Proceedings of the Royal Society of Edinburgh, **135A**, 999–1015, *2005*

Global continuation of nonlinear waves in a ring of neurons

Shangjiang Guo and **Lihong Huang**
College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China
(shangjguo@etang.com)

(MS received 14 May 2003; accepted 23 March 2005)

In this paper, we consider a ring of neurons with self-feedback and delays. As a result of our approach based on global bifurcation theorems of delay differential equations coupled with representation theory of Lie groups, the coexistence of its asynchronous periodic solutions (i.e. *mirror-reflecting waves*, *standing waves* and *discrete waves*), bifurcated simultaneously from the trivial solution at some critical values of the delay, will be established for delay not only near to but also far away from the critical values. Therefore, we can obtain wave solutions of large amplitudes. In addition, we consider the coincidence of these periodic solutions.

1. Introduction

In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behaviours of recurrent networks [17, 22]. Moreover, ring networks belong to the class of cyclic feedback systems whose asymptotic behaviour has been investigated in more detail [1, 2, 5, 8, 13, 15, 19, 24, 25, 27, 29, 30]. These theoretical results help in better understanding the system's dynamics and are important complements to experimental and numerical investigations using analog circuits and digital computers. On the other hand, time delays are unavoidably encountered in the implementation of artificial neural networks, due to the finite speeds of switching and transmission of signals. It is well known that time delays in the response of neurons can influence the stability of a network creating oscillatory and unstable characteristics, and can dramatically increase the number of co-existent attractors, even in the simplest case of a recurrent inhibitory loop composed of a single excitatory and inhibitory neuron [9, 10].

By means of the general symmetric local Hopf bifurcation theorem [30, theorem 2.1] coupled with representation theory of standard dihedral groups, Guo and Huang [13, 15] studied the influence of the delay on the behaviour of a ring of coupled oscillators in which each neuron receives self-excitatory feedback and from its nearest neighbours, inhibitory feedback:

$$\dot{u}_i(t) = -u_i(t) + f(u_i(t-\tau)) - [g(u_{i-1}(t-\tau)) + g(u_{i+1}(t-\tau))], \tag{1.1}$$

where $i (\operatorname{mod} N), f, g \in C^1(\mathbb{R};\mathbb{R})$ with $f(0) = g(0) = 0$. Such a network has been found in a variety of neural structures, such as the hippocampus [3], cerebellum [7], neocortex [28], and even in chemistry and electrical engineering. System (1.1) can be regarded as a special example of the general Hopfield model [14,18] for artificial neural networks with electronic circuit implementation. According to the Cohen–Grossberg–Hopfield convergence theorem [6,18], under some standard assumptions on the transfer functions, a network modelled by (1.1) without delay (namely, $\tau = 0$) relaxes towards the set of equilibria. However, the presence of a large delay τ may cause some stable nonlinear oscillations and lead to a completely different computational performance of the network [4,12,13,16,20,22,23,25,26,30]. Moreover, most of these nonlinear oscillations may appear in the form of periodic solutions with certain spatio-temporal structures and, if they are stable under small perturbation, may represent memory of the network to be stored and retrieved. Therefore, it is of great interest in many applications to discuss the spatio-temporal patterns of these periodic solutions and to describe the mode interaction along multiple branches of such periodic solutions.

In [13], Guo and Huang not only investigated the effect of synaptic delay of signal transmission on the pattern formation, but also obtained some important results about the spontaneous bifurcation of multiple branches of asynchronous periodic solutions and their spatio-temporal patterns:

(i) *mirror-reflecting waves* of the form $x_j(t) = x_{N+2-j}(t)$, $t \in \mathbb{R}$, $j (\operatorname{mod} N)$;

(ii) *standing waves* of the form $x_j(t) = x_{N+2-j}(t - \frac{1}{2}\omega)$, $t \in \mathbb{R}$, $j (\operatorname{mod} N)$, where $\omega > 0$ is a period of x;

(iii) *discrete waves* of the form $x_j(t) = x_{j+1}(t \pm k\omega/N)$, $t \in \mathbb{R}$, $j (\operatorname{mod} N)$, where $\omega > 0$ is a period of x.

In particular, the discrete waves are also called *phase-locked oscillations*, as each neuron oscillates just like the others, except that they are not necessarily in phase with each other. These wave solutions are special cases of the so-called 'coherent oscillation' observed by Marcus and Westervelt [22]. Depending on the value of the neurons' gain and the topology and size of the interconnection matrix, the aforementioned oscillations can be either stable or unstable. In [22], it has been observed that some systems of neural networks with delay possess multiple basins of attraction for coexisting equilibria and oscillatory attractors. Based on the normal form approach and the centre manifold theory, Guo and Huang [15] derived some formulae for determining the properties of Hopf bifurcating periodic orbit for a ring of neurons with delays, such as the direction of Hopf bifurcation and stability of the Hopf bifurcating periodic orbits.

The purpose of this paper is to study the global continuation of the aforementioned wave solutions, i.e. their coexistence for delay not only near to but also far away from the critical values. Therefore, we can obtain wave solutions of large amplitudes.

Throughout this paper, we always assume the following hypothesis holds.

