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A SIMPLIFIED ORDINAL ANALYSIS OF FIRST-ORDER REFLECTION

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Abstract. In this note we give a simplified ordinal analysis of first-order reflection. An ordinal notation system OT is introduced based on ψ-functions. Provable Σ_1-sentences on $L_{\omega_1^{CK}}$ are bounded through cut-elimination on operator controlled derivations.

§1. Introduction. Let ORD denote the class of all ordinals, $A \subset ORD$ and α a limit ordinal. α is said to be Π_n-*reflecting* on A iff for any Π_n-formula $\phi(x)$ and any $b \in L_\alpha$, if $\langle L_\alpha, \in\rangle \models \phi(b)$, then there exists a $\beta \in A \cap \alpha$ such that $b \in L_\beta$ and $\langle L_\beta, \in\rangle \models \phi(b)$. Let us write $\alpha \in rM_n(A) :\Leftrightarrow \alpha$ is Π_n-reflecting on A. Also α is said to be Π_n-*reflecting* iff α is Π_n-reflecting on ORD.

It is not hard for us to show that the assumption that the universe is Π_n-reflecting is proof-theoretically reducible to iterabilities of the lower operation rM_{n-1} (and Mostowski collapsings), cf. [4].

In this paper we aim at an ordinal analysis of Π_n-reflection. Such an analysis was done by Pohlers and Stegert [8] using reflection configurations introduced in Rathjen [10], and an alternative analysis was given in [2, 3, 5] with the complicated combinatorial arguments of ordinal diagrams and finite proof figures. Our approach is simpler in view of combinatorial arguments. In [2], a Π_n-reflecting universe is resolved into ramified hierarchies of lower Mahlo operations, and ultimately into iterations of recursively Mahlo operations. Our ramification process is akin to a tower, i.e., has an exponential structure. It is natural that an exponential structure emerges in lowering and eliminating first-order formulas (in reflections), cf. ordinal analysis for the fragments $I\Sigma_{n-3}$ of the first-order arithmetic. The Mahlo classes $Mh_k(\xi)$ defined in Definition 2.5 to resolve or approximate Π_n-reflection are based on a similar structure. As in Rathjen's analysis for Π_3-reflection in [9], thinning operations are applied on the Mahlo classes $Mh_k(\xi)$, and this yields an exponential structure similar to the one in [2] as follows.

Let us consider the simplest case $N = 4$. Let $\Lambda := \varepsilon_{\mathbb{K}+1}$, the next epsilon number above the least Π_4-reflecting ordinal $\mathbb{K}$. Roughly $\pi \in Mh_3(\xi)$ designates the fact that an ordinal π is Π_3-reflecting on $Mh_3(\nu)$ for any $\nu < \xi < \Lambda$. Suppose a Π_3-sentence θ on L_π is derived from the assumption $\pi \in Mh_3(\xi)$. We need to find an ordinal $\kappa < \pi$ for which $L_\kappa \models \theta$ holds. It turns out that $\kappa \in Mh_2(\Lambda^\xi a)$ suffices for an ordinal $a < \Lambda$, where the ordinal κ in the class $Mh_2(\Lambda^\xi a)$ is Π_2-reflecting on classes $Mh_2(\Lambda^\xi b) \cap Mh_3(\nu)$ for any $b < a$ and any $\nu < \xi$. Note that the class $Mh_2(\Lambda^\xi a)$ is not obtained through iterations of recursively Mahlo operations since it involves

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Π_4-definable classes $Mh_3(\nu)$. The classes $Mh_3(\nu)\,(\nu < \xi)$ for the assumption $\pi \in Mh_3(\xi)$ are thinned out with the new classes $Mh_2(\Lambda^{\xi} b)\,(b < \Lambda)$, cf. Lemma 5.1.

Our theorem runs as follows. Let $\mathsf{KP}\Pi_N$ denote the set theory for Π_N-reflecting universes, and $\mathsf{KP}\omega$ the Kripke–Platek set theory with the axiom of infinity. OT is a computable notation system for ordinals defined in §3, $\Omega = \omega_1^{CK}$ and ψ_Ω is a collapsing function such that $\psi_\Omega(\alpha) < \Omega$. $\mathbb{K}_N$ is an ordinal term denoting the least Π_N-reflecting ordinal in the theorems.

THEOREM 1.1. *Suppose* $\mathsf{KP}\Pi_N \vdash \theta$ *for a* $\Sigma_1(\Omega)$*-sentence* θ*. Then we can find an* $n < \omega$ *such that for* $\alpha = \psi_\Omega(\omega_n(\mathbb{K}_N + 1))$, $L_\alpha \models \theta$.

Actually the bound is seen to be tight, cf. [6].

THEOREM 1.2. $\mathsf{KP}\Pi_N$ *proves that each initial segment* $\{\alpha \in OT : \alpha < \psi_\Omega(\omega_n(\mathbb{K}_N + 1))\}\,(n = 1, 2, \ldots)$ *is well-founded.*

Thus the ordinal $\psi_\Omega(\varepsilon_{\mathbb{K}_N+1})$ is seen to be the proof-theoretic ordinal of $\mathsf{KP}\Pi_N$.

THEOREM 1.3.
$\psi_\Omega(\varepsilon_{\mathbb{K}_N+1}) = |\mathsf{KP}\Pi_N|_{\Sigma_1^\Omega} := \min\{\alpha \leq \omega_1^{CK} : \forall \theta \in \Sigma_1(\mathsf{KP}\Pi_N \vdash \theta^{L_\Omega} \Rightarrow L_\alpha \models \theta)\}$.

$A \subset ORD$ is Π^1_n*-indescribable* in α iff for any Π^1_n-formula $\phi(X)$ and any $B \subset ORD$, if $\langle L_\alpha, \in; B \cap \alpha\rangle \models \phi(B \cap \alpha)$, then there exists a $\beta \in A \cap \alpha$ such that $\langle L_\beta, \in; B \cap \beta\rangle \models \phi(B \cap \beta)$. A regular cardinal π is Π^1_n*-indescribable* ifff ORD is Π^1_n-indescribable in π.

Let us mention the contents of this paper. In the next §2 we define simultaneously iterated Skolem hulls $\mathcal{H}_\alpha(X)$ of sets X of ordinals, ordinals $\psi_\kappa^{\vec{\xi}}(\alpha)$ for regular cardinals κ, $\alpha < \varepsilon_{\mathbb{K}+1}$ and sequences $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ of ordinals $\xi_i < \varepsilon_{\mathbb{K}+2}$, and classes $Mh_k^\alpha(\xi)$ under the *assumption* that a Π^1_{N-2}-indescribable cardinal $\mathbb{K}$ exists. It is shown that for $2 \leq k < N$, $\alpha < \varepsilon_{\mathbb{K}+1}$ and each $\xi < \varepsilon_{\mathbb{K}+2}$, ($\mathbb{K}$is a Π^1_{N-2}-indescribable cardinal) $\to \mathbb{K} \in Mh_k^\alpha(\xi)$ in $\mathsf{ZF} + (V = L)$.

In §3 a computable notation system OT of ordinals is extracted. Following Buchholz [7], operator controlled derivations for $\mathsf{KP}\Pi_N$ are introduced in §4, and inference rules for Π_N-reflection are eliminated from derivations in §5. This completes a proof of Theorem 1.1 for an upper bound.

IH denotes the Induction Hypothesis, MIH the Main IH and SIH the Subsidiary IH. We are assuming tacitly the axiom of constructibility $V = L$. Throughout this paper $N \geq 3$ is a fixed integer.

§2. Ordinals for Π_N-reflection. In this section we work in the set theory ZFLK_N obtained from $\mathsf{ZFL} = \mathsf{ZF} + (V = L)$ by adding the axiom stating that $\exists \mathbb{K}(\mathbb{K}$ is Π^1_{N-2}-indescribable) for a fixed integer $N \geq 3$. For ordinals α, $\varepsilon(\alpha)$ denotes the least epsilon number above α.

Let $ORD \subset V$ denote the class of ordinals, $\mathbb{K} = \mathbb{K}_N$ the least Π^1_{N-2}-indescribable cardinal, and Reg the set of regular ordinals below $\mathbb{K}$. Θ denotes finite sets of ordinals $\leq \mathbb{K}$. $u, v, w, x, y, z, \ldots$ range over sets in the universe, $a, b, c, \alpha, \beta, \gamma, \ldots$ range over ordinals $< \Lambda = \varepsilon(\mathbb{K})$, $\xi, \zeta, \nu, \mu, \iota, \ldots$ range over ordinals $< \varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$, $\vec{\xi}, \vec{\zeta}, \vec{\nu}, \vec{\mu}, \vec{\iota}, \ldots$ range over finite sequences over ordinals $< \varepsilon(\Lambda)$, and $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$ range over regular ordinals. θ denotes formulas.

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Let $\vec{\xi} = (\xi_0, \dots, \xi_{m-1})$ be a sequence of ordinals. The *length* $lh(\vec{\xi}) := m$. Sequences consisting of a single element (ξ) are identified with the ordinal ξ, and $\emptyset$ denotes the *empty sequence*. $\vec{0}$ denotes ambiguously a zero-sequence $(0, \dots, 0)$ with its length $0 \leq lh(\vec{0}) \leq N-1$. $\vec{\xi} * \vec{\mu} = (\xi_0, \dots, \xi_{m-1}) * (\mu_0, \dots, \nu_{n-1}) = (\xi_0, \dots, \xi_{m-1}, \mu_0, \dots, \mu_{n-1})$ denotes the *concatenated* sequence of $\vec{\xi}$ and $\vec{\mu}$.

$\Lambda = \varepsilon(\mathbb{K}) = \varepsilon_{\mathbb{K}+1}$ denotes the next epsilon number above the least Π_{N-2}-indescribable cardinal $\mathbb{K}$, and $\varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$ the next epsilon number above Λ.

Definition 2.1. For a nonzero ordinal $\xi < \varepsilon(\Lambda)$, its Cantor normal form with base Λ is uniquely determined as

$$\xi =_{NF} \sum_{i \leq m} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_0} a_0 \tag{1}$$

where $\xi_m > \cdots > \xi_0, 0 < a_i < \Lambda$.

1. $K(\xi) = \{a_i : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$ is the set of *components* of ξ with $K(0) = \emptyset$. For a sequence $\vec{\xi} = (\xi_0, \dots, \xi_{n-1})$ of ordinals $\xi_i < \varepsilon(\Lambda)$, $K(\vec{\xi}) := \bigcup\{K(\xi_i) : i < n\}$.
2. For $\xi > 1$, $te(\xi) = \xi_0$ in (1) is the *tail exponent*, and $he(\xi) = \xi_m$ is the *head exponent* of ξ, resp. The *head* $Hd(\xi) := \Lambda^{\xi_m} a_m$, and the *tail* $Tl(\xi) := \Lambda^{\xi_0} a_0$ of ξ.
3. $he^{(i)}(\xi)$ is the *ith head exponent* of ξ, defined recursively by $he^{(0)}(\xi) = \xi$, $he^{(i+1)}(\xi) = he(he^{(i)}(\xi))$.
   The *i-th tail exponent* $te^{(i)}(\xi)$ is defined similarly.
4. ζ is a *part* of ξ, denoted by $\zeta \leq_{pt} \xi$ iff $\zeta =_{NF} \sum_{i \geq n} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_n} a_n$ for an $n\,(0 \leq n \leq m+1)$.
   $\zeta <_{pt} \xi :\Leftrightarrow \zeta \leq_{pt} \xi \,\&\, \zeta \neq \xi$.
5. A sequence $\vec{\mu} = (\mu_0, \dots, \mu_n)$ is an *iterated tail parts* of ξ, denoted by $\vec{\mu} \subset_{pt} \xi$ iff $\mu_0 \leq_{pt} \xi \,\&\, \forall i < n(\mu_{i+1} \leq_{pt} te(\mu_i))$.
6. $\vec{\nu} = (\nu_0, \dots, \nu_n) * \vec{0} < \xi$ iff there exists a sequence $\vec{\mu} = (\mu_0, \dots, \mu_n)$ such that $\vec{\mu} \subset_{pt} \xi$ and $\nu_i < \mu_i$ for every $i \leq n$.
7. Let $\vec{\nu} = (\nu_0, \dots, \nu_n)$ and $\vec{\xi} = (\xi_0, \dots, \xi_n)$ be sequences of ordinals in the same length, and $0 \leq k \leq n$.
   $\vec{\nu} <_k \vec{\xi} :\Leftrightarrow \forall i < k(\nu_i \leq \xi_i) \wedge (\nu_k, \dots, \nu_n) < \xi_k$.
8. ζ is a *step-down* of ξ, denoted by $\zeta <_{sd} \xi$ iff $\zeta = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_1} a_1 + \Lambda^{\xi_0} b + \nu$ for some ordinals $b < a_0$ and $\nu < \Lambda^{\xi_0}$.
9. $\vec{\nu} = (\nu_0, \dots, \nu_n) * \vec{0} <_{sd} \xi$ iff $\nu_i <_{sd} te^{(i)}(\xi)$ for every $i \leq n$.
10. $\zeta \leq_{sp} \xi :\Leftrightarrow \exists \mu \leq_{pt} \xi(\zeta \leq_{sd} \mu)$, and $\zeta <_{sp} \xi :\Leftrightarrow \exists \mu \leq_{pt} \xi(\zeta <_{sd} \mu)$.
11. $\vec{\nu} <_{sp} \xi$ iff $\vec{\nu} <_{sd} \mu$ for a $\mu \leq_{pt} \xi$.

Note that $(\nu) * \vec{0} < \xi \Leftrightarrow \nu < \xi$, and $(\xi, te(\xi), te^{(2)}(\xi), \dots) \subset_{pt} \xi$. Also $\zeta <_{sd} \xi \Leftrightarrow \zeta < \xi$ if $\xi < \Lambda$.

Proposition 2.2. $\xi < \mu < \varepsilon(\Lambda) \Rightarrow te(\xi) \leq he(\xi) \leq he(\mu)$.

Proposition 2.3. $\vec{\nu} < \xi \leq \zeta \Rightarrow \vec{\nu} < \zeta$.

Proof. By induction on the lengths $n = lh(\vec{\nu})$. Let $\vec{\mu} = (\mu_0, \dots, \mu_{n-1})$ be a sequence for $\vec{\nu} = (\nu_0, \dots, \nu_{n-1})$ such that $\vec{\mu} \subset_{pt} \xi$ and $\forall i \leq n-1(\nu_i < \mu_i)$, cf. Definition 2.1.6.

If $n = 1$, then $\nu_0 < \mu_0 \leq_{pt} \xi \leq \zeta$. $\nu_0 < \zeta \leq_{pt} \zeta$ yields $\vec{\nu} = (\nu_0) < \zeta$.

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Let $n > 1$. We have $(\nu_1, \dots, \nu_{n-1}) < te(\mu_0)$ with $(\mu_1, \dots, \mu_{n-1}) \subset_{pt} te(\mu_0)$. We show the existence of a λ such that $\mu_0 \leq \lambda \leq_{pt} \zeta$ and $te(\mu_0) \leq te(\lambda)$. Then IH yields $(\nu_1, \dots, \nu_{n-1}) < te(\lambda)$, and $\vec{\nu} < \zeta$ follows.

If $\mu_0 \leq_{pt} \zeta$, then $\lambda = \mu_0$ works. Suppose $\mu_0 \not\leq_{pt} \zeta$. On the other hand we have $\mu_0 \leq_{pt} \xi \leq \zeta$. This means that $\xi < \zeta$ and there exists a $\lambda \leq_{pt} \zeta$ such that $\mu_0 < \lambda$ and $te(\mu_0) \leq te(\lambda)$. ⊣

2.1. Ordinals.

Definition 2.4. 1. For $i < \omega$ and $\xi < \varepsilon(\Lambda)$, $\Lambda_i(\xi)$ is defined recursively by $\Lambda_0(\xi) = \xi$ and $\Lambda_{i+1}(\xi) = \Lambda^{\Lambda_i(\xi)}$.
2. For $A \subset ORD$, limit ordinals α and $i \geq 0$, let $\alpha \in M_{2+i}(A)$ iff $A \cap \alpha$ is Π^1_i-indescribable in α. $M_{2+i} := M_{2+i}(ORD)$.
3. κ^+ denotes the next regular ordinal above κ.
4. $\Omega_\alpha := \omega_\alpha$ for $\alpha > 0$, $\Omega_0 := 0$, and $\Omega = \Omega_1$.

