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SOME REMARKS ON CRITICAL POINT THEORY FOR LOCALLY LIPSCHITZ FUNCTIONS

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Abstract. In this paper, a dual version of the Mountain Pass Theorem and the Generalized Mountain Pass Theorem are extended to functions that are locally Lipschitz only. An application involving elliptic hemivariational inequalities is next examined.

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1. Introduction. The critical point theory for smooth functions in a Banach space is by now well established and excellent monographs devoted to various aspects of it are already available; see for instance [**10**, **12**, **7**, **3**, **11**]. Starting with the famous Mountain Pass Theorem (briefly, MPT) by Ambrosetti–Rabinowitz [**10**, Theorem 2.2], several other meaningful results have been obtained. Let us mention here

(a) some 'dual versions' of the MPT; vide [**10**, Theorem 3.2] and, as regards the more general case of linking subsets, [**6**, Theorem 2.2].

(b) the Saddle Point Theorem [**10**, Theorem 4.6].

(c) the Generalized MPT [**10**, Theorem 5.3].

(d) the results where the strict inequality occurring in the MPT is weakened to allow also equality; see e.g. [**6**, Theorem 2.1].

(e) Benci–Rabinowitz's theorem [**10**, Theorem 5.29], which unifies (b) and (c).

In 1981, through techniques of nonsmooth analysis previously introduced by Clarke [**5**], K.-C. Chang developed a critical point theory for locally Lipschitz functions in a Banach space, extending both the MPT and (b) to this more general framework; vide Theorems 3.4 and 3.3 of [**4**], respectively. Later on, in 1997, D. Motreanu and C. Varga made the same for Du's result mentioned in (d); see [**9**, Theorem 2.1]. However, to the best of our knowledge, no locally Lipschitz version of (a) and (c) can be found in the literature. The main purpose of the present paper is to simply fill in such a gap, thus improving the analogy between the two theories. The approach of [**9**] is adopted here. Consequently, we work in the case of linkings and with the strict inequality weakened

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to permit also equality; vide Theorems 3.1 and 4.1 below. The latter result is then employed to solve an elliptic hemivariational inequality patterned after Problem 5.1 in [**10**]. Let us also note that Theorem 4.1 might actually be reformulated for a wider class of nonsmooth functions by using [**8**, Theorem 3.1]. Finally, possible extensions of (e) as well as applications will be examined in a future paper.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a real Banach space. If $\rho > 0$, we define $B_\rho = \{x \in X : \|x\| < \rho\}$, $\bar{B}_\rho = \{x \in X : \|x\| \leq \rho\}$, and $\partial B_\rho = \{x \in X : \|x\| = \rho\}$. Given $x, z \in X$, the symbol $[x, z]$ indicates the line segment joining x to z, i.e.

$$[x, z] = \{(1-t)x + tz : t \in [0, 1]\}.$$

We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $g : X \to \mathbb{R}$ is called locally Lipschitz when to every $x \in X$ there correspond a neighbourhood U_x of x besides a constant $L_x \geq 0$ such that

$$|g(z) - g(w)| \leq L_x \|z - w\| \quad \forall z, w \in U_x.$$

If $x, z \in X$, the symbol $g^0(x; z)$ indicates the generalized directional derivative of g at the point x along the direction z, namely

$$g^0(x; z) = \limsup_{w \to x, t \to 0+} \frac{g(w + tz) - g(w)}{t}.$$

It is known [**5**, Proposition 2.1.1] that g^0 turns out upper semicontinuous on $X \times X$. Moreover, for locally Lipschitz $g_1, g_2 : X \to \mathbb{R}$ one evidently has

$$(g_1 + g_2)^0(x; z) \leq g_1^0(x; z) + g_2^0(x; z) \quad x, z \in X.$$

We denote by $\partial g(x)$ the generalized gradient of g at x, i.e.

$$\partial g(x) = \{x^* \in X^* : \langle x^*, z \rangle \leq g^0(x; z) \, \forall z \in X\}.$$

Proposition 2.1.2 in [**5**] ensures that the set $\partial g(x)$ is nonempty, convex, and weak* compact. Hence, it makes sense to write

$$m_g(x) = \min\{\|x^*\|_{X^*} : x^* \in \partial g(x)\}.$$

The following compactness condition (see [**4**, Definition 2]) of Palais–Smale type will be adopted throughout the paper.

