslant C \|u_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)} + \|\theta_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)},
\end{aligned}$$

where we have used integration by parts in z and the property that b and ∇u^0 are uniformly bounded. Similarly, multiplying (2.12) by θ_b^0, integrating over $\Omega \times \mathbb{R}_1^+$ and using the boundary condition (2.13) yield

$$\tfrac{1}{2}\partial_t \|\theta_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)} + \iint_{\Omega \times \mathbb{R}_1^+} |\partial_z \theta_b^0|^2 \, \mathrm{d}x \, \mathrm{d}z \leqslant C \|u_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)} + C \|\theta_b^0\|^2_{L^2(\Omega \times \mathbb{R}_1^+)}.$$

Thus,

$$\partial_t(\|u_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} + \|\theta_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)}) + \iint_{\Omega\times\mathbb{R}_1^+} (|\partial_z u_b^0|^2 + |\partial_z \theta_b^0|^2)\,\mathrm{d}x\,\mathrm{d}z \leqslant C\|u_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} + C\|\theta_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)}.$$

Set $\xi(t) = \|u_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} + \|\theta_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} \geqslant 0$. Then from the above inequality we get

$$\partial_t \xi(t) - C\xi(t) \leqslant 0,$$

which implies that $\mathrm{e}^{-Ct}\xi(t)$ is decreasing in t. Thus,

$$\xi(t) \leqslant \mathrm{e}^{-Ct}\xi(t) \leqslant \xi(0) = 0 \quad \forall t \geqslant 0,$$

where we have used $u_b^0|_{t=0} = \theta_b^0|_{t=0} = 0$. It then follows that

$$\|u_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} + \|\theta_b^0\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} = 0, \tag{2.16}$$

which leads to (2.14). Now we come to determine the next order of boundary layer functions.

2.2. The orthogonal property of u_b^1

It follows from (2.6), (2.8) and the inviscid equation (1.13) that

$$\partial_t u_b^1 + u_b^1\nabla u^0 + u^0\nabla_x u_b^1 + u_b^0\nabla_x u_b^1 + u_b^1\nabla_x u_b^0 + b(t,x)z\partial_z u_b^1 + u_b^1\cdot n\partial_z u_b^1 = \partial_z^2 u_b^1 + \theta_b^1 e_n - \nabla_x p_b^1 - n\cdot\partial_z q,$$

which is derived from the $\mathcal{O}(\sqrt{\varepsilon})$ terms. Instead of studying this equation directly, we turn to the following problem, which consists of

$$u_b^1\cdot n = 0, \tag{2.17}$$

$$\partial_t u_b^1 + (u_b^1\nabla u^0 + u^0\nabla_x u_b^1 + u_b^0\nabla_x u_b^1 + u_b^1\nabla_x u_b^0 - \theta_b^1 e_n)_{\tan} + b(t,x)z\partial_z u_b^1 = \partial_z^2 u_b^1, \tag{2.18}$$

$$(u_b^1\nabla u^0 + u^0\nabla_x u_b^1 + u_b^0\nabla_x u_b^1 + u_b^1\nabla_x u_b^0 - \theta_b^1 e_n)_{\mathrm{normal}} = -n\cdot\partial_z q - \nabla_x p_b^1. \tag{2.19}$$

Taking the scalar product of (2.18) with n implies that

$$\partial_t(u_b^1\cdot n) + b(t,x)z\partial_z(u_b^1\cdot n) - \partial_z^2(u_b^1\cdot n) = 0. \tag{2.20}$$

It then follows from (2.11) that

$$\begin{aligned}\partial_t\|u_b^1\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} + 2\|\partial_z(u_b^1\cdot n)\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} &= \iint_{\Omega\times\mathbb{R}_1^+} b(t,x)|u_b^1\cdot n|^2\,\mathrm{d}x\,\mathrm{d}z \\ &\leqslant \|b(t,x)\|_{L^\infty([0,T]\times\Omega)}\|u_b^1\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} \\ &\leqslant C\|u_b^1\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_1^+)}.\end{aligned}$$

Since

$$u_b^1(0,x,z) = 0 \quad \forall (x,z)\in\Omega\times\mathbb{R}+, \tag{2.21}$$

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similar to (2.16) we have

$$\|u_b^1(t)\cdot n\|^2_{L^2(\Omega\times\mathbb{R}_1^+)} \leqslant 0. \tag{2.22}$$

Consequently,

$$u_b^1\cdot n = 0 \quad \forall (t,x,z)\in[0,T]\times\Omega\times\mathbb{R}_1^+. \tag{2.23}$$

2.3. Basic L^2 estimate of the boundary layer functions u_b^1 and θ_b^1

For $k,m,p\in\mathbb{N}$, we introduce the following weighted anisotropic semi-norm of a function $v(t,x,z)$ (z is a fast variable):

$$\|v\|_{k,m,p} = \biggl(\sum_{|\alpha|\leqslant m}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|\partial_x^\alpha\partial_z^p v|^2\,\mathrm{d}x\,\mathrm{d}z\biggr)^{1/2}$$

and the weighted anisotropic Sobolev space with norm given by

$$\|v\|^2_{H^{k,m,p}} = \sum_{j=0}^{p}\|v\|^2_{k,m,j} = \sum_{|\alpha|\leqslant m, j\leqslant p}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|\partial_x^\alpha\partial_z^j v|^2\,\mathrm{d}x\,\mathrm{d}z.$$

In view of (2.14), (2.18) can be reduced to

$$\partial_t u_b^1 + (u_b^1\nabla u^0 + u^0\nabla_x u_b^1 - \theta_b^1 e_n)_{\tan} + b(t,x)z\partial_z u_b^1 = \partial_z^2 u_b^1. \tag{2.24}$$

Substituting (1.22) into the boundary condition (1.4) gives

$$\begin{aligned}[D(u^\varepsilon)\cdot n+\alpha u^\varepsilon]_{\tan} &= [D(u^0)\cdot n+\alpha u^0]_{\tan} + \frac{1}{2}[\partial_z u_b^1]_{\tan} + \frac{1}{2}\biggl[\sum_j \partial_{x_i}\varphi\cdot\partial_{x_j}\varphi\biggr]_{\tan} \\ &\quad + \mathcal{O}(\sqrt{\varepsilon}) \\ &= [D(u^0)\cdot n+\alpha u^0]_{\tan} + \frac{1}{2}[\partial_z u_b^1]_{\tan} + \frac{1}{2}\biggl[\sum_j \partial_{x_i}\varphi\cdot\partial_{x_j}\varphi\cdot\tau_i\biggr]\boldsymbol{\tau} \\ &\quad + \mathcal{O}(\sqrt{\varepsilon}),\end{aligned}$$

where $\boldsymbol{\tau}$ denotes the tangent vector of the boundary $\partial\Omega$ at x. Notice that $n=\nabla\varphi$, so

$$\biggl[\sum_j \partial_{x_i}\varphi\cdot\partial_{x_j}\varphi\cdot\tau_i\biggr]\boldsymbol{\tau} = 0.$$

It then follows from (1.4) that

$$[\partial_z u_b^1]_{\tan}(t,x,0) = -2[D(u^0(t,x))\cdot n + \alpha u^0(t,x)]_{\tan} \triangleq f(t,x) \quad \forall (t,x)\in[0,T]\times\Omega, \tag{2.25}$$

that is,

$$(\partial_z u_b^1)|_{z=0} = f(t,x) \quad \forall (t,x)\in[0,T]\times\Omega \tag{2.26}$$

due to (2.23). By [8, lemma 4], we have

$$f(t,x)\in C(0,T;H^2(\Omega))\cap C^1(0,T;H^1(\Omega)). \tag{2.27}$$

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On the other hand, we derive from the $\mathcal{O}(\sqrt{\varepsilon})$ terms that

$$\partial_t\theta_b^1 + u^0\nabla_x\theta_b^1 + b(t,x)z\partial_z\theta_b^1 + u_b^1\nabla\theta^0 = h\partial_z^2\theta_b^1 \tag{2.28}$$

with the boundary conditions

$$\partial_z\theta_b^1(t,x,0) = -\nabla\theta_0\cdot n \triangleq g(t,x), \qquad \theta_b^1(t,x,+\infty) = 0 \tag{2.29}$$

and the initial data $\theta_b^1(t=0,x,z)=0$. It is easy to obtain

$$g(t,x)\in C(0,T;H^2(\Omega))\cap C^1(0,T;H^1(\Omega)). \tag{2.30}$$

Next we carry out the energy estimate of the equations of u_b^1 and θ_b^1 given in (2.24) and (2.28). We multiply by $(1+z^{2k})u_b^1$, integrate in x and z and remember that $u_b^1\cdot n = 0$ to obtain

$$\begin{aligned}\tfrac12\partial_t\|u_b^1\|_{k,0,0}^2 &+ \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla u^0\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &+\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\theta_b^1\cdot e_n u_b^1\,\mathrm{d}x\,\mathrm{d}z + \iint_{\Omega\times\mathbb{R}_1^+}(z+z^{2k+1})b(t,x)\partial_z u_b^1\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &-\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_z^2 u_b^1\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z = 0.\end{aligned}$$