Define simultaneously classes $\mathcal{H}_\alpha(X)$, $Mh^\alpha_k(\xi)$, and ordinals $\psi^{\vec{\xi}}_\kappa(\alpha)$ as follows. We see that these are Σ_1-definable as a fixed point in ZFL, cf. Proposition 2.7.

Let $a < \Lambda$, and φ denote the binary Veblen function. Let us define a Skolem hull $\mathcal{H}_a(X)$ of $\{0, \mathbb{K}\} \cup X$ under the functions $+, \alpha \mapsto \omega^\alpha, (\alpha, \beta) \mapsto \varphi\alpha\beta\, (\alpha, \beta < \mathbb{K}), \alpha \mapsto \Omega_\alpha\, (\alpha < \mathbb{K})$ and ψ-functions. *Reg* denotes the set of regular ordinals $\leq \mathbb{K}$.

Definition 2.5. $\mathcal{H}_a[Y](X) := \mathcal{H}_a(Y \cup X)$ for sets $Y \subset \mathbb{K}$.

1. (Inductive definition of $\mathcal{H}_a(X)$).
   (a) $\{0, \mathbb{K}\} \cup X \subset \mathcal{H}_a(X)$.
   (b) $x, y \in \mathcal{H}_a(X) \Rightarrow x + y \in \mathcal{H}_a(X)$, $x \in \mathcal{H}_a(X) \Rightarrow \omega^x \in \mathcal{H}_a(X)$, and $x, y \in \mathcal{H}_a(X) \cap \mathbb{K} \Rightarrow \varphi xy \in \mathcal{H}_a(X)$.
   (c) $\mathbb{K} > \alpha \in \mathcal{H}_a(X) \Rightarrow \Omega_\alpha \in \mathcal{H}_a(X)$.
   (d) Let $\{\pi, b\} \subset \mathcal{H}_a(X)$ with $\pi \in Reg$, and $\vec{\nu} = (\nu_2, \dots, \nu_{N-1})$ be a sequence of ordinals $< \varepsilon(\Lambda)$ such that $K(\vec{\nu}) \subset \mathcal{H}_a(X)$ and $\max K(\vec{\nu}) \leq b < a$.Then $\kappa = \psi^{\vec{\nu}}_\pi(b) \in \mathcal{H}_a(X)$.
2. (Definitions of $Mh^a_k(\xi)$ and $Mh^a_k(\vec{\xi})$)First let $\mathbb{K} \in Mh^a_N(0) :\Leftrightarrow \mathbb{K} \in M_N \Leftrightarrow$ $\mathbb{K}$ is Π^1_{N-2}-indescribable.The classes $Mh^a_k(\xi)$ are defined for $2 \leq k < N$, and ordinals $a < \Lambda$, $\xi < \varepsilon(\Lambda)$. Let π be a regular ordinal $\leq \mathbb{K}$. Then for $\xi > 0$

$$\begin{aligned} \pi \in Mh^a_k(\xi) :\Leftrightarrow & \{\pi, a\} \cup K(\xi) \subset \mathcal{H}_a(\pi) \,\& \\ & \forall \vec{\nu} < \xi \left(K(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh^a_k(\vec{\nu})) \right) \end{aligned} \tag{2}$$

   where $\vec{\nu} = (\nu_k, \dots, \nu_n)\, (2 \leq k \leq n \leq N - 1)$ varies through nonempty sequences of ordinals $< \varepsilon(\Lambda)$ and

$$\pi \in Mh^a_k(\vec{\nu}) :\Leftrightarrow \pi \in \bigcap_{k \leq i \leq n} Mh^a_i(\nu_i).$$

   By convention, let for $2 \leq k < N$, $\pi \in Mh^a_k(0) :\Leftrightarrow \pi \in Mh^a_2(\emptyset) :\Leftrightarrow$ π is a limit ordinal. Note that by letting $\vec{\nu} = (0)$, $\pi \in Mh^a_k(\xi) \Rightarrow \pi \in M_k$ for $\xi > 0$. Also $\vec{0} < 1$, and $Mh^a_k(1) = M_k$.

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3. (Definition of $\psi_\pi^{\vec{\xi}}(a)$)Let $a < \Lambda$ be an ordinal, $\pi \leq \mathbb{K}$ a regular ordinal and $\vec{\xi}$ a sequence of ordinals $< \varepsilon(\Lambda)$ such that $lh(\vec{\xi}) = N - 2$. Then let

$$\psi_\pi^{\vec{\xi}}(a) := \min(\{\pi\} \cup \{\kappa \in Mh_2^a(\vec{\xi}) \cap \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, K(\vec{\xi}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)\}). \tag{3}$$

Let $\psi_\pi a := \psi_\pi^{\vec{0}} a$, where $lh(\vec{0}) = N - 2$, $Mh_2^a(\vec{0}) = Lim$, and $\pi \in M_2$, i.e., π is a regular ordinal.

Note that $\pi \in Mh_k^a(\xi) \Rightarrow \forall \nu < \xi\,(\pi \in M_k(Mh_k^a(\nu)))$, since $(\nu) < \xi$ holds with $(\xi) \subset_{pt} \xi$ for $\nu < \xi$.

Proposition 2.6. *$b + c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$, and $\omega^c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$.*

The following Proposition 2.7 is easy to see.

Proposition 2.7. *Each of $x = \mathcal{H}_a(y)\,(a < \Lambda, y < \mathbb{K})$, $x \in Mh_k^a(\xi)$ and $x = \psi_\kappa^{\vec{\xi}}(a)$, is a Σ_1-predicate as fixed points in* ZFL.

Proof. This is seen from the facts that there exists a universal Π_n^1-formula, and by using it, $\alpha \in M_n(x)$ iff $\langle L_\alpha, \in\rangle \models m_n(x \cap L_\alpha)$ for some Π_{n+1}^1-formula $m_n(R)$ with a unary predicate R. ⊣

Let $A(a)$ denote the conjunction of $\forall u < \mathbb{K} \exists! x[x = \mathcal{H}_a(u)]$, and $\forall \vec{\xi} \forall x(\max K(\vec{\xi}) \leq a \,\&\, K(\vec{\xi}) \cup \{\kappa, a\} \subset x = \mathcal{H}_a(\kappa) \to \exists! b \leq \kappa (b = \psi_\kappa^{\vec{\xi}}(a)))$, where $lh(\vec{\xi}) = N - 2$.
Since the cardinality of the set $\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi)$ is π for any infinite cardinal $\pi \leq \mathbb{K}$, pick an injection $f : \mathcal{H}_\Lambda(\mathbb{K}) \to \mathbb{K}$ so that $f``\mathcal{H}_\Lambda(\pi) \subset \pi$ for any weakly inaccessibles $\pi \leq \mathbb{K}$.

Lemma 2.8. 1. $\forall a < \Lambda\, A(a)$.
2. *$\pi \in Mh_k^a(\xi)$ is a Π_{k-1}^1-class on L_π uniformly for weakly inaccessible cardinals $\pi \leq \mathbb{K}$ and a, ξ. This means that for each k there exists a Π_{k-1}^1-formula $mh_k^a(x)$ such that $\pi \in Mh_k^a(\xi)$ iff $L_\pi \models mh_k^a(\xi)$ for any weakly inaccessible cardinals $\pi \leq \mathbb{K}$ with $f``(\{a\} \cup K(\xi)) \subset L_\pi$.*
3. $\mathbb{K} \in Mh_{N-1}^\alpha(\Lambda) \cap M_{N-1}(Mh_{N-1}^\alpha(\Lambda))$.

Proof. 2.8.1. We show that $A(a)$ is progressive, i.e., $\forall a < \Lambda[\forall c < a\, A(c) \to A(a)]$.
Assume $\forall c < a\, A(c)$ and $a < \Lambda$. $\forall b < \mathbb{K} \exists! x[x = \mathcal{H}_a(b)]$ follows from IH in ZFL. $\exists! b \leq \kappa (b = \psi_\kappa^{\vec{\xi}} a)$ follows from this.
2.8.2. Let π be a weakly inaccessible cardinal with $f``(\{a\} \cup K(\xi)) \subset L_\pi$. Let $f$ be an injection such that $f``\mathcal{H}_\Lambda(\pi) \subset L_\pi$. Then for $\forall \alpha \in K(\xi)(f(\alpha) \in f``\mathcal{H}_\alpha(\pi))$, $\pi \in Mh_k^a(\xi)$ iff for any $f(\vec{\nu}) = (f(\nu_k), \ldots, f(\nu_{N-1}))$, each of $f(\nu_i) \in L_\pi$, if $\forall \alpha \in K(\vec{\nu})(f(\alpha) \in f``\mathcal{H}_a(\pi))$ and $\vec{\nu} < \xi$, then $\pi \in M_k(Mh_k^a(\vec{\nu}))$, where $f``\mathcal{H}_a(\pi) \subset L_\pi$ is a class in L_π.
2.8.3. We show the following $B(a)$ is progressive in $a < \Lambda$:

$$B(a) :\Leftrightarrow \mathbb{K} \in Mh_{N-1}^\alpha(a) \cap M_{N-1}(Mh_{N-1}^\alpha(a)).$$

Note that $a \in \mathcal{H}_a(\mathbb{K})$ holds for any $a < \Lambda$.
Suppose $\forall b < a\, B(b)$. We have to show that $Mh_{N-1}^\alpha(a)$ is Π_{N-3}^1-indescribable in $\mathbb{K}$. It is easy to see that if $\pi \in M_{N-1}(Mh_{N-1}^\alpha(a))$, then $\pi \in Mh_{N-1}^\alpha(a)$ by induction on π. Let $\theta(u)$ be a Π_{N-3}^1-formula such that $L_\mathbb{K} \models \theta(u)$.

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By IH we have $\forall b < a[\mathbb{K} \in M_{N-1}(Mh^{\alpha}_{N-1}(b))]$. In other words, $\mathbb{K} \in Mh^{\alpha}_{N-1}(a)$, i.e., $L_{\mathbb{K}} \models mh^{\alpha}_{N-1}(a)$, where $mh^{\alpha}_{N-1}(a)$ is a Π^1_{N-2}-sentence in Proposition 2.8.2. Since the universe $L_{\mathbb{K}}$ is Π^1_{N-2}-indescribable, pick a $\pi < \mathbb{K}$ such that L_π enjoys the Π^1_{N-2}-sentence $\theta(u) \wedge mh^{\alpha}_{N-1}(a)$, and $\{f(\alpha), f(a)\} \subset L_\pi$. Therefore $\pi \in Mh^{\alpha}_{N-1}(a)$ and $L_\pi \models \theta(u)$. Thus $\mathbb{K} \in M_{N-1}(Mh^{\alpha}_{N-1}(a))$. ⊣

2.2. Normal forms in ordinal notations. In this subsection we introduce an *irreducibility* of sequences, which is needed to define a normal form in ordinal notations.

PROPOSITION 2.9. $\pi \in Mh^a_k(\zeta) \,\&\, \xi \leq \zeta \Rightarrow \pi \in Mh^a_k(\xi)$.

PROOF. (2) for $\pi \in Mh^a_k(\xi)$ in Definition 2.5.2 follows from $\pi \in Mh^a_k(\zeta)$ and Proposition 2.3. ⊣

LEMMA 2.10. (Cf. Lemma 3 in [2].) *Assume* $\mathbb{K} \geq \pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ *with* $2 \leq k \leq N-1$, $he(\mu) \leq \xi_0$ *and* $K(\mu) \subset \mathcal{H}_a(\pi)$. *Then* $\pi \in Mh^a_k(\xi + \mu)$ *holds. Moreover if* $\pi \in M_{k+1}$, *then* $\pi \in M_{k+1}(Mh^a_k(\xi + \mu))$ *holds.*

PROOF. Suppose $\pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ and $K(\mu) \subset \mathcal{H}_a(\pi)$ with $he(\mu) \leq \xi_0$. We show $\pi \in Mh^a_k(\xi + \mu)$ by induction on ordinals μ. First note that if $b \in \mathcal{H}_a(\pi)$, then $f(b) \in f``\mathcal{H}_\Lambda(\pi) \subset L_\pi$. We have $K(\xi + \mu) \subset \mathcal{H}_a(\pi)$. $\pi \in M_{k+1}(Mh^a_k(\xi + \mu))$ follows from $\pi \in Mh^a_k(\xi + \mu)$ and $\pi \in M_{k+1}$.

Let $(\zeta) * \vec{\nu} < \xi + \mu$ and $K(\zeta) \cup K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$ for $\vec{\nu} = (\nu_0, \dots, \nu_{n-1})$. We need to show that $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$. By Definition 2.1.6, let $(\zeta_0) * (\mu_0, \dots, \mu_{n-1})$ be a sequence such that $\zeta < \zeta_0 \leq_{pt} \xi + \mu$, $\mu_0 \leq_{pt} te(\zeta_0)$, $\forall i \leq n-1(\nu_i < \mu_i)$, and $\forall i < n-1(\mu_{i+1} \leq_{pt} te(\mu_i))$.

If $\zeta_0 \leq_{pt} \xi$, then $(\zeta) * \vec{\nu} < \xi$, and $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$ by $\pi \in Mh^a_k(\xi)$.

Let $\zeta_0 = \xi + \zeta_1$ with $0 < \zeta_1 \leq_{pt} \mu$. If $\zeta_1 <_{pt} \mu$, then by IH with $he(\zeta_1) = he(\mu)$ we have $\pi \in Mh^a_k(\zeta_0)$. On the other hand we have $(\zeta) * \vec{\nu} < \zeta_0$. Hence $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$.

Finally consider the case when $0 < \zeta_1 = \mu$. Then we obtain $\vec{\nu} < te(\xi + \mu) = te(\mu) \leq he(\mu) \leq \xi_0$. $\pi \in Mh^a_{k+1}(\xi_0)$ with Proposition 2.9 yields $\pi \in M_{k+1}(Mh^a_{k+1}(\vec{\nu}))$.

On the other side we see $\pi \in Mh^a_k(\zeta)$ as follows. We have $\zeta < \xi + \mu$. If $\zeta \leq \xi$, then this follows from $\pi \in Mh^a_k(\xi)$ and Proposition 2.9, and if $\zeta = \xi + \lambda < \xi + \mu$, then IH yields $\pi \in Mh^a_k(\zeta)$.

Since $\pi \in Mh^a_k(\zeta)$ is a Π^1_{k-1}-sentence holding on L_π by Lemma 2.8.2 and $\{a\} \cup K(\zeta) \subset \mathcal{H}_a(\pi)$, we obtain $\pi \in M_{k+1}(Mh^a_k((\zeta) * \vec{\nu}))$, a fortiori $\pi \in M_k(Mh^a_k((\zeta) * \vec{\nu}))$. ⊣

DEFINITION 2.11. For sequences of ordinals $\vec{\xi} = (\xi_k, \dots, \xi_{N-1})$ and $\vec{\nu} = (\nu_k, \dots, \nu_{N-1})$ and $2 \leq k, m, n \leq N-1$,

$$Mh^a_m(\vec{\nu}) \prec_k Mh^a_n(\vec{\xi}) :\Leftrightarrow \forall \pi \in Mh^a_n(\vec{\xi})(\{a, \pi\} \cup K(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh^a_m(\vec{\nu}))).$$

COROLLARY 2.12. *Let* $\vec{\nu}$ *be a sequence defined from a sequence* $\vec{\xi}$ *as follows.* $\forall i < k(\nu_i = \xi_i)$, $\forall i > k(\nu_i = 0)$, *and* $\nu_k = \xi_k + \Lambda^{\xi_{k+1}} b$, *where* $2 \leq k < N$, $b < \Lambda$ *and* $\xi_{k+1} \neq 0$. *Then* $Mh^a_2(\vec{\nu}) \prec_{k+1} Mh^a_2(\vec{\xi})$ *holds. In particular if* $\pi \in Mh^a_2(\vec{\xi})$ *and* $a \geq b \in \mathcal{H}_a(\pi)$, *then* $\psi^{\vec{\nu}}_\pi(a) < \pi$.

PROOF. $Mh^a_2(\vec{\nu}) \prec_{k+1} Mh^a_2(\vec{\xi})$ is seen from Lemma 2.10.