$(\mathrm{PS})_g$ *Every sequence* $\{x_n\} \subseteq X$ *satisfying* $g(x_n) \to d \in \mathbb{R}$ *and* $m_g(x_n) \to 0$ *possesses a convergent subsequence*.

We say that $x \in X$ is a critical point of g when $0 \in \partial g(x)$, which clearly means $g^0(x; z) \geq 0$ for all $z \in X$. Finally, given a real number d, define

$$g_d = \{x \in X : g(x) \leq d\} \quad g^d = \{x \in X : g(x) \geq d\},$$
$$K_d(g) = \{x \in X : g(x) = d, x \text{ is a critical point of } g\}.$$

We conclude with the deformation results below. The first of them was established by K.-C. Chang [**4**, Theorem 3.1], while the other is due to Motreanu–Varga [**9**, Theorem 1.1].

LEMMA 2.1. *Let X be reflexive and let $g : X \to \mathbb{R}$ be locally Lipschitz. Assume g fulfils* $(\mathrm{PS})_g$, *$d \in \mathbb{R}$, while U denotes any neighbourhood of $K_d(g)$. Then to every $\varepsilon_0 > 0$ there correspond $\varepsilon \in]0, \varepsilon_0[$ besides a homeomorphism $\eta : X \to X$ such that*

1. $\eta(x) = x$ *for all* $x \in X \backslash g^{-1}(]d - \varepsilon_0, d + \varepsilon_0])$.
2. $\eta(g_{d+\varepsilon} \backslash U) \subseteq g_{d-\varepsilon}$.
3. $\eta(g_{d+\varepsilon}) \subseteq g_{d-\varepsilon}$ *provided* $K_d(g) = \emptyset$.

LEMMA 2.2. *Let X, g, and d be as in Lemma* 2.1. *Suppose A, B are nonempty closed subsets of X with the following properties:*

$$A \cap B = \emptyset; \quad A \subseteq g^d; \quad B \subseteq g_d; \quad B \cap K_d(g) = \emptyset.$$

Then there exist $\varepsilon > 0$ and a homeomorphism $\eta : X \to X$ such that

1. $\eta(x) = x$ *for every* $x \in A$.
2. $\eta(B) \subseteq g_{d-\varepsilon}$.

3. A dual version of the MPT with possibly 'zero altitude'. It is known that the Mountain Pass Theorem [**10**, Theorem 2.2] has a dual version, which also holds when the 'mountain ring' possesses a 'zero altitude'; see for instance [**10**, Chapter 4] and, as regards the more general framework of linking subsets, [**6**, Theorem 2.2]. In this section we shall prove that the same is true for locally Lipschitz functions.

The next definition of linking is adopted here; vide [**9**, Section 3]. Let $(X, \|\cdot\|)$ be a real reflexive Banach space and let Q, Q_0, S be nonempty closed subsets of X such that $Q_0 \subseteq Q$. Write

$$\Gamma := \left\{\gamma \in C^0(Q, X) : \gamma|_{Q_0} = id|_{Q_0}\right\}.$$

The pair (Q, Q_0) is said to link with S provided $Q_0 \cap S = \emptyset$ and for every $\gamma \in \Gamma$ it results $\gamma(Q) \cap S \neq \emptyset$.

Now, as in [**6**, Theorem 2.2], we put

$$\Gamma^* := \left\{\gamma^* \in C^0(X, X) : \gamma^* \text{ is a homeomorphism, } \gamma^*|_{Q_0} = id|_{Q_0}\right\}.$$

THEOREM 3.1. *Suppose (Q, Q_0) links with S while the locally Lipschitz function $f : X \to \mathbb{R}$ satisfies the following assumptions in addition to* $(\mathrm{PS})_f$.

(f_1) $\sup_{x\in Q} f(x) < +\infty$.

(f_2) $Q_0 \subseteq f_a$ *and* $S \subseteq f^a$ *for some* $a \in \mathbb{R}$.