Because u^0 is divergence free and tangent to the boundary, we have

$$\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla u^0\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z = 0.$$

Integrating by parts in z and using the condition (2.23), (2.26) and (2.27) yield that

$$\begin{aligned}\partial_t\|u_b^1\|_{k,0,0}^2 &+ 2\|u_b^1\|_{k,0,1}^2\\ &= -2\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla u^0\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &\quad+\iint_{\Omega\times\mathbb{R}_1^+}(1+(2k+1)z^{2k})b(t,x)\cdot|u_b^1|^2\,\mathrm{d}x\,\mathrm{d}z - 2\int_\Omega u_b^1(x,0)f(t,x)\,\mathrm{d}x\\ &\quad-4k\iint_{\Omega\times\mathbb{R}_1^+}z^{2k-1}\partial_z u_b^1\cdot u_b^1\,\mathrm{d}x\,\mathrm{d}z - 2\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\theta_b^1\cdot e_n u_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &\leqslant C|\nabla u^0|_{L^\infty}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|u_b^1|^2\,\mathrm{d}x\,\mathrm{d}z + 4k(2k-1)\iint_{\Omega\times\mathbb{R}_1^+}z^{2k-2}|u_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\\ &\quad+2\iint_{\Omega\times\mathbb{R}_1^+}\partial_z u_b^1\cdot f(t,x)\,\mathrm{d}x\,\mathrm{d}z + C|b(t,x)|_{L^\infty}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|u_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\\ &\leqslant C\|u_b^1\|_{k,0,0}^2 + \|u_b^1\|_{k,0,1}\|(1+z^{2k})^{-1/2}f\|_{L^2(\Omega\times\mathbb{R}_1^+)} + \|\theta_b^1\cdot e_n\|_{k,0,0}^2\\ &\leqslant C\|u_b^1\|_{k,0,0}^2 + \|u_b^1\|_{k,0,1}^2 + \|\theta_b^1\|_{k,0,0}^2 + C,\end{aligned} \tag{2.31}$$

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where we have used that ∇u^0 is uniformly bounded. Similarly, multiplying (2.28) by $(1+z^{2k})\theta_b^1$ we obtain

$$\begin{aligned}\tfrac{1}{2}\|\theta_b^1\|_{k,0,0}^2 &+ \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u^0\nabla_x\theta_b^1\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z \\ &+ \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla\theta^0\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z + \iint_{\Omega\times\mathbb{R}_1^+}(z+z^{2k+1})b(t,x)\partial_z\theta_b^1\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z \\ &- \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_z^2\theta_b^1\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z = 0.\end{aligned}$$

Since u^0 is divergence free and tangent to the boundary, we have

$$\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u^0\nabla_x\theta_b^1\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z = 0.$$

Integrating by parts in z, and using the boundary condition (2.29) give that

$$\begin{aligned}\partial_t\|\theta_b^1\|_{k,0,0}^2 &+ 2\|\theta_b^1\|_{k,0,1}^2 \\ &= \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k+1})b(t,x)|u_b^1|^2\,\mathrm{d}x\,\mathrm{d}z - \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla\theta^0\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z \\ &\quad - \int_\Omega\theta_b^1(t,x,0)g(t,x)\,\mathrm{d}x + \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|\partial_z\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z \\ &\quad + 2k\iint_{\Omega\times\mathbb{R}_1^+}z^{2k-1}\partial_z\theta_b^1\cdot\theta_b^1\,\mathrm{d}x\,\mathrm{d}z \\ &\leqslant C\|u_b^1\|_{k,0,0}^2 + C\|\theta_b^1\|_{k,0,0}^2 + \|\theta_b^1\|_{k,0,1}\|(1+z^{2k})^{-1/2}g\|_{L^2(\Omega\times\mathbb{R}_1^+)} \\ &\leqslant C\|u_b^1\|_{k,0,0}^2 + C\|\theta_b^1\|_{k,0,0}^2 + \|\theta_b^1\|_{k,0,1}^2 + C, \end{aligned} \tag{2.32}$$

where we have used (2.30) and $\nabla\theta^0$ is uniformly bounded. Adding (2.31) to (2.32), we have

$$\partial_t(\|u_b^1\|_{k,0,0}^2 + \|\theta_b^1\|_{k,0,0}^2) + (\|u_b^1\|_{k,0,1}^2 + \|\theta_b^1\|_{k,0,1}^2) \leqslant C\|u_b^1\|_{k,0,0}^2 + C\|\theta_b^1\|_{k,0,0}^2 + C.$$

Set $\zeta(t) = \|u_b^1\|_{k,0,0}^2 + \|\theta_b^1\|_{k,0,0}^2$. We then obtain from the above inequality that $\mathrm{e}^{-Ct}\zeta(t) - Ct$ is decreasing in t. It follows from $u_b^1|_{t=0} = \theta_b^1|_{t=0} = 0$ that

$$\zeta(t) \leqslant Ct\mathrm{e}^{Ct} \leqslant C, \quad t\in[0,T].$$

Therefore, for $t\in[0,T]$, it follows that

$$(\|u_b^1\|_{k,0,0}^2 + \|\theta_b^1\|_{k,0,0}^2) + \int_0^t(\|u_b^1\|_{k,0,1}^2(\tau) + C\|\theta_b^1\|_{k,0,1}^2(\tau))\,\mathrm{d}\tau \leqslant C, \quad k\in\mathbb{N}. \tag{2.33}$$

2.4. H_x^m, $m=1,2$ estimates for the boundary layer functions

For a vector field v and a multi-index α, we define

$$\mathrm{D}_x^m(v) = \{D_x^\alpha(v),\ |\alpha| = m\}.$$

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Thus,

$$\partial^\alpha(v_{\tan}) = \partial^\alpha\{\chi(x)[v-(v\cdot n)n]\} = \chi(x)[\partial^\alpha v - (\partial^\alpha v\cdot n)n] + \mathcal{Q} = (\partial^\alpha v)_{\tan} + \mathcal{Q}, \tag{2.34}$$

where $\mathcal{Q}$ is the linear combination of $\mathrm{D}_x^\gamma(v)$, $|\gamma| \leqslant |\alpha|-1$. We first look at the temperature equation (2.28) for $|\alpha|=m$, $m=1,2$. Applying ∂_x^α to (2.28), multiplying by $(1+z^{2k})\partial_x^\alpha\theta_b^1$ and integrating over $\Omega\times\mathbb{R}_1^+$ give that

$$\begin{aligned}\frac{1}{2}\partial_t\iint_{\Omega\times\mathbb{R}_1^+}&(1+z^{2k})|\partial_x^\alpha\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\\ &= -\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_x^\alpha(u^0\nabla_x\theta_b^1)\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &\quad -\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_x^\alpha(bz\partial_z\theta_b^1)\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &\quad -\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_x^\alpha(u_b^1\cdot\theta^0)\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &\quad +\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_z^2\partial_x^\alpha\theta_b^1\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &:= \sum_{i=1}^4 I_i.\end{aligned}$$

Now, we treat I_i, $i=1,2,3,4$, term by term. First, for $m=1$, we have

$$\begin{aligned}|I_1| \leqslant \bigg|\iint_{\Omega\times\mathbb{R}_1^+}&(1+z^{2k})\mathrm{D}_x(u^0)\mathrm{D}_x\theta_b^1\cdot\mathrm{D}_x\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\bigg|\\ &+\bigg|\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u^0\nabla_x(\partial_x^\alpha\theta_b^1)\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\bigg|.\end{aligned}$$

Since u^0 is divergence free and tangent to the boundary, we have

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}_1^+}&(1+z^{2k})u^0\nabla_x(\partial_x^\alpha\theta_b^1)\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\\ &= -\frac{1}{2}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\operatorname{div}_x(u^0\cdot|\partial_x^\alpha\theta_b^1|^2)\,\mathrm{d}x\,\mathrm{d}z\\ &\quad -\frac{1}{2}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\operatorname{div}_x u^0\cdot|\partial_x^\alpha\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\\ &= 0.\end{aligned}$$

Therefore,

$$|I_1| \leqslant |\mathrm{D}_x(u^0)|_{L^\infty}\iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|\mathrm{D}_x\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z \leqslant C\|\theta_b^1\|_{k,1,0}^2.$$

For $m = 2$, we have

$$
\begin{aligned}
I_1 = &\iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k}) D^2(u^0) \mathrm{D}_x \theta_b^1 \cdot \mathrm{D}_x^2 \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&+ \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k}) u^0 \nabla_x \mathrm{D}_x^2 \theta_b^1 \cdot \mathrm{D}_x^2 \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&+ \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k}) \mathrm{D}_x(u^0) \mathrm{D}_x^2 \theta_b^1 \cdot \mathrm{D}_x^2 \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z,
\end{aligned}
$$

where the second the term vanishes after integration by parts and by using (1.14) and (1.16). The last term satisfies

$$
\iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k}) \mathrm{D}_x(u^0) \mathrm{D}_x^2 \theta_b^1 \cdot \mathrm{D}_x^2 \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z \leqslant C \|\theta_b^1\|_{k,2,0}^2.
$$