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Suppose $\pi \in Mh_2^a(\vec{\xi})$ and $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$. The set $C = \{\kappa < \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, K(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)\}$ is a club subset of the regular cardinal π. This shows the existence of a $\kappa \in Mh_2^a(\vec{\nu}) \cap C \cap \pi$, and hence $\psi_\pi^{\vec{\nu}}(a) < \pi$ by the definition (3). ⊣

Proposition 2.13. *Let $\vec{\nu} = (\nu_2, \dots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \dots, \xi_{N-1})$ be sequences of ordinals $< \varepsilon(\Lambda)$ such that $\vec{\nu} <_k \vec{\xi}$ for an integer k with $2 \leq k \leq N-1$. Then $Mh_2^a(\vec{\nu}) \prec_k Mh_2^a(\vec{\xi})$. In particular if $\pi \in Mh_2^a(\vec{\xi})$, $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$, and $\max K(\vec{\nu}) \leq a$, then $\psi_\pi^{\vec{\nu}}(a) < \pi$.*

Proof. Assume $\pi \in Mh_2^a(\vec{\xi})$ and $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$. We have $\pi \in Mh_k^a(\xi_k)$. By the definition (2) and $(\nu_k, \dots, \nu_{N-1}) < \xi_k$, we obtain $\pi \in M_k(\bigcap_{k \leq i \leq N-1} Mh_i^a(\nu_i))$.

On the other hand we have $\pi \in \bigcap_{i<k} Mh_i^a(\xi_i)$, and hence $\pi \in \bigcap_{i<k} Mh_i^a(\nu_i)$ by $\forall i < k(\nu_i \leq \xi_i)$ and Proposition 2.9. Since $\pi \in \bigcap_{i<k} Mh_i^a(\nu_i)$ is a Π^1_{k-2}-sentence holding in L_π, we obtain $\pi \in M_k(\bigcap_{i \leq N-1} Mh_i^a(\nu_i)) = M_k(Mh_2^a(\vec{\nu}))$, a fortiori $\pi \in M_2(Mh_2^a(\vec{\nu}))$.

If $\pi \in Mh_2^a(\vec{\xi})$, $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$, and $\max K(\vec{\nu}) \leq a$, then $\psi_\pi^{\vec{\nu}}(a) < \pi$ is seen as in Corollary 2.12. ⊣

Proposition 2.14. *Let $\vec{\xi} = (\xi_2, \dots, \xi_{N-1})$ be a sequence of ordinals $< \varepsilon(\Lambda)$ such that $\{\pi, a\} \cup K(\vec{\xi}) \subset \mathcal{H}_a(\pi)$. Assume $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$ for some $i < N-1$ and $k > 0$. Then $\pi \in Mh_2^a(\vec{\xi}) \Leftrightarrow \pi \in Mh_2^a(\vec{\mu})$, where $\vec{\mu} = (\mu_2, \dots, \mu_{N-1})$ with $\mu_i = \xi_i - Tl(\xi_i)$ and $\mu_j = \xi_j$ for $j \neq i$.*

Proof. When $0 < \xi_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1 + \Lambda^{\gamma_0} a_0$ with $\gamma_m > \cdots > \gamma_1 > \gamma_0$, $0 < a_i < \Lambda$, $\mu_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1$ for $Tl(\xi_i) = \Lambda^{\gamma_0} a_0$. If $\xi_i = 0$, then so is $\mu_i = 0$.

Let $\pi \in Mh_2^a(\vec{\mu})$ and $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$. We obtain $\forall j \leq k(he^{(j)}(Tl(\xi_i)) < \Lambda_{k-j}(\xi_{i+k}+1))$, and $he^{(k)}(Tl(\xi_i)) \leq \xi_{i+k}$. On the other hand we have $\pi \in Mh_{i+k}^a(\xi_{i+k})$. From Lemma 2.10 we see inductively that for any $j < k$, $\pi \in Mh_{i+j}^a(he^{(j)}(Tl(\xi_i)))$. In particular $\pi \in Mh_{i+1}^a(he(Tl(\xi_i)))$, and once again by Lemma 2.10 and $\pi \in Mh_i^a(\mu_i)$ we obtain $\pi \in Mh_i^a(\xi_i)$. Hence $\pi \in Mh_2^a(\vec{\xi})$. ⊣

Definition 2.15. A sequence of ordinals $\vec{\xi} = (\xi_2, \dots, \xi_{N-1})$ is said to be *irreducible* iff $\forall i < N-1 \forall k > 0(\xi_i > 0 \Rightarrow Tl(\xi_i) \geq \Lambda_k(\xi_{i+k}+1))$.

Proposition 2.16. *Let $\vec{\nu} = (\nu_k, \dots, \nu_{N-1}) \neq \vec{0}$ be an irreducible sequence, and $k_0 \geq k$ be the least number such that $\nu_{k_0} \neq 0$. Assume $\nu_{k_0} < he^{(k_0-k)}(\xi)$. Then $\vec{\nu} < \xi$ holds in the sense of* Definition 2.1.6.

Proof. Let $\ell < N-k$ be the largest number such that $\nu_{k+\ell} \neq 0$. We show $(\nu_k, \dots, \nu_{k+\ell}) < \xi$. Since $\vec{\nu}$ is irreducible, we have $\Lambda_i(\nu_{k_0+i}+1) \leq Tl(\nu_{k_0})$. From $\nu_{k_0} < he^{(k_0-k)}(\xi)$ and $te(\mu) \leq he(\mu)$ we obtain $\nu_{k_0+i} < \nu_{k_0+i}+1 \leq he^{(i)}(\nu_{k_0}) \leq he^{(k_0-k+i)}(\xi)$. Let $(\mu_k, \dots, \mu_{N-1}) \subset_{pt} \xi$ such that $\mu_k = Hd(\xi)$ and $\mu_{i+1} = he(\mu_i) = te(Hd(\mu_i))$. Then $te(\mu_{k+i}) = he(\mu_{k+i})$ and $\mu_{k_0+i} = he(\mu_{k_0+i-1}) = he^{(k_0-k+i)}(\xi)$ for $k_0 - k + i > 0$. Therefore $(\mu_k, \dots, \mu_{k+\ell}) \subset_{pt} \xi$ witnesses $(\nu_k, \dots, \nu_{k+\ell}) < \xi$. ⊣

Definition 2.17. Let $\vec{\xi} = (\xi_k, \dots, \xi_{N-1})$, $\vec{\nu} = (\nu_k, \dots, \nu_{N-1})$ and $\vec{\nu} \neq \vec{\xi}$. Let $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. Suppose $(\xi_i, \dots, \xi_{N-1}) \neq \vec{0}$, and let $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:

1. $(\nu_i, \dots, \nu_{N-1}) = \vec{0}$.

2. In what follows assume $(\nu_i, \dots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0\,(i = \min\{k_0, k_1\})$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:
   (a) $i = k_0 < k_1$ and $he^{(k_1-k_0)}(\nu_{k_0}) \leq \xi_{k_1}$.
   (b) $k_0 \geq k_1 = i$ and $\nu_{k_0} < he^{(k_0-k_1)}(\xi_{k_1})$.

Proposition 2.18. *Suppose that both of $\vec{\nu}$ and $\vec{\xi}$ are irreducible. Then $\vec{\nu} <_{lx,k} \vec{\xi} \Rightarrow Mh^a_k(\vec{\nu}) \prec_k Mh^a_k(\vec{\xi})$.*

Proof. Let $\pi \in Mh^a_k(\vec{\xi})$, $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$, and $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. We have $\pi \in \bigcap_{k \leq j < i} Mh^a_j(\nu_j)$, which is a Π^1_{i-2}-sentence holding on L_π. In the case $\xi_i \neq 0$, it suffices to show that $\pi \in M_i(\bigcap_{j \geq i} Mh^a_j(\nu_j))$, since then we obtain $\pi \in M_i(Mh^a_k(\vec{\nu}))$ by $\pi \in Mh^a_i(\xi_i) \subset M_i$, a fortiori $\pi \in M_k(Mh^a_k(\vec{\nu}))$.

If $(\nu_i, \dots, \nu_{N-1}) = \vec{0}$, then $\xi_i \neq 0$ and $\bigcap_{j \geq i} Mh^a_j(\nu_j)$ denotes the class of limit ordinals. Obviously $\pi \in M_i(\bigcap_{j \geq i} Mh^a_j(\nu_j))$.

In what follows assume $(\nu_i, \dots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$, and $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$.

Case 1. $k_0 \geq k_1 = i$: Then we have $\nu_{k_0} < he^{(k_0-k_1)}(\xi_{k_1})$. Proposition 2.16 yields $(\nu_{k_0}, \dots, \nu_{N-1}) < \xi_{k_1} = \xi_i$, which in turn yields $\pi \in M_i(\bigcap_{j \geq i} Mh^a_j(\nu_j))$ by the definition (2) of $\pi \in Mh^a_i(\xi_i)$.

Case 2. $i = k_0 < k_1$: Then we have $he^{(k_1-i)}(\nu_i) \leq \xi_{k_1}$. Also $\nu_{i+p} < he^{(p)}(\nu_i)$ for any $p > 0$ since $\vec{\nu}$ is irreducible and $\nu_i \neq 0$. Let $j \geq k_1$. Then $\nu_j < he^{(j-i)}(\nu_i) \leq he^{(j-k_1)}(\xi_{k_1})$. In particular $\nu_{k_1} < \xi_{k_1}$. Proposition 2.16 yields $(\nu_{k_1}, \dots, \nu_{N-1}) < \xi_{k_1}$. $\pi \in Mh^a_{k_1}(\xi_{k_1})$ yields $\pi \in M_{k_1}(\bigcap_{j \geq k_1} Mh^a_j(\nu_j))$. Moreover for any $p < k_1 - i$, $he^{(k_1-i-p)}(\nu_{i+p}) \leq \xi_{k_1}$ by Proposition 2.2. Lemma 2.10 yields $\pi \in \bigcap_{k_1 > j \geq i} Mh^a_j(\nu_j)$. Therefore $\pi \in M_{k_1}(Mh^a_k(\vec{\nu}))$, a fortiori $\pi \in M_k(Mh^a_k(\vec{\nu}))$. ⊣

Proposition 2.19. (Cf. Proposition 4.20 in [9])
Let $\vec{\nu} = (\nu_2, \dots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \dots, \xi_{N-1})$ be irreducible sequences of ordinals $< \varepsilon(\Lambda)$. Assume that $\psi^{\vec{\nu}}_\pi(b) < \pi$ with $K(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_b(\psi^{\vec{\nu}}_\pi(b))$ and $\max K(\vec{\nu}) \leq b$. Also assume that $\psi^{\vec{\xi}}_\kappa(a) < \kappa$ with $K(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a))$ and $\max K(\vec{\xi}) \leq a$.

Then $\beta_1 = \psi^{\vec{\nu}}_\pi(b) < \psi^{\vec{\xi}}_\kappa(a) = \alpha_1$ iff one of the following cases holds:

1. $\pi \leq \psi^{\vec{\xi}}_\kappa(a)$.
2. $b < a$, $\psi^{\vec{\nu}}_\pi(b) < \kappa$ *and* $K(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a))$.
3. $b > a$ *and* $K(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\psi^{\vec{\nu}}_\pi(b))$.
4. $b = a$, $\kappa < \pi$ *and* $\kappa \notin \mathcal{H}_b(\psi^{\vec{\nu}}_\pi(b))$.
5. $b = a$, $\pi = \kappa$, $K(\vec{\nu}) \subset \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a))$, *and* $\vec{\nu} <_{lx,2} \vec{\xi}$.
6. $b = a$, $\pi = \kappa$, $K(\vec{\xi}) \not\subset \mathcal{H}_b(\psi^{\vec{\nu}}_\pi(b))$.

Proof. If the case (2) holds, then $\psi^{\vec{\nu}}_\pi(b) \in \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a)) \cap \kappa \subset \psi^{\vec{\xi}}_\kappa(a)$.

If one of the cases (3) and (4) holds, then $K(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_a(\psi^{\vec{\nu}}_\pi(b))$. On the other hand we have $K(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a))$. Hence $\psi^{\vec{\nu}}_\pi(b) < \psi^{\vec{\xi}}_\kappa(a)$.

If the case (5) holds, then Proposition 2.18 yields $Mh^a_2(\vec{\nu}) \prec_2 Mh^a_2(\vec{\xi}) \ni \psi^{\vec{\xi}}_\kappa(a)$. Hence $\psi^{\vec{\xi}}_\kappa(a) \in M_2(Mh^a_2(\vec{\nu}))$. Since $K(\vec{\nu}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi^{\vec{\xi}}_\kappa(a))$, the set $\{\rho < \psi^{\vec{\xi}}_\kappa(a) :$

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$\mathcal{H}_a(\rho)\cap\kappa\subset\rho, K(\vec{\nu})\cup\{\kappa,a\}\subset\mathcal{H}_a(\rho)\}$ is club in $\psi_\kappa^{\vec{\xi}}(a)$. Therefore $\psi_\pi^{\vec{\nu}}(b)=\psi_\kappa^{\vec{\nu}}(a)<\psi_\kappa^{\vec{\xi}}(a)$ by (3) in Definition 2.5.3.

Finally assume that the case (6) holds. Since $K(\vec{\xi})\subset\mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, $\psi_\pi^{\vec{\nu}}(b)<\psi_\kappa^{\vec{\xi}}(a)$ holds.

Conversely assume that $\psi_\pi^{\vec{\nu}}(b)<\psi_\kappa^{\vec{\xi}}(a)$ and $\psi_\kappa^{\vec{\xi}}(a)<\pi$.

First consider the case $b<a$. Then we have $K(\vec{\nu})\cup\{\pi,b\}\subset\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))\subset\mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$. Hence (2) holds.

Next consider the case $b>a$. $K(\vec{\xi})\cup\{\kappa,a\}\subset\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$ would yield $\psi_\kappa^{\vec{\xi}}(a)\in\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))\cap\pi\subset\psi_\pi^{\vec{\nu}}(b)$, a contradiction $\psi_\kappa^{\vec{\xi}}(a)<\psi_\pi^{\vec{\nu}}(b)$. Hence (3) holds.

Finally assume $b=a$. Consider the case $\kappa<\pi$. $\kappa\in\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))\cap\pi$ would yield $\psi_\kappa^{\vec{\xi}}(a)<\kappa<\psi_\pi^{\vec{\nu}}(b)$, a contradiction. Hence $\kappa\not\in\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$, and (4) holds. If $\pi<\kappa$, then $\pi\in\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))\cap\kappa\subset\mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))\cap\kappa$, and $\pi<\psi_\kappa^{\vec{\xi}}(a)$, a contradiction, or we should say that (1) holds. Finally let $\pi=\kappa$. We can assume that $K(\vec{\xi})\subset\mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$, otherwise (6) holds. If $\vec{\xi}<_{lx,2}\vec{\nu}$, then by (5) $\psi_\kappa^{\vec{\xi}}(a)<\psi_\pi^{\vec{\nu}}(b)$ would follow. If $K(\vec{\nu})\not\subset\mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, then by (6) again $\psi_\kappa^{\vec{\xi}}(a)<\psi_\pi^{\vec{\nu}}(b)$ would follow. Hence $K(\vec{\nu})\subset\mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$ and $\vec{\nu}\leq_{lx,2}\vec{\xi}$. If $\vec{\nu}=\vec{\xi}$, then $\psi_\kappa^{\vec{\xi}}(a)=\psi_\pi^{\vec{\nu}}(b)$. Therefore (5) must be the case. ⊣

Definition 2.20 is utilized to define a computable notation system in the next section 3.

Definition 2.20. A set SD of sequences $\vec{\xi}=(\xi_2,\dots,\xi_{N-1})$ of ordinals $\xi_i<\varepsilon(\Lambda)$ is defined recursively as follows.

1. $\vec{0}*(a)\in SD$ for each $a<\Lambda$.
2. (Cf.Definition 2.1.9.) Let $\vec{\xi}=(\xi_2,\dots,\xi_{N-1})\in SD$, $1\leq k<N-1$, $\zeta<\varepsilon(\Lambda)$ be an ordinal such that $(\xi_{k+1},\dots,\xi_{N-1})<_{sd}\zeta$, and $(\xi_2,\dots,\xi_{k-1},\xi_k,\zeta)*\vec{0}\in SD$. Then for $\zeta_k=\xi_k+\Lambda^\zeta a$ with an ordinal $a<\Lambda$, $(\xi_2,\dots,\xi_{k-1})*(\zeta_k)*(\xi_{k+1},\dots,\xi_{N-1})\in SD$ and $(\xi_2,\dots,\xi_{k-1})*(\zeta_k)*\vec{0}\in SD$.

Proposition 2.21. *Let* $\vec{\xi}=(\xi_2,\dots,\xi_{N-1})\in SD$.