Then, setting

$$b := \sup_{\gamma^*\in\Gamma^*} \inf_{x\in S} f(\gamma^*(x)), \qquad c := \inf_{\gamma\in\Gamma} \sup_{x\in Q} f(\gamma(x))$$

one has

(i_1) $a \leq b \leq c$,

(i_2) $K_b(f) \backslash Q_0 \neq \emptyset$,

(i_3) $K_b(f) \cap S \neq \emptyset$ *if* $a = b$.

Proof. We first note that $c < +\infty$ because the function $\gamma = id|_Q$ belongs to Γ while (f_1) gives $\sup_{x\in Q} f(\gamma(x)) < +\infty$. Moreover, $a \leq b$. In fact, $\gamma^* = id \in \Gamma^*$ and through

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(f_2) we get

$$a \le \inf_{x\in S} f(\gamma^*(x)) \le b.$$

Since (Q, Q_0) links with S, for every $\gamma \in \Gamma, \gamma^* \in \Gamma^*$ there exists $z \in Q$ such that $\gamma^{*-1}(\gamma(z)) \in S$. This forces

$$\inf_{x\in S} f(\gamma^*(x)) \le f(\gamma^*(\gamma^{*-1}(\gamma(z)))) = f(\gamma(z)) \le \sup_{x\in Q} f(\gamma(x)).$$

As γ, γ^* were arbitrary, we actually have $b \le c$, and (i_1) follows.

When $a < b$, by (f_2) again it results $K_b(f)\setminus Q_0 = K_b(f)$. To show ($i_2$) suppose on the contrary that $K_b(f) = \emptyset$ and define $g = -f$, $d = -b$, $\varepsilon_0 = (b-a)/2$. Evidently, the function g fulfils $(\mathrm{PS})_g$ while $K_d(g) = \emptyset$. Thus, using Lemma 2.1 we can find $\varepsilon \in]0, \varepsilon_0[$ besides a homeomorphism $\eta : X \to X$ such that

$$\eta(x) = x \quad \forall x \in X\setminus f^{-1}([(a+b)/2, (-a+3b)/2[), \tag{1}$$
$$b + \varepsilon \le f(\eta(x)) \quad \forall x \in f^{b-\varepsilon}. \tag{2}$$

The definition of b produces $\gamma_\varepsilon^*(S) \subseteq f^{b-\varepsilon}$ for some $\gamma_\varepsilon^* \in \Gamma^*$. Hence, owing to (2),

$$b + \varepsilon \le \inf_{x\in S} f(\eta(\gamma_\varepsilon^*(x))).$$

Since (f_2) and (1) yield $\eta \circ \gamma_\varepsilon^* \in \Gamma^*$, the preceding inequality leads to $b + \varepsilon \le b$, which is clearly impossible.

Finally, let $a = b$. The conclusion will be achieved once we verify (i_3), because $Q_0 \cap S = \emptyset$. Suppose on the contrary that $K_b(f) \cap S = \emptyset$ and put, as before, $g = -f$, $d = -b$, $A = Q_0$, $B = S$. One immediately has $A \subseteq g^d$, $B \subseteq g_d$, $K_d(g) \cap B = \emptyset$, while the function g satisfies $(\mathrm{PS})_g$. Then, by Lemma 2.2, there exist $\varepsilon > 0$ and a homeomorphism $\eta : X \to X$ such that

$$\eta(x) = x \quad \forall x \in Q_0, \tag{3}$$
$$b + \varepsilon \le f(\eta(x)) \quad \forall x \in S. \tag{4}$$

Through (3) we get $\eta \in \Gamma^*$. Therefore, due to (4),

$$b + \varepsilon \le \inf_{x\in S} f(\eta(x)) \le b,$$

a contradiction. This completes the proof. □

REMARK 3.1. The preceding result improves Theorem 2.2 in [**6**]. Moreover, it can be regarded as the dual version of [**9**, Theorem 2.1].

REMARK 3.2. For bounded S, the conclusion of Theorem 3.1 remains true also when condition $(\mathrm{PS})_f$ is replaced by the following weaker one:

$(\mathrm{C})_f$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{f(x_n)\}$ *turns out bounded and*

$$\lim_{n\to+\infty} (1 + \|x_n\|) m_f(x_n) = 0$$

possesses a convergent subsequence.

