Proceedings of the Royal Society of Edinburgh, **142A**, 585–594, *2012*

On singular quasi-monotone (p, q)-Laplacian systems

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(MS received 19 August 2010; accepted 12 May 2011)

We combine the sub- and supersolution method and perturbation arguments to obtain positive solutions of singular quasi-monotone (p, q)-Laplacian systems.

1. Introduction

Consider the (p, q)-Laplacian system

$$\left.\begin{aligned} -\Delta_p u &= f(x, u, v) && \text{in } \Omega, \\ -\Delta_q v &= g(x, u, v) && \text{in } \Omega \\ u, v &> 0 && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $1 < p, q < \infty$, and f and g are Carathéodory functions on $\Omega \times (0, \infty) \times (0, \infty)$, i.e. $f(x, s, t)$ and $g(x, s, t)$ are measurable in x for all (s, t) and continuous in (s, t) for almost all x. We assume the following:

(A_1) (1.1) is quasi-monotone, i.e. $f(x, s, t)$ is increasing in t for almost all x and all s, and $g(x, s, t)$ is increasing in s for almost all x and all t,

(A_2) for all $0 < s_0 \leqslant s_1$ and $0 < t_0 \leqslant t_1$, f is bounded from above on $\Omega \times [s_0, s_1] \times (0, t_1]$, g is bounded from above on $\Omega \times (0, s_1] \times [t_0, t_1]$ and f and g are bounded on $\Omega \times [s_0, s_1] \times [t_0, t_1]$.

We allow f and g to be singular as $s \to 0$ or $t \to 0$, and seek solutions $(u, v) \in W^{1,p}_{\text{loc}}(\Omega) \times W^{1,q}_{\text{loc}}(\Omega)$ with $u, v \in C(\bar{\Omega})$, that satisfy the first two equations in the

https://doi.org/10.1017/S0308210510001356 Published online by Cambridge University Press

sense of distributions, i.e.

$$\left.\begin{aligned} \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi &= \int_\Omega f(x,u,v)\varphi, \\ \int_\Omega |\nabla v|^{q-2}\nabla v\cdot\nabla\psi &= \int_\Omega g(x,u,v)\psi \quad \text{for all } \varphi,\psi\in C_0^\infty(\Omega). \end{aligned}\right\} \tag{1.2}$$

Then $f(x,u(x),v(x)), g(x,u(x),v(x)) \in L^\infty_{\mathrm{loc}}(\Omega)$ by (A_2) and hence $u,v\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ by the local regularity results of DiBenedetto [1]. We will combine the sub- and supersolution method and perturbation arguments to obtain such solutions of (1.1).

For example, our results give a positive solution of

$$\left.\begin{aligned} -\Delta_p u &= u^{-\alpha_1} + \mu v^{\alpha_2} && \text{in } \Omega, \\ -\Delta_q v &= v^{-\beta_1} + \mu u^{\beta_2} && \text{in } \Omega, \\ u,v &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1.3}$$

for all $\alpha_1,\beta_1>0$, $\alpha_2,\beta_2\geqslant 0$, and sufficiently small $\mu\geqslant 0$, and a positive solution of

$$\left.\begin{aligned} -\Delta_p u &= -u^{-\alpha_1} + v^{\alpha_2} + \lambda && \text{in } \Omega, \\ -\Delta_q v &= -v^{-\beta_1} + u^{\beta_2} + \lambda && \text{in } \Omega, \\ u,v &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1.4}$$

for $0<\alpha_1,\beta_1<1$, $\alpha_2,\beta_2\geqslant 0$ with $\alpha_2\beta_2<(p-1)(q-1)$, and sufficiently large $\lambda>0$.

We refer the reader to [2, 3] for related results on singular semipositone systems with nonlinearities that satisfy a combined sublinear condition at infinity.

2. Preliminaries

Consider the problem

$$\left.\begin{aligned} -\Delta_p u &= f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where f is a Carathéodory function on $\Omega\times[0,\infty)$. Denoting by $\lambda_{1,p}>0$ the first Dirichlet eigenvalue of $-\Delta_p$ on Ω, we have the following well-known result.

PROPOSITION 2.1. *If there are positive constants $C_1<\lambda_{1,p}$ and C_2 such that*

$$0\leqslant f(x,s)\leqslant C_1 s^{p-1} + C_2 \quad \textit{for all } (x,s)\in\Omega\times[0,\infty) \tag{2.2}$$

and $f(x,0)$ is non-trivial, then (2.1) has a weak solution $u>0$ in $C_0^{1,\alpha}(\bar\Omega)$ for some $\alpha\in(0,1)$.