1. $(\xi_2,\dots,\xi_i)*\vec{0}\in SD$ *for each* i *with* $1\leq i<N$.
2. *For* $2\leq i<j<k<N$, *if* $\xi_i\neq 0$ *and* $\xi_k\neq 0$, *then* $\xi_j\neq 0$.
3. *Let* $\xi_i\neq 0$. *Then* $(\xi_{i+1},\dots,\xi_{N-1})<_{sd}te(\xi_i)$.
4. $\vec{\xi}$ *is irreducible.*

Proof. Let $1\leq k<N-1$, $\zeta<\varepsilon(\Lambda)$ be an ordinal such that $(\xi_{k+1},\dots,\xi_{N-1})<_{sd}\zeta$, and $(\xi_2,\dots,\xi_{k-1},\xi_k,\zeta)*\vec{0}\in SD$. Also let $\zeta_k=\xi_k+\Lambda^\zeta a$ with an ordinal $a<\Lambda$.

2.21.1 is seen by induction on the recursive definition of $\vec{\xi}\in SD$.

2.21.2 is seen by induction on the recursive definition of $\vec{\xi}\in SD$. Suppose $\xi_i\neq 0$ for an $i<k$. From $(\xi_2,\dots,\xi_{k-1},\xi_k,\zeta)*\vec{0}\in SD$ and $\zeta\neq 0$, IH yields $\xi_k\neq 0$.

2.21.3 and 2.21.4. We show these by simultaneous induction on the recursive definition of $\vec{\xi}\in SD$.

2.21.3. We show Proposition 2.21.3 for the sequence $(\xi_2,\dots,\xi_{k-1})*(\zeta_k)*(\xi_{k+1},\dots,\xi_{N-1})\in SD$. The proposition holds for the sequence $\vec{\xi}$, and we can assume $a\neq 0$. We obtain $(\xi_{i+1},\dots,\xi_{N-1})<_{sd}te(\xi_i)$ for $i>k$ if $\xi_i\neq 0$, and $(\xi_{k+1},\dots,\xi_{N-1})<_{sd}te(\zeta_k)=\zeta$ by the assumption. Let $2\leq i<k$ and $\xi_i\neq 0$. We show $(\xi_{i+1},\dots,\xi_{k-1})*(\zeta_k)*(\xi_{k+1},\dots,\xi_{N-1})<_{sd}te(\xi_i)$. It suffices to show that $\zeta_k<_{sd}te^{(k-i)}(\xi_i)$. By IH we have

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$\xi_k <_{sd} te^{(k-i)}(\xi_i)$. On the other hand we have $\xi_k \neq 0$ by $(\xi_2,\dots,\xi_{k-1},\xi_k,\zeta) * \vec{0} \in SD$, $\zeta \neq 0$, and Proposition 2.21.2. Moreover $(\xi_2,\dots,\xi_{k-1},\xi_k,\zeta) * \vec{0}$ is irreducible by Proposition 2.21.4, and hence $Tl(\xi_k) \geq \Lambda^{\zeta+1}$. Therefore $te(\xi_k) > \zeta$. This means that $\zeta_k =_{NF} \xi_k + \Lambda^{\zeta} a$, and $\xi_k <_{sd} te^{(k-i)}(\xi_i)$ yields $\zeta_k <_{sd} te^{(k-i)}(\xi_i)$ by Definition 2.1.8.

2.21.4. If $(\xi_{i+1},\dots,\xi_{N-1}) <_{sd} te(\xi_i)$ for $\xi_i \neq 0$, then $\xi_{i+k} <_{sd} te^{(k)}(\xi_i)$ for $k > 0$, and $\xi_{i+k} + 1 \leq te^{(k)}(\xi_i)$. Hence $\Lambda_k(\xi_{i+k}+1) \leq \Lambda^{te(\xi_i)} \leq Tl(\xi_i)$, and $\vec{\xi}$ is irreducible. ⊣

§3. Computable notation system *OT*. In this section (except Proposition 3.6 and Lemma 3.8) we work in a weak fragment of arithmetic, e.g., in the fragment $I\Sigma_1$ or even in the bounded arithmetic S^1_2. Referring to Proposition 2.19 the sets of ordinal terms $OT \subset \Lambda = \varepsilon_{\mathbb{K}+1}$ and $E \subset \varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$ over symbols $\{0,\mathbb{K},\Lambda,+,\varphi,\Omega,\psi\}$ are defined recursively. OT is isomorphic to a subset of $\mathcal{H}_\Lambda(0)$. Simultaneously we define finite sets $K_\delta(\alpha) \subset OT$ for $\delta,\alpha \in OT$, and sequences $(m_k(\alpha))_{2\leq k\leq N-1}$ for $\alpha \in OT \cap \mathbb{K}$, where in $\alpha = \psi_\pi^{\vec{\nu}}(a)$, $m_k(\alpha) = \nu_k$, i.e., $\vec{\nu} = (\nu_2,\dots,\nu_{N-1}) = (m_2(\alpha),\dots,m_{N-1}(\alpha)) = (m_k(\alpha))_k = \vec{m}(\alpha)$. For $\{\alpha_0,\dots,\alpha_m,\beta\} \subset OT$, $K_\delta(\alpha_0,\dots,\alpha_m) := \bigcup_{i\leq m} K_\delta(\alpha_i)$, $K_\delta(\alpha_0,\dots,\alpha_m) < \beta$ iff $\forall\gamma \in K_\delta(\alpha_0,\dots,\alpha_m)(\gamma < \beta)$, and $\beta \leq K_\delta(\alpha_0,\dots,\alpha_m)$ iff $\exists\gamma \in K_\delta(\alpha_0,\dots,\alpha_m)(\beta \leq \gamma)$.

First let us define a set $\overline{OT}$ of terms over symbols $\{0,\mathbb{K},\Lambda,+,\varphi,\Omega,\psi\}$. Second a relation $\alpha < \beta$ on $\overline{OT}$ is defined. Third a subset $OT \subset \overline{OT}$ is defined to be the set of terms in normal form. The relation $\alpha < \beta$ on OT is defined to be the restriction of $\alpha < \beta$ to OT.

Definition 3.1. Sets $\overline{OT}$ and $\overline{E}$ of terms are defined simultaneously. Also a finite set $K(\nu) \subset \overline{OT}$ is defined for $\nu \in \overline{E}$, and subsets SC, P of $\overline{OT}$ are defined. SC [P] is intended to be the set of strongly critical numbers [the set of additive principal numbers], resp.

1. (a) $0 \in \overline{E}$.
   (b) If $0 \neq a \in \overline{OT}$, then $a \in \overline{E}$. $K(a) = \{a\}$.
   (c) If $\{\xi_i : i \leq m\} \subset \overline{E}$ and $0 \neq b_i \in \overline{OT}$, then $\sum_{i\leq m} \Lambda^{\xi_i} b_i = \Lambda^{\xi_0} b_0 + \cdots + \Lambda^{\xi_m} b_m \in \overline{E}$. $K(\sum_{i\leq m} \Lambda^{\xi_i} b_i) = \{b_i : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$.
   (d) For sequences $\vec{\nu} = (\nu_2,\dots,\nu_{N-1})$, let $K(\vec{\nu}) = \bigcup_{2\leq i\leq N-1} K(\nu_i)$.
2. (a) $SC \subset P \subset \overline{OT}$.
   (b) $0 \in \overline{OT}$, and $\mathbb{K} \in SC$.
   (c) If $\{\alpha_i : i \leq m\} \subset P\,(m > 0)$, then $\alpha_0 + \cdots + \alpha_m \in \overline{OT}$.
   (d) If $\{\beta,\gamma\} \subset \overline{OT}$, then $\varphi\beta\gamma \in P$.
   (e) If $\beta \in \overline{OT}$, then $\Omega_\beta \in SC$.
   (f) Let $\pi \in SC$, $a \in \overline{OT}$ and $\vec{\nu} = (\nu_2,\dots,\nu_{N-1})$ be a sequence of terms $\nu_i \in \overline{E}$. Then $\psi_\pi^{\vec{\nu}}(a) \in SC$.

Definition 3.2. A finite set $K_\delta(\alpha) \subset \overline{OT}$ for $\delta,\alpha \in \overline{OT}$, a relation $\alpha < \beta$ on $\overline{OT}$, and a relation $\nu < \xi$ on $\overline{E}$ are defined simultaneuously as follows. $\alpha \leq \beta :\Leftrightarrow \alpha < \beta \vee \alpha = \beta$.

1. $K_\delta(0) = K_\delta(\mathbb{K}) = \emptyset$ $K_\delta(\alpha_0 + \cdots + \alpha_m) = K_\delta(\alpha_0,\dots,\alpha_m)$. $K_\delta(\varphi\beta\gamma) = K_\delta(\beta,\gamma)$. $K_\delta(\omega^\beta) = K_\delta(\beta)$. $K_\delta(\Omega_\beta) = K_\delta(\beta)$. $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \emptyset$ if $\psi_\pi^{\vec{\nu}}(a) < \delta$. Otherwise $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \{a\} \cup K_\delta(a,\pi) \cup \bigcup\{K_\delta(b) : b \in K(\vec{\nu})\}$.

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2. (a) $0 < \alpha$ for $0 \neq \alpha \in \overline{OT}$.
   (b) $\alpha_0 + \cdots + \alpha_m < \beta_0 + \cdots + \beta_n$ iff either there exists a $p \leq \min\{n,m\}$ such that $\forall i < p(\alpha_i = \beta_i)$ and $\alpha_p < \beta_p$, or $n < m$ and $\forall i \leq n(\alpha_i = \beta_i)$. For $\beta \in P$, $\alpha_0 + \cdots + \alpha_m < \beta$ iff $\forall i \leq m(\alpha_i < \beta)$, and $\beta < \alpha_0 + \cdots + \alpha_m$ iff $\exists i \leq m(\beta \leq \alpha_i)$.
   (c) $\alpha_0 = \varphi\beta_0\gamma_0 < \varphi\beta_1\gamma_1 = \alpha_1$ iff $\beta_0 < \beta_1 \,\&\, \gamma_0 < \alpha_1$ or $\beta_0 = \beta_1 \,\&\, \gamma_0 < \gamma_1$ or $\beta_1 < \beta_0 \,\&\, \alpha_0 < \gamma_1$.For $\alpha \in SC$, $\varphi\beta\gamma < \alpha$ iff $\beta, \gamma < \alpha$, and $\alpha < \varphi\beta\gamma$ iff $\alpha \leq \beta \vee \alpha \leq \gamma$.
   (d) $\Omega_\alpha < \Omega_\beta$ iff $\alpha < \beta$. $\Omega_\alpha < \psi_\pi^{\vec{\nu}}(a)$ iff $\alpha < \psi_\pi^{\vec{\nu}}(a)$, and $\psi_\pi^{\vec{\nu}}(a) < \Omega_\alpha$ iff $\psi_\pi^{\vec{\nu}}(a) \leq \alpha$.
   (e) $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$ iff one of the following cases holds:
      (i) $\pi \leq \psi_\kappa^{\vec{\xi}}(a)$.
      (ii) $b < a$, $\psi_\pi^{\vec{\nu}}(b) < \kappa$, and $K_{\psi_\kappa^{\vec{\xi}}(a)}(\{\pi, b\} \cup K(\vec{\nu})) < a$.
      (iii) $b \geq a$, and $b \leq K_{\psi_\pi^{\vec{\nu}}(b)}(\{\kappa, a\} \cup K(\vec{\xi}))$.
      (iv) $b = a$, $\pi = \kappa$, $K_{\psi_\kappa^{\vec{\xi}}(a)}(K(\vec{\nu})) < a$, and $\vec{\nu} <_{lx,2} \vec{\xi}$, cf.Definition 2.17.
3. $\nu < \xi$ for $\nu, \xi \in \overline{E}$ is defined. $0 < a < \sum_{i \leq m} \Lambda^{\xi_i} b_i$. $\sum_{i \leq n} \Lambda^{\nu_i} a_i < \sum_{i \leq m} \Lambda^{\xi_i} b_i$ iff either there exists a $p \leq \min\{n,m\}$ such that $\forall i < p[(\nu_i, a_i) = (\xi_i, b_i)]$ and $(\nu_p, a_p) < (\xi_p, b_p)$ lexicographically, or $n < m$ and $\forall i \leq n[(\nu_i, a_i) = (\xi_i, b_i)]$.

Proposition 3.3. *$(\overline{OT}, <)$ is a computable linear ordering.*

Proof. $\ell\alpha$ denotes the number of occurrences of symbols $\{0, \mathbb{K}, \Lambda, +, \varphi, \Omega, \psi\}$ in terms $\alpha \in \overline{OT} \cup \overline{E}$. Note that $\ell\beta < \ell\alpha$ for any $\beta \in K_\delta(\alpha)$. It is clear that both $\alpha \in \overline{OT}$ and $\alpha < \beta$ are decidable for terms over symbols $\{0, \mathbb{K}, \Lambda, +, \varphi, \Omega, \psi\}$.

For $\alpha, \beta, \gamma \in \overline{OT}$, $\alpha \not< \alpha$, $\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha$, and $\alpha < \beta < \gamma \Rightarrow \alpha < \gamma$ are seen simultaneously by induction on $\ell\alpha + \ell\beta + \ell\gamma$ as in [1]. ⊣

An ordinal term is said to be a *regular* term if it is one of the form $\mathbb{K}$, $\Omega_{\beta+1}$ or $\psi_\pi^{\vec{\nu}}(a)$ with a nonzero sequence $\vec{\nu} \neq \vec{0}$. $\mathbb{K}$ and the latter terms $\psi_\pi^{\vec{\nu}}(a)$ are *Mahlo* terms.

$\alpha =_{NF} \alpha_0 + \cdots + \alpha_m$ means that $\alpha = \alpha_0 + \cdots + \alpha_m$, $\alpha_0 \geq \cdots \geq \alpha_m$, and each α_i is a nonzero additive principal number. $\alpha =_{NF} \varphi\beta\gamma$ means that $\alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$. $\alpha =_{NF} \omega^\beta$ means that $\alpha = \omega^\beta > \beta$. $\alpha =_{NF} \Omega_\beta$ means that $\alpha = \Omega_\beta > \beta$.

Let $pd(\psi_\pi^{\vec{\nu}}(a)) = \pi$ (even if $\vec{\nu} = \vec{0}$). Moreover for n, $pd^{(n)}(\alpha)$ is defined recursively by $pd^{(0)}(\alpha) = \alpha$ and $pd^{(n+1)}(\alpha) \simeq pd(pd^{(n)}(\alpha))$.

For terms π, κ, $\pi \prec \kappa$ denotes the transitive closure of the relation $\{(\pi, \kappa) : \exists\vec{\xi}\exists b[\pi = \psi_\kappa^{\vec{\xi}}(b)]\}$, and its reflexive closure $\pi \preceq \kappa :\Leftrightarrow \pi \prec \kappa \vee \pi = \kappa \Leftrightarrow \exists n(\kappa = pd^{(n)}(\pi))$.

For each ordinal term $\alpha = \psi_\pi^{\vec{\nu}}(a)$, a series $(\pi_i)_{i \leq L}$ of ordinal terms is uniquely determined as follows: $\pi_L = \alpha$, $\pi_i = pd(\pi_{i+1})$ and $\pi_0 = \mathbb{K}$. Let us call the series $(\pi_i)_{i \leq L}$ the *collapsing series* of $\alpha = \pi_L$.

Then we see that an ordinal term $\alpha = \psi_\pi^{\vec{\nu}}(a)$ with $\vec{\nu} \neq \vec{0}$ is constructed by Definition 3.4.2g below iff $L = 1$. α is constructed by Definition 3.4.2i iff $L \equiv 1(\text{mod } (N-2))$. Otherwise α is constructed by Definition 3.4.2h.

Definition 3.4. Subsets $OT \subset \overline{OT}$ and $E \subset \overline{E}$ are defined recursively as follows. Also we define sequences $(m_k(\alpha))_{2 \leq k \leq N-1}$ for $\alpha \in OT \cap \mathbb{K}$.

1. (a) $0 \in E$.
   (b) If $0 \neq a \in OT$, then $a \in E$.

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(c) If $\{\xi_i : i \leq m\} \subset E$, $\xi_0 > \cdots > \xi_m \neq 0$ and $0 \neq b_i \in OT$, then $\sum_{i\leq m} \Lambda^{\xi_i} b_i = \Lambda^{\xi_0} b_0 + \cdots + \Lambda^{\xi_m} b_m \in E$.