(H1) There exists some $k \in \{1, 2, \dots, [\frac{1}{2}(N-1)]\}$ such that $|\gamma + 4\eta \sin^2(k\pi/N)| > 1$, where $\gamma = f'(0) - 2g'(0)$, $\eta = g'(0)$, and $[\cdot]$ is the greatest integer function.

For convenience, we use the transformation $x_i(t) = u_i(\tau t)$ for $i (\operatorname{mod} N)$ and $h = f - 2g$, and then rewrite (1.1) as the system of delay differential equations

$$\tau^{-1}\dot{x}_i(t) = -x_i(t) + h(x_i(t-1)) - [g(x_{i-1}(t-1)) + g(x_{i+1}(t-1)) - 2g(x_i(t-1))], \quad (1.2)$$

where $i (\operatorname{mod} N)$.

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

The rest of this paper is organized as follows. In § 2, we discuss the associated characteristic equation and collect some results from [13, 20]. By using the global bifurcation theory established by Krawcewicz and Wu [20], we show in § 3 that these bifurcations of periodic solutions exist for large delays (global continuation). Section 4 is devoted to the coincidence of these periodic solutions.

2. Preliminaries

Let $C([-1,0],\mathbb{R}^N)$ denote the Banach space of continuous mapping from $[-1,0]$ into $\mathbb{R}^N$ equipped with the supremum norm

$$\|\phi\| = \sup_{-1\leqslant\theta\leqslant 0} |\phi(\theta)| \quad \text{for } \phi \in C([-1,0],\mathbb{R}^N).$$

In what follows, if $\sigma \in \mathbb{R}$, $A \geqslant 0$ and $x : [\sigma-1,\sigma+A] \to \mathbb{R}^N$ is a continuous mapping, then $x_t \in C([-1,0],\mathbb{R}^N)$, $t \in [\sigma,\sigma+A]$, is defined by $x_t(\theta) = x(t+\theta)$ for $-1 \leqslant \theta \leqslant 0$. We denote a symmetric circulant matrix by $J = \operatorname{circ}(a_1, a_2, \dots, a_N)$, where $J_{ij} = a_{j-i+1}$ and $a_i = a_{N-i+2}$, $i (\operatorname{mod} N)$.

We introduce three compact Lie groups in order to explore the possible (spatial) symmetry of the system (1.2). One is the cycle group S^1, another is $\mathbb{Z}_n$, the cyclic group of order n (the order of a finite group is the number of the elements it contains) and the third is the dihedral group $\mathbb{D}_n$ of order $2n$, which is generated by $\mathbb{Z}_n$ together with the flip κ of order 2 (see [11] for more details). Denote by ρ the generator of the cyclic subgroup $\mathbb{Z}_n$, and by κ the flip. Define the action of $\mathbb{D}_N$ on $\mathbb{R}^N$ by

$$(\rho x)_i = x_{i+1}, \quad (\kappa x)_i = x_{N+2-i} \quad \text{for all } i (\operatorname{mod} N) \text{ and } x \in \mathbb{R}^N. \tag{2.1}$$

In [13], we showed that system (1.2) is $\mathbb{D}_N$-equivariant.

Next, the linearization of (1.2) at the origin leads to

$$\dot{x}_i = -\tau x_i(t) + \tau\gamma x_i(t-1) - \tau\eta[x_{i-1}(t-1) + x_{i+1}(t-1) - 2x_i(t-1)], \tag{2.2}$$

where $i (\operatorname{mod} N)$. It is well known that the associated characteristic equation of (2.2) takes the form

$$\det \Delta(\tau,\lambda) = 0,$$

where the characteristic matrix $\Delta(\tau,\lambda)$ is given by

$$\Delta(\tau,\lambda) = (\lambda+\tau)\operatorname{Id} - \tau M \mathrm{e}^{-\lambda}, \quad \lambda \in \mathbb{C}, \tag{2.3}$$

with Id denoting the identity matrix and $M = \operatorname{circ}(\gamma+2\eta, -\eta, 0, \dots, -\eta)$.

We put $\chi = \mathrm{e}^{2\mathrm{i}\pi/N}$ and

$$v_k = (1, \chi^k, \chi^{2k}, \dots, \chi^{(N-1)k})^{\mathrm{T}}, \quad 0 \leqslant k \leqslant N-1.$$

Clearly, $v_0 = (1, 1, \dots, 1)^{\mathrm{T}}$ and $v_k = \bar{v}_{N-k}$. Let

$$\mathbb{C}_k = \{v_k z; z \in \mathbb{C}\}, \quad k = 0, 1, 2, \dots, N-1.$$

Then

$$\mathbb{C}^N = \mathbb{C}_0 \oplus \mathbb{C}_1 \oplus \cdots \oplus \mathbb{C}_{N-1} \tag{2.4}$$

and

$$\begin{aligned}(\Delta(\tau,\lambda)v_k)_j &= (\lambda+\tau)\chi^{(j-1)k} - \tau e^{-\lambda}[(\gamma+2\eta)\chi^{(j-1)k} - \eta(\chi^{(j-2)k}+\chi^{jk})]\\ &= \{\lambda+\tau-\tau e^{-\lambda}[\gamma+2\eta-\eta(\chi^{-k}+\chi^{k})]\}\chi^{(j-1)k}\\ &= \left[\lambda+\tau-\left(\gamma+4\eta\sin^2\frac{k\pi}{N}\right)\tau e^{-\lambda}\right](v_k)_j.\end{aligned}$$

Thus, we obtain the following results.