In view of the regularity of the inviscid solution (1.17), the first term on the right hand side can be treated as

$$
\begin{aligned}
&\left| \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k}) D^2(u^0) \mathrm{D}_x \theta_b^1 \cdot \mathrm{D}_x^2 \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z \right| \\
&\qquad \leqslant \int_{\mathbb{R}_1^+} (1+z^{2k}) \|D^2(u^0)\|_{L^4(\Omega)} \cdot \|\mathrm{D}_x \theta_b^1\|_{L^4(\Omega)} \cdot \|\mathrm{D}_x^2 \theta_b^1\|_{L^2(\Omega)} \,\mathrm{d}z \\
&\qquad \leqslant C \int_{\mathbb{R}_1^+} (1+z^{2k}) \|D^2(u^0)\|_{L^2(\Omega)}^{1/4} \cdot \|D^2(u^0)\|_{L^6(\Omega)}^{3/4} \cdot \|\mathrm{D}_x \theta_b^1\|_{L^2(\Omega)}^{1/4} \\
&\qquad\qquad \cdot \|\mathrm{D}_x \theta_b^1\|_{L^6(\Omega)}^{3/4} \cdot \|\mathrm{D}_x^2 \theta_b^1\|_{L^2(\Omega)} \,\mathrm{d}z \\
&\qquad \leqslant C \int_{\mathbb{R}_1^+} (1+z^{2k}) \|D^2(u^0)\|_{H^1(\Omega)}^{1/4} \cdot \|D^2(u^0)\|_{H^1(\Omega)}^{3/4} \cdot \|\mathrm{D}_x \theta_b^1\|_{H^1(\Omega)}^{1/4} \\
&\qquad\qquad \cdot \|\mathrm{D}_x \theta_b^1\|_{H^1(\Omega)}^{3/4} \cdot \|\mathrm{D}_x^2 \theta_b^1\|_{L^2(\Omega)} \,\mathrm{d}z \\
&\qquad \leqslant C \|D^2(u^0)\|_{H^1(\Omega)} \int_{\mathbb{R}_1^+} (1+z^{2k}) \|\theta_b^1\|_{H^2(\Omega)}^2 \,\mathrm{d}z \\
&\qquad \leqslant C \|\theta_b^1\|_{k,2,0}^2.
\end{aligned}
$$

Consequently, for $m = 1, 2$, we have

$$
|I_1| \leqslant C \|\theta_b^1\|_{k,m,0}^2. \tag{2.35}
$$

We now come to I_2. For $m = 1$, we have

$$
\begin{aligned}
I_2 = &\iint_{\Omega\times\mathbb{R}_1^+} (z+z^{2k+1}) b(t,x) \partial_z \partial_x^\alpha \theta_b^1 \cdot \partial_x^\alpha \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&+ \iint_{\Omega\times\mathbb{R}_1^+} (z+z^{2k+1}) \partial_x^\alpha b \cdot \partial_z \theta_b^1 \cdot \partial_x^\alpha \theta_b^1 \,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}
$$

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The first term can be controlled by $C\|\theta_b^1\|^2_{k,1,0}$ due to (2.11). We can estimate the second term as follows:

$$\begin{aligned}
&\left|\iint_{\Omega\times\mathbb{R}_1^+}(z+z^{2k+1})\partial_x^\alpha b\cdot\partial_z\theta_b^1\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z\right|\\
&\quad\leqslant\int_{\mathbb{R}_1^+}(z+z^{2k+1})\|\partial_x^\alpha b\|_{L^6(\Omega)}\cdot\|\partial_z\theta_b^1\|_{L^3(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\
&\quad\leqslant C\|b(t,x)\|_{H^2(\Omega)}\int_{\mathbb{R}_1^+}\|(1+z^{2k+4})^{1/2}\partial_z\theta_b^1\|^{1/2}_{L^2(\Omega)}\cdot\|(1+z^{2k})^{1/2}\partial_z\theta_b^1\|^{1/2}_{H^1(\Omega)}\\
&\qquad\qquad\qquad\qquad\cdot\|(1+z^{2k})^{1/2}\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\
&\quad\leqslant C\|\theta_b^1\|^{1/2}_{k+2,0,1}\|\theta_b^1\|^{1/2}_{k,1,1}\|\theta_b^1\|_{k,1,0},
\end{aligned}$$

where we have used the Sobolev interpolation inequalities (see [5])

$$\|g(x)\|_{L^q(\Omega)}\leqslant C\|g(x)\|_{H^1(\Omega)}$$

with $2\leqslant q<+\infty$ for $\Omega\subseteq\mathbb{R}^2$ and $q=6$ for $\Omega\subseteq\mathbb{R}^3$, and

$$\|g(x)\|_{L^3(\Omega)}\leqslant C\|g(x)\|^{1/2}_{L^2(\Omega)}\|g(x)\|^{1/2}_{H^1(\Omega)}.$$

For the case in which $m=2$, I_2 is the sum of

$$\begin{aligned}
J_1&=\iint_{\Omega\times\mathbb{R}_1^+}z(1+z^{2k})\partial_x^\alpha f\cdot\partial_z\theta_b^1\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z,\\
J_2&=\iint_{\Omega\times\mathbb{R}_1^+}z(1+z^{2k})\mathrm{D}_x f\cdot\partial_z\mathrm{D}_x\theta_b^1\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z
\end{aligned}$$

and

$$J_3=\iint_{\Omega\times\mathbb{R}_1^+}z(1+z^{2k})f\cdot\partial_x^\alpha\partial_z\theta_b^1\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z.$$

Obviously, J_3 is bounded by $C\|\theta_b^1\|^2_{k,2,0}$. Using the interpolation inequality we obtain

$$\begin{aligned}
|J_1|&\leqslant\iint_{\Omega\times\mathbb{R}_1^+}z(1+z^{2k})\|\partial_x^\alpha f\|_{L^2(\Omega)}\cdot\|\partial_z\theta_b^1\|_{L^\infty(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\
&\leqslant C\|f\|_{H^2}\int_{\mathbb{R}_1^+}\|(1+z^{2k+4})^{1/2}\partial_z\theta_b^1\|^{1/2}_{H^1(\Omega)}\cdot\|(1+z^{2k})\partial_z\theta_b^1\|^{1/2}_{H^2(\Omega)}\\
&\qquad\qquad\qquad\cdot\|(1+z^{2k})^{1/2}\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\
&\leqslant C\|\theta_b^1\|^{1/2}_{k+2,1,1}\cdot\|\theta_b^1\|^{1/2}_{k,2,1}\cdot\|\theta_b^1\|_{k,2,0}
\end{aligned}$$

and

$$\begin{aligned}
|J_2|&\leqslant\int_{\mathbb{R}_1^+}z(1+z^{2k})\|\mathrm{D}_x f\|_{L^6}\cdot\|\partial_z\mathrm{D}_x\theta_b^1\|_{L^3}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2}\,\mathrm{d}x\,\mathrm{d}z\\
&\leqslant C\|f\|_{H^2}\int_{\mathbb{R}_1^+}\|(1+z^{2k+4})^{1/2}\partial_z\mathrm{D}_x\theta_b^1\|^{1/2}_{L^2(\Omega)}\cdot\|(1+z^{2k})^{1/2}\partial_z\mathrm{D}_x\theta_b^1\|^{1/2}_{H^1(\Omega)}\\
&\qquad\qquad\qquad\cdot\|(1+z^{2k})^{1/2}\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z
\end{aligned}$$

$$\leqslant C\left(\int_{\mathbb{R}_1^+} \|(1+z^{2k+4})^{1/2}\partial_z \mathrm{D}_x\theta_b^1\|_{L^2(\Omega)}^2\,\mathrm{d}z\right)^{1/4} \cdot \left(\int_{\mathbb{R}_1^+} \|(1+z^{2k})^{1/2}\partial_z \mathrm{D}_x\theta_b^1\|_{H^1(\Omega)}^2\,\mathrm{d}z\right)^{1/4} \cdot \left(\int_{\mathbb{R}_1^+} \|(1+z^{2k})^{1/2}\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}^2\,\mathrm{d}z\right)^{1/2}$$
$$\leqslant C\|\theta_b^1\|_{k+2,1,1}^{1/2}\|\theta_b^1\|_{k,2,1}^{1/2}\|\theta_b^1\|_{k,2,0}.$$

Consequently, for $m=1,2$ we obtain, by using Young's inequality,

$$\begin{aligned}|I_2| &\leqslant C\|\theta_b^1\|_{k,m,0}^2 + C\|\theta_b^1\|_{k+2,m-1,1}^{1/2}\|\theta_b^1\|_{k,m,1}^{1/2}\|\theta_b^1\|_{k,m,0}\\ &\leqslant C\|\theta_b^1\|_{k,m,0}^2 + C\|\theta_b^1\|_{k+2,m-1,1}^2 + \eta\|\theta_b^1\|_{k,m,1}^2\end{aligned} \tag{2.36}$$

with $\eta>0$. We now look at the term I_3. For $m=1$,

$$\iint_{\Omega\times\mathbb{R}_1^+} |(1+z^{2k})\partial_x^\alpha u_b^1\cdot\nabla\theta^0\cdot\partial_x^\alpha\theta_b^1|\,\mathrm{d}x\,\mathrm{d}z \leqslant C\|\theta_b^1\|_{k,1,0}^2 + C\|u_b^1\|_{k,1,0}^2,$$

where we have used that $\nabla\theta^0$ is uniformly bounded. It follows from the interpolation inequality and (1.17) that