2. (a) $0, \mathbb{K} \in OT$. $m_k(0) = 0$ for any k.
(b) If $\alpha =_{NF} \alpha_m + \cdots + \alpha_0\,(m > 0)$ with $\{\alpha_i : i \leq m\} \subset OT$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k.
(c) If $\alpha =_{NF} \varphi\beta\gamma$ with $\{\beta, \gamma\} \subset OT \cap \mathbb{K}$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k.
(d) If $\alpha =_{NF} \omega^\beta :\equiv \varphi 0 \beta$ with $\mathbb{K} < \beta \in OT$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k.
(e) If $\alpha =_{NF} \Omega_\beta$ with $\beta \in OT \cap \mathbb{K}$, then $\alpha \in OT$. $m_2(\alpha) = 1, m_k(\alpha) = 0$ for any $k > 2$ if β is a successor ordinal. Otherwise $m_k(\alpha) = 0$ for any k.
(f) Let $\alpha = \psi_\pi(a) := \psi_\pi^{\vec{0}}(a)$ with $\pi, a \in OT$ where π is a regular term, i.e., either $\pi = \mathbb{K}$ or $\vec{m}(\pi) \neq \vec{0}$, and $K_\alpha(\pi, a) < a$.Then $\alpha = \psi_\pi(a) \in OT$. Let $m_k(\alpha) = 0$ for any k.
(g) Let $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a)$ with $\vec{\nu} = \vec{0} * (b)\,(lh(\vec{\nu}) = N-2)$ and $b, a \in OT$ such that $0 < b \leq a$ and $K_\alpha(b, a) < a$.Then $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a) \in OT$. Let $m_{N-1}(\alpha) = b$, $m_k(\alpha) = 0$ for $k < N-1$.
(h) Let $\pi \in OT \cap \mathbb{K}$ be such that $m_{k+1}(\pi) \neq 0$ and $\forall i > k+1 (m_i(\pi) = 0)$ for a $k\,(2 \leq k \leq N-2)$, and $b, a \in OT$ such that $0 \leq b \leq a$. Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ be a sequence defined by $\forall i < k(\nu_i = m_i(\pi))$, $\nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)} b$, and $\forall i > k(\nu_i = 0)$.Then $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$ if $K_\alpha(\pi, a, b) \cup K_\alpha(K(\vec{m}(\pi))) < a$. Let $m_i(\alpha) = \nu_i$ for each i.
(i) Let $\pi \in OT \cap \mathbb{K}$ be such that $m_2(\pi) \neq 0$ and $\forall i > 2(m_i(\pi) = 0)$, and $a \in OT$. Let $\vec{0} \neq \vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) \in SD$ be a sequence of ordinal terms $\nu_i \in E$ such that $\vec{\nu} <_{sp} m_2(\pi)$.Then $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$ if $K_\alpha(\pi, a) < a$ and

$$\forall k (K_\alpha(\nu_k) < \max K(\nu_k)). \tag{4}$$

Let $m_i(\alpha) = \nu_i$ for each i.

Let $\{\pi, a, \xi\} \subset \mathcal{H}_a(\pi)$. Then $\xi = m_k(\pi)$ is intended to be equivalent to $\pi \in Mh_k^a(\xi)$.

Proposition 3.5. *For each Mahlo term $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, $\vec{m}(\alpha) = \vec{\nu} \in SD$ for the class SD in Definition 2.20.*

Proposition 3.6. *For any $\alpha \in OT$ and any δ such that $\delta = 0, \mathbb{K}$ or $\delta = \psi_\pi^{\vec{\nu}}(b)$ for some $\pi, b, \vec{\nu}$, $\alpha \in \mathcal{H}_\gamma(\delta) \Leftrightarrow K_\delta(\alpha) < \gamma$.*

Proof. By induction on $\ell\alpha$. ⊣

Proposition 3.7. 1. *Let $\beta = \psi_\pi^{\vec{\nu}}(b)$ with $\pi = \psi_\kappa^{\vec{\xi}}(a)$. Then $a < b$.*
2. *For $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, $\max K(\vec{\nu}) \leq a$ holds.*

Proof. 3.7.1. Let $\beta = \psi_\pi^{\vec{\nu}}(b)$ with $\pi = \psi_\kappa^{\vec{\xi}}(a)$. Then $K_\beta(\{\pi, b\} \cup K(\vec{\nu})) < b$. On the other hand we have $\beta < \pi$. Hence $a \in K_\beta(\pi) < b$.

3.7.2. This is seen by induction on $\ell\alpha$. We have $c < a$ by Proposition 3.7.1 when $\pi = \psi_\sigma^{\vec{\mu}}(c)$. When α is constructed by Definition 3.4.2h, $\nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)} b$ holds for $b \leq a$. By IH we have $\max K(\vec{m}(\pi)) \leq c < a$ when $\pi = \psi_\sigma^{\vec{\mu}}(c)$.

Suppose α is constructed by Definition 3.4.2i. We obtain $\vec{\nu} <_{sp} m_2(\pi)$, and hence $\max K(\vec{\nu}) \leq \max K(m_2(\pi)) \leq c < a$ by IH. ⊣

LEMMA 3.8. *OT is isomorphic to a subset of $\mathcal{H}_\Lambda(0)$. $(OT,<)$ is a computable notation system for ordinals. In particular the order type of the initial segment $\{\alpha \in OT : \alpha < \Omega_1\}$ is less than ω_1^{CK}.*

PROOF. This is seen referring to Propositions 2.19, 3.6, and 3.7.2. To see $\psi_\pi^{\vec{\nu}}(a) < \pi$ in Definition 3.4.2h, see Corollary 2.12, and in Definition 3.4.2i, see Proposition 2.13. ⊣

§4. Operator controlled derivations.

In this section, operator controlled derivations are defined, which are introduced by Buchholz [7].

In this and the next sections except otherwise stated $\alpha,\beta,\gamma,\ldots,a,b,c,d,\ldots$ range over ordinal terms in $OT \subset \mathcal{H}_\Lambda(0)$, $\xi,\zeta,\nu,\mu,\iota,\ldots$ range over ordinal terms in E, $\vec{\xi},\vec{\zeta},\vec{\nu},\vec{\mu},\vec{\iota},\ldots$ range over finite sequences over ordinal terms in E, and $\pi,\kappa,\rho,\sigma,\tau,\lambda,\ldots$ range over regular ordinal terms $\mathbb{K}$, $\Omega_{\beta+1}$, $\psi_\pi^{\vec{\nu}}(a)$ with $\vec{\nu} \neq \vec{0}$. *Reg* denotes the set of regular ordinal terms. We write $\alpha \in \mathcal{H}_a(\beta)$ for $K_\beta(\alpha) < a$, cf. Proposition 3.6.

4.1. Classes of sentences.

Following Buchholz [7] let us introduce a language for ramified set theory RS.

DEFINITION 4.1. *RS-terms* and their *levels* are inductively defined.

1. For each $\alpha \in OT \cap \mathbb{K}$, L_α is an RS-term of level α.
2. If $\phi(x,y_1,\ldots,y_n)$ is a set-theoretic formula in the language $\{\in\}$, and $a_1,\ldots,a_n$ are RS-terms of levels $< \alpha$, then $[x \in L_\alpha : \phi^{L_\alpha}(x,a_1,\ldots,a_n)]$ is an RS-term of level α.

Each ordinal term α is denoted by the ordinal term $[x \in L_\alpha : x \text{ is an ordinal}]$, whose level is α.

DEFINITION 4.2.
1. $|a|$ denotes the level of RS-terms a, and $Tm(\alpha)$ the set of RS-terms of level $< \alpha$. $Tm = Tm(\mathbb{K})$ is then the set of RS-terms, which are denoted by $a,b,c,d,\ldots$
2. *RS-formulas* are constructed from *literals* $a \in b, a \notin b$ by propositional connectives $\vee,\wedge$, bounded quantifiers $\exists x \in a, \forall x \in a$ and unbounded quantifiers $\exists x, \forall x$. Unbounded quantifiers $\exists x,\forall x$ are denoted by $\exists x \in L_\mathbb{K}, \forall x \in L_\mathbb{K}$, resp.
3. For RS-terms and RS-formulas ι, $\mathsf{k}(\iota)$ denotes the set of ordinal terms α such that the constant L_α occurs in ι.
4. For a set-theoretic Σ_n-formula $\psi(x_1,\ldots,x_m)$ in $\{\in\}$ and $a_1,\ldots,a_m \in Tm(\kappa)$, $\psi^{L_\kappa}(a_1,\ldots,a_m)$ is a *$\Sigma_n(\kappa)$-formula*, where $n = 0,1,2,\ldots$ and $\kappa \leq \mathbb{K}$. $\Pi_n(\kappa)$-formulas are defined dually.
5. For $\theta \equiv \psi^{L_\kappa}(a_1,\ldots,a_m) \in \Sigma_n(\kappa)$ and $\lambda < \kappa$, $\theta^{(\lambda,\kappa)} :\equiv \psi^{L_\lambda}(a_1,\ldots,a_m)$.

Note that the level $|t| = \max(\{0\} \cup \mathsf{k}(t))$ for RS-terms t. In what follows we need to consider *sentences*. Sentences are denoted A, C possibly with indices.

The assignment of disjunctions and conjunctions to sentences is defined as in [7].

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DEFINITION 4.3. 1. For $b, a \in Tm(\mathbb{K})$ with $|b| < |a|$,

$$(b \, \varepsilon \, a) :\equiv \begin{cases} A(b) & \text{if } a \equiv [x \in L_\alpha : A(x)], \\ b \notin L_0 & \text{if } a \equiv L_\alpha \end{cases}$$

and $(a = b) :\equiv (\forall x \in a(x \in b) \wedge \forall x \in b(x \in a))$.

2. For $b, a \in Tm(\mathbb{K})$ and $J := Tm(|a|)$

$$(b \in a) :\simeq \bigvee (c \varepsilon a \wedge c = b)_{c \in J} \text{ and } (b \notin a) :\simeq \bigwedge (c \not\varepsilon a \vee c \neq b)_{c \in J}.$$

3. $(A_0 \vee A_1) :\simeq \bigvee (A_\iota)_{\iota \in J}$ and $(A_0 \wedge A_1) :\simeq \bigwedge (A_\iota)_{\iota \in J}$ for $J := 2$.
4. For $a \in Tm(\mathbb{K}) \cup \{L_\mathbb{K}\}$ and $J := Tm(|a|)$

$$\exists x \in a \, A(x) :\simeq \bigvee (b \varepsilon a \wedge A(b))_{b \in J} \text{ and } \forall x \in a \, A(x) :\simeq \bigwedge (b \not\varepsilon a \vee A(b))_{b \in J}.$$

The rank $\text{rk}(\iota)$ of sentences or terms ι is defined as in [7].

DEFINITION 4.4. 1. $\text{rk}(\neg A) := \text{rk}(A)$.
2. $\text{rk}(L_\alpha) = \omega\alpha$.
3. $\text{rk}([x \in L_\alpha : A(x)]) = \max\{\omega\alpha + 1, \text{rk}(A(L_0)) + 2\}$.
4. $\text{rk}(a \in b) = \max\{\text{rk}(a) + 6, \text{rk}(b) + 1\}$.
5. $\text{rk}(A_0 \vee A_1) := \max\{\text{rk}(A_0), \text{rk}(A_1)\} + 1$.
6. $\text{rk}(\exists x \in a \, A(x)) := \max\{\omega \text{rk}(a), \text{rk}(A(L_0)) + 2\}$ for $a \in Tm(\mathbb{K}) \cup \{L_\mathbb{K}\}$.

PROPOSITION 4.5. *Let A be a sentence with $A \simeq \bigvee (A_\iota)_{\iota \in J}$ or $A \simeq \bigwedge (A_\iota)_{\iota \in J}$.*

1. $\text{rk}(A) < \mathbb{K} + \omega$.
2. $|A| \leq \text{rk}(A) \in \{\omega |A| + i : i \in \omega\}$.
3. $\forall \iota \in J(\text{rk}(A_\iota) < \text{rk}(A))$.
4. $\text{rk}(A) < \lambda \Rightarrow A \in \Sigma_0(\lambda)$

4.2. Operator controlled derivations. By an *operator* we mean a map $\mathcal{H}$, $\mathcal{H} : \mathcal{P}(OT) \to \mathcal{P}(OT)$, such that

1. $\forall X \subset OT[X \subset \mathcal{H}(X)]$.
2. $\forall X, Y \subset OT[Y \subset \mathcal{H}(X) \Rightarrow \mathcal{H}(Y) \subset \mathcal{H}(X)]$.

For an operator $\mathcal{H}$ and $\Theta, \Theta_1 \subset OT$, $\mathcal{H}[\Theta](X) := \mathcal{H}(X \cup \Theta)$, and $\mathcal{H}[\Theta][\Theta_1] := (\mathcal{H}[\Theta])[\Theta_1]$, i.e., $\mathcal{H}[\Theta][\Theta_1](X) = \mathcal{H}(X \cup \Theta \cup \Theta_1)$.

Obviously $\mathcal{H}_\alpha$ in Definition 2.5.1 is an operator for any α, and if $\mathcal{H}$ is an operator, then so is $\mathcal{H}[\Theta]$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus. Let $\mathcal{H} = \mathcal{H}_\gamma \, (\gamma \in OT)$ be an operator, Θ a finite set of $\mathbb{K}$, Γ a sequent, $a \in OT$ and $b \in OT \cap (\mathbb{K} + \omega)$.

We define a relation $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$, which is read 'there exists an infinitary derivation of Γ which is Θ-*controlled* by $\mathcal{H}_\gamma$, and whose height is at most a and its cut rank is less than b'.

$\kappa, \lambda, \sigma, \tau, \pi$ ranges over regular ordinal terms.

DEFINITION 4.6. $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$ holds if

$$\mathsf{k}(\Gamma) \cup \{a\} \subset \mathcal{H}_\gamma[\Theta] \tag{5}$$

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and one of the following cases holds:

($\bigvee$) $A \simeq \bigvee\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and there exist $\iota \in J$ and $a(\iota) < a$ such that

$$|\iota| < a \tag{6}$$

and $(\mathcal{H}_\gamma, \Theta) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

($\bigwedge$) : $A \simeq \bigwedge\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for every $\iota \in J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \{\mathsf{k}(\iota)\}) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

(cut) : There exist $a_0 < a$ and C such that $\mathrm{rk}(C) < b$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b C, \Gamma$.

$(\Omega \in M_2)$: There exist ordinals a_ℓ, $a_r(\alpha)$ and a sentence $C \in \Pi_2(\Omega)$ such that $\sup\{a_\ell + 1, a_r(\alpha) + 1 : \alpha < \Omega\} \leq a$, $b \geq \Omega$, $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_b \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta \cup \{\omega\alpha\}) \vdash^{a_r(\alpha)}_b \neg C^{(\alpha,\Omega)}, \Gamma$ for any $\alpha < \Omega$.

$(\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{\nu}))$: There exist a Mahlo ordinal $\mathbb{K} \geq \pi \in \mathcal{H}_\gamma[\Theta] \cap (b+1)$, an integer $2 \leq k \leq N$ and sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}), \vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) \in SD$ of ordinals $\nu_i, \xi_i \in E$, ordinals $a_\ell, a_r(\rho), a_0$, and a finite set Δ of $\Sigma_k(\pi)$-sentences enjoying the following conditions: When $\pi = \mathbb{K}$, $k = N$ and $\vec{\nu} = \vec{0}$ with $lh(\vec{\nu}) = N - 1$ hold. Also let $\vec{\xi} = \vec{0}$ in this case. When $\pi < \mathbb{K}$, $\xi_k \neq 0$ with $k < N$, $\vec{0} \neq \vec{\xi}$, and $\forall i(\xi_i \leq_{sp} m_i(\pi))$.

1. When $\pi < \mathbb{K}$, cf.Definitions 2.1.9,

$$\forall i < k(\nu_i = \xi_i) \,\&\, (\nu_k, \ldots, \nu_{N-1}) <_{sd} \xi_k \,\&\, K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta] \tag{7}$$

and

$$\forall \mu \in \vec{\nu} \cup \vec{\xi} \cup \vec{m}(\pi)(K(\mu) \subset \mathcal{H}_{\max K(\mu)}[\Theta]) \tag{8}$$

cf.(4).

2. For each $\delta \in \Delta$, $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_b \Gamma, \neg\delta$.
3. $H(\vec{\nu}, \pi, \gamma, \Theta)$ denotes the *resolvent class* for $\vec{\nu}$, π, γ and Θ defined as follows:

$$C(\pi, \gamma, \Theta) := \{\rho < \pi : \mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho\} \tag{9}$$
$$\rho \in H(\vec{\nu}, \pi, \gamma, \Theta) :\Leftrightarrow \forall i(\nu_i \leq_{sp} m_i(\rho) \wedge K(m_i(\rho)) \subset \mathcal{H}_{\max K(m_i(\rho))}(\rho))$$

for $\rho \in Reg \cap C(\pi, \gamma, \Theta)$.