Lemma 2.1.

$$\det\Delta(\tau,\lambda) = \prod_{k=0}^{N-1}\left[\lambda+\tau-\left(\gamma+4\eta\sin^2\frac{k\pi}{N}\right)\tau e^{-\lambda}\right].$$

Thus, $\lambda\in\mathbb{C}$ is a zero of $\det\Delta(\tau,\lambda)$ if and only if there exists a $k\in\{0,1,2,\dots,N-1\}$ such that

$$p_k(\tau,\lambda) := \lambda+\tau-\left(\gamma+4\eta\sin^2\frac{k\pi}{N}\right)\tau e^{-\lambda} = 0. \tag{2.5}$$

Lemma 2.2 (Guo and Huang [13]). *Let $A(\tau)$ denote the infinitesimal generator of the semigroup generated by system (2.2). Assume that hypothesis (H1) holds. Define*

$$\beta_{k,s} = \begin{cases} 2s\pi+\arccos\left(\gamma+4\eta\sin^2\dfrac{k\pi}{N}\right)^{-1} & \text{if } \gamma+4\eta\sin^2\dfrac{k\pi}{N} < -1,\\ 2(s+1)\pi-\arccos\left(\gamma+4\eta\sin^2\dfrac{k\pi}{N}\right)^{-1} & \text{if } \gamma+4\eta\sin^2\dfrac{k\pi}{N} > 1;\end{cases}$$

$$\tau_{k,s} = \beta_{k,s}\left\{\left(\gamma+4\eta\sin^2\frac{k\pi}{N}\right)^2-1\right\}^{-1/2}$$

for all $s\in\mathbb{N}_0 := \{0,1,2,\dots\}$. Then we have the following results.

(i) *At (and only at) $\tau=\tau_{k,s}$, $s\in\mathbb{N}_0$, $A(\tau)$ has purely imaginary eigenvalues. These eigenvalues are given by $\pm i\beta_{k,s}$ with $\beta_{k,s}\in((2s+\frac{3}{2})\pi, 2(s+1)\pi)$ provided that $\gamma+4\eta\sin^2 k\pi/N > 1$ and $\beta_{k,s}\in((2s+\frac{1}{2})\pi,(2s+1)\pi)$ provided that $\gamma+4\eta\sin^2 k\pi/N < -1$.*

(ii) *All other eigenvalues of $A(\tau_{k,s})$ are not integer multiples of $\pm i\beta_{k,s}$.*

(iii) *For each fixed $s\in\mathbb{N}_0$, there exist $\delta_{k,s}>0$ and C^1-mapping $\lambda_{k,s}:(\tau_{k,s}-\delta_{k,s},\tau_{k,s}+\delta_{k,s})\to\mathbb{C}$ such that $\lambda_{k,s}(\tau_{k,s}) = i\beta_{k,s}$ and $\det\Delta(\tau,\lambda_{k,s}(\tau))=0$ for all $\tau\in(\tau_{k,s}-\delta_{k,s},\tau_{k,s}+\delta_{k,s})$. Moreover,*

$$\frac{d}{d\tau}\operatorname{Re}\{\lambda_{k,s}(\tau)\}\bigg|_{\tau=\tau_{k,s}} > 0.$$

Under hypothesis (H1), Guo and Huang [13] show that near $\tau=\tau_{k,s}$, for each $s\in\mathbb{N}_0$, there exist $2(n+1)$ branches of asynchronous periodic solutions of period near $(2\pi/\beta_{k,s})$, bifurcated simultaneously from the zero solution of system (1.2) and these are two phase-locked oscillations, N mirror-reflecting waves and N standing waves. In what follows, we need the general global symmetric Hopf bifurcation theorem developed in [20] to show that these bifurcations of periodic solutions exist for all $\tau > \tau_{k,s}$ (i.e. these bifurcations are supercritical and global). Namely, we consider the following one-parameter family of retarded functional differential equations (FDEs)

$$\dot{x}(t) = \tau F(x_t), \tag{2.6}$$

where $x \in \mathbb{R}^N$, $\tau \in (0, \infty)$ and $F : C([-\tau, 0]; \mathbb{R}^N) \to \mathbb{R}^N$ is continuously differentiable and completely continuous. Furthermore, we make the following assumptions.

(A1) There exists some positive integer n such that the cyclic group $\Gamma := \mathbb{Z}_n$ acts on $\mathbb{R}^N$ and $F : C([-\tau, 0]; \mathbb{R}^N) \to \mathbb{R}^N$ is Γ-equivariant.

(A2) For every $x_0 \in M^\Gamma := \{x \in \mathbb{R}^N; \gamma x = x \text{ for } \gamma \in \Gamma, F(\bar{x}) = 0\}$, where $\bar{x} \in C$ is the constant mapping with the constant value $x \in \mathbb{R}^N$, $\det D\hat{F}(x_0) \neq 0$, where $\hat{F}$ is the C^1 mapping from $\mathbb{R}^N$ into $\mathbb{R}^N$, induced by F according to $\hat{F}(x) = F(\bar{x})$ for $x \in \mathbb{R}^N$.