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}_1^+} |(1+z^{2k})u_b^1\partial_x^\alpha(\nabla\theta^0)\cdot\partial_x^\alpha\theta_b^1|\,\mathrm{d}x\,\mathrm{d}z\\ &\qquad\leqslant \int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{L^4(\Omega)}\cdot\|D^2(\theta^0)\|_{L^4(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\qquad\leqslant C\|\theta^0\|_{H^3(\Omega)}\int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{L^4(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\qquad\leqslant \int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{H^1(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\qquad\leqslant C\|u_b^1\|_{k,1,0}^2 + C\|\theta_b^1\|_{k,1,0}^2.\end{aligned}$$

For $m=2$, I_3 can be written as the sum of $\bar{J}_i$, $i=1,2,3$, with

$$\bar{J}_1 = \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\partial_x^\alpha u_b^1\cdot\nabla\theta^0\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z,$$
$$\bar{J}_2 = \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})\mathrm{D}_x(u_b^1)D_x^2(\theta^0)\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z$$

and

$$\bar{J}_3 = \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})u_b^1\cdot\nabla D^3(\theta^0)\cdot\partial_x^\alpha\theta_b^1\,\mathrm{d}x\,\mathrm{d}z.$$

It is easy to show that

$$|\bar{J}_1| \leqslant C\|u_b^1\|_{k,2,0}^2 + C\|\theta_b^1\|_{k,2,0}^2.$$

The interpolation inequality gives that

$$\begin{aligned}|\bar{J}_2| &\leqslant \int_{\mathbb{R}_1^+}(1+z^{2k})\|\mathrm{D}_x u_b^1\|_{L^4(\Omega)}\cdot\|\mathrm{D}_x^2(\theta^0)\|_{L^4(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\leqslant C\|\theta^0\|_{H^3(\Omega)}\int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{H^3(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\leqslant C\|u_b^1\|_{k,1,0}^2+C\|\theta_b^1\|_{k,1,0}^2\end{aligned}$$

and

$$\begin{aligned}|\bar{J}_3| &\leqslant \int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{L^\infty(\Omega)}\cdot\|D^3(\theta^0)\|_{L^2(\Omega)}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\leqslant C\|\theta^0\|_{H^3(\Omega)}\int_{\mathbb{R}_1^+}(1+z^{2k})\|u_b^1\|_{H^1(\Omega)}^{1/2}\cdot\|u_b^1\|_{H^2(\Omega)}^{1/2}\cdot\|\partial_x^\alpha\theta_b^1\|_{L^2(\Omega)}\,\mathrm{d}z\\ &\leqslant C\|u_b^1\|_{k,1,0}^2+C\|u_b^1\|_{k,2,0}^2+C\|\theta_b^1\|_{k,2,0}^2.\end{aligned}$$

So, for $m=1,2$, we have

$$|I_3|\leqslant C\|u_b^1\|_{k,m,0}^2+C\|\theta_b^1\|_{k,m,0}^2. \tag{2.37}$$

The last term I_4 satisfies

$$\begin{aligned}I_4 &+ \iint_{\Omega\times\mathbb{R}_1^+}(1+z^{2k})|\partial_z\partial_x^\alpha\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\\ &= -\int_\Omega \partial_z\partial_x^\alpha\theta_b^1(t,x,0)\cdot\partial_x^\alpha\theta_b^1(t,x,0)\,\mathrm{d}x - k\iint_{\Omega\times\mathbb{R}_1^+} z^{2k-1}\partial_z|\partial_x^\alpha\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

It follows from (2.29) and (2.30) that

$$\begin{aligned}-\int_\Omega &\partial_z\partial_x^\alpha\theta_b^1(t,x,0)\cdot\partial_x^\alpha\theta_b^1(t,x,0)\,\mathrm{d}x\\ &= \iint_{\Omega\times\mathbb{R}_1^+}\partial_x^\alpha g(t,x)\cdot\partial_z\partial_x^\alpha\theta_b^1(t,x,z)\,\mathrm{d}x\,\mathrm{d}z\\ &\leqslant \eta\|\theta_b^1\|_{k,m,1}^2 + C\|\partial_x^\alpha g(t,x)\|_{L^2(\Omega)}\|(1+z^{2k})^{-1/2}\|_{L^2(\Omega)}\\ &\leqslant \eta\|\theta_b^1\|_{k,m,1}^2 + C\end{aligned}$$

with $\eta>0$. Integrating by parts yields that

$$k\iint_{\Omega\times\mathbb{R}_1^+} z^{2k-1}\partial_z|\partial_x^\alpha\theta_b^1|^2\,\mathrm{d}x\,\mathrm{d}z\leqslant C\|\theta_b^1\|_{k,m,0}^2.$$

Consequently, we have

$$I_4+\|\theta_b^1\|_{k,m,1}^2\leqslant C\|\theta_b^1\|_{k,m,0}^2+\eta\|\theta_b^1\|_{k,m,1}^2+C.$$

Thus, collecting all the estimates of I_i, $i=1,2,3,4$, gives

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\theta_b^1\|_{k,m,0}^2 &+ \|\theta_b^1\|_{k,m,1}^2\\ &\leqslant C\|u_b^1\|_{k,m,0}^2+C\|\theta_b^1\|_{k,m,0}^2+C\|\theta_b^1\|_{k+2,m-1,1}^2+2\eta\|\theta_b^1\|_{k,m,1}^2+C.\end{aligned}$$

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Choosing $\eta > 0$ sufficiently small we have

$$\partial_t \|\theta_b^1\|_{k,m,0}^2 + \|\theta_b^1\|_{k,m,1}^2 \leqslant C\|u_b^1\|_{k,m,0}^2 + C\|\theta_b^1\|_{k,m,0}^2 + C\|\theta_b^1\|_{k+2,m-1,1}^2 + C. \quad (2.38)$$

Similarly, we have from (2.24) that

$$\begin{aligned}
\frac{1}{2}\partial_t \iint_{\Omega\times\mathbb{R}_1^+} & (1+z^{2k})|\partial_x^\alpha u_b^1|^2 \,\mathrm{d}x\,\mathrm{d}z \\
&= \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k})\partial_z^2 \partial_x^\alpha u_b^1 \cdot \partial_x^\alpha u_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&\quad - \iint_{\Omega\times\mathbb{R}_1^+} z(1+z^{2k})\partial_x^\alpha (b(t,x)\partial_z u_b^1)\cdot \partial_x^\alpha u_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&\quad - \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k})\partial_x^\alpha [(u_b^1\cdot\nabla u^0 + u^0\cdot\nabla_x u_b^1 + \theta_b^1\cdot e_n)_{\tan}]\cdot \partial_x^\alpha u_b^1 \,\mathrm{d}x\,\mathrm{d}z \\
&= \sum_{i=1}^{3} \tilde{I}_i.
\end{aligned}$$

Then, carrying out a similar analysis as above, we have for $m = 1, 2$ that

$$\tilde{I}_1 + \tfrac{1}{2}\|u_b^1\|_{k,m,1}^2 \leqslant C\|u_b^1\|_{k,m,0}^2 + C \quad (2.39)$$

and

$$|\tilde{I}_2| \leqslant C\|u_b^1\|_{k,m,0}^2 + C\|u_b^1\|_{k+2,m-1,1}^2 + \eta\|u_b^1\|_{k,m,1}^2 \quad (2.40)$$

for $\eta > 0$. In view of (2.34) one has

$$\begin{aligned}
\tilde{I}_3 = \iint_{\Omega\times\mathbb{R}_1^+} (1+z^{2k})\{&\partial_x^\alpha (u_b^1\cdot\nabla u^0 + u^0\cdot\nabla_x u_b^1 + \theta_b^1\cdot e_n) + \mathcal{Q}_* \\
&- [\partial_x^\alpha (u_b^1\cdot\nabla u^0 + u^0\cdot\nabla_x u_b^1 + \theta_b^1\cdot e_n)\cdot n]n\}\partial_x^\alpha u_b^1 \,\mathrm{d}x\,\mathrm{d}z,
\end{aligned}$$

where $\mathcal{Q}_*$ is the linear combination of $\mathrm{D}_x^\gamma(u_b^1\cdot\nabla u^0 + u^0\cdot\nabla_x u_b^1 + \theta_b^1\cdot e_n)$ with $|\gamma| \leqslant m-1$. Applying ∂_x^α to $n\cdot u_b^1 = 0$, we have that $n\cdot\partial_x^\alpha u_b^1$ is equal to the linear combination of $\mathrm{D}_x^\gamma(u_b^1)$, $|\gamma| \leqslant m-1$. For $m = 1$, using (1.17) we obtain

$$\begin{aligned}
|\tilde{I}_3| &\leqslant C\int_{\mathbb{R}_1^+} (1+z^{2k})\|\mathrm{D}_x(u_b^1)\|_{L^2(\Omega)}^2 \,\mathrm{d}z \\
&\quad + C\int_{\mathbb{R}_1^+} (1+z^{2k})\|u_b^1\|_{L^4(\Omega)}\|\mathrm{D}_x^2(u^0)\|_{L^4(\Omega)}\|\mathrm{D}_x(u_b^1)\|_{L^2(\Omega)} \,\mathrm{d}z \\
&\quad + C\int_{\mathbb{R}_1^+} (1+z^{2k})(\|u_b^1\|_{L^2(\Omega)}\|\theta_b^1\|_{H^1(\Omega)} + \|u_b^1\|_{H^1(\Omega)}\|\theta_b^1\|_{L^2(\Omega)} \\
&\qquad\qquad + \|u_b^1\|_{H^1(\Omega)}\|\theta_b^1\|_{H^1(\Omega)}) \,\mathrm{d}z \\
&\leqslant C\|u_b^1\|_{k,0,0}^2 + C\|u_b^1\|_{k,1,0}^2 + C\|\theta_b^1\|_{k,0,0}^2 + C\|\theta_b^1\|_{k,1,0}^2.
\end{aligned}$$