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In fact, by using the same arguments introduced in the proof of [**4**, Theorem 3.1] we see that Lemma 2.1 holds with $(C)_f$ in place of $(PS)_f$. Further, since S is bounded, $(C)_f$ implies the Palais–Smale condition adopted in [**9**, Theorem 2.1].

4. The Generalized MPT and application. This section begins with a version of the Generalized Mountain Pass Theorem [**10**, Theorem 5.3] for locally Lipschitz functions.

Keep the same notation used in Section 3 and suppose further that $X = V \oplus E$, where V is finite dimensional, $S \subseteq E$, while $Q = Q_V \oplus [0, Re]$ with Q_V closed subset of V, $R > 0$, $e \in \partial B_1 \cap E$.

THEOREM 4.1. *Under the assumptions of Theorem* 3.1 *assertions* $(i_1) - (i_3)$ *hold and, moreover,*

(i_4) $K_c(f)\setminus Q_0 \neq \emptyset$,

(i_5) $K_c(f) \cap S \neq \emptyset$ *if* $a = c$.

Proof. Since (i_5) immediately follows from (i_1) and (i_3), it remains to show (i_4) only. When $a = c$, conclusion (i_4) is a consequence of (i_5) because $Q_0 \cap S = \emptyset$. So, let $a < c$. In this case $K_c(f)\setminus Q_0 = K_c(f)$ by (f_2). If $K_c(f) = \emptyset$ then, using Lemma 2.1, we can find $\varepsilon \in]0, (c-a)/2[$ besides a homeomorphism $\eta : X \to X$ such that

$$\eta(x) = x \quad \forall x \in X\setminus f^{-1}(](a+c)/2, (-a+3c)/2]), \tag{5}$$

$$f(\eta(x)) \leq c - \varepsilon \quad \forall x \in f_{c+\varepsilon}. \tag{6}$$

The definition of c produces $\gamma_\varepsilon(Q) \subseteq f_{c+\varepsilon}$ for some $\gamma_\varepsilon \in \Gamma$. Hence, owing to (6),

$$\sup_{x\in Q} f(\eta(\gamma_\varepsilon(x)) \leq c - \varepsilon.$$

As (f_2) and (5) yield $\eta \circ \gamma_\varepsilon \in \Gamma$, the above inequality leads to $c \leq c - \varepsilon$, which is clearly impossible. □

REMARK 4.1. A meaningful special case of Theorem 4.1 occurs when $Q_V = V \cap \bar{B}_R$, $Q_0 = \partial Q$ (the boundary of Q relative to $V \oplus$ span $\{e\}$), while $S = \partial B_\rho \cap E$ with $0 < \rho < R$; vide [**10**, Theorem 5.3]. If $V = \{0\}$, namely $X = E$, $Q_0 = \{0, Re\}$, $\rho \in]0, R[$, and $S = \partial B_\rho$ then the preceding result reduces to [**9**, Corollary 2.2].

REMARK 4.2. By means of [**8**, Theorem 3.1] we can easily reformulate Theorem 4.1 for functions f on X satisfying the structural hypothesis

$(H)_f$ $f = \Phi + \alpha$, *where* $\Phi : X \to \mathbb{R}$ *is locally Lipschitz while* $\alpha : X \to]-\infty, +\infty]$ *is convex, proper, besides lower semicontinuous.*

We now apply Theorem 4.1 to solve an elliptic hemivariational inequality patterned after Problem (5.1) in [**10**].

Let Ω be a nonempty, bounded, open subset of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, having a smooth boundary $\partial\Omega$. The symbol $|\Omega|$ stands for the Lebesgue measure of Ω, while $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 \, dx\right)^{1/2}.$$

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Denote by 2^* the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subseteq L^p(\Omega)$. Recall that $2^* = \frac{2N}{N-2}$, if $p \in [1, 2^*]$ then there exists a constant $c_p > 0$ fulfilling

$$\|u\|_{L^p(\Omega)} \leq c_p \|u\| \quad \forall u \in H_0^1(\Omega), \tag{7}$$

and the embedding is compact whenever $p \in [1, 2^*[$; see e.g. [**10**, Proposition B.7].