(A3) For every $\tau_0 > 0$ and $x_0 \in M^\Gamma$ such that the generator $A(\tau_0, x_0)$ of the linearized system of (2.6) with $\tau = \tau_0$ at $x = x_0$ has a pair of purely imaginary eigenvalues $\pm \mathrm{i}\beta_0$, there exist positive constants b, c and δ such that:

  (i) the only possible eigenvalue $u + \mathrm{i}v$ of $A(\tau_0, x_0)$ with $(u, v) \in \partial\Omega$ is $\mathrm{i}\beta_0$, where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c)$;

  (ii) for $(\tau, \beta) \in [\tau_0 - \delta, \tau_0 + \delta] \times [\beta_0 - c, \beta_0 + c]$, $\mathrm{i}\beta$ is an eigenvalue of $A(\tau, x_0)$ if and only if $\tau = \tau_0$ and $\beta = \beta_0$.

(A4) $M^* := \{(\tau, x, \beta) \in (0, \infty) \times M^\Gamma \times (0, \infty); \pm\mathrm{i}\beta \text{ are eigenvalues of } A(\tau, x)\}$ is a discrete set.

Note that the action of Γ on $\mathbb{R}^N$ induces an action on $\mathbb{C}^N = \mathbb{R}^N + \mathrm{i}\mathbb{R}^N$, with respect to which we have the isotypical decomposition

$$\mathbb{C}^N = \mathbb{C}_0^N \oplus \mathbb{C}_1^N \oplus \cdots \oplus \mathbb{C}_j^N \oplus \cdots,$$

where $\mathbb{C}_j^N$, $j \geqslant 0$, is the direct sum of all one-dimensional Γ-irreducible subspaces V of $\mathbb{C}^N$ such that the restricted action Γ on V is isomorphic to the Γ-action on $\mathbb{C}$ defined by $\rho \cdot z = \rho^j z$ for the generator $\rho \in \mathbb{Z}_n \leqslant S^1$ and for $z \in \mathbb{C}$. Let

$$\Delta_{x_0}(\tau, \lambda) := \lambda I_N - \tau D_\phi F(\bar{x}_0)(\mathrm{e}^{\lambda \cdot} I_N) \tag{2.7}$$

for $\tau > 0$, $x_0 \in M^\Gamma$ and $\lambda \in \mathbb{C}$. By assumption (A1), we have $\Delta_{x_0}(\tau, \lambda)\mathbb{C}_j^N \subset \mathbb{C}_j^N$ for $j \geqslant 0$ and for $\lambda \in \mathbb{C}$. Put

$$\Delta_{x_0,j}(\tau, \lambda) = \Delta_{x_0}(\tau, \lambda)|_{\mathbb{C}_j^N}, \quad j \geqslant 0. \tag{2.8}$$

Clearly, $\Delta_{x_0}(\tau, \lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous in $\tau > 0$. So, under assumption (A3), we may assume that $\det \Delta_{x_0}(\tau_0 \pm \delta, u + \mathrm{i}v) \neq 0$ for $(u, v) \in \partial\Omega$. Therefore, $\det \Delta_{x_0,j}(\tau_0 \pm \delta, u + \mathrm{i}v) \neq 0$ for $(u, v) \in \partial\Omega$ and for $j \geqslant 0$. Consequently, the following integers are well defined:

$$c_j(x_0, \tau_0, \beta_0) = \deg_B(\det \Delta_{x_0,j}(\tau_0 - \delta, \cdot), \Omega) - \deg_B(\det \Delta_{x_0,j}(\tau_0 + \delta, \cdot), \Omega),$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

where $\deg_B$ is the Brouwer degree. Let

$$\epsilon(x_0) = (-1)^N \operatorname{sgn} \det D\hat{F}(x_0). \tag{2.9}$$

We have the following global symmetric Hopf bifurcation theorem due to [20].

LEMMA 2.3. *Assume that (A1)–(A4) are satisfied and that* $c_j(x_0, \tau_0, \beta_0) \neq 0$ *for some integer* $j \geqslant 0$ *and some* $(\tau_0, x_0, \beta_0) \in (0, \infty) \times M^{\Gamma} \times (0, \infty)$. *Let* S_j *denote the closure in* $[0, \infty) \times C(\mathbb{R}; \mathbb{R}^n) \times [0, \infty)$ *of the set of all* $(\tau, z, \beta) \in [0, \infty) \times C(\mathbb{R}; \mathbb{R}^n) \times \mathbb{R} \setminus M^*$ *such that*

$$x(t) := z\left(\frac{\beta}{2\pi} t\right)$$

is a $2\pi/\beta$*-periodic solution of (2.6) with*

$$\rho x(t) = x\left(t - \frac{2\pi}{\beta}\frac{j}{n}\right) \quad \textit{for } t \in \mathbb{R}.$$

Then for each connected component E_j *of* S_j, *at least one of the following conditions holds.*

(i) E_j *is unbounded, i.e.*

$$\sup\left\{\tau + \beta + \beta^{-1} + \sup_{t \in \mathbb{R}} |x(t)|; (\tau, x, \beta) \in E_j\right\} = \infty.$$

(ii) $(\Gamma \times S^1)E_j \cap M^*$ *is finite and is composed of a finite number of disjoint* Γ*-orbits. Moreover,*

$$\sum_{(\tau, x, \beta) \in (\Gamma \times S^1)E_j \cap M^*} \epsilon(x) c_j(x, \tau, \beta) = 0. \tag{2.10}$$

3. Global continuation of waves

To obtain large-amplitude periodic solutions of system (1.2) when τ is far away from $\tau_{k,s}$, we need the following assumptions:

(H2) $\sup_{y \in \mathbb{R}} |h'(y)| < 1$;

(H3) $g'(x) > 0$ for all $x \in \mathbb{R}$;

(H4) there exist positive constants α_1 and α_2 satisfying $\alpha_1 + 4\alpha_2 < 1$, and σ_i $(i = 1, 2)$ such that $|h(y)| < \alpha_1 |y| + \sigma_1$ and $|g(y)| < \alpha_2 |y| + \sigma_2$ for all $y \in \mathbb{R}$.