For the case in which $m = 2$ it follows from the interpolation inequalities that

$$|\tilde{I}_3| \leqslant C\|u_b^1\|_{k,1,0}^2 + C\|u_b^1\|_{k,2,0}^2 + C\|\theta_b^1\|_{k,1,0}^2 + C\|\theta_b^1\|_{k,2,0}^2.$$

Therefore,

$$|\tilde{I}_3| \leqslant C\|u_b^1\|_{k,m,0}^2 + C\|\theta_b^1\|_{k,m,0}^2, \quad m = 1, 2. \tag{2.41}$$

Collecting the above estimates (2.39)–(2.41) and choosing $\eta > 0$ sufficiently small yield

$$\partial_t\|u_b^1\|_{k,m,0}^2 + \|u_b^1\|_{k,m,1}^2 \leqslant C\|u_b^1\|_{k,m,0}^2 + C\|u_b^1\|_{k+2,m-1,1}^2 + C. \tag{2.42}$$

Adding (2.42) to (2.38) we obtain

$$\begin{aligned}\partial_t(\|u_b^1\|_{k,m,0}^2 + \|\theta_b^1\|_{k,m,0}^2) + (\|u_b^1\|_{k,m,1}^2 + \|\theta_b^1\|_{k,m,1}^2) \\ \leqslant C(\|u_b^1\|_{k,m,0}^2 + \|\theta_b^1\|_{k,m,0}^2) + C(\|u_b^1\|_{k+2,m-1,1}^2 + \|\theta_b^1\|_{k+2,m-1,1}^2) + C.\end{aligned}$$

Gronwall's inequality (see [5, p. 624]) then implies that

$$\begin{aligned}(\|u_b^1\|_{k,m,0}^2 + \|\theta_b^1(\tau)\|_{k,m,0}^2) + \int_0^t (\|u_b^1(\tau)\|_{k,m,1}^2 + \|\theta_b^1(\tau)\|_{k,m,1}^2)\,\mathrm{d}\tau \\ \leqslant Ct + C\int_0^t (\|u_b^1(\tau)\|_{k+2,m-1,1}^2 + \|\theta_b^1(\tau)\|_{k+2,m-1,1}^2)\,\mathrm{d}\tau \quad \forall t \in [0, T].\end{aligned} \tag{2.43}$$

Therefore, for the case in which $m = 1$, (2.43) gives

$$\begin{aligned}(\|u_b^1\|_{k,1,0}^2 + \|\theta_b^1\|_{k,1,0}^2) + \int_0^t (\|u_b^1(\tau)\|_{k,1,1}^2 + \|\theta_b^1(\tau)\|_{k,1,1}^2)\,\mathrm{d}\tau \\ \leqslant Ct + C\int_0^t (\|u_b^1(\tau)\|_{k+2,0,1}^2 + \|\theta_b^1(\tau)\|_{k+2,0,1}^2)\,\mathrm{d}\tau \quad \forall t \in [0, T],\end{aligned}$$

where the right-hand side of the above inequality is bounded due to the L^2 estimate (2.33). Consequently, $(u_b^1, \theta_b^1)(t, x, z) \in L^\infty(0, T; H^{1,0}(\Omega \times \mathbb{R}_1^+)) \cap L^2(0, T; H^{1,1}(\Omega \times \mathbb{R}_1^+))$. Similarly, applying (2.43) to $m = 2$ we obtain

$$(u_b^1, \theta_b^1)(t, x, z) \in L^\infty(0, T; H^{2,0}(\Omega \times \mathbb{R}_1^+)) \cap L^2(0, T; H^{2,1}(\Omega \times \mathbb{R}_1^+)),$$

which verifies (1.20) in theorem 1.3. In the next section we turn to the proof of (1.21).

2.5. Uniform estimates for $\partial_z u_b^1$ and $\partial_z \theta_b^1$

Set

$$\beta = 4\|\nabla u^0\|_{L^\infty([0,T]\times\Omega)} + 2\|b(t, x)\|_{L^\infty([0,T]\times\Omega)} + 2\|\nabla\theta^0\|_{L^\infty([0,T]\times\Omega)} + 2 \tag{2.44}$$

and

$$\begin{aligned}\psi_1(t, x, z) &= \partial_z u_b^1(t, x, z)\mathrm{e}^{-\beta t}, \\ \psi_2(t, x, z) &= \partial_z \theta_b^1(t, x, z)\mathrm{e}^{-\beta t}.\end{aligned}$$

It follows from (2.18), (2.28) and the orthogonality property $u_b^1 \cdot n = 0$ that the function

$$\psi(t, x, z) = |\psi_1|^2 + |\psi_2|^2$$

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satisfies

$$\partial_t\psi + u^0\nabla_x\psi + 2b(t,x)\psi + b(t,x)z\partial_z\psi + 2(\psi_1\nabla u^0)\cdot\psi_1 + 2\psi_2 e_n\cdot\psi_1 + 2(\psi_1\nabla\theta^0)\cdot\psi_2 - \partial_z^2\psi + 2(|\partial_z\psi_1|^2 + |\partial_z\psi_2|^2) + 2\beta\psi = 0.$$

Then $\psi(t,x,z)$ can only attain its maximum on $t=0$ or $z=0$. In fact, if we assume that ψ attains its maximum at a point $(t_0,x_0,z_0)\in(0,T]\times\bar{\Omega}\times\mathbb{R}_1^+$, then at this point

$$\partial_t\psi\geqslant 0,\qquad \partial_z\psi = 0,\qquad \partial_z^2\psi\leqslant 0.$$

If $x_0\in\Omega$, then $\nabla_x\psi = 0$. If $x_0\in\partial\Omega$, then

$$u^0\nabla_x\psi = u^0(\nabla_x\psi)_{\tan}$$

since u^0 is tangent to the boundary $\partial\Omega$. As a consequence, we obtain by (2.44) that

$$\begin{aligned}2\beta\psi &\leqslant -2(\psi_1\nabla u^0)\cdot\psi_1 - 2(\psi_1\nabla\theta^0)\cdot\psi_2 - 2\psi_2 e_n\cdot\psi_1 - 2b(t,x)\psi\\ &\leqslant 2|\nabla u^0|_{L^\infty([0,T]\times\Omega)}|\psi_1|^2 + \psi + |\nabla\theta^0|_{L^\infty([0,T]\times\Omega)}\psi + 2|b|_{L^\infty([0,T]\times\Omega)}\psi\\ &\leqslant 2|\nabla u^0|_{L^\infty([0,T]\times\Omega)}\psi + \psi + |\nabla\theta^0|_{L^\infty([0,T]\times\Omega)}\psi + 2|b|_{L^\infty([0,T]\times\Omega)}\psi\\ &\leqslant \beta\psi.\end{aligned}\tag{2.45}$$

Since $\psi\geqslant 0$, (2.44) implies that $\psi\equiv 0$, and therefore $\psi_1=\psi_2=0$, which is not true. Thus, ψ can only attain its maximum on $t=0$ or $z=0$, that is,

$$\psi_1(t,x,z),\psi_2(t,x,z)\in L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}_1^+).$$

Consequently, we have

$$\partial_z u_b^1\in L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}_1^+)\tag{2.46}$$

and

$$\partial_z\theta_b^1\in L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}_1^+).\tag{2.47}$$

2.6. Higher-order boundary layer functions

We now determine the next order functions ω and q in the solution expansions given in (1.22) and (1.24). Let

$$\bar{\omega}(t,x,z) = n\cdot\omega$$

be a scalar function. Then (2.3) gives

$$\bar{\omega}(t,x,z) = \int_z^{+\infty}\operatorname{div}_x u_b^1(t,x,y)\,\mathrm{d}y,\tag{2.48}$$

where ω vanishes for $x\in\Omega\setminus\mathcal{V}$. In view of (2.9) and (2.14), we have

$$p_b^1(t,x,z) = 0\quad\forall(t,x,z)\in[0,T]\times\Omega\times\mathbb{R}_1^+.\tag{2.49}$$