Given a function $a \in L^\infty(\Omega)$ with

$$\operatorname*{ess\,inf}_{x \in \Omega} a(x) > 0, \tag{8}$$

consider the Sturm–Liouville eigenvalue problem

$$\begin{cases} -\Delta u = \lambda a(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

It is well known (we refer for instance to [**1**, Theorem 0.6]) that (9) possesses a sequence of eigenvalues $\{\lambda_n\}$ which satisfies $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ (the number of times an eigenvalue appears in the sequence equals its multiplicity) besides $\lim_{n \to +\infty} \lambda_n = +\infty$. Let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$\begin{gathered} \|\varphi_n\|^2 = 1 = \lambda_n \int_\Omega a(x) \varphi_n(x)^2 \, dx, \quad n \in \mathbb{N}; \\ \int_\Omega \nabla \varphi_m(x) \cdot \nabla \varphi_n(x) \, dx = \int_\Omega a(x) \varphi_m(x) \varphi_n(x) \, dx = 0 \quad \text{provided } m \neq n. \end{gathered} \tag{10}$$

If $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfils the conditions

(j_1) *j is measurable with respect to each variable separately,*

(j_2) *there exist $a_1 > 0$, $p \in]2, 2^*[$ such that*

$$|j(x, t)| \leq a_1(1 + |t|^{p-1}) \quad \forall (x, t) \in \Omega \times \mathbb{R}, \tag{11}$$

then the function $J : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$J(x, \xi) = \int_0^\xi -j(x, t) \, dt, \quad (x, \xi) \in \Omega \times \mathbb{R},$$

turns out well defined, $J(\cdot, \xi)$ is measurable, while $J(x, \cdot)$ is locally Lipschitz. So it makes sense to consider its generalized directional derivative J_x^0 with respect to the variable ξ. For our application, we will further assume

(j_3) $\lim_{\xi \to 0} \frac{j(x,\xi)}{\xi} = 0$ *uniformly in $x \in \Omega$,*

(j_4) *there are constants $\mu > 2$, $a_2 \in \mathbb{R}$ such that*

$$J_x^0(\xi; \xi) \leq \mu J(x, \xi) + a_2 \quad \forall (x, \xi) \in \Omega \times \mathbb{R},$$

(j_5) $J(x, \xi) \leq \min\{0, a_3(1 - |\xi|^\mu)\}$ *in $\Omega \times \mathbb{R}$ where $a_3 > 0$.*

Given $\lambda \in \mathbb{R}$, denote by (P_λ) the following elliptic hemivariational inequality: *Find $u \in H_0^1(\Omega)$ satisfying*

$$-\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx + \lambda \int_\Omega a(x) u(x) v(x) \, dx \leq \int_\Omega J_x^0(u(x); v(x)) \, dx$$

for all $v \in H_0^1(\Omega)$.

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REMARK 4.3. When $j \in C^0(\Omega \times \mathbb{R})$ the above inequality takes the form

$$-\int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx + \lambda \int_\Omega a(x)u(x)v(x)\,dx = \int_\Omega -j(x, u(x))v(x)\,dx, \qquad v \in H_0^1(\Omega).$$

Therefore, in this case, a function $u \in H_0^1(\Omega)$ solves (P_λ) if and only if it is a weak solution to the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda a(x)u + j(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

namely Problem (5.1) of [**10**].

THEOREM 4.2. *Suppose* (j_1)–(j_5) *hold. Then for every* $\lambda \in \mathbb{R}$, (P_λ) *possesses a nontrivial solution* $u \in H_0^1(\Omega)$.