It should be noted that the assumptions on the activation functions in (H4) are very general. In particular, if the activation functions in system (1.2) are all bounded on $\mathbb{R}$, i.e. there exist positive constants σ_i $(i = 1, 2)$ such that $|h(x)| < \sigma_1$ and $|g(x)| < \sigma_2$ for all $x \in \mathbb{R}$, then (H4) holds with $\alpha_1 = \alpha_2 = 0$. For example, in cellular neural network models, the activation function takes the form $f(u) = \frac{1}{2}(|u + 1| - |u - 1|)$, which is bounded. Hence, it is *wrong* to think that condition (H4) implies that the zero solution is globally asymptotically stable. In fact, if (H1) holds, then the zero solution is unstable.

As a illustrating example, we consider (1.2) with $n = 3$ and $h(x) = 0.5\tanh(x)$ and $g(x) = \tanh(x)$. Then, (H1) holds because $\gamma + 3\eta = 3.5 > 1$; (H4) holds with $\alpha_1 = \alpha_2 = 0$ and $\sigma_1 = \sigma_2 = 3$.

LEMMA 3.1. *Assume that (H1)–(H3) are satisfied. Then system (1.2) has no non-constant* 2*-periodic solutions.*

Proof. By way of contradiction, assume that $x(t)$ is a 2-periodic solution. Constructing a Lyapunov functional,

$$V(x_1, x_2, \dots, x_N)(t) = \frac{1}{\tau}\sum_{i=1}^{N}|x_i(t) - x_i(t-1)|, \tag{3.1}$$

and calculating the upper-right Dini derivative of V along the solutions of (1.2), we have

$$\begin{aligned} &D^+V(x_1, x_2, \dots, x_N)(t) \\ &\quad \leqslant \sum_{i=1}^{N}[-|x_i(t) - x_i(t-1)| - 2|g(x_i(t)) - g(x_i(t-1))| + |h(x_i) - h(x_i(t-1))| \\ &\qquad\qquad + |g(x_{i+1}(t)) - g(x_{i+1}(t-1))| + |g(x_{i-1}(t)) - g(x_{i-1}(t-1))|] \\ &\quad \leqslant -\Big[1 - \sup_{\theta\in\mathbb{R}}|h'(\theta)|\Big]\sum_{i=1}^{N}|x_i(t) - x_i(t-1)|. \end{aligned}$$

This implies that

$$\lim_{t\to\infty} V(x_1, x_2, \dots, x_N)(t) = 0.$$

Therefore, for a 2-periodic solution x of (1.2), we must have $x_i(t) = x_i(t-1)$ for all $i (\operatorname{mod} N)$. Then we obtain a system of ordinary differential equations

$$\frac{1}{\tau}\dot{x}_i(t) = -x_i(t) + h(x_i(t)) + 2g(x_i(t)) - g(x_{i+1}(t)) - g(x_{i-1}(t)). \tag{3.2}$$

We define an energy function associated with (3.2) as follows:

$$V(x_1, x_2, \dots, x_n) = -\frac{1}{2}\sum_{1\leqslant i,j\leqslant N} T_{ij}g(x_i)g(x_j) + \sum_{k=1}^{N}\int_0^{x_k}[s - h(s)]g'(s)\,\mathrm{d}s,$$

where T_{ij} is the connection weight, i.e. $T_{ij} = 2$ if $i = j$, and -1 if $i = j + 1$ or $i = j - 1$, and 0 otherwise. Calculating the derivative of V along the solutions of (3.2), we find that

$$\begin{aligned} \dot{V}(x_1, x_2, \dots, x_N) &= \sum_{i=1}^{N}\{g'(x_i)\dot{x}_i[g(x_{i-1}) + g(x_{i+1}) - 2g(x_i) + x_i - h(x_i)]\} \\ &= -\tau^{-1}\sum_{i=1}^{N}g'(x_i)(\dot{x}_i)^2 \\ &\leqslant 0. \end{aligned}$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

Therefore, a standard Lyapunov stability theorem implies that every solution of (3.2) converges to an equilibrium as $t \to \infty$. In particular, every 2-periodic solution of (1.2) must be constant. This completes the proof. □

We now make an *a priori* estimate of the periodic solutions of (1.2).

