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THE PRODUCT OF TWO (UNBOUNDED) DERIVATIONS

BY

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ABSTRACT. We characterize when the product of two (unbounded) derivations of a C^*-algebra is a derivation.

1. **Introduction**. The purpose of this note is to show that if δ_1, δ_2 and $\delta_1\delta_2$ are derivations, defined on a dense subalgebra of a C^*-algebra, then $\delta_1\delta_2 = 0$. To achieve this we need to impose a technical condition on δ_1, that δ_1 generate a strongly continuous group of automorphisms of A will suffice. Other sufficient conditions may be found in section 3. In [4] Mathieu proved the theorem for bounded derivations, using (*i*) a stronger result [6], valid for derivations of prime rings, and (*ii*) that bounded derivations are inner [5] in the double dual. These results are not available for unbounded derivations, although the result in section 2 below imply a version of (*i*). The main technical tool used in this note is a result, due to Fong and Sourour, about elementary operators on $B(H)$ [2], [3]. We refer to [1] for background material about unbounded derivations, and to [5] for the theory of C^*-algebras.

2. **The characterization**. We show how information about elementary operators on $B(H)$ (from [2]) can be patched together to obtain a global result via the (reduced) atomic representation.

THEOREM. *Let δ_1 and δ_2 be derivations of a C^*-algebra A. Assume that D is a subalgebra of A and that D is a subset of the domains of δ_1, δ_2 and $\delta_1\delta_2$. If $\delta_1\delta_2$ is a derivation, then there exist unique orthogonal central projections e_1, e_2 and e_3 in $\pi_a(A)''$ (the weak closure of the image of A under the atomic representation) such that $e_1 + e_2 + e_3 = 1$ and*

$$\pi_a(\delta_1(b))e_1 = 0,\ b \in D;\ \pi_a(\delta_2(b))e_1 \neq 0,\ \text{some } b \in D$$

$$\pi_a(\delta_2(b))e_2 = 0,\ b \in D;\ \pi_a(\delta_1(b))e_2 \neq 0,\ \text{some } b \in D$$

$$\pi_a(\delta_1(b))e_3 = 0,\ b \in D;\ \pi_a(\delta_2(b))e_3 = 0,\ b \in D.$$

PROOF. Expanding $\delta_1\delta_2(ab)$ twice, first using that $\delta_1\delta_2$ is a derivation, and secondly using that δ_1 and δ_2 are derivations, will lead to

$$\delta_1(a)\delta_2(b) + \delta_2(a)\delta_1(b) = 0$$

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for a and b in D. Substituting ac for a yields

$$(1)\qquad \delta_1(a)c\delta_2(b)+\delta_2(a)c\delta_1(b)=0$$

for a, b, and c in D. Now let π be the (reduced) atomic representation of A, that is

$$\pi=\bigoplus_{t\in\hat{A}}\pi_t \text{ on } H=\bigoplus_{t\in\hat{A}}H_t.$$

It is well known that π is faithful, and that

$$\pi(A)''=\prod_{t\in\hat{A}}B(H_t)$$

See e.g. [5] for more details. From (1) and the density of D in A we get

$$(2)\qquad (\pi_t\delta_1(a))c(\pi_t\delta_2(b))+(\pi_t\delta(a))c(\pi_t\delta_1(b))=0$$

for a and b in D and c in $B(H_t)$. We will apply Theorem 1 of [2] to (2). If $\pi_t\delta_1(b)$ and $\pi_t\delta_2(b)$ are linearly independent for some b in D, then [2] give

$$\pi_t\delta_1(a)=\pi_t\delta_2(a)=0$$

for all a in D, a contradiction. Hence $\pi_t\delta_1(b)$ and $\pi_t\delta_2(b)$ are linearly dependent for all b in D. Now take b in D with $\pi_t\delta_2(b)\neq 0$ (if possible). Then

$$(3)\qquad \pi_t\delta_1(b)=\lambda_b\pi_t\delta_2(b)$$

for some complex number λ_b, a second application of [1] results in

$$(4)\qquad \pi_t\delta_1(a)=-\lambda_b\pi_t\delta_2(a)$$

for all a in D. Taking $a=b$ and comparing (3) and (4) yields $\lambda_b=0$; so that

$$\pi_t\delta_1(a)=0$$

for all a in D by (4). It is now easy to complete the proof. □

Note we could replace the atomic representation by any faithful direct sum of disjoint irreducible representations as in [4]. Also we did not use that D is dense in A, but only that $\pi_a(D)$ is weakly dense in $\pi_a(A)''$.

3. **Consequences**. If γ is an operator on A with domain D, denote by γ^a the operator on $\pi_a(A)$ with domain $\pi_a(D)$ given by

$$\gamma^a(\pi_a(b))=\pi_a(\gamma(b))$$

for b in D.

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COROLLARY. $\delta_1\delta_2 = 0$ *provided either (i) e_1 is in $\pi_a(D)$; (ii) δ_1^a is (σ-weakly) closable and e_1 is in the domain of the closure; or (iii) δ_1^a is σ-weakly closable derivation and the closure generate a σ-weakly continuous one-parameter group of automorphisms of $\pi_a(A)''$. By the closure of δ_1^a, we understand the closure of the restriction of δ_1^a to $\pi_a(D)$.*

PROOF. We will work entirely in the atomic representation, so let us drop the superscript designating this. First note that we can take b in the domain of the closure of δ_1, in the conclusions (that involve δ_1) of the theorem.
If $e_1 \in D$, then

$$\delta_1\delta_2(b) = \delta_1(e_1\delta_2(b)) = \delta_1(e_1)\delta_2(b) + e_1\delta_1\delta_2(b)$$

for b in D. But both terms in this sum are zero by the theorem. This proves (i) and (ii). Now let us prove (iii). Let α denote the automorphism group generated by δ_1. Then

$$\alpha_t(a) = e_2\alpha_t(a) + (e_1 + e_3)a \tag{5}$$

for a in $\pi(A)''$, because it is true if a is analytic for δ_1 by the usual series expansion of $\alpha_t(a)$, in fact

$$\alpha_t(a) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\delta_1^n(a) = a + e_2 \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\delta_1^n(a)$$

since $\delta_1(a) = e_2\delta_1(a)$. By (5) and the theorem

$$\alpha_t\delta_2(b) = e_2\alpha_t\delta_2(b) + e_1\delta_2(b) \tag{6}$$

for b in $\pi(D)$, since $e_3\delta_2(b) = 0$. Applying α_{-t} to (6) yields

$$\delta_2(b) = \alpha_{-t}(e_2)\delta_2(b) + \alpha_{-t}(e_1\delta_2(b)) \text{ for } b \text{ in } \pi(D). \tag{7}$$

It is easy to see that the first term in (7) is zero, indeed take $a = e_2$ in (5) and get

$$\alpha_{-t}(e_2)\delta_2(b) = e_2\alpha_{-t}(e_2)e_1\delta_2(b) = 0$$

since $\delta_2(b) = e_1\delta_2(b)$. Hence (7) reduces to

$$\alpha_t\delta_2(b) = \delta_2(b)$$

for $b \in \pi(D)$. □

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