Then for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$, $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_b \Gamma, \Delta^{(\rho,\pi)}$.

4.

$$\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{\nu}, \pi, \gamma, \Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta] \cap a. \tag{10}$$

In the inference rule $(\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{\nu}))$ for $\pi = \psi^{\vec{\xi}}_\sigma(c) < \mathbb{K}$, we have $\pi \in Mh^c_2(\vec{\xi})$. In particular, $\pi \in \bigcap_{i<k} Mh^c_i(\xi_i) \cap Mh^c_k(\xi_k)$. Also we are assuming $(\nu_k, \ldots, \nu_{N-1}) <_{sd} \xi_k$, a fortiori $(\nu_k, \ldots, \nu_{N-1}) < \xi_k$. Since $\pi \in \bigcap_{i<k} Mh^c_i(\nu_i)$ is a Π_k-sentence holding on L_π, we obtain $\pi \in M_k(Mh^c_2(\vec{\nu}))$. Thus the reflection rule $(\mathrm{rfl}(\pi, k, \vec{\nu}))$ says that π is Π_k-reflecting on the class $H(\vec{\nu}, \pi, \gamma, \gamma_0, \Theta)$ for the club subset $C(\pi, \gamma, \Theta)$ of π, cf. Proposition 2.13. On the other side we see $\rho \in Mh^a_2(\vec{\nu})$ from Proposition 2.9 if $\forall i(\nu_i \leq m_i(\rho))$ for $\rho \in Mh^a_2(\vec{m}(\rho))$.

We will state some lemmas for the operator controlled derivations. These can be shown as in [7]. In what follows by an operator $\mathcal{H}$ we mean an $\mathcal{H}_\gamma$ for an ordinal γ.

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LEMMA 4.7. *Let* $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$.

1. $(\mathcal{H}_{\gamma'}, \Theta \cup \Theta_0) \vdash^{a'}_{b'} \Gamma, \Delta$ *for any* $\gamma' \geq \gamma$, *any* Θ_0, *and any* $a' \geq a$, $b' \geq b$ *such that* $\mathsf{k}(\Delta) \cup \{a'\} \subset \mathcal{H}_{\gamma'}[\Theta \cup \Theta_0]$.
2. *Assume* $\Theta_1 \cup \{c\} = \Theta$, $c \in \mathcal{H}_\gamma[\Theta_1]$. *Then* $(\mathcal{H}_\gamma, \Theta_1) \vdash^a_b \Gamma$.

LEMMA 4.8 (Tautology). $(\mathcal{H}, \mathsf{k}(\Gamma \cup \{A\})) \vdash^{2rk(A)}_0 \Gamma, \neg A, A$.

LEMMA 4.9 (Inversion). *Let* $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, *and* $(\mathcal{H}, \Theta) \vdash^a_b \Gamma$ *with* $A \in \Gamma$. *Then for any* $\iota \in J$, $(\mathcal{H}, \Theta \cup \mathsf{k}(\iota)) \vdash^a_b \Gamma, A_\iota$ *holds.*

LEMMA 4.10 (Boundedness). *Suppose* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C$ *for a* $C \in \Sigma_1(\lambda)$, *and* $a \leq b \in \mathcal{H} \cap \lambda$. *Then* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C^{(b,\lambda)}$.

LEMMA 4.11 (Persistency). *Suppose* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C^{(b,\lambda)}$ *for a* $C \in \Sigma_1(\lambda)$ *and a* $b < \lambda \in \mathcal{H}[\Theta]$. *Then* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C$.

LEMMA 4.12 (Predicative Cut-elimination). *Suppose* $(\mathcal{H}, \Theta) \vdash^b_{c+\omega^a} \Gamma$, $a \in \mathcal{H}[\Theta]$ *and* $]c, c+\omega^a] \cap Reg = \emptyset$. *Then* $(\mathcal{H}, \Theta) \vdash^{\varphi ab}_c \Gamma$.

LEMMA 4.13 (Embedding of Axioms).
For each axiom A *in* $\mathsf{KP}\Pi_N$, *there is an* $m < \omega$ *such that for any operator* $\mathcal{H} = \mathcal{H}_\gamma$, $(\mathcal{H}, \emptyset) \vdash^{\mathbb{K}\cdot 2}_{\mathbb{K}+m} A$ *holds.*

PROOF. The axiom $\neg A, \exists z\, A^{(z)}$ for Π_N-reflection follows from $A, \neg A$ and $\exists z\, A^{(z)}, \neg A^{(\rho)}$ for regular ordinals $\rho < \mathbb{K}$ by an inference $(\mathrm{rfl}(\mathbb{K}, N, \vec{0}, \vec{0}))$. ⊣

LEMMA 4.14 (Embedding). *If* $\mathsf{KP}\Pi_N \vdash \Gamma$ *for a set* Γ *of sentences, there are* $m, k < \omega$ *such that for any operator* $\mathcal{H} = \mathcal{H}_\gamma$, $(\mathcal{H}, \emptyset) \vdash^{\mathbb{K}\cdot 2+k}_{\mathbb{K}+m} \Gamma$ *holds*

§5. Lowering and eliminating higher Mahlo operations. In this section inferences $(\mathrm{rfl}(\mathbb{K}, N, \vec{0}, \vec{0}))$ for Π_N-reflecting ordinals $\mathbb{K}$ are eliminated from operator controlled derivations of Σ_1-sentences φ^{L_Ω} over Ω.

$\alpha \# \beta$ denotes the natural (commutative) sum of ordinal terms α, β.

LEMMA 5.1. *For a Mahlo term* $\pi \in OT$, $\vec{\xi} \in SD$ *denotes a sequence with* $lh(\vec{\xi}) = N-2$, *and* $2 \leq k \leq N-1$ *an integer for which the following hold*: *When* $\pi = \mathbb{K}$, *let* $\vec{\xi} = \vec{0}$ *and* $k = N-1$. $\vec{\xi} = (\xi_2, \dots, \xi_{k+1}) * \vec{0}$ *with* $\xi_{k+1} \neq 0$ *such that* $\forall i \leq k+1 (\xi_i \leq_{sp} m_i(\pi))$.

For ordinal terms $\gamma, a \in OT$ *let us define a sequence* $\vec{\zeta}(a) := (\zeta_2(a), \dots, \zeta_k(a)) * \vec{0}$ *with* $lh(\vec{\zeta}(a)) = N-2$ *as follows.* $\vec{\zeta}(a) = \vec{0} * (\gamma + a)$ *when* $\pi = \mathbb{K}$. *Otherwise* $\zeta_k(a) = \xi_k + \Lambda^{\xi_{k+1}}(\gamma + a)$ *and* $\zeta_i(a) = \xi_i$ *for* $i < k$.

Let $\kappa \in H(\vec{\zeta}(a), \pi, \gamma, \Theta)$ *for a finite set* $\Theta \subset OT$.

Now suppose $(\mathcal{H}_\gamma, \Theta) \vdash^a_\pi \Gamma$ *where* $\{\gamma, \pi\} \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, $\Theta \subset \pi$, $\forall i (K(\xi_i) \subset \mathcal{H}_{\max K(\xi_i)}[\Theta])$, *and* $\Gamma \subset \Pi_{k+1}(\pi)$.

Let $\gamma(a,b) = \gamma \# a \# b$, $\beta(a,b) = \psi_\pi(\gamma(a,b))$, *and* $c > \gamma(a,\kappa)$. *Then the following holds*:

$$(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a,\kappa)}_\kappa \Gamma^{(\kappa,\pi)}. \tag{11}$$

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PROOF. By induction on a. Let $\kappa \in H(\vec{\zeta}(a),\pi,\gamma,\Theta)$. We see $\vec{\zeta}(a) \in SD$, and from (5) and $\Theta \subset \kappa$ that

$$\mathsf{k}(\Gamma) \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa. \tag{12}$$

For any $a \in \mathcal{H}_\gamma[\Theta]$, we obtain $\{\gamma,\pi,a,\kappa\} \subset \mathcal{H}_\gamma(\pi)$ by $\Theta \cup \{\kappa\} \subset \pi$. Hence for $\gamma(a,\kappa) = \gamma\#a\#\kappa$, $\{\gamma(a,\kappa),\pi\} \subset \mathcal{H}_\gamma(\pi)$, and $\{\gamma(a,\kappa),\pi\} \subset \mathcal{H}_{\gamma(a,\kappa)}(\beta(a,\kappa))$ by the definition (3). Therefore $\kappa \in \mathcal{H}_{\gamma(a,\kappa)}(\beta(a,\kappa)) \cap \pi \subset \beta(a,\kappa)$ by Proposition 2.6, and $\Theta \subset \beta(a,\kappa) < \pi$. Thus we obtain

$$\{a_0,a_1\} \subset \mathcal{H}_\gamma[\Theta \cup \Theta_0] \,\&\, a_0 < a_1 \,\&\, \Theta_0 \subset \kappa \Rightarrow \beta(a_0,\kappa) < \beta(a_1,\kappa).$$

CASE 1. First consider the case when the last inference is a $(\mathrm{rfl}(\pi,k+1,\vec{\xi},\vec{\nu}))$.

We have $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$, and a finite set Δ of $\Sigma_{k+1}(\pi)$-sentences. We have for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma,\Theta) \vdash^{a_\ell}_{\pi} \Gamma, \neg\delta \tag{13}$$

and for each $\rho \in H(\vec{\nu},\pi,\gamma,\Theta)$

$$(\mathcal{H}_\gamma,\Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)}. \tag{14}$$

When $\pi < \mathbb{K}$, $\vec{\nu} = (\nu_2,\dots,\nu_{N-1}) \in SD$ is a sequence such that $\forall i < k+1(\nu_i = \xi_i)$, $(\nu_{k+1},\dots,\nu_{N-1}) <_{sd} \xi_{k+1}$, $K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, and $\forall i(K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}[\Theta])$, cf. (7) and (8).

Let $\Gamma_0 = \Gamma \cap \Sigma_k(\pi)$ and $\{\forall x \in L_\pi\, \theta_i(x) : i = 1,\dots,n\}\, (n \geq 0) = \Gamma \setminus \Gamma_0$ for $\Sigma_k(\pi)$-formulas $\theta_i(x)$. Let us fix $\vec{d} = \{d_1,\dots,d_n\} \subset Tm(\kappa)$ arbitrarily. Put $\mathsf{k}(\vec{d}) = \bigcup\{\mathsf{k}(d_i) : i = 1,\dots,n\}$ and $\Gamma(\vec{d}) = \Gamma_0 \cup \{\theta_i(d_i) : i = 1,\dots,n\}$.

By Inversion lemma 4.9 from (13) we obtain for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma,\Theta \cup \mathsf{k}(\vec{d})) \vdash^{a_\ell}_{\pi} \Gamma(\vec{d}), \neg\delta. \tag{15}$$

Let $\rho \in C(\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. We see $\rho < \kappa$, and $\mathsf{k}(\vec{d}) < \rho$ from $\mathsf{k}(\vec{d}) < \kappa$. By $\Theta \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$ and $\gamma \leq c$ we obtain $C(\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \subset C(\pi,\gamma,\Theta)$. Namely, cf. (9)

$$\rho \in H(\vec{\nu},\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \Rightarrow \rho \in H(\vec{\nu},\pi,\gamma,\Theta). \tag{16}$$

For each $\rho \in H(\vec{\nu},\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$, IH with (14) and (16) yields for $c > \gamma(a_r(\rho),\kappa)$ and $\kappa \in H(\vec{\zeta}(a_r(\rho)),\pi,\gamma,\Theta \cup \{\rho\})$

$$(\mathcal{H}_c,\Theta \cup \{\rho,\kappa\}) \vdash^{\beta(a_r(\rho),\kappa)}_{\kappa} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}. \tag{17}$$

Let $\rho \in M_\ell := \{\rho \in Reg : \forall i(\zeta_i(a_\ell) \leq_{sp} m_i(\rho))\} \cap H(\vec{\nu},\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. Then $M_\ell \subset H(\vec{\zeta}(a_\ell),\pi,\gamma,\Theta \cup \mathsf{k}(\vec{d}))$ and $\Theta \cup \mathsf{k}(\vec{d}) \subset \rho$. For each $\delta \in \Delta$, IH with (15) yields for $c > \gamma(a_\ell,\rho)$

$$(\mathcal{H}_c,\Theta \cup \mathsf{k}(\vec{d}) \cup \{\rho\}) \vdash^{\beta(a_\ell,\rho)}_{\rho} \Gamma(\vec{d})^{(\rho,\pi)}, \neg\delta^{(\rho,\pi)}. \tag{18}$$

From (17) and (18) by several (cut)'s of $\delta^{(\rho,\pi)}$ with $\mathrm{rk}(\delta^{(\rho,\pi)}) < \kappa$ we obtain for $a(\rho) = \max\{a_\ell, a_r(\rho)\}$ and some $p < \omega$

$$\{(\mathcal{H}_c,\Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa,\rho\}) \vdash^{\beta(a(\rho),\kappa)+p}_{\kappa} \Gamma(\vec{d})^{(\rho,\pi)}, \Gamma^{(\kappa,\pi)} : \rho \in M_\ell\}. \tag{19}$$

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On the other hand we have by Tautology lemma 4.8 for each $\theta(\vec{d})^{(\kappa,\pi)} \in \Gamma(\vec{d})^{(\kappa,\pi)}$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa\}) \vdash_0^{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)})} \Gamma(\vec{d})^{(\kappa,\pi)}, \neg\theta(\vec{d})^{(\kappa,\pi)} \tag{20}$$

where $2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}) \leq \kappa + p$ for some $p < \omega$.

Moreover we have $\sup\{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}), \beta(a(\rho),\kappa)+p : \rho \in M_\ell\} \leq \beta(a_0,\kappa)+p \in \mathcal{H}_\gamma[\Theta \cup \{\kappa\}]$, where $\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{\nu},\pi,\gamma,\Theta)\} \leq a_0 < a$ by (10).

Now let $\vec{\mu} = (\mu_2,\dots,\mu_{N-1}) = \max\{\vec{\zeta}(a_\ell), \vec{\nu}\}$ with $\mu_i = \max\{\zeta_i(a_\ell), \nu_i\}$. Since $\nu_i = \xi_i \leq_{pt} \zeta_i(a_\ell)$ for $i < k+1$, we obtain $\mu_i = \begin{cases} \zeta_i(a_\ell) & i \leq k, \\ \nu_i & i > k. \end{cases}$ We see that $M_\ell = H(\vec{\mu},\kappa,c,\Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. Moreover we have $\forall i < k(\mu_i = \xi_i = \zeta_i(a))$ and $(\mu_k,\dots,\mu_{N-1}) = (\zeta_k(a_\ell)) * (\nu_{k+1},\dots,\nu_{N-1}) <_{sd} \zeta_k(a)$. Also $\forall i(K(\zeta_i(a)) \subset \mathcal{H}_{\max K(\zeta_i(a))}[\Theta])$ and $\forall i(K(\mu_i) \subset \mathcal{H}_{\max K(\mu_i)}[\Theta])$. For $\neg\Gamma(\vec{d})^{(\kappa,\pi)} \subset \Pi_k(\kappa)$, by an inference rule $(\mathrm{rfl}(\kappa,k,\vec{\zeta}(a),\vec{\mu}))$ with its resolvent class M_ℓ, we conclude from (20) and (19) that $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \vdash_\kappa^{\beta(a_0,\kappa)+p+1} \Gamma(\vec{d})^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}$. Since $\vec{d} \subset Tm(\kappa)$ is arbitrary, several $(\bigwedge)$'s yield (11).

Case 2. Second consider the case when the last inference is a $(\mathrm{rfl}(\pi,j,\vec{\xi},\vec{\nu}))$ for a $j < k+1$. We have $(\mathcal{H}_\gamma,\Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ for each $\delta \in \Delta \subset \Sigma_j(\pi)$ with $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, and $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\pi)}$ for each $\rho \in H(\vec{\nu},\pi,\gamma,\Theta)$ with $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$. $\vec{\nu} \in SD$ is a sequence such that $\forall i < j(\nu_i = \xi_i)$ and $(\nu_j,\dots,\nu_{N-1}) <_{sd} \xi_j$.