Then, together with (2.8), the associated pressure has the form

$$p^a(t,x) = p^0(t,x) + \varepsilon q(t,x,z),\tag{2.50}$$

where $q(t,x,z)$ satisfies

$$(u_b^1\nabla u^0 + u^0\nabla_x u_b^1 - \theta_b^1 e_n)_{\text{normal}} = -n\cdot\partial_z q \quad \forall (t,x,z)\in[0,T]\times\Omega\times\mathbb{R}_1^+. \tag{2.51}$$

Then,

$$\begin{aligned}
\|\nabla_x q(t,x,z)\|_{L(\Omega)} &\leqslant \int_{\mathbb{R}_1^+}\|u_b^1\|_{H_x^2(\Omega)}\,\mathrm{d}z\\
&\leqslant \biggl(\int_{\mathbb{R}_1^+}(1+z)^{-2}\,\mathrm{d}z\biggr)^{1/2}\cdot\biggl(\int_{\mathbb{R}_1^+}(1+z)^2\|u_b^1\|_{H_x^2(\Omega)}^2\,\mathrm{d}z\biggr)^{1/2}\\
&\leqslant C\biggl(\int_{\mathbb{R}_1^+}\|(1+z)u_b^1\|_{H_x^2(\Omega)}^2\,\mathrm{d}z\biggr)^{1/2}\\
&\leqslant C\|u_b^1\|_{1,2,0}.
\end{aligned}\tag{2.52}$$

Furthermore, q vanishes for $x\in\Omega\setminus\mathcal{V}$.

3. Estimates of the error terms

In this section we first derive the initial boundary-value problem of the error terms and then, by an energy estimate, we show the L^2 estimate of the error terms, which readily yields the stability result. Let $R^\varepsilon(t,x)$, $S^\varepsilon(t,x)$ and $\pi^\varepsilon(t,x)$ be the remainder terms. In our context, the function $\omega(t,x,z)$ is only a part of the $\mathcal{O}(\varepsilon)$ boundary layer functions, so $(R^\varepsilon,S^\varepsilon,\pi^\varepsilon)$ and $q(t,x,z)$ are of the order of ε. That is,

$$\begin{aligned}
u^\varepsilon(t,x) &= u^0(t,x)+\sqrt{\varepsilon}u_b^1(t,x,z)+\varepsilon\omega(t,x,z)+\varepsilon R^\varepsilon(t,x),\\
\theta^\varepsilon(t,x) &= \theta^0(t,x)+\sqrt{\varepsilon}\theta_b^1(t,x,z)+\varepsilon S^\varepsilon(t,x),\\
p^\varepsilon(t,x) &= p^0(t,x)+\varepsilon q(t,x,z)+\varepsilon\pi^\varepsilon(t,x).
\end{aligned}$$

Then R^ε and S^ε satisfy

$$\begin{aligned}
\partial_t R^\varepsilon - \varepsilon\Delta R^\varepsilon + u^\varepsilon\nabla R^\varepsilon + R^\varepsilon\cdot\nabla u^0 + R^\varepsilon\cdot n\partial_z u_b^1&\\
- S^\varepsilon\cdot e_n + \sqrt{\varepsilon}R^\varepsilon\cdot n\partial_z\omega + \sqrt{\varepsilon}R^\varepsilon\cdot\nabla_x u_b^1 &= K_1, \qquad (3.1)\\
\operatorname{div} R^\varepsilon &= -\operatorname{div}_x\omega, \qquad (3.2)\\
\partial_t S^\varepsilon - \varepsilon\Delta S^\varepsilon + u^\varepsilon\nabla S^\varepsilon + R^\varepsilon\cdot\nabla\theta^0 + R^\varepsilon\cdot n\partial_z\theta_b^1 + \sqrt{\varepsilon}R^\varepsilon\cdot\nabla_x\theta_b^1 &= K_2 \qquad (3.3)
\end{aligned}$$

with the boundary conditions

$$R^\varepsilon\cdot n = -\omega(t,x,0)\cdot n,\quad x\in\partial\Omega, \tag{3.4}$$

$$[D(R^\varepsilon)n + \tfrac12\nabla_x(n\cdot\omega)+\alpha R^\varepsilon]_{\tan} = \zeta(t,x),\quad x\in\partial\Omega, \tag{3.5}$$

$$\frac{\partial S^\varepsilon}{\partial n} = -\frac{1}{\sqrt{\varepsilon}}\frac{\partial\theta_b^1}{\partial n}(t,x,0),\quad x\in\partial\Omega, \tag{3.6}$$

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where

$$\begin{aligned}K_1 &= -\partial_t\omega + \Delta u^0 + 2n\cdot\nabla_x\partial_z u_b^1 - u^\varepsilon\cdot\nabla_x\omega - \omega\cdot\nabla u^0 - \omega\cdot n\partial_z u_b^1 - u_b^1\cdot\nabla_x u_b^1\\ &\quad + \Delta\varphi\partial_z u_b^1 - b(t,x)z\partial_z\omega + \sqrt{\varepsilon}\Delta_x u_b^1 - \sqrt{\varepsilon}\omega\cdot n\partial_z\omega - \sqrt{\varepsilon}\omega\cdot\nabla_x u_b^1 + \varepsilon\Delta_x\omega\\ &\quad + 2\sqrt{\varepsilon}n\cdot\nabla_x\partial_z\omega + \sqrt{\varepsilon}\Delta\varphi\partial_z\omega + \partial_z^2\omega + \nabla_x q + \nabla_x\pi^\varepsilon,\\ K_2 &= \Delta\theta^0 + 2n\cdot\nabla_x\partial_z\theta_b^1 + \Delta\varphi\partial_z\theta_b^1 - \omega\nabla\theta^0 - u_b^1\nabla_x\theta_b^1\\ &\qquad\qquad - \omega\cdot n\partial_z\theta_b^1 + \sqrt{\varepsilon}\Delta_x\theta_b^1 - \sqrt{\varepsilon}\omega\nabla_x\theta_b^1\end{aligned}$$

and

$$\begin{aligned}\zeta(t,x) = \bigg[&-\bar{\omega}(t,x,0)D(n)n - \alpha\omega(t,x,0)\\ &- \frac{\alpha}{\sqrt{\varepsilon}}u_b^1(t,x,0) - \frac{1}{\sqrt{\varepsilon}}\mathrm{D}_x u_b^1(t,x,0)n\bigg]_{\tan}, \quad x\in\Omega.\end{aligned}$$

Furthermore, the error terms satisfy the initial condition that

$$R^\varepsilon(t=0,x) = S^\varepsilon(t=0,x) = 0, \quad x\in\Omega. \tag{3.7}$$

We now carry out the estimate of the remainder terms R^ε and S^ε in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. We decompose R^ε as

$$R^\varepsilon = \mathbb{P}R^\varepsilon + (I-\mathbb{P})R^\varepsilon = \mathbb{P}R^\varepsilon + \nabla\rho,$$

where $\mathbb{P}$ denotes the Leray projector (see [8]). We adapt the estimate in [8] so that for some $\delta_0 > 0$ there holds

$$\begin{aligned}&\|\mathbb{P}R^\varepsilon(t)\|_{L^2(\Omega)}^2 + \varepsilon\delta_0\int_0^t \|R^\varepsilon(\tau)\|_{H^1(\Omega)}^2\,\mathrm{d}\tau\\ &\qquad\leqslant C + \int_0^t f_1^\varepsilon(\tau)\|R^\varepsilon(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau \quad \forall t\in[0,T],\end{aligned} \tag{3.8}$$

where f_1^ε is bounded and independent of ε in $L^1(0,T)$. The Leray projector $\mathbb{P}$ satisfies

$$(I-\mathbb{P})R^\varepsilon = \nabla\rho$$

with ρ being the solution of the following problem:

$$\begin{aligned}\Delta\rho &= -\operatorname{div}_x\omega \quad &&\text{in } \Omega,\\ \frac{\partial\rho}{\partial n} &= -\bar{\omega}(t,x,0) \quad &&\text{on } \partial\Omega.\end{aligned}$$

Thus, a standard elliptic estimate gives that

$$\|\nabla\rho\|_{H^1} \leqslant C\|\operatorname{div}_x\omega\|_{L^2} + C\|\bar{\omega}(t,x,0)\|_{H^1} \leqslant C\|u_b^1\|_{1,2,0} \leqslant C.$$

This implies that

$$\|(I-\mathbb{P})R^\varepsilon(t)\|_{L^2} \leqslant C. \tag{3.9}$$

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On the other hand, we multiply (3.3) by S^ε and integrate over $\Omega \times [0,T]$ to obtain

$$\tfrac{1}{2}\|S^\varepsilon\|^2_{L^2(\Omega)} - \varepsilon \int_0^t \int_\Omega \Delta S^\varepsilon \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau \leqslant \sum_{i=1}^{5} Q_i,$$

where

$$\begin{aligned}
Q_1 &= \int_0^t \int_\Omega u^\varepsilon \nabla S^\varepsilon \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau,\\
Q_2 &= \int_0^t \int_\Omega R^\varepsilon \nabla \theta^0 \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau,\\
Q_3 &= \sqrt{\varepsilon}\int_0^t \int_\Omega R^\varepsilon \nabla_x \theta_b^1 \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau,\\
Q_4 &= \int_0^t \int_\Omega R^\varepsilon \cdot n \nabla_z \theta_b^1 \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau,\\
Q_5 &= \int_0^t \int_\Omega K_2 S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