For the case when f is defined only on $\Omega\times(0,\infty)$ (and possibly singular as $s\to 0$), the following estimate was proved in [5].

PROPOSITION 2.2. *If $p\leqslant n$ and there are $\varepsilon>0$, positive constants C_1 and C_2, and $1<r<np/(n-p)$ such that*

$$f(x,s)\leqslant C_1 s^{r-1} + C_2 \quad \textit{for all } (x,s)\in\Omega\times[\varepsilon,\infty) \tag{2.3}$$

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and $u > 0$ in $W_0^{1,p}(\Omega)$ is a solution of (2.1), then $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leqslant C \tag{2.4}$$

for some $C > 0$ depending only on Ω, ε, C_1, C_2, and $\|(u-\varepsilon)^+\|_{1,p}$.

Now consider the system

$$\left.\begin{aligned} -\Delta_p u &= f(x,u,v) && \text{in } \Omega, \\ -\Delta_q v &= g(x,u,v) && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.5}$$

where f and g are Carathéodory functions on $\Omega \times \mathbb{R} \times \mathbb{R}$ satisfying the following:

(A_3) $f(x,s,t)$ is increasing in t for almost all x and all s, and $g(x,s,t)$ is increasing in s for almost all x and all t.

Recall that $(\underline{u}, \underline{v}) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ is a subsolution of (2.5) if $f(x,\underline{u},\underline{v}) \in L^{p'}(\Omega)$ and $g(x,\underline{u},\underline{v}) \in L^{q'}(\Omega)$, where $p' = p/(p-1)$ is the Hölder conjugate of p, and

$$\left.\begin{aligned} -\Delta_p \underline{u} &\leqslant f(x,\underline{u},\underline{v}) && \text{in } \Omega, \\ -\Delta_q \underline{v} &\leqslant g(x,\underline{u},\underline{v}) && \text{in } \Omega, \\ \underline{u}, \underline{v} &\leqslant 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.6}$$

A supersolution $(\bar{u}, \bar{v})$ is defined similarly by reversing all inequalities in (2.6). We write $(\underline{u}, \underline{v}) \leqslant (\bar{u}, \bar{v})$ if $\underline{u} \leqslant \bar{u}$ and $\underline{v} \leqslant \bar{v}$ a.e. The following result is well known (see, for example, [4]).

PROPOSITION 2.3. *Assume that (A_3) holds and (2.5) has a subsolution $(\underline{u}, \underline{v})$ and a supersolution $(\bar{u}, \bar{v})$ in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that $(\underline{u}, \underline{v}) \leqslant (\bar{u}, \bar{v})$ and, for almost all x, all $s \in [\underline{u}(x), \bar{u}(x)]$, and all $t \in [\underline{v}(x), \bar{v}(x)]$,*

$$|f(x,s,t)|, |g(x,s,t)| \leqslant C \tag{2.7}$$

for some $C > 0$. Then (2.5) has a solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ between $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$, with $u, v \in C_0^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$.

3. Regularization

To obtain a solution of the system (1.1) using proposition 2.3, first we regularize it. Writing $s \wedge t = \min\{s,t\}$ and $s \vee t = \max\{s,t\}$, define Carathéodory functions f_j and g_j on $\Omega \times \mathbb{R} \times \mathbb{R}$ such that $f_j \to f$ and $g_j \to g$ on $\Omega \times (0,\infty) \times (0,\infty)$ by

$$f_j(x,s,t) = f(x, s \vee \varepsilon_j, t \vee \varepsilon_j), \qquad g_j(x,s,t) = g(x, s \vee \varepsilon_j, t \vee \varepsilon_j), \tag{3.1}$$

where $\varepsilon_j \searrow 0$, and consider the sequence of systems

$$\left.\begin{aligned} -\Delta_p u &= f_j(x,u,v) && \text{in } \Omega, \\ -\Delta_q v &= g_j(x,u,v) && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

THEOREM 3.1. *Assume that* (A_1) *and* (A_2) *hold and that, for each* j, *(3.2) has a subsolution* $(\underline{u}_j, \underline{v}_j)$ *and a supersolution* $(\bar{u}_j, \bar{v}_j)$ *in* $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ *such that* $(\underline{u}_j, \underline{v}_j) \leqslant (\bar{u}_j, \bar{v}_j)$,

$$\inf_j \operatorname{ess\,inf}_{\Omega'} (\underline{u}_j \wedge \underline{v}_j) > 0 \quad \textit{for all } \Omega' \subset\subset \Omega \tag{3.3}$$

and

$$\sup_j \operatorname{ess\,sup}_{\Omega} (\bar{u}_j \vee \bar{v}_j) < \infty. \tag{3.4}$$

Then (1.1) has a solution (u, v) *with* $u, v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$.