We see that the resolvent class $H(\vec{\nu},\kappa,c_1,\Theta \cup \{\kappa\})$ is a subclass of $H(\vec{\nu},\pi,\gamma,\Theta)$. By IH we have $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a_\ell,\kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta^{(\kappa,\pi)}$ for each $\delta \in \Delta$, and $(\mathcal{H}_c, \Theta \cup \{\kappa,\rho\}) \vdash_\kappa^{\beta(a_r(\rho),\kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}$ for each $\rho \in H(\vec{\nu},\kappa,c,\Theta \cup \{\kappa\})$ with $\Delta^{(\rho,\pi)} = (\Delta^{(\kappa,\pi)})^{(\rho,\kappa)}$. We claim that $\forall i \leq j(\xi_j \leq_{sp} m_i(\kappa))$. Consider the case when $i = j = k$. Then we have $\xi_k \leq_{sp} m_k(\pi)$ and $\zeta_k(a) \leq_{sp} m_k(\kappa)$ with $\xi_k <_{pt} \zeta_k(a)$. We obtain $\xi_k \leq_{sp} m_k(\kappa)$. Hence by an inference rule $(\mathrm{rfl}(\kappa,j,\vec{\xi}(j),\vec{\nu}))$ for the sequence $\vec{\xi}(j) = (\xi_2,\dots,\xi_j) * \vec{0} \in SD$, cf. Proposition 2.21.1, we obtain (11).

Case 3. Third consider the case when the last inference is a $(\mathrm{rfl}(\sigma,j,\vec{\mu},\vec{\nu}))$ for a $\sigma < \pi$. We have $(\mathcal{H}_\gamma,\Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ for each $\delta \in \Delta \subset \Sigma_j(\sigma)$, and $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\sigma)}$ for each $\rho \in H(\vec{\nu},\sigma,\gamma,\Theta)$. We obtain $\sigma < \kappa$ by (12) for $\sigma \in \mathcal{H}_\gamma[\Theta]$. Hence $\Delta \subset \Sigma_0^1(\sigma) \subset \Sigma_0(\kappa)$ and $\delta^{(\kappa,\pi)} \equiv \delta$ for any $\delta \in \Delta$. Let $H(\vec{\nu},\sigma,c,\Theta \cup \{\kappa\})$ be the resolvent class for σ, $\vec{\nu}$, c and $\Theta \cup \{\kappa\}$. Then $H(\vec{\nu},\sigma,c,\Theta \cup \{\kappa\}) \subset H(\vec{\nu},\sigma,\gamma,\Theta)$.

From IH we have $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a_\ell,\kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta$ for each $\delta \in \Delta$, and $(\mathcal{H}_c, \Theta \cup \{\kappa,\rho\}) \vdash_\kappa^{\beta(a_r(\rho),\kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\sigma)}$ for each $\rho \in H(\vec{\nu},\sigma,c,\Theta \cup \{\kappa\})$. We obtain (11) by an inference rule $(\mathrm{rfl}(\sigma,j,\vec{\mu},\vec{\nu}))$ with the resolvent class $H(\vec{\nu},\sigma,c,\Theta \cup \{\kappa\})$.

Case 4. Fourth consider the case when the last inference $(\bigwedge)$ introduces a $\Pi_{k+1}(\pi)$-sentence $(\forall x \in L_\pi\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a(d)} \Gamma, \theta(d)$ for each $d \in Tm(\pi)$. For each $d \in Tm(\kappa)$, IH with $\mathsf{k}(d) < \kappa$ yields $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(d)) \vdash_\kappa^{\beta(a(d),\kappa)} \Gamma^{(\kappa,\pi)}, \theta(d)^{(\kappa,\pi)}$. $(\bigwedge)$ yields (11) for $\forall x \in L_\kappa\, \theta(x)^{(\kappa,\pi)} \equiv (\forall x \in L_\pi\, \theta(x))^{(\kappa,\pi)} \in \Gamma^{(\kappa,\pi)}$.

Case 5. Fifth consider the case when the last inference $(\bigwedge)$ introduces a $\Sigma_0(\pi)$-sentence $(\forall x \in c\, \theta(x)) \in \Gamma$ for a $c \in Tm(\pi)$. We have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a(d)} \Gamma, \theta(d)$ for each $d \in Tm(|c|)$. Then we have $|d| < |c| < \kappa$ by (12). IH yields $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(d) \vdash_\kappa^{\beta(a(d),\kappa)} \Gamma^{(\kappa,\pi)}, \theta(d)$, and we obtain (11) by an inference $(\bigwedge)$.

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CASE 6. Sixth consider the case when the last inference $(\bigvee)$ introduces a $\Sigma_k(\pi)$-sentence $(\exists x \in L_\pi\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \theta(d)$ for a $d \in Tm(\pi)$. Without loss of generality we can assume that $\mathsf{k}(d) \subset \mathsf{k}(\theta(d))$. Then we see that $|d| < \kappa$ from (12), and $d \in Tm(\kappa)$. Also $|d| < \kappa < \beta(a,\kappa)$ for (6). IH yields with $(\exists x \in L_\pi\, \theta(x))^{(\kappa,\pi)} \equiv (\exists x \in L_\kappa\, \theta(x)^{(\kappa,\pi)}) \in \Gamma^{(\kappa,\pi)}$, $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_{\kappa} \Gamma^{(\kappa,\pi)}, \theta(d)^{(\kappa,\pi)}$, and we obtain (11) by an inference $(\bigvee)$.

CASE 7. Seventh consider the case when the last inference is a (cut). We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} C, \Gamma$ for $a_0 < a$ with $\mathrm{rk}(C) < \pi$. Then $C \in \Sigma_0(\pi)$ by Proposition 4.5.4. On the other side $\mathsf{k}(C) \subset \pi$ holds by Proposition 4.5.2. Then $\mathsf{k}(C) \subset \kappa$ by (12). Hence $C^{(\kappa,\pi)} \equiv C$ and $\mathrm{rk}(C^{(\kappa,\pi)}) < \kappa$ again by Proposition 4.5.2. IH yields $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_{\kappa} \Gamma^{(\kappa,\pi)}, \neg C^{(\kappa,\pi)}$ and $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_{\kappa} C^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}$. Hence by a (cut) we obtain (11).

CASE 8. Eighth consider the case when the last inference is an $(\Omega \in M_2)$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta \cup \{\omega\alpha\}) \vdash^{a_r(\alpha)}_{\pi} \neg C^{(\alpha,\Omega)}, \Gamma$ for each $\alpha < \Omega$ with $\sup\{a_\ell + 1, a_r(\alpha) + 1 : \alpha < \Omega\} \leq a$ and $C \in \Pi_2(\Omega)$.

We obtain $\omega\alpha < \kappa$ for $\alpha < \Omega$. IH with $C^{(\kappa,\pi)} \equiv C$ yields for each $\alpha < \Omega$, $(\mathcal{H}_c, \Theta \cup \{\kappa, \omega\alpha\}) \vdash^{\beta(a_r(\alpha),\kappa)}_{\kappa} \neg C^{(\alpha,\Omega)}, \Gamma^{(\kappa,\pi)}$, and $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_\ell,\kappa)}_{\kappa} \Gamma^{(\kappa,\pi)}, C$. An $(\Omega \in M_2)$ yields (11)

All other cases are seen easily from IH. ⊣

LEMMA 5.2. *Let* $\lambda \leq \pi$ *be a regular ordinal term such that* $\forall i (K(m_i(\pi)) \subset \mathcal{H}_{\max K(m_i(\pi))}[\Theta]))$, *and* $\Gamma \subset \Sigma_1(\lambda)$.

Suppose for an ordinal term $a \in OT$

$$(\mathcal{H}_\gamma, \Theta) \vdash^{a}_{\pi} \Gamma$$

where $\{\gamma, \lambda, \pi\} \subset \mathcal{H}_\gamma[\Theta]$.

Assume

$$\forall \rho \in [\lambda, \pi] \forall d [\Theta \subset \psi_\rho(\gamma \# d)]. \tag{21}$$

Let $\hat{a} = \gamma \# \omega^{\pi + a + 1}$ *and* $\beta = \psi_\lambda(\hat{a})$. *Then the following holds*

$$(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta}_{\beta} \Gamma. \tag{22}$$

PROOF. By main induction on π with subsidiary induction on a. We can assume $a > 0$.

We see that $\Theta \subset \beta = \psi_\lambda(\hat{a})$ from (21). Hence

$$a_0 \in \mathcal{H}_\gamma[\Theta] \cap a \Rightarrow \psi_\lambda(\widehat{a_0}) < \psi_\lambda(\hat{a})$$

Let $\vec{\xi} \in SD$ be a sequence of ordinals and k a number for which the following hold: If $\pi = \mathbb{K}$, then let $\vec{\xi} = \vec{0}$ with $lh(\vec{\xi}) = N - 1$ and $k = N - 1$. Let $\pi < \mathbb{K}$. If $\vec{m}(\pi) \neq \vec{0}$, then $K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, $\vec{\xi} \leq \vec{m}(\pi)$ and $k = \max\{k \leq N - 2 : \xi_{k+1} > 0\}$. Otherwise let $\vec{\xi} = \vec{0}$ and $k = 1$. By the assumption (21), and (5) we obtain

$$\forall \rho \in [\lambda, \pi] \forall b \in K(\vec{\xi}) \forall d [\mathsf{k}(\Gamma) \cup \{\gamma, \lambda, a, \pi, b\} \subset \mathcal{H}_\gamma(\psi_\rho(\gamma \# d))] \tag{23}$$

CASE 1. First consider the case when $k \geq 2$.

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Let $\vec{\xi} = \vec{m}(\pi)$, and $\vec{\zeta}(a) := (\zeta_2(a), \dots, \zeta_k(a)) * \vec{0}$ be the sequence defined as in Lemma 5.1 from γ, a: $\vec{\zeta}(a) = \vec{0} * (\gamma + a)$ when $\pi = \mathbb{K}$, otherwise $\zeta_k(a) = \xi_k + \Lambda^{\xi_{k+1}}(\gamma + a)$ and $\zeta_i(a) = \xi_i$ for $i < k$. Also let $\gamma(a,b) = \gamma\#a\#b$ and $\beta(a,b) = \psi_\pi\gamma(a,b)$.

Let $\kappa := \psi_\pi^{\vec{\zeta}(a)}(\gamma(a,0))$. By the assumption (21) we have $\Theta \subset \psi_\pi(\gamma\#a)$. On the other hand we have $\psi_\pi(\gamma\#a) = \psi_\pi(\gamma(a,0)) \leq \kappa$, and $\Theta \subset \kappa$. $\pi \in \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \pi$ yields $K(\vec{\xi}) = K(\vec{m}(\pi)) \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\gamma(a,0)}(\kappa)$. Hence $K(\vec{\xi}) \cup \{\pi, \gamma(a,0)\} \subset \mathcal{H}_{\gamma(a,0)}(\kappa)$, and $\kappa \in OT$ by $\gamma(a,0) = \gamma\#a > 0$ and Definition 3.4.2h such that $\kappa < \pi$ and $\mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$. Moreover we have $\forall i(K(\zeta_i(a)) \subset \mathcal{H}_{\max K(\zeta_i(a))}[\Theta])$ by $\forall i(K(m_i(\pi)) \subset \mathcal{H}_{\max K(m_i(\pi))}[\Theta])$ and $\{\gamma, a\} \subset \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \kappa$. In other words, $\kappa \in H(\vec{\zeta}(a), \pi, \gamma, \Theta)$.

By Lemma 5.1 we obtain $(\mathcal{H}_{\gamma(a,\kappa)+1}, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a,\kappa)} \Gamma^{(\kappa,\pi)}$, and Lemma 4.7.2 with $\kappa \in \mathcal{H}_{\gamma(a,0)+1}[\Theta]$

$$(\mathcal{H}_{\gamma(a,\kappa)+1}, \Theta) \vdash_\kappa^{\beta(a,\kappa)} \Gamma^{(\kappa,\pi)}. \tag{24}$$

If $\lambda = \pi$, then $\Gamma^{(\kappa,\pi)} \subset \Sigma_1(\kappa) \subset \Sigma_0(\lambda)$. We have $\Theta \subset \psi_\pi(\hat{a}) = \beta$, and $\kappa \in \mathcal{H}_{\hat{a}}(\beta)$. Hence $\{\gamma, \pi, a, \kappa\} \subset \mathcal{H}_{\hat{a}}(\beta)$, and $\gamma(a,\kappa) = \gamma\#a\#\kappa < \gamma\#\omega^{\pi+a+1} = \hat{a}$. Therefore $\kappa < \beta(a,\kappa) \leq \psi_\pi(\hat{a}) = \beta$. We obtain (22) by Persistency lemma 4.11.

Next consider the case when $\lambda < \pi$. Then $\lambda < \kappa$ and $\Gamma^{(\kappa,\pi)} = \Gamma$. We have for (21), $\forall d \forall \rho \in [\lambda, \kappa](\Theta \subset \psi_\rho(\gamma(a,\kappa) + 1\#d))$. By MIH on (24) we obtain $(\mathcal{H}_{b_0+1}, \Theta) \vdash_{\beta_0}^{\beta_0} \Gamma$ for $\beta_0 = \psi_\lambda(b_0)$ with $b_0 = (\gamma(a,\kappa)+1)\#\omega^{\kappa+\beta(a,\kappa)+1}$. We have $b_0 = \gamma\#a\#\kappa\#1\#\omega^{\beta(a,\kappa)+1} < \gamma\#\omega^{\pi+a+1} = \hat{a}$ by $\beta(a,\kappa) < \pi$. This yields $\psi_\lambda(b_0) = \beta_0 < \beta = \psi_\lambda(\hat{a})$ by $\Theta \subset \beta$ and $\{\gamma, \kappa, \pi, a\} \subset \mathcal{H}_{\hat{a}}(\beta)$. Hence (22) follows.

In what follows suppose $k = 1$.

CASE 2. Consider the case when the last inference rule is a $(\mathrm{rfl}(\pi, 2, \vec{\xi}, \vec{\nu}))$.

We have an ordinal term $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, and a finite set Δ of $\Sigma_2(\pi)$-sentences for which $(\mathcal{H}_\gamma, \Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ holds for each $\delta \in \Delta$. On the other hand we have sequences $\vec{\nu}, (\xi_2) * \vec{0} \in SD$ such that $\vec{\nu} <_{sd} \xi_2$ and $K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$ by (7), and an ordinal term $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$ for which $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\pi)}$ holds for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$, where $\xi_2 \leq_{sp} m_2(\pi)$.

Let $\rho := \psi_\pi^{\vec{\nu}}(\widehat{a_\ell}\#\pi)$ for $\widehat{a_\ell} = \gamma\#\omega^{\pi+a_\ell+1}$. By the assumption (21) we have $\Theta \subset \psi_\pi(\widehat{a_\ell}) \subset \rho$. $K(\vec{\nu}) \cup \{\pi, \gamma, a\} \subset \mathcal{H}_\gamma[\Theta]$ yields $K(\vec{\nu}) \cup \{\pi, \widehat{a_\ell}\} \subset \mathcal{H}_{\widehat{a_\ell}\#\pi}(\rho)$. Next consider the condition (4). We have $\forall i(K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}[\Theta])$ by (8), and hence $\forall i(K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}(\rho))$ by $\Theta \subset \rho$. Therefore $\rho \in OT$ by Definition 3.4.2i. Moreover $\rho \in C(\pi, \gamma, \Theta)$, i.e., $\mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho$. Hence $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$.

By Inversion lemma 4.9 we obtain for each $\delta \equiv (\exists x \in L_\pi \delta_1(x)) \in \Delta$ and each $d \in Tm(\rho)$ with $|d| = \max(\{0\} \cup \mathsf{k}(d))$, $(\mathcal{H}_{\gamma\#|d|}, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a_\ell} \Gamma, \neg\delta_1(d)$.