It follows from the divergence theorem and the boundary condition (3.6) that

$$\begin{aligned}
-\varepsilon \int_0^t \int_\Omega \Delta S^\varepsilon \cdot S^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau &= \sqrt{\varepsilon}\int_0^t \left(\int_{\partial\Omega} (\nabla_x \theta_b^1(x,0)\cdot n) S^\varepsilon \,\mathrm{d}s\right)\mathrm{d}\tau\\
&\quad + \varepsilon \int_0^t \|\nabla S^\varepsilon\|^2_{L^2(\Omega)} \,\mathrm{d}\tau\\
&= \sqrt{\varepsilon}\int_0^t \int_\Omega \mathrm{div}_x(\nabla_x \theta_b^1 \cdot S^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon \int_0^t \|\nabla S^\varepsilon\|^2_{L^2(\Omega)} \,\mathrm{d}\tau.
\end{aligned} \tag{3.10}$$

For some $\eta > 0$, we have

$$\begin{aligned}
&\sqrt{\varepsilon}\int_0^t \int_\Omega \mathrm{div}_x(\nabla_x \theta_b^1 \cdot S^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad \leqslant C\int_0^t \|\theta_b^1\|^2_{1,2,1}\,\mathrm{d}\tau + C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon \int_0^t \|S^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau.
\end{aligned} \tag{3.11}$$

Furthermore, there exists $\delta_1 > 0$ such that

$$\varepsilon \int_0^t \|\nabla S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \geqslant \delta_1 \varepsilon \int_0^t \|S^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau - C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \tag{3.12}$$

Because u^ε is divergence free and tangent to the boundary, we have $Q_1 = 0$. By Cauchy's inequality, we obtain

$$|Q_2| \leqslant C\int_0^t \|R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau, \tag{3.13}$$

where we have used that $\nabla\theta^0$ is uniformly bounded. Since for any boundary layer function $v(t, x, z)$ we have

$$v(t,x,z) = -\int_z^{+\infty} \partial_z v(t,x,y)\,\mathrm{d}y,$$

for $2 \leqslant p < +\infty$ the Sobolev embedding implies that

$$\begin{aligned}\|v(t,x,z)\|_{L^p(\Omega)} &\leqslant \int_{\mathbb{R}^1_+} \biggl(\int_\Omega |\partial_z v|^p\,\mathrm{d}x\biggr)^{1/p}\mathrm{d}z \\ &\leqslant \biggl(\int_{\mathbb{R}^1_+} (1+z)^{-2}\,\mathrm{d}z\biggr)^{1/2} \cdot \biggl(\int_{\mathbb{R}^1_+} (1+z)^2 \|\partial_z v\|_{L^p(\Omega)}\,\mathrm{d}z\biggr)^{1/2} \\ &\leqslant C\|v\|_{1,m,1}, \end{aligned} \tag{3.14}$$

where $m \geqslant 3/2 - 3/p$. Similarly,

$$\|v(t,x,z)\|_{L^\infty(\Omega)} \leqslant C\|v\|_{1,2,1}. \tag{3.15}$$

It then follows for $\eta > 0$ that

$$\begin{aligned}|Q_3| &\leqslant \sqrt{\varepsilon}\int_0^t \|\nabla_x\theta_b^1\|_{L^6(\Omega)} \cdot \|R^\varepsilon\|_{L^3(\Omega)} \cdot \|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\nabla_x\theta_b^1\|^2_{L^6(\Omega)} \cdot \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \varepsilon\int_0^t \|R^\varepsilon\|^2_{L^3(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\theta_b^1\|^2_{1,2,1} \cdot \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C\varepsilon\int_0^t \|R^\varepsilon\|_{L^2(\Omega)} \cdot \|R^\varepsilon\|_{H^1(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\theta_b^1\|^2_{1,2,1} \cdot \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C\varepsilon\int_0^t \|R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon\int_0^t \|R^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau. \end{aligned} \tag{3.16}$$

In view of (2.47), we obtain

$$|Q_4| \leqslant C\int_0^t \|R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \tag{3.17}$$

Next, we estimate Q_5 term by term. First, due to (1.17), we have

$$\int_0^t\int_\Omega \Delta\theta^0 \cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \tag{3.18}$$

In view of (3.14), one has

$$\begin{aligned}\sqrt{\varepsilon}\int_0^t\int_\Omega \Delta_x\theta_b^1 \cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant C\int_0^t \|\Delta_x\theta_b^1\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\theta_b^1\|^2_{1,2,1}\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \end{aligned} \tag{3.19}$$

Thanks to [8, lemma 3], for any $h(x,z)$ in $L^2_z(\mathbb{R}^+_1; H^1_x(\Omega))$ that vanishes for x outside the neighbourhood $\mathcal{V}$, there exists a constant C independent of ε such that

$$\|h(x,z)\|_{L^2(\Omega)} \leqslant C\|h\|_{L^2_z(\mathbb{R}^+_1;H^1_x(\Omega))}. \tag{3.20}$$

Hence,

$$\begin{aligned} 2\int_0^t\int_\Omega n\nabla_x\partial_z\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \int_0^t \|\partial_n\partial_z\theta_b^1\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\theta_b^1\|^2_{0,2,1}\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \end{aligned} \tag{3.21}$$

and

$$\int_0^t\int_\Omega \Delta\varphi\partial_z\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C + \int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \tag{3.22}$$

In view of (2.48) and (2.52), it follows that

$$\|\omega(t,x,z)\|_{L^6(\Omega)} \leqslant C\|\nabla u_b^1\|_{L^1_z(\mathbb{R}^+_1;L^6_x(\Omega))} \leqslant C\int_{\mathbb{R}^+_1} \|u_b^1\|_{H^2_x(\Omega)}\,\mathrm{d}z \leqslant C\|u_b^1\|_{1,2,0}, \tag{3.23}$$

where we used the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Thus, by (1.17) and (1.20), we have

$$\begin{aligned} \int_0^t\int_\Omega \omega\nabla\theta^0 S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \int_0^t \|\omega\|_{L^6(\Omega)}\cdot\|\nabla\theta^0\|_{L^3(\Omega)}\cdot\|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\omega\|_{L^6(\Omega)}\cdot\|\nabla\theta^0\|^{1/2}_{L^2(\Omega)}\cdot\|\nabla\theta^0\|^{1/2}_{H^1(\Omega)}\cdot\|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|u_b^1\|_{1,2,0}\cdot\|\theta^0\|_{H^2(\Omega)}\cdot\|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leqslant C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C. \end{aligned} \tag{3.24}$$

Thanks to (3.14), we have

$$\begin{aligned} \int_\Omega u_b^1\cdot\nabla_x\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x &\leqslant \|u_b^1\|_{L^3(\Omega)}\cdot\|\nabla_x\theta_b^1\|_{L^6(\Omega)}\cdot\|S^\varepsilon\|_{L^2(\Omega)} \\ &\leqslant C\|\nabla_x\theta_b^1\|^2_{L^6(\Omega)}\cdot\|S^\varepsilon\|^2_{L^2(\Omega)} + C\|u_b^1\|^2_{L^3(\Omega)} \\ &\leqslant C\|\theta_b^1\|^2_{1,2,1}\|S^\varepsilon\|^2_{L^2(\Omega)} + C\|u_b^1\|_{L^2(\Omega)}\|u_b^1\|_{H^1(\Omega)} \\ &\leqslant C\|\theta_b^1\|^2_{1,2,1}\|S^\varepsilon\|^2_{L^2(\Omega)} + C\|u_b^1\|^2_{1,2,1}. \end{aligned}$$

Consequently,

$$\int_0^t\int_\Omega u_b^1\cdot\nabla_x\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C\int_0^t \|\theta_b^1\|^2_{1,2,1}\|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C$$

due to (1.20). Similarly, using (3.23) and (1.20) we obtain

$$\begin{aligned}\sqrt{\varepsilon}\int_0^t\int_\Omega \omega\nabla_x\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant C\int_0^t \|\omega\|_{L^6(\Omega)}\cdot\|\nabla_x\theta_b^1\|_{L^3(\Omega)}\cdot\|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\leqslant C\int_0^t \|u_b^1\|_{1,2,0}\|\theta_b^1\|_{1,2,1}\|S^\varepsilon\|_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\leqslant C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C.\end{aligned}$$

Due to the uniform boundedness of $\partial_z\theta_b^1$ given in (2.47), it is easy to obtain

$$\begin{aligned}\int_0^t\int_\Omega \omega\cdot n\partial_z\theta_b^1\cdot S^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant C\int_0^t \|\omega\|^2_{L^2}\,\mathrm{d}\tau + C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\leqslant C\int_0^t \|u_b^1\|^2_{1,2,0}\,\mathrm{d}\tau + C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\leqslant C + C\int_0^t \|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau. \end{aligned}\tag{3.25}$$