Under the assumptions of theorem 3.1, (3.2) has a solution

$$(u_j, v_j) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$$

such that

$$\varepsilon_{\Omega'} := \inf_j \operatorname{ess\,inf}_{\Omega'} (u_j \wedge v_j) > 0 \quad \text{for all } \Omega' \subset\subset \Omega \tag{3.5}$$

and

$$M := \sup_j \operatorname{ess\,sup}_{\Omega} (u_j \vee v_j) < \infty \tag{3.6}$$

by proposition 2.3, so it suffices to prove the following compactness result.

PROPOSITION 3.2. *Assume that* (A_1) *and* (A_2) *hold and that, for each* j, *(3.2) has a solution* $(u_j, v_j) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ *such that (3.5) and (3.6) hold. Then a subsequence of* (u_j, v_j) *converges a.e. to a solution* (u, v) *of (1.1), with* $u, v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$.

Proof. Take a sequence (Ω_k) of subdomains of Ω such that $\Omega_k \subset\subset \Omega_{k+1}$ and $\bigcup_k \Omega_k = \Omega$. For all j so large that $\varepsilon_j \leqslant \varepsilon_{\Omega_1}$, taking

$$\varphi = (u_j - \varepsilon_{\Omega_1})^+, \qquad \psi = (v_j - \varepsilon_{\Omega_1})^+$$

as the test functions in

$$\left.\begin{aligned} \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla \varphi &= \int_\Omega f_j(x, u_j, v_j)\varphi, \\ \int_\Omega |\nabla v_j|^{q-2} \nabla v_j \cdot \nabla \psi &= \int_\Omega g_j(x, u_j, v_j)\psi \end{aligned}\right\} \tag{3.7}$$

gives

$$\left.\begin{aligned} \int_{\Omega_1} |\nabla u_j|^p \leqslant \int_{u_j > \varepsilon_{\Omega_1}} |\nabla u_j|^p &= \int_{u_j > \varepsilon_{\Omega_1}} f(x, u_j, v_j \vee \varepsilon_j)(u_j - \varepsilon_{\Omega_1}), \\ \int_{\Omega_1} |\nabla v_j|^q \leqslant \int_{v_j > \varepsilon_{\Omega_1}} |\nabla v_j|^q &= \int_{v_j > \varepsilon_{\Omega_1}} g(x, u_j \vee \varepsilon_j, v_j)(v_j - \varepsilon_{\Omega_1}) \end{aligned}\right\} \tag{3.8}$$

since $u_j, v_j \geqslant \varepsilon_{\Omega_1}$ a.e. in Ω_1. The far right-hand sides are bounded from above by (A_2) since u_j and v_j are essentially bounded, so (u_j, v_j) is bounded in $W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_1)$. Hence, a subsequence (u^1_j, v^1_j) converges to some (u^1, v^1) weakly in

$W^{1,p}(\Omega_1)\times W^{1,q}(\Omega_1)$, strongly in $L^p(\Omega_1)\times L^q(\Omega_1)$, and a.e. in $\Omega_1\times\Omega_1$. Repeating with further and further subsequences, for each k we get a subsequence (u_j^k,v_j^k) that converges to some (u^k,v^k) weakly in $W^{1,p}(\Omega_k)\times W^{1,q}(\Omega_k)$, strongly in $L^p(\Omega_k)\times L^q(\Omega_k)$, and a.e. in $\Omega_k\times\Omega_k$ such that (u_j^{k+1},v_j^{k+1}) is a subsequence of (u_j^k,v_j^k). Then $(u^{k+1},v^{k+1})|_{\Omega_k\times\Omega_k}=(u^k,v^k)$, so

$$(u,v):=\begin{cases}(u^1,v^1) & \text{on } \Omega_1\times\Omega_1,\\ (u^{k+1},v^{k+1}) & \text{on } (\Omega_{k+1}\setminus\Omega_k)\times(\Omega_{k+1}\setminus\Omega_k),\ k\geqslant 1\end{cases} \tag{3.9}$$

is a well-defined function in $W^{1,p}_{\text{loc}}(\Omega)\times W^{1,q}_{\text{loc}}(\Omega)$ with $0<u,v\leqslant M$ a.e., to which the diagonal subsequence (u_k^k,v_k^k) converges a.e.