We have $\{\pi, \gamma, |d|\} \subset \mathcal{H}_{\gamma\#|d|}(\pi)$ by $|d| < \rho < \pi$, and this yields $|d| \in \mathcal{H}_{\gamma\#|d|}(\psi_\pi(\gamma\#|d|)) \cap \pi \subset \psi_\pi(\gamma\#|d|)$. Hence $|d| < \psi_\pi(\gamma\#|d|)$, and $\forall e(\Theta \cup \mathsf{k}(d) \subset \psi_\pi(\gamma\#|d|\#e))$, i.e., (21) holds for $\lambda = \pi$ and $\gamma\#|d|$. Let $\beta_d = \psi_\pi(\widehat{a_d})$ for $\widehat{a_d} = \gamma\#|d|\#\omega^{\pi+a_\ell+1} = \widehat{a_\ell}\#|d|$. SIH yields $(\mathcal{H}_{\widehat{a_d}+1}, \Theta \cup \mathsf{k}(d)) \vdash_{\beta_d}^{\beta_d} \Gamma, \neg\delta_1(d)$, which in turn Boundedness lemma 4.10 yields $(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \mathsf{k}(d)) \vdash_{\beta_d}^{\beta_d} \Gamma, \neg\delta_1^{(\beta_d,\pi)}(d)$ for $\widehat{a_\pi} = \gamma\#\pi\#\omega^{\pi+a_\ell+1} = \widehat{a_\ell}\#\pi$. By persistency we obtain $(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \mathsf{k}(d)) \vdash_\rho^{\beta_d} \Gamma, \neg\delta_1^{(\rho,\pi)}(d)$ for $\beta_d < \psi_\pi(\widehat{a_\pi}) = \rho \in \mathcal{H}_\gamma[\Theta]$. Since $d \in Tm(\rho)$ is arbitrary, $(\bigwedge)$ yields

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta) \vdash_\rho^\rho \Gamma, \neg\delta^{(\rho,\pi)}. \tag{25}$$

Now pick the ρth branch from the right upper sequents

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta \cup \{\rho\} \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)}.$$

By $\rho \in \mathcal{H}_{\widehat{a_\pi}+1}[\Theta]$ and Lemma 4.7.2 we obtain

$$(\mathcal{H}_{\widehat{a_\pi}+1}, \Theta) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\pi)}. \tag{26}$$

CASE 2.1. First consider the case $\lambda = \pi$. Then $\Delta^{(\rho,\pi)} \subset \Sigma_0(\lambda)$. Let $\beta_\rho = \psi_\pi(b_\rho)$ with $b_\rho = \widehat{a_\pi} \# 1 \# \omega^{\pi + a_r(\rho)+1} = \gamma \# \omega^{\pi + a_\ell + 1} \# \omega^{\pi + a_r(\rho)+1} \# \pi \# 1$. Then $\beta_\rho > \rho$ and $\forall d[\Theta \cup \{\rho\} \subset \psi_\pi(\widehat{a_\pi} + 1 \# d)]$. SIH yields for (26)

$$(\mathcal{H}_{b_\rho+1}, \Theta) \vdash^{\beta_\rho}_{\beta_\rho} \Gamma, \Delta^{(\rho,\pi)}. \tag{27}$$

Several (cut)'s with (27), (25) yield $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_\rho + p}_{\beta_\rho} \Gamma$ for $\beta_\rho \geq \rho$, $\widehat{a_\pi} < b_\rho < \hat{a}$ and some $p < \omega$, where $\beta_\rho < \beta = \psi_\pi(\hat{a})$ by $b_\rho < \hat{a}$. (22) follows.

CASE 2.2. Next consider the case when $\lambda < \pi$. Then $\lambda < \rho$ and $\Delta^{(\rho,\pi)} \subset \Sigma_1(\rho^+)$ with $\rho^+ = \Omega_{\rho+1}$. SIH with (26) yields $(\mathcal{H}_{b_\rho+1}, \Theta \cup \{\rho\}) \vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}} \Gamma, \Delta^{(\rho,\pi)}$ for $\beta_{\rho^+} = \psi_{\rho^+}(b_\rho) > \rho$, and by Lemma 4.7.2 we obtain

$$(\mathcal{H}_{b_\rho+1}, \Theta) \vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}} \Gamma, \Delta^{(\rho,\pi)}. \tag{28}$$

Several (cut)'s with (25), (28) yield $(\mathcal{H}_{b_0+1}, \Theta) \vdash^{\beta_{\rho^+}+p}_{\beta_{\rho^+}} \Gamma$ for $\beta_{\rho^+} > \rho$ and $b_0 = \gamma \# (\omega^{\pi + a_\ell + 1} \cdot 2) \# \omega^{\pi + a_r(\rho)+1} \# 1 \geq \max\{b_\ell, b_\rho\}$. Predicative cut-elimination lemma 4.12 yields for $\beta_1 = \varphi(\beta_{\rho^+})(\beta_{\rho^+} + p) < \rho^+$

$$(\mathcal{H}_{b_0+1}, \Theta) \vdash^{\beta_1}_{\rho} \Gamma. \tag{29}$$

We obtain $\lambda < \rho \in \mathcal{H}_{b_0+1}[\Theta]$ by $\gamma < \widehat{a_\ell} < b_0$. MIH with (29) yields $(\mathcal{H}_{c+1}, \Theta) \vdash^{\psi_\lambda c}_{\psi_\lambda c} \Gamma$ for $c = b_0 \# 1 \# \omega^{\rho + \beta_1 + 1}$. We obtain $c = b_0 \# \omega^{\rho + \beta_1 + 1} \# 1 = \gamma \# (\omega^{\pi + a_\ell + 1} \cdot 2) \# \omega^{\pi + a_r(\rho)+1} \# \omega^{\rho + \beta_1 + 1} \# 2 < \gamma \# \omega^{\pi + a + 1} = \hat{a}$ since $a_\ell, a_r(\rho) < a$ and $\rho, \beta_1 < \rho^+ < \pi$. Hence $\psi_\lambda c < \psi_\lambda(\hat{a}) = \beta$, and (22) follows.

CASE 3. Third consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\forall x \in c\, \theta(x)) \in \Gamma$ for $c \in Tm(\lambda)$. We have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash^{a(d)}_{\pi} \Gamma, \theta(d)$ for each $d \in Tm(|c|)$. Then we see from (23) that $|d| < |c| \in \mathcal{H}_\gamma(\psi_\rho(\gamma \# e)) \cap \rho \subset \psi_\rho(\gamma \# e)$ for any $\rho \in [\lambda, \pi]$ and any e. Hence $|d| \in \psi_\rho(\gamma \# e)$. (21) is enjoyed for $\Theta \cup \mathsf{k}(d)$. SIH yields $(\mathcal{H}_{\hat{a}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\beta_d} \Gamma, \theta(d)$ for $\beta_d = \psi_\lambda(\widehat{a(d)})$. $(\bigwedge)$ yields (22) for $\beta = \psi_\lambda(\hat{a}) > \beta_d$.

CASE 4. Fourth consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\exists x \in L_\lambda\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \theta(d)$ for a $d \in Tm(\lambda)$. SIH yields $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \theta(d)$ for $\beta = \psi_\lambda(\hat{a}) > \psi_\lambda(\widehat{a_0}) = \beta_0$. Without loss of generality we can assume that $\mathsf{k}(d) \subset \mathsf{k}(\theta(d))$. Then we see from (23) that [10] $|d| \in \mathcal{H}_\gamma(\psi_\lambda(\gamma + 1)) \cap \lambda \subset \psi_\lambda(\gamma + 1) < \beta$. Thus is enjoyed in the following inference rule $(\bigvee)$. We obtain $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta}_{\beta} \Gamma$ by a $(\bigvee)$, which enjoys (6).

CASE 5. Fifth consider the case when the last inference is a $(\mathrm{rfl}(\tau, j, \vec{\mu}, \vec{\nu}))$ for a $\tau \in \mathcal{H}_\gamma[\Theta] \cap \pi$. We have an $a_\ell < a$ and a finite set Δ of $\Sigma_j(\tau)$-sentences such that $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, \neg\delta$ for each $\delta \in \Delta$. On the other hand we have a sequence $\vec{\nu}$ and

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an ordinal term $a_r(\rho) < a$ for each $\rho \in H(\vec{v},\tau,\gamma,\Theta)$ such that $(\mathcal{H}_\gamma, \Theta\cup\{\rho\}) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\tau)}$. By (23), for any $\rho \in H(\vec{v},\tau,\gamma,\Theta)$ we obtain

$$\forall e \forall \kappa [\max\{\tau+1,\lambda\} \leq \kappa \leq \pi \Rightarrow \rho < \tau \in \mathcal{H}_\gamma(\psi_\kappa(\gamma\# e)) \cap \kappa \subset \psi_\kappa(\gamma\# e)]. \tag{30}$$

Case 5.1. First consider the case when $\tau < \lambda$. Then $\rho < \psi_\kappa(\gamma\# e)$ for any $\kappa \in [\lambda,\pi]$ and e. From SIH with (30) we obtain $(\mathcal{H}_{\hat{a}+1},\Theta) \vdash^{\beta_\ell}_{\beta_\ell} \Gamma, \neg\delta$ for each $\delta\in\Delta$ with $\beta_\ell = \psi_\lambda(\widehat{a_\ell})$, and $(\mathcal{H}_{\hat{a}+1},\Theta\cup\{\rho\}) \vdash^{\beta_r(\rho)}_{\beta_r(\rho)} \Gamma, \Delta^{(\rho,\tau)}$ for each $\rho \in H(\vec{v},\tau,\gamma,\Theta)$ with $\beta_r(\rho) = \psi_\lambda(\widehat{a_r(\rho)})$. We see $\max\{\beta_\ell, \beta_r(\rho), \tau\} < \beta = \psi_\lambda(\hat{a})$, and an inference rule $(\text{rfl}(\tau,j,\vec{\mu},\vec{v}))$ yields $(\mathcal{H}_{\hat{a}+1},\Theta) \vdash^{\beta}_{\beta} \Gamma$.

Case 5.2. Second consider the case when $\lambda \leq \tau$. Then $\Delta \cup \Delta^{(\rho,\tau)} \subset \Sigma_1(\tau^+)$, and $\rho < \psi_\kappa(\gamma\# e)$ for $\tau < \kappa \leq \pi$ and e by (30). SIH yields $(\mathcal{H}_{\widehat{a_\ell}+1},\Theta) \vdash^{\beta_2}_{\beta_2} \Gamma, \neg\delta$ for each $\delta\in\Delta$, where $\beta_2 = \psi_{\tau^+}(\widehat{a_\ell})$. On the other side SIH yields $(\mathcal{H}_{\widehat{a_r(\rho)}+1},\Theta\cup\{\rho\}) \vdash^{\beta_\rho}_{\beta_\rho} \Gamma, \Delta^{(\rho,\tau)}$ for each $\rho \in H(\vec{v},\tau,\gamma,\Theta)$, where $\beta_\rho = \psi_{\tau^+}\left(\widehat{a_r(\rho)}\right)$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{\widehat{a_\ell}+1},\Theta) \vdash^{\delta_2}_{\tau} \Gamma, \neg\delta$ and $(\mathcal{H}_{\widehat{a_r(\rho)}+1},\Theta\cup\{\rho\}) \vdash^{\delta_\rho}_{\tau} \Gamma, \Delta^{(\rho,\tau)}$ for $\delta_2 = \varphi(\beta_2)(\beta_2)$ and $\delta_\rho = \varphi(\beta_\rho)(\beta_\rho)$. From these with the inference rule $(\text{rfl}(\tau,j,\vec{\mu},\vec{v}))$ we obtain

$$(\mathcal{H}_{\widehat{a_0}+1},\Theta) \vdash^{\delta_0+1}_{\tau} \Gamma \tag{31}$$

where $\sup\{\delta_2,\delta_\rho : \rho \in H(\vec{v},\tau,\widehat{a_0}+1,\Theta)\} \leq \delta_0 := \varphi(\beta_0)(\beta_0) \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ with $\sup\{\beta_2,\beta_\rho : \rho\in H(\vec{v},\tau,\gamma,\Theta)\} \leq \beta_0 := \psi_{\tau^+}(\widehat{a_0})$, and $\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{v},\tau,\gamma,\Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta]\cap a$, cf. (10).

MIH with (31) yields $(\mathcal{H}_{\hat{a}+1},\Theta) \vdash^{\delta}_{\delta} \Gamma$ for $\delta = \psi_\lambda((\widehat{a_0}+1)\#\omega^{\tau+\delta_0+2})$ and $(\widehat{a_0}+1)\#\omega^{\tau+\delta_0+2} < \hat{a}$. We have $\delta = \psi_\lambda(\widehat{a_0}\#1\#\omega^{\tau+\delta_0+2}) < \psi_\lambda(\hat{a}) = \beta$ by $\widehat{a_0} < \hat{a}$ and $\tau,\delta_0 < \tau^+ < \pi$ and $\tau \in \mathcal{H}_\gamma[\Theta]$. (22) follows.

Case 6. Sixth consider the case when the last inference is a (cut). For an $a_0 < a$ and a C with $\text{rk}(C) < \pi$, we have $(\mathcal{H}_\gamma,\Theta) \vdash^{a_0}_{\pi} \Gamma, \neg C$ and $(\mathcal{H}_\gamma,\Theta) \vdash^{a_0}_{\pi} C, \Gamma$.

Case 6.1. First consider the case when $\text{rk}(C) < \lambda$. Then $C \in \Sigma_0(\lambda)$. SIH yields the lemma.

Case 6.2. Second consider the case when $\lambda \leq \text{rk}(C) < \pi$. Let $\rho^+ = (\text{rk}(C))^+ = \min\{\kappa \in Reg : \text{rk}(C) < \kappa\}$. Then $C \in \Sigma_0(\rho^+)$ and $\lambda \leq \rho \in \mathcal{H}_\gamma[\Theta]\cap\pi$. SIH yields $(\mathcal{H}_{\widehat{a_0}+1},\Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \neg C$ and $(\mathcal{H}_{\widehat{a_0}+1},\Theta) \vdash^{\beta_0}_{\beta_0} C, \Gamma$ for $\beta_0 = \psi_{\rho^+}(\widehat{a_0}) \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$. By a (cut) we obtain $(\mathcal{H}_{\widehat{a_0}+1},\Theta) \vdash^{\beta_1}_{\beta_1} \Gamma$ for $\beta_1 = \max\{\beta_0, \text{rk}(C)\}+1$ with $\rho < \beta_1 < \rho^+$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{\widehat{a_0}+1},\Theta) \vdash^{\delta_1}_{\rho} \Gamma$ for $\delta_1 = \varphi(\beta_1)(\beta_1)$, where $\widehat{a_0} \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$, and $\forall e \forall \tau \in [\lambda,\rho][\Theta \subset \psi_\tau(\widehat{a_0}\# e)]$ hold. Hence MIH with $\rho \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ yields $(\mathcal{H}_{b+1},\Theta) \vdash^{\psi_\lambda(b)}_{\psi_\lambda(b)} \Gamma$ for $b = \widehat{a_0}\#1\#\omega^{\rho+\delta_1+1}$. We see $b < \hat{a}$ and $\psi_\lambda(b) < \psi_\lambda(\hat{a}) = \beta$, and (22) follows.

Case 7. Seventh consider the case when the last inference is an $(\Omega \in M_2)$. We have $(\mathcal{H}_\gamma,\Theta) \vdash^{a_\ell}_{\pi} \Gamma, C$ for an $a_\ell < a$, and $(\mathcal{H}_\gamma,\Theta\cup\{\alpha\}) \vdash^{a_r(\alpha)}_{\pi} \neg C^{(\alpha,\Omega)}, \Gamma$ for an $a_r(\alpha) < a$ for each $\alpha < \Omega$, where $C \in \Pi_2(\Omega)$.

The case $\lambda > \Omega$ is seen as in Case 5.1. The case $\lambda = \Omega$ is seen as in Case 5.2. ⊣

Let us conclude Theorem 1.1. Let $\Omega = \Omega_1$.

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PROOF OF THEOREM 1.1. Let $\mathsf{KP}\Pi_N \vdash \theta$ for a $\Sigma_1(\Omega)$-sentence θ. By Embedding lemma 4.14 pick an m so that $(\mathcal{H}_0, \emptyset) \vdash^{\mathbb{K}\cdot 2+m}_{\mathbb{K}+m} \theta$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_0, \emptyset) \vdash^{\omega_{m+1}(\mathbb{K}+1)}_{\mathbb{K}} \theta$ for $\omega_m(\mathbb{K}\cdot 2+m) < \omega_{m+1}(\mathbb{K}+1)$. Lemma 5.2 yields $(\mathcal{H}_{a+1}, \emptyset) \vdash^{\beta}_{\beta} \theta$ for $a = \omega^{\mathbb{K}+\omega_{m+1}(\mathbb{K}+1)+1}$ and $\beta = \psi_\Omega(a)$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{a+1}, \emptyset) \vdash^{\varphi(\beta)(\beta)}_{0} \theta$. We obtain $\varphi(\beta)(\beta) < \alpha := \psi_\Omega(\omega_n(\mathbb{K}+1))$ for $n = m+3$, and hence $(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^{\alpha}_{0} \theta$. Boundedness lemma 4.10 yields $(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^{\alpha}_{0} \theta^{(\alpha,\Omega)}$. Since each inference rule other than reflection rules $(\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{v}))$ and $(\Omega \in M_2)$ is sound, we see by induction up to $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$ that $L_\alpha \models \theta$.

This completes a proof of Theorem 1.1.

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