LEMMA 3.2. *Assume that (H1)–(H4) are satisfied. There then exists $H = H(h, g) > 0$ such that $\sum_{i=1}^{N} |x_i(t)| \leqslant H$ for all $t \in \mathbb{R}$ and for all periodic solutions x of (1.2).*

Proof. Let $x(t) = (x_1(t), x_2(t), \dots, x_N(t))^{\mathrm{T}}$ be an arbitrary periodic solution of system (1.2). $x_j(t)$, $j = 1, 2, \dots, N$, as the components of $x(t)$, are all continuously differentiable. Thus, there exist t_j such that $|x_j(t_j)| = \max_{t\in\mathbb{R}} |x_j(t)|$. Then $\dot{x}_j(t_j) = 0$. That is,

$$x_j(t_j) = h(x_j(t_j - 1)) - [g(x_{j-1}(t_j - 1)) + g(x_{j+1}x(t_j - 1)) - 2g(x_j(t_j - 1))].$$

Thus, we have

$$\begin{aligned} |x_j(t_j)| &\leqslant |h(x_j(t_j - 1))| + |g(x_{j-1}(t_j - 1))| \\ &\quad + |g(x_{j+1}(t_j - 1))| + 2|g(x_j(t_j - 1))| \\ &\leqslant \alpha_1|x_j(t_j)| + \alpha_2|x_{j-1}(t_{j-1})| \\ &\quad + \alpha_2|x_{j+1}(t_{j+1})| + 2\alpha_2|x_j(t_j)| + \sigma_1 + 4\sigma_2. \end{aligned}$$

Namely,

$$|x(t^*)| \leqslant W|x(t^*)| + D,$$

where $|x(t^*)| = (|x_1(t_1)|, |x_2(t_2)|, \dots, |x_N(t_N)|)^{\mathrm{T}}$, $D = (\sigma_1 + 4\sigma_2)v_0$, and W is an $n \times n$ matrix given by $W = \operatorname{circ}(\alpha_1 + 2\alpha_2, \alpha_2, 0, \dots, \alpha_2)$. In view of $\alpha_1 + 4\alpha_2 < 1$, we have $\rho(W) < 1$, and hence $(\mathrm{Id} - W)^{\mathrm{T}} \geqslant 0$ and $B = (\mathrm{Id} - W)^{-1}D \geqslant 0$. Therefore, $|x(t^*)| \leqslant B$, i.e. $|x_j(t_j)| \leqslant B_j$ for all $j = 1, 2, \dots, N$, where B_j is the jth component of B. Thus,

$$\sum_{i=1}^{N} |x_i(t)| \leqslant \sum_{i=1}^{N} |x_i(t_i)| \leqslant \sum_{i=1}^{N} B_i := H.$$

Obviously, H is independent of the choice of periodic solution $x(t)$. This completes the proof. □

We now start to apply the global symmetric Hopf bifurcation theorem (lemma 2.3) to investigate the global continuation of standing, mirror-reflecting and discrete waves.

First of all, note that near $\tau = \tau_{k,s}$ system (1.2) has two bifurcations of discrete waves satisfying $x_{j-1}(t) = x_j(t \pm k\omega/N)$, where ω is a period. To look at the global continuation of such local bifurcations, we regard system (1.2) as an FDE equivariant with respect to the action of $\Gamma = \mathbb{Z}_N$, where the action is cyclic permutation. We obtain

$$M^{\Gamma} = \{x \in \mathbb{R}^N; x_1 = x_2 = \cdots = x_N \text{ and } x_1 = h(x_1)\} = \{0\}.$$

Under assumption (H1), lemma 2.2 implies that

$$M^* = \{(\tau_{k,s}, 0, \beta_{k,s}); s \in \mathbb{N}_0\}.$$

Therefore, M^* is discrete in $\mathbb{R}^N$.

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

Using the definition of $\Delta(\tau,\lambda)$ (see (2.3)) and lemma 2.2(iii), for a fixed integer $s\in\mathbb{N}_0$, we can show that $\Delta(\tau,\lambda)$ is analytic in $\lambda\in\mathbb{C}$ and continuous in $\tau\in[\tau_{k,s}-\delta_0,\tau_{k,s}+\delta_0]$, where δ_0 is an appropriate positive constant. Moreover, it follows from lemma 2.2(i), (ii) that there exist small positive constants $b,c,\delta\in(0,\delta_0)$ so that the only possible eigenvalue $u+\mathrm{i}v$ of $A(\tau_{k,s})$ with $(u,v)\in\partial\Omega$ is $\mathrm{i}\beta_{k,s}$, where $\Omega=(0,b)\times(\beta_{k,s}-c,\beta_{k,s}+c)$ and if $(\tau,\beta)\in[\tau_{k,s}-\delta,\tau_{k,s}+\delta]\times[\beta_{k,s}-c,\beta_{k,s}+c]$, then $\mathrm{i}\beta$ is an eigenvalue of $A(\tau)$ if and only if $\tau=\tau_{k,s}$ and $\beta=\beta_{k,s}$. Then, in view of lemma 2.2, we can conclude that $p_k(\tau,\lambda)$ has no zero in $\bar{\Omega}$ for $\tau=\tau_{k,s}\pm\delta$. Moreover, the above b, c and δ can be chosen so that $p_k(\tau_{k,s}-\delta,\lambda)$ has no zero in $\bar{\Omega}$, while $p_k(\tau_{k,s}+\delta,\lambda)$ has exactly one zero in $\bar{\Omega}$ and this zero is simple and is in the interior of $\bar{\Omega}$. Therefore,

$$\deg_B(p_k(\tau_{k,s}-\delta,\cdot),\Omega)=0\quad\text{and}\quad\deg_B(p_k(\tau_{k,s}+\delta,\cdot),\Omega)=1.$$