For any $\varphi,\psi\in C_0^\infty(\Omega)$,

$$\left.\begin{aligned}\int_{\Omega_k}|\nabla u_j^k|^{p-2}\nabla u_j^k\cdot\nabla\varphi&=\int_{\Omega_k}f(x,u_j^k,v_j^k)\varphi,\\ \int_{\Omega_k}|\nabla v_j^k|^{q-2}\nabla v_j^k\cdot\nabla\psi&=\int_{\Omega_k}g(x,u_j^k,v_j^k)\psi\end{aligned}\right\} \tag{3.10}$$

for a fixed k so large that $\Omega_k\supset\operatorname{supp}\varphi,\operatorname{supp}\psi$ and all j so large that $\varepsilon_j^k\leqslant\varepsilon_{\Omega_k}$, where (ε_j^k) is the subsequence of (ε_j) that corresponds to (u_j^k,v_j^k). Passing to the limit in j gives

$$\left.\begin{aligned}\int_{\Omega_k}|\nabla u^k|^{p-2}\nabla u^k\cdot\nabla\varphi&=\int_{\Omega_k}f(x,u^k,v^k)\varphi,\\ \int_{\Omega_k}|\nabla v^k|^{q-2}\nabla v^k\cdot\nabla\psi&=\int_{\Omega_k}g(x,u^k,v^k)\psi,\end{aligned}\right\} \tag{3.11}$$

which reduces to (1.2) since $(u^k,v^k)=(u,v)|_{\Omega_k\times\Omega_k}$ and $\varphi,\psi=0$ outside Ω_k. Then $u,v\in C^{1,\alpha}_{\text{loc}}(\Omega)$ since $f(x,u(x),v(x)),g(x,u(x),v(x))\in L^\infty_{\text{loc}}(\Omega)$, so $u,v>0$.

To prove that $u,v\in C(\bar\Omega)$ with $u,v=0$ on $\partial\Omega$, we will show that, given any $\varepsilon\in(0,2M]$, there is a neighbourhood U of $\partial\Omega$ such that $u,v<\varepsilon$ in $U\cap\Omega$. We only give the proof for u as the argument for v is similar. By (A_2), there is a $C>0$ such that $f\leqslant C$ on $\Omega\times[\frac12\varepsilon,M]\times(0,M]$. Let $u_\varepsilon>0$ in $C_0^{1,\alpha}(\bar\Omega)$ be the solution of the problem

$$\left.\begin{aligned}-\Delta_p u&=C \quad \text{in } \Omega,\\ u&=0 \quad \text{on } \partial\Omega\end{aligned}\right\} \tag{3.12}$$

given by proposition 2.1. Taking $\varphi=(u_j^k-u_\varepsilon-\frac12\varepsilon)^+$ in

$$\left.\begin{aligned}\int_\Omega|\nabla u_j^k|^{p-2}\nabla u_j^k\cdot\nabla\varphi&=\int_\Omega f(x,u_j^k\vee\varepsilon_j^k,v_j^k\vee\varepsilon_j^k)\varphi,\\ \int_\Omega|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla\varphi&=\int_\Omega C\varphi\end{aligned}\right\} \tag{3.13}$$

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gives

$$\left.\begin{aligned}\int_{u_j^k>u_\varepsilon+\varepsilon/2}&|\nabla u_j^k|^{p-2}\nabla u_j^k\cdot\nabla(u_j^k-u_\varepsilon-\tfrac{1}{2}\varepsilon)\\&\leqslant\int_{u_j^k>u_\varepsilon+\varepsilon/2}C(u_j^k-u_\varepsilon-\tfrac{1}{2}\varepsilon)\\&=\int_{u_j^k>u_\varepsilon+\varepsilon/2}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla(u_j^k-u_\varepsilon-\tfrac{1}{2}\varepsilon),\end{aligned}\right\}\tag{3.14}$$

which reduces to

$$\int_{u_j^k>u_\varepsilon+\varepsilon/2}(|\nabla u_j^k|^{p-2}\nabla u_j^k-|\nabla(u_\varepsilon+\tfrac{1}{2}\varepsilon)|^{p-2}\nabla(u_\varepsilon+\tfrac{1}{2}\varepsilon))\cdot\nabla(u_j^k-u_\varepsilon-\tfrac{1}{2}\varepsilon)\leqslant 0.\tag{3.15}$$