Proof. Choose $X = H_0^1(\Omega)$ and define

$$f(u) = \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 - \lambda a(x)u(x)^2)\,dx + \int_\Omega J(x, u(x))\,dx \quad \forall u \in X.$$

By (j_2) the function f turns out locally Lipschitz. Moreover, one has

$$\lim_{u \to 0} \frac{1}{\|u\|^2}\int_\Omega J(x, u(x))\,dx = 0. \tag{12}$$

In fact, integrating (11) yields

$$|J(x, \xi)| \le a_1\left(|\xi| + \frac{|\xi|^p}{p}\right) \le a_1(|\xi| + |\xi|^p) \le 2a_1 \max\{|\xi|, |\xi|^p\} \tag{13}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}$. On account of (j_3), given any $\varepsilon > 0$ there is a $\delta \in]0, 1[$ such that $|\xi| \le \delta$ implies that

$$|J(x, \xi)| \le \frac{\varepsilon}{2}|\xi|^2 \quad \forall x \in \Omega. \tag{14}$$

Since (13) easily leads to

$$|J(x, \xi)| < \frac{2a_1}{\delta^p}|\xi|^p \quad \text{whenever } |\xi| > \delta,$$

gathering the above inequality and (14) together we obtain

$$|J(x, \xi)| \le \frac{\varepsilon}{2}|\xi|^2 + \frac{2a_1}{\delta^p}|\xi|^p \quad \forall (x, \xi) \in \Omega \times \mathbb{R}.$$

Consequently, by (7),

$$\left|\int_\Omega J(x, u(x))\,dx\right| \le \|u\|^2\left(\frac{\varepsilon}{2}c_2^2 + \frac{2a_1 c_p^p}{\delta^p}\|u\|^{p-2}\right), \qquad u \in X,$$

which forces (12) because $p > 2$.

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Now, fix $\lambda \in \mathbb{R}$. If $\lambda < \lambda_1$ then

$$\|u\|_* = \left[\int_\Omega (|\nabla u(x)|^2 - \lambda a(x)u(x)^2)\,dx\right]^{1/2}, \quad u \in X,$$

is a norm on X equivalent to the usual one and it results

$$f(u) = \frac{1}{2}\|u\|_*^2 + \int_\Omega J(x, u(x))\,dx, \quad u \in X.$$

Thus, through (12) we can find $\rho > 0$ such that $f(u) > 0$ provided $\|u\|_* = \rho$. Since for every $u \in X \setminus \{0\}$ hypothesis (j_5) implies

$$f(tu) \le \frac{1}{2}t^2\|u\|_*^2 + a_3\big(|\Omega| - t^\mu \|u\|^\mu_{L^\mu(\Omega)}\big) \to -\infty$$

as $t \to +\infty$, there obviously exist $R > \rho$ besides $e \in X$ with $\|e\|_* = 1$ satisfying $f(Re) < 0$. So, once we set

$$V = \{0\}, \quad Q = [0, Re], \quad Q_0 = \{0, Re\}, \quad S = \{u \in X : \|u\|_* = \rho\},$$

assumptions (f_1) and (f_2) of Theorem 4.1 hold. Let us next verify $(PS)_f$. To this end, pick a sequence $\{u_n\} \subseteq X$ such that

$$\lim_{n \to +\infty} f(u_n) = c, \tag{15}$$

$$\lim_{n \to +\infty} m_f(u_n) = 0. \tag{16}$$

By (15) one has $|f(u_n)| \le M (n \in \mathbb{N})$ for some $M > 0$. Exploiting (16) as well as the definition of generalized gradient produces a sequence $\{v_n\} \subseteq X$ fulfilling

$$\langle v_n, w\rangle \le f^0(u_n; w) \quad \forall n \in \mathbb{N}, w \in X, \tag{17}$$

$$\lim_{n \to +\infty} \|v_n\| = 0. \tag{18}$$

Consequently, due to (j_4) and formula (2) at p. 77 in [5],

$$\begin{aligned} M + \frac{1}{\mu}\|u_n\|_* &\ge M - \frac{1}{\mu}\langle v_n, u_n\rangle \ge f(u_n) - \frac{1}{\mu} f^0(u_n; u_n) \\ &\ge f(u_n) - \frac{1}{\mu}\left(\|u_n\|_*^2 + \int_\Omega J_x^0(u_n(x); u_n(x))\,dx\right) \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_*^2 - \frac{a_2}{\mu}|\Omega| \end{aligned} \tag{19}$$

for all sufficiently large n, which guarantees that the sequence $\{u_n\}$ is bounded. Passing to a subsequence if necessary, we may thus suppose $u_n \rightharpoonup u$ in X besides $u_n \to u$ in $L^p(\Omega)$. From (17) with $w = u - u_n$ it follows

$$\begin{aligned} -\|v_n\|_*\|u - u_n\|_* + \|u_n\|_*^2 &\le f^0(u_n; u - u_n) + \langle u_n, u_n\rangle_* \\ &= \langle u_n, u\rangle_* + h^0(u_n; u - u_n) \quad \forall n \in \mathbb{N}, \end{aligned}$$