Summing up the above estimates of Q_i, $i = 1,\dots,5$, yields

$$\begin{aligned}&\|S^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\delta_1\int_0^t \|S^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau\\ &\quad\leqslant \int_0^T f_2^\varepsilon\cdot\|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon\int_0^t \|S^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau\\ &\qquad + \int_0^T f_3^\varepsilon\cdot\|R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon\int_0^t \|R^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau + C,\end{aligned}$$

where f_2^ε, f_3^ε are bounded independent of ε in $L^1(0,T)$ due to (1.20). Thus, together with (3.8), we have

$$\begin{aligned}&\|S^\varepsilon\|^2_{L^2(\Omega)} + \|\mathbb{P}R^\varepsilon(t)\|^2_{L^2(\Omega)} + \underline{\delta}\varepsilon\int_0^t (\|R^\varepsilon(t)\|^2_{H^1(\Omega)} + \|S^\varepsilon\|^2_{H^1(\Omega)})\,\mathrm{d}\tau\\ &\quad\leqslant \int_0^t f_2^\varepsilon\cdot\|S^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon\int_0^t \|S^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau + \int_0^t f_3^\varepsilon\cdot\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\qquad + \int_0^t f_3^\varepsilon\cdot\|R^\varepsilon - \mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \eta\varepsilon\int_0^t \|R^\varepsilon\|^2_{H^1(\Omega)}\,\mathrm{d}\tau\\ &\qquad + \int_0^t f_1^\varepsilon\cdot\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \int_0^t f_1^\varepsilon\cdot\|R^\varepsilon - \mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\\ &\qquad + \int_0^t \|S^\varepsilon(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + C \quad \forall t\in[0,T], \end{aligned}\tag{3.26}$$

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where $\underline{\delta} = \min\{\delta_0, \delta_1\}$. Then, for $\eta > 0$ sufficiently small, it follows from (3.26) that

$$(\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)} + \|S^\varepsilon\|^2_{L^2(\Omega)}) + \varepsilon \int_0^t (\|S^\varepsilon\|^2_{H^1(\Omega)} + \|R^\varepsilon\|^2_{H^1(\Omega)})\,\mathrm{d}\tau \leqslant \int_0^T f_4^\varepsilon \cdot (\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)} + \|S^\varepsilon\|^2_{L^2(\Omega)})\,\mathrm{d}\tau + C, \quad (3.27)$$

where f_4^ε is also bounded independent of ε in $L^1(0,T)$. Set

$$Q(t) = \int_0^t f_4^\varepsilon \cdot (\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)} + \|S^\varepsilon\|^2_{L^2(\Omega)})\,\mathrm{d}\tau.$$

Then (3.27) implies that

$$\partial_t Q(t) \leqslant f_4^\varepsilon(t)Q(t) + Cf_4^\varepsilon(t).$$

According to the differential form of Gronwall's inequality (see [5]), we have

$$Q(t) \leqslant \exp\left\{\int_0^t f_4^\varepsilon\,\mathrm{d}\tau\right\}\left[Q(0) + C\int_0^t f_4^\varepsilon\,\mathrm{d}\tau\right] \leqslant C.$$

It then follows from (3.27) that

$$(\|\mathbb{P}R^\varepsilon\|^2_{L^2(\Omega)} + \|S^\varepsilon\|^2_{L^2(\Omega)}) + \varepsilon \int_0^t (\|S^\varepsilon\|^2_{H^1(\Omega)} + \|R^\varepsilon\|^2_{H^1(\Omega)})\,\mathrm{d}\tau \leqslant C \quad \forall t \in [0,T].$$

Together with (3.9), we have finished the proof of theorem 1.3.

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References

1 L. C. Berselli. Some results on the Navier–Stokes equations with Navier boundary conditions. *Riv. Mat. Univ. Parma* **1** (2010), 1–75.

2 J. R. Cannon and E. DiBenedetto. The initial value problem for the Boussinesq equations with data in L^p. In *Approximation Methods for Navier–Stokes Problems: Proc. Symp. Int. Union of Theoretical and Applied Mechanics, University of Paderborn, Paderborn, Germany, September 9–15, 1979.* Lecture Notes in Mathematics, vol. 771, pp. 129–144 (Springer, 1980).

3 D. Chae. Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Adv. Math.* **203** (2006), 497–513.

4 D. Chae and O. Y. Imanuvilov. Generic solvability of the axisymmetric 3D Euler equations and the 2D Boussinesq equations. *J. Diff. Eqns* **156** (1999), 1–17.

5 L. C. Evans. *Partial differential equations.* Graduate Studies in Mathematics, vol. 19 (Providence, RI: American Mathematical Society, 1998).

6 J. S. Fan, G. Nakamura and H. B. Wang. Blow-up criteria of smooth solutions to the 3D Boussinesq system with zero viscosity in a bounded domain. *Nonlin. Analysis* **75** (2012), 3436–3442.

7 M. H. Holmes. *Introduction to perturbation methods*. Texts in Applied Mathematics, vol. 20 (Springer, 1995).

8 D. Iftimie and F. Sueur. Viscous boundary layers for the Navier–Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Analysis* **199** (2011), 145–175.

9 N. Ishimura and H. Morimoto. Remarks on the blow-up criterion for the 3-D Boussinesq equations. *Math. Models Meth. Appl. Sci.* **9** (1999), 1323–1332.

10 S. Jiang, J. W. Zhang and J. N. Zhao. Boundary-layer effects for the 2-D Boussinesq equations with vanishing diffusivity limit in the half plane. *J. Diff. Eqns* **250** (2011), 3907–3936.

11 T. T. Li and W. C. Yu. *Boundary value problems for quasilinear hyperbolic systems*. Duke University Mathematics Series, vol. V (Durham, NC: Duke University Mathematics Department, 1985).

12 S. A. Lorca and J. L. Boldrini. The initial value problem for a generalized Boussinesq model. *Nonlin. Analysis* **36** (1999), 457–480.

13 A. Majda. *Introduction to PDEs and waves for the atmosphere and ocean*. Courant Lecture Notes in Mathematics, vol. 9 (New York: American Mathematical Society/Courant Institute of Mathematical Sciences, 2003).

14 H. K. Moffatt. Some remarks on topological fluid mechanics. In *An introduction to the geometry and topology of fluid flows* (ed. R. L. Ricca), pp. 3–10 (Dordrecht: Kluwer Academic, 2001).

15 O. A. Oleinik and V. N. Samokhin. *Mathematical models in boundary layer theory*. Applied Mathematics and Mathematical Computation, vol. 15 (Boca Raton, FL: Chapman and Hall/CRC, 1999).

16 J. Pedlosky. *Geophysical fluid dynamics* (Springer, 1987).

17 Y. M. Qin, X. G. Yang, Y. Z. Wang and X. Liu. Blow-up criteria of smooth solutions to the 3D Boussinesq equations. *Math. Meth. Appl. Sci.* **35** (2012), 278–285.

18 F. Rousset. Stability of small amplitude boundary layers for mixed hyperbolic–parabolic systems. *Trans. Am. Math. Soc.* **355** (2003), 2991–3008.

19 F. Rousset. Characteristic boundary layers in real vanishing viscosity limits. *J. Diff. Eqns* **210** (2005), 25–64.

20 M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Commun. Math. Phys.* **192** (1998), 433–461.

21 M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions of the Navier–Stokes equation on a half-space. II. Construction of the Navier–Stokes solution. *Commun. Math. Phys.* **192** (1998), 463–491.

22 H. Schlichting. *Boundary layer theory* (McGraw-Hill, 1979).

23 D. Serre and K. Zumbrun. Boundary layer stability in real vanishing viscosity limit. *Commun. Math. Phys.* **221** (2001), 267–292.

24 R. Temam and X. M. Wang. Asymptotic analysis of the linearized Navier–Stokes equations in a general 2D domain. *Asymp. Analysis* **14** (1997), 293–321.

25 J. Wang. Boundary layers for compressible Navier–Stokes equations with outflow boundary condition. *J. Diff. Eqns* **248** (2010), 1143–1174.

26 J. Wang. Zero dissipation limit of the 1D linearized Navier–Stokes equations for a compressible fluid. *J. Math. Analysis Applic.* **374** (2011), 693–721.

27 J. Wang and L. Tong. Stability of boundary layers for the inflow compressible Navier–Stokes equations. *Discrete Contin. Dynam. Syst.* B **17** (2012), 2595–2613.

28 J. Wang and F. Xie. Characteristic boundary layers for parabolic perturbations of quasilinear hyperbolic problems. *Nonlin. Analysis* **73** (2010), 2504–2523.

29 Z. P. Xin and T. Yanagisawa. Zero-viscosity limit of the linearized Navier–Stokes equations for a compressible viscous fluid in the half-plane. *Commun. Pure Appl. Math.* **52** (1999), 479–541.

30 K. Zhao. 2D inviscid heat conductive Boussinesq equations on a bounded domain. *Michigan Math. J.* **59** (2010), 329–352.

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