This implies that $u_j^k\leqslant u_\varepsilon+\frac{1}{2}\varepsilon$ and hence $u\leqslant u_\varepsilon+\frac{1}{2}\varepsilon$. Since u_ε is continuous up to the boundary, there is a neighbourhood U of $\partial\Omega$ such that $u_\varepsilon<\frac{1}{2}\varepsilon$ in $U\cap\Omega$. □

4. Positone-type singular systems

Now we apply theorem 3.1 to obtain a solution of the system

$$\left.\begin{aligned}-\Delta_p u&=f_1(x,u,v)+\mu f_2(x,u,v)&&\text{in }\Omega,\\-\Delta_q v&=g_1(x,u,v)+\mu g_2(x,u,v)&&\text{in }\Omega,\\u,v&>0&&\text{in }\Omega,\\u,v&=0&&\text{on }\partial\Omega,\end{aligned}\right\}\tag{4.1}$$

where f_1, f_2, g_1 and g_2 are Carathéodory functions on $\Omega\times(0,\infty)\times(0,\infty)$ satisfying

(B_1) $f_1(x,s,t)$ and $f_2(x,s,t)$ are increasing in t for almost all x and all s, and $g_1(x,s,t)$ and $g_2(x,s,t)$ are increasing in s for almost all x and all t,

(B_2) for all $0<s_0\leqslant s_1$ and $0<t_0\leqslant t_1$, f_1 is bounded from above on $\Omega\times[s_0,s_1]\times(0,t_1]$, g_1 is bounded from above on $\Omega\times(0,s_1]\times[t_0,t_1]$, f_1 and g_1 are bounded on $\Omega\times[s_0,s_1]\times[t_0,t_1]$, and f_2 and g_2 are bounded on $\Omega\times(0,s_1]\times(0,t_1]$,

(B_3) there are $s_1,t_1>0$ and non-trivial functions $a,b\geqslant 0$ in $L^\infty(\Omega)$ such that $f_1\geqslant a$, $g_1\geqslant b$, and $f_2,g_2\geqslant 0$ on $\Omega\times(0,s_1]\times(0,t_1]$,

(B_4) for each $s_0>0$, there are positive constants $C_1<\lambda_{1,p}$ and C_2 such that

$$f_1(x,s,t)\leqslant C_1 s^{p-1}+C_2\quad\text{for all }(x,s,t)\in\Omega\times[s_0,\infty)\times(0,\infty),\tag{4.2}$$

and, for each $t_0>0$, there are positive constants $D_1<\lambda_{1,q}$ and D_2 such that

$$g_1(x,s,t)\leqslant D_1 t^{q-1}+D_2\quad\text{for all }(x,s,t)\in\Omega\times(0,\infty)\times[t_0,\infty)\tag{4.3}$$

and $\mu\geqslant 0$ is a parameter.

THEOREM 4.1. *Assume that* (B_1)–(B_4) *hold. Then there is a* $\mu_0>0$ *such that (4.1) has a solution* (u,v) *with* $u,v\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$ *for each* $\mu\in[0,\mu_0)$.

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Proof. We apply theorem 3.1 with $f = f_1 + \mu f_2$ and $g = g_1 + \mu g_2$. Define f_{1j}, f_{2j}, g_{1j}, and g_{2j} as in (3.1). We may assume that each $\varepsilon_j \leqslant s_1 \wedge t_1$, so $f_{1j} \geqslant a$, $g_{1j} \geqslant b$, and $f_{2j}, g_{2j} \geqslant 0$ on $\Omega \times (0, s_1] \times (0, t_1]$.

First we construct a subsolution $(\underline{u}_j, \underline{v}_j)$ of (3.2) satisfying (3.3). Let $u, v > 0$ in $C_0^{1,\alpha}(\bar{\Omega})$ be the solutions of the problems

$$\left.\begin{aligned} -\Delta_p u &= a(x) && \text{in } \Omega, \\ -\Delta_q v &= b(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.4}$$

given by proposition 2.1, let $c = 1 \wedge (s_1/\max u)$, $d = 1 \wedge (t_1/\max v)$, and let $\underline{u}_j = cu$, $\underline{v}_j = dv$. Then $0 < c, d \leqslant 1$ and $0 < \underline{u}_j \leqslant s_1$, $0 < \underline{v}_j \leqslant t_1$, so

$$-\Delta_p \underline{u}_j = c^{p-1} a(x) \leqslant a(x) \leqslant f_{1j}(x, \underline{u}_j, \underline{v}_j) + \mu f_{2j}(x, \underline{u}_j, \underline{v}_j), \tag{4.5}$$

and similarly $-\Delta_q \underline{v}_j \leqslant g_{1j}(x, \underline{u}_j, \underline{v}_j) + \mu g_{2j}(x, \underline{u}_j, \underline{v}_j)$.