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where $\langle\cdot,\cdot\rangle_*$ denotes the scalar product related to $\|\cdot\|_*$ while

$$h(w) = \int_\Omega J(x, w(x))\,dx,\ w \in L^p(\Omega).$$

The upper semicontinuity of h^0 and (18) then yield $\limsup_{n\to+\infty}\|u_n\|_* \le \|u\|_*$, namely $u_n \to u$ in X; vide e.g. [**2**, Proposition III.30].

Since Proposition 5.9 of [**10**] ensures that the pair (Q, Q_0) links with S, Theorem 4.1 can be applied, and we obtain a point $u \in X\setminus\{0, Re\}$ satisfying $f^0(u; v) \ge 0$ for all $v \in X$. Finally, by formula (2) at p. 77 in [**5**] it results

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \lambda\int_\Omega a(x)u(x)v(x)\,dx + \int_\Omega J^0_x(u(x); v(x))\,dx \ge 0, \qquad v \in X,$$

i.e. the function u turns out a nontrivial solution to Problem (P_λ).

Now, let $\lambda \in [\lambda_k, \lambda_{k+1}[$ for some integer $k \ge 1$. We define

$$V = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}, \quad E = V^\perp, \quad e = \varphi_{k+1}, \quad W = V \oplus \operatorname{span}\{e\}.$$

It is evident that $X = V \oplus E$ while $\dim(V) = k < +\infty$. Moreover, if $u \in E$ then $u = \sum_{i=k+1}^{+\infty} t_i\varphi_i$, where $t_i \in \mathbb{R}$, $i \ge k+1$. On account of (10) we get

$$\begin{aligned} f(u) &= \frac{1}{2}\sum_{i=k+1}^{+\infty} t_i^2\left(1 - \frac{\lambda}{\lambda_i}\right) + \int_\Omega J(x, u(x))\,dx \\ &\ge \frac{\|u\|^2}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}} + \frac{2}{\|u\|^2}\int_\Omega J(x, u(x)\,dx\right). \end{aligned} \tag{20}$$

So, through (12), the condition $\lambda < \lambda_{k+1}$, and (20) one can find $\rho, a > 0$ fulfilling

$$\partial B_\rho \cap E \subseteq f^a \subseteq f^0.$$

Our next goal is to prove that, for a suitable $R > \rho$,

$$f(u) \le 0 \quad \forall u \in W\setminus B_R. \tag{21}$$

To see this we first fix $q \in]2, \min\{2^*, \mu\}[$ and note that W is a (finite dimensional) subspace of $L^q(\Omega)$. Hence,

$$\|u\|_{L^q(\Omega)} \ge a_4\|u\|, \quad u \in W,$$

where $a_4 > 0$. If $u \in W$ then $u = \sum_{i=1}^{k+1} t_i\varphi_i$, with $t_1, \ldots, t_{k+1} \in \mathbb{R}$. From (10), (j_5), besides the above inequality, it follows

$$\begin{aligned} f(u) &= \frac{1}{2}\sum_{i=1}^{k+1} t_i^2\left(1 - \frac{\lambda}{\lambda_i}\right) + \int_\Omega J(x, u(x))\,dx \\ &\le \frac{1}{2}t_{k+1}^2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) + a_3\left(|\Omega| - \|u\|^\mu_{L^\mu(\Omega)}\right) \\ &\le \frac{1}{2}\|u\|^2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) + a_3|\Omega|\left(1 - |\Omega|^{-\mu/q}a_4^\mu\|u\|^\mu\right), \end{aligned}$$

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which immediately leads to (21) because $\mu > 2$. Finally, write

$$Q = (V \cap \bar{B}_R) \oplus [0, Re], \quad Q_0 = \partial Q, \quad S = \partial B_\rho \cap E.$$