For the isotypical decomposition (2.3) of the complexification of the above $\Gamma=\mathbb{Z}_N$ action on $\mathbb{R}^N$, we have

$$\Delta_{0,j}:=\Delta_0(\tau,\lambda)|_{\mathbb{C}_j}=p_j(\tau,\lambda),\quad j=0,1,2,\dots,N-1.$$

Therefore, from the above discussions we obtain

$$\begin{aligned}c_j(0,\tau_{k,s},\beta_{k,s})&=\deg_B(p_j(\tau_{k,s}-\delta,\cdot),\Omega)-\deg_B(p_j(\tau_{k,s}+\delta,\cdot),\Omega)\\&=\begin{cases}0, & \text{for } j\neq k \text{ and } j\neq N-k,\\-1, & \text{for } j=k,N-k,\end{cases}\end{aligned}$$

Let S_j, $j=k,N-k$, denote the closure in $[0,\infty)\times C(\mathbb{R};\mathbb{R}^N)\times[0,\infty)$ of the set of all triples $(\tau,z,\beta)\notin M^*$ such that

$$x(t):=z\left(\frac{\beta}{2\pi}t\right)$$

is a $2\pi/\beta$-periodic solution of (1.2) with

$$x_{i+1}(t)=x_i\left(t-\frac{2\pi}{\beta}\frac{j}{N}\right)\quad\text{for } t\in\mathbb{R} \text{ and } i(\operatorname{mod} N).$$

Then lemma 2.3 implies that S_j must have a non-empty connected component E_j passing through $(\tau_{k,s},0,\beta_{k,s})$ and this component must be unbounded, for otherwise the summation (2.10) holds, which is clearly impossible because

$$\epsilon(0)=(-1)^N\operatorname{sgn}\left[\prod_{j=0}^{N-1}\left(\gamma-1+4\beta\sin^2\frac{j\pi}{N}\right)\right]=(-1)^{N+1}$$

and $c_j(0,\tau_{k,s},\beta_{k,s})=-1$, $j=k,N-k$.

The projection of E_j onto the space $C(\mathbb{R};\mathbb{R}^n)$ is bounded due to lemma 3.2. Assumptions (H1) and (H2) mean that $\gamma+4\eta\sin^2(k\pi/N)>1$. This, together with lemma 2.2(i), implies that

$$\frac{2\pi}{\beta}\in\left(\frac{2\pi}{2(s+1)\pi},\frac{2\pi}{(2s+3/2)\pi}\right)\subset\left(\frac{1}{s+1},\frac{1}{s+3/4}\right)\subset\left(\frac{1}{s+1},\frac{4}{3}\right)\subset\left(\frac{1}{s+1},2\right)$$

for $(\tau,z,\beta)\in E_j$.

On the other hand, because system (1.2) has no non-constant periodic solution of period 2, it has no non-constant periodic solution of period $1/(s+1)$ for all $s \in \mathbb{N}_0$. Therefore, lemma 3.1 implies that the projection of E_j onto the β-plane can never reach the lines

$$\frac{2\pi}{\beta} = \frac{1}{s+1} \quad \text{and} \quad \frac{2\pi}{\beta} = 2.$$

Therefore, the projection of E_j onto the β-plane always satisfies $\pi < \beta < 2(s+1)\pi$. Moreover, the result of [21] shows that there exists $\alpha^* > 0$ such that any period p of a periodic solution of (1.1) must satisfy $p \geqslant \alpha^*$. Consequently, for $(\tau, z, \beta) \in E_j$, we must have $2\pi\tau/\beta \geqslant \alpha^*$. That is, $\tau \geqslant \beta\alpha^*/(2\pi) > \frac{1}{2}\alpha^*$ for every $\tau \in I$, the projection of E_j onto the τ-axis which must be an interval. Therefore, I must be unbounded from above. Clearly, I contains $\tau_{k,s}$. This proves the following theorem.

THEOREM 3.3. *Assume that hypothesis (H1)–(H4) hold. Then, for each $\tau > \tau_{k,s}$, $s \in \mathbb{N}_0$, system (1.2) always has two discrete waves satisfying*

$$x_{j+1}(t) = x_j\left(t \pm \frac{k\omega}{N}\right) \quad \textit{for } t \in \mathbb{R} \textit{ and } j (\operatorname{mod} N),$$

where ω is a period of $x(t)$ and satisfies the condition that $1/(s+1) < \omega < 2$.

Next, near $\tau = \tau_{k,s}$, system (1.2) has several bifurcations of mirror-reflecting waves satisfying $x_j(t) = x_{N+2-j}(t)$ and standing waves satisfying

$$x_j(t) = x_{N+2-j}(t - \tfrac{1}{2}\omega),$$

where ω is a period. Similarly, in order to study the global continuation of such local bifurcations, we regard system (1.2) as an FDE equivariant with respect to the action of $\Gamma = \mathbb{Z}_2$, where the action is the flip. Namely, the action of $\Gamma = \mathbb{Z}_2$ on $\mathbb{R}^N$ is defined by

$$(\rho x)_j = x_{N+2-j} \quad \text{for } j (\operatorname{mod} N), \ x \in \mathbb{R}^N.$$

In this case,

$$M^\Gamma = \{x \in \mathbb{R}^N; x_j = h(x_j) - g(x_{j-1}) - g(x_{j+1}) + 2g(x_j) \\ \text{and } x_j = x_{N+2-j} \text{ for } j (\operatorname{mod} N)\}.$$

Obviously, M^Γ contains at least one element 0. Moreover, the following assumption plays an important role in describing the structure of M^Γ.