Now we construct a supersolution $(\bar{u}_j, \bar{v}_j) \geqslant (\underline{u}_j, \underline{v}_j)$ of (3.2) satisfying (3.4) for sufficiently small μ. Let C_{1j}, D_{1j}, C_{2j} and D_{2j} be the constants in (B_4) that correspond to s_0, $t_0 = \varepsilon_j$. Then

$$f_{1j}(x, s, t) \leqslant C_{1j} s^{p-1} + C'_{2j}, \quad g_{1j}(x, s, t) \leqslant D_{1j} t^{q-1} + D'_{2j} \quad \text{for all } (x, s, t) \in \Omega \times (0, \infty) \times (0, \infty), \tag{4.6}$$

where $C'_{2j} = C_{1j}\varepsilon_j^{p-1} + C_{2j}$, $D'_{2j} = D_{1j}\varepsilon_j^{q-1} + D_{2j}$. By proposition 2.1, the problems

$$\left.\begin{aligned} -\Delta_p u &= C_{1j} u^{p-1} + C'_{2j} + 1 \text{ in } \Omega, && u = 0 \text{ on } \partial\Omega, \\ -\Delta_q v &= D_{1j} v^{q-1} + D'_{2j} + 1 \text{ in } \Omega, && v = 0 \text{ on } \partial\Omega \end{aligned}\right\} \tag{4.7}$$

have solutions $u, v > 0$ in $C_0^{1,\alpha}(\bar{\Omega})$. By (4.6), (u, v) is a supersolution of the system

$$\left.\begin{aligned} -\Delta_p u &= f_{1j}(x, u, v) + 1 && \text{in } \Omega, \\ -\Delta_q v &= g_{1j}(x, u, v) + 1 && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.8}$$

As in (4.5), $(\underline{u}_j, \underline{v}_j)$ is also a subsolution of (4.8). On the set where $u < \underline{u}_j$,

$$-\Delta_p u \geqslant f_1(x, u \vee \varepsilon_j, v \vee \varepsilon_j) \geqslant f_1(x, u \vee \varepsilon_j, \varepsilon_j) \geqslant a(x) \geqslant -\Delta_p \underline{u}_j, \tag{4.9}$$

so $u \geqslant \underline{u}_j$, and similarly $v \geqslant \underline{v}_j$. So (4.8) has a solution $(\bar{u}_j, \bar{v}_j) \geqslant (\underline{u}_j, \underline{v}_j)$ with $\bar{u}_j, \bar{v}_j \in C_0^{1,\alpha}(\bar{\Omega})$ by proposition 2.3.

Note that $\bar{u}_j$ is a solution of (2.1) with $f(x, s) = f_{1j}(x, s, \bar{v}_j(x)) + 1$. Fix $\varepsilon > 0$ and let C_1 and C_2 be the constants in (B_4) that correspond to $s_0 = \varepsilon$. We may assume that each $\varepsilon_j \leqslant \varepsilon$, so

$$f(x, s) = f_1(x, s, \bar{v}_j(x) \vee \varepsilon_j) + 1 \leqslant C_1 s^{p-1} + C'_2 \quad \text{for all } (x, s) \in \Omega \times [\varepsilon, \infty), \tag{4.10}$$

where $C'_2 = C_2 + 1$. Taking $\varphi = (\bar{u}_j - \varepsilon)^+$ in

$$\int_\Omega |\nabla \bar{u}_j|^{p-2} \nabla \bar{u}_j \cdot \nabla \varphi = \int_\Omega f(x, \bar{u}_j)\varphi \tag{4.11}$$

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and using (4.10) gives

$$\int_\Omega |\nabla(\bar{u}_j - \varepsilon)^+|^p \leqslant \int_\Omega (C_1 \bar{u}_j^{p-1} + C_2')(\bar{u}_j - \varepsilon)^+. \tag{4.12}$$

Since

$$C_1 < \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}, \tag{4.13}$$

this implies that $\|(\bar{u}_j - \varepsilon)^+\|_{1,p}$ is bounded. Then $\|\bar{u}_j\|_\infty$ is bounded by proposition 2.2 if $p \leqslant n$ and bounded by the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ if $p > n$, and similarly so is $\|\bar{v}_j\|_\infty$.