Since each $u \in V \cap B_R$ can be written as $u = \sum_{i=1}^k t_i \varphi_i$, where $t_1, \dots, t_k \in \mathbb{R}$, by (10) and ($j_5$) again one has

$$f(u) = \frac{1}{2} \sum_{i=1}^k t_i^2 \left(1 - \frac{\lambda}{\lambda_i}\right) + \int_\Omega J(x, u(x))\, dx \le \frac{1}{2} \sum_{i=1}^k t_i^2 \left(1 - \frac{\lambda}{\lambda_i}\right) \le 0.$$

Bearing in mind (21) this obviously forces $\partial Q \subseteq f_0$. Therefore, the function f satisfies assumption (f_2) of Theorem 4.1, while (f_1) is an immediate consequence of the compactness of Q. As Proposition 5.9 in **[10]** ensures that the pair (Q, Q_0) links with S, it remains to verify $(PS)_f$ only. Pick a sequence $\{u_n\} \subseteq X$ fulfilling (15), (16) and choose $\beta \in]\mu^{-1}, 2^{-1}[$. The same arguments adopted in showing (19) provide here

$$\begin{aligned} M + \beta \|u_n\| \ge \left(\frac{1}{2} - \beta\right)\|u_n\|^2 - \lambda\left(\frac{1}{2} - \beta\right)\|a\|_{L^\infty(\Omega)}\|u_n\|^2_{L^2(\Omega)} \\ + (1 - \beta\mu)\int_\Omega J(x, u_n(x))\, dx - a_2\beta|\Omega| \end{aligned}$$

for all sufficiently large n. Thanks to (j_5) it implies

$$\begin{aligned} M + \beta \|u_n\| \ge \left(\frac{1}{2} - \beta\right)\|u_n\|^2 - \lambda\left(\frac{1}{2} - \beta\right)\|a\|_{L^\infty(\Omega)}\|u_n\|^2_{L^2(\Omega)} \\ + (\beta\mu - 1)a_3\|u_n\|^\mu_{L^\mu\Omega} - a_5, \end{aligned}$$

where $a_5 = |\Omega|(a_2\beta + (\beta\mu - 1)a_3)$. Because of Hölder and Young's inequalities, we also have

$$\|u\|^2_{L^2(\Omega)} \le \frac{2}{\mu}\varepsilon^{\mu/2}\|u\|^\mu_{L^\mu(\Omega)} + \frac{\mu - 2}{\mu}|\Omega|\varepsilon^{-\mu/(\mu-2)}, \quad u \in X,$$

for any $\varepsilon > 0$. Hence,

$$\begin{aligned} M + \beta\|\mu_n\| &\ge \left(\frac{1}{2} - \beta\right)\|\mu_n\|^2 \\ &+ \left[(\beta\mu - 1)a_3 - \lambda\left(\frac{1}{2} - \beta\right)\|a\|_{L^\infty(\Omega)}\frac{2}{\mu}\varepsilon^{\mu/2}\right]\|u_n\|^\mu_{L^\mu(\Omega)} - a_6, \end{aligned} \tag{22}$$

with $a_6 = a_5 + \lambda(2^{-1} - \beta)\|a\|_{L^\infty(\Omega)}(\mu - 2)\mu^{-1}|\Omega|\varepsilon^{-\mu/(\mu-2)}$. Since $\beta\mu > 1$, choosing ε so small that

$$(\beta\mu - 1)a_3 - \lambda\left(\frac{1}{2} - \beta\right)\|a\|_{L^\infty(\Omega)}\frac{2}{\mu}\varepsilon^{\mu/2} > 0,$$

from (22) it follows

$$M + \beta\|u_n\| \ge \left(\frac{1}{2} - \beta\right)\|u_n\|^2 - a_6$$

for all sufficiently large n, namely the sequence $\{u_n\}$ is bounded. The same technique exploited in the case $\lambda < \lambda_1$ then ensures that it possesses a strongly convergent subsequence. At this point Theorem 4.1 can be applied, and the proof goes on exactly as before. □

REMARK 4.4. Theorem 4.1 actually gives two nontrivial solutions to Problem (P_λ) whenever $b < c$.

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