Let M be the left-hand side of (3.4) and let

$$\mu_0 = \frac{1}{\sup_{\Omega\times(0,M]\times(0,M]}(|f_2| \vee |g_2|)} \leqslant \infty. \tag{4.14}$$

We may assume that each $\varepsilon_j \leqslant M$, so, for all $\mu \in [0, \mu_0)$,

$$-\Delta_p \bar{u}_j = f_{1j}(x, \bar{u}_j, \bar{v}_j) + 1 \geqslant f_{1j}(x, \bar{u}_j, \bar{v}_j) + \mu f_{2j}(x, \bar{u}_j, \bar{v}_j), \tag{4.15}$$

and similarly $-\Delta_q \bar{v}_j \geqslant g_{1j}(x, \bar{u}_j, \bar{v}_j) + \mu g_{2j}(x, \bar{u}_j, \bar{v}_j)$. □

5. Semipositone-type singular systems

Finally, we apply theorem 3.1 to obtain a solution of

$$\left.\begin{aligned} -\Delta_p u &= f_1(x,u,v) + \lambda + \mu f_2(x,u,v) && \text{in } \Omega, \\ -\Delta_q v &= g_1(x,u,v) + \lambda + \mu g_2(x,u,v) && \text{in } \Omega, \\ u, v &> 0 && \text{in } \Omega, \\ u, v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.1}$$

where f_1, f_2, g_1 and g_2 are Carathéodory functions on $\Omega\times(0,\infty)\times(0,\infty)$ satisfying the following:

(G_1) $f_1(x,s,t)$ and $f_2(x,s,t)$ are increasing in t for almost all x and all s, and $g_1(x,s,t)$ and $g_2(x,s,t)$ are increasing in s for almost all x and all t;

(G_2) there are $0 < \alpha_1, \beta_1 < 1$, $\alpha_2, \beta_2 > 0$ with $\alpha_2\beta_2 < (p-1)(q-1)$, $1 < p_1 < p, 1 < q_1 < q$, and positive constants C and D such that

$$\left.\begin{aligned} -Cs^{-\alpha_1} \leqslant f_1(x,s,t) \leqslant C(s^{p_1-1} + t^{\alpha_2} + 1), \\ -Dt^{-\beta_1} \leqslant g_1(x,s,t) \leqslant D(t^{q_1-1} + s^{\beta_2} + 1) \end{aligned}\right\} \tag{5.2}$$

for all $(x,s,t) \in \Omega\times(0,\infty)\times(0,\infty)$;

(G_3) for all $s_1, t_1 > 0$, f_2 and g_2 are bounded on $\Omega\times(0,s_1]\times(0,t_1]$

and $\lambda > 0$ and $\mu \geqslant 0$ are parameters.

https://doi.org/10.1017/S0308210510001356 Published online by Cambridge University Press

THEOREM 5.1. *Assume that* (G_1)–(G_3) *hold. Then there is a* $\lambda_0 > 0$ *such that for each* $\lambda \geqslant \lambda_0$ *there is a* $\mu_0(\lambda) > 0$ *for which (5.1) has a solution* (u,v) *with* $u, v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ *whenever* $\mu \in [0, \mu_0(\lambda))$.

Proof. We apply theorem 3.1 with $f = f_1 + \lambda + \mu f_2$ and $g = g_1 + \lambda + \mu g_2$. Define f_{1j}, f_{2j}, g_{1j} and g_{2j} as in (3.1). We may assume that each $\varepsilon_j \leqslant 1$, so

$$\left.\begin{aligned} -Cs^{-\alpha_1} \leqslant f_{1j}(x,s,t) \leqslant C(s^{p_1-1} + t^{\alpha_2} + 3),\\ -Dt^{-\beta_1} \leqslant g_{1j}(x,s,t) \leqslant D(t^{q_1-1} + s^{\beta_2} + 3) \end{aligned}\right\} \tag{5.3}$$

for all $(x,s,t) \in \Omega \times (0,\infty) \times (0,\infty)$.

First we construct a subsolution $(\underline{u}_j, \underline{v}_j)$ of (3.2) satisfying (3.3). Let $0 < \varphi_{1,p} \leqslant 1$ be an eigenfunction associated with $\lambda_{1,p}$, let $1 < \alpha_2 < p/(p-1+\alpha_1), 1 < \beta_2 < q/(q-1+\beta_1)$, and let $\underline{u}_j = \varphi_{1,p}^{\alpha_2}/\alpha_2, \underline{v}_j = \varphi_{1,q}^{\beta_2}/\beta_2$. Then

$$\begin{aligned} -\Delta_p \underline{u}_j &= \varphi_{1,p}^{(\alpha_2-1)(p-1)}(-\Delta_p \varphi_{1,p}) - \frac{(\alpha_2-1)(p-1)|\nabla\varphi_{1,p}|^p}{\varphi_{1,p}^{1-(\alpha_2-1)(p-1)}} \\ &= a(x) - b(x)\underline{u}_j^{-\alpha_1}, \end{aligned} \tag{5.4}$$

where

$$a(x) = \lambda_{1,p}\varphi_{1,p}^{\alpha_2(p-1)}, \qquad b(x) = \frac{(\alpha_2-1)(p-1)|\nabla\varphi_{1,p}|^p}{\alpha_2^{\alpha_1}\varphi_{1,p}^{p-\alpha_2(p-1+\alpha_1)}}. \tag{5.5}$$

Since $\varphi_{1,p} = 0$ and $\nabla\varphi_{1,p} \neq 0$ on $\partial\Omega$, in some neighbourhood $\Omega' \subset \Omega$ of $\partial\Omega$, $b(x) \geqslant C$, and hence

$$-\Delta_p \underline{u}_j \leqslant \lambda_{1,p} - C\underline{u}_j^{-\alpha_1} \leqslant f_{1j}(x, \underline{u}_j, \underline{v}_j) + \lambda - 1 \tag{5.6}$$

for $\lambda \geqslant \lambda_{1,p} + 1$ by (5.3). On $\Omega \setminus \Omega'$, $-\Delta_p \underline{u}_j \leqslant \lambda_{1,p}$ and $f_{1j}(x, \underline{u}_j, \underline{v}_j)$ is bounded since $\varphi_{1,p}$ is uniformly positive, so $-\Delta_p \underline{u}_j \leqslant f_{1j}(x, \underline{u}_j, \underline{v}_j) + \lambda - 1$ still holds for λ sufficiently large. Now take μ so small that $\mu f_{2j}(x, \underline{u}_j, \underline{v}_j) \geqslant -1$. Similarly,

$$-\Delta_q \underline{v}_j \leqslant g_{1j}(x, \underline{u}_j, \underline{v}_j) + \lambda + \mu g_{2j}(x, \underline{u}_j, \underline{v}_j)$$

for λ large and μ small.

Now we construct a supersolution $(\bar{u}_j, \bar{v}_j) \geqslant (\underline{u}_j, \underline{v}_j)$ of (3.2) satisfying (3.4). Let $u, v > 0$ in $C_0^{1,\alpha}(\bar{\Omega})$ be the solutions of the problems

$$\left.\begin{aligned} -\Delta_p u &= 1 \quad \text{in } \Omega,\\ -\Delta_q v &= 1 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.7}$$

given by proposition 2.1, let $c > 1/(p-1)$ and $d > 1/(q-1)$ with $\alpha_2/(p-1) < c/d < (q-1)/\beta_2$, and let $\bar{u}_j = \lambda^c u, \bar{v}_j = \lambda^d v$. For λ large and μ small,

$$\begin{aligned} -\Delta_p \bar{u}_j &= \lambda^{c(p-1)} \\ &\geqslant C(\bar{u}_j^{p_1-1} + \bar{v}_j^{\alpha_2} + 3) + \lambda + 1 \\ &\geqslant f_{1j}(x, \bar{u}_j, \bar{v}_j) + \lambda + \mu f_{2j}(x, \bar{u}_j, \bar{v}_j) \end{aligned} \tag{5.8}$$

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by (5.3) and $\bar{u}_j \geqslant \underline{u}_j$ since

$$-\Delta_p \bar{u}_j \geqslant \lambda_{1,p} \geqslant -\Delta_p \underline{u}_j. \tag{5.9}$$

Similarly, $-\Delta_q \bar{v}_j \geqslant g_{1j}(x, \bar{u}_j, \bar{v}_j) + \lambda + \mu g_{2j}(x, \bar{u}_j, \bar{v}_j)$ and $\bar{v}_j \geqslant \underline{v}_j$. □

Acknowledgements

S.E.M. was supported by Al-Imam University Project no. 290101202, Riyadh, KSA.

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(*Issued 8 June 2012*)