(H5) If system (1.2) has a non-zero stationary point $x^* = (x_1^*, x_2^*, \dots, x_N^*)^{\mathrm{T}}$ satisfying $x_j^* = x_{N+2-j}^*$ for $j (\operatorname{mod} N)$, then the spectral radius $\rho(DF(x^*)) < 1$, where $F(x) = (F_1(x), F_2(x), \dots, F_N(x))^{\mathrm{T}}$ and

$$F_j(x) = h(x_j) + 2g(x_j) - g(x_{j-1}) - g(x_{j+1}), \quad j = 1, 2, \dots, N.$$

Thus, if M^Γ contains a non-zero element $x^* = (x_1^*, x_2^*, \dots, x_N^*)^{\mathrm{T}}$, then x^* is also a non-zero stationary point of system (1.2). The linearization of (1.2) at $x^* =$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

$(x_1^*, x_2^*, \dots, x_N^*)^{\mathrm{T}}$ takes the form

$$\begin{aligned}\dot{x}_i = -\tau x_i(t) + \tau h'(x_i^*)x_i(t-1) \\ - \tau[g'(x_{i-1}^*)x_{i-1}(t-1) + g'(x_{i+1}^*)x_{i+1}(t-1) - 2g'(x_i^*)x_i(t-1)],\end{aligned}$$

where $i(\bmod N)$, and the characteristic matrix becomes

$$\Delta_{x^*}(\tau, \lambda) = (\lambda + \tau)\,\mathrm{Id}_N - \tau DF(x^*)\mathrm{e}^{-\lambda}.$$

We may assume that q_j, $j = 1, 2, \dots, N$ are eigenvalues of matrix $DF(x^*)$. Then

$$\det \Delta_{x^*}(\tau, \lambda) = \prod_{j=1}^{N}[\lambda + \tau - \tau q_j \mathrm{e}^{-\lambda}].$$

Recall that $\rho(DF(x^*)) < 1$, i.e. $|q_j| < 1$ for $j = 1, 2, \dots, N$. We see that every zero of $\lambda + \tau - \tau q_j \mathrm{e}^{-\lambda}$ has a negative real part, and hence all zeros of $\det \Delta_{x^*}(\tau, \lambda)$ have negative real parts. This shows that (A2) is satisfied. Therefore, when $\Gamma = \mathbb{Z}_2$, we have

$$M^* = \{(\tau_{k,s}, 0, \beta_{k,s}); s \in \mathbb{N}_0\}.$$

Thus, M^* is discrete in $\mathbb{R}^N$.

The isotypical decomposition of $\mathbb{C}^N$ with respect to the above $\Gamma = \mathbb{Z}_2$ action is

$$\mathbb{C}^N = \mathbb{C}_0^N \oplus \mathbb{C}_1^N,$$

where

$$\begin{aligned}\mathbb{C}_0^N &= \{(x_1, x_2, \dots, x_N)^{\mathrm{T}};\ x_j \in \mathbb{C},\ x_j = x_{N+2-j} \text{ for } j(\bmod N)\},\\ \mathbb{C}_1^N &= \{(x_1, x_2, \dots, x_N)^{\mathrm{T}};\ x_j \in \mathbb{C},\ x_j = -x_{N+2-j} \text{ for } j(\bmod N)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_{0,0}(\tau, \lambda) &:= \Delta_0(\tau, \lambda)|_{\mathbb{C}_0^N} = (\lambda + \tau)\,\mathrm{Id}_{[N/2]+1} - \tau M_1 \mathrm{e}^{-\lambda},\\ \Delta_{0,1}(\tau, \lambda) &:= \Delta_0(\tau, \lambda)|_{\mathbb{C}_1^N} = (\lambda + \tau)\,\mathrm{Id}_{[(N-1)/2]} - \tau M_2 \mathrm{e}^{-\lambda},\end{aligned}$$

where M_1 and M_2 are $([\frac{1}{2}N]+1) \times ([\frac{1}{2}N]+1)$ and $[\frac{1}{2}(N-1)] \times [\frac{1}{2}(N-1)]$ matrices, respectively, i.e.

$$M_1 = \begin{pmatrix} \gamma + 2\eta & -2\eta & 0 & \cdots & 0 & 0 \\ -\eta & \gamma + 2\eta & -\eta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{2}\eta[3 + (-1)^N] & \gamma + \frac{1}{2}\eta[3 + (-1)^N] \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} \gamma + 2\eta & -\eta & 0 & \cdots & 0 & 0 \\ -\eta & \gamma + 2\eta & -\eta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\eta & \gamma + [2 - (-1)^N]\eta \end{pmatrix}.$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

Moreover, it is easy to see that

$$\mathbb{C}_0^N = \operatorname{span}\{v_j + v_{N-j};\ 0 \leqslant j \leqslant [\tfrac{1}{2}N]\},$$
$$\mathbb{C}_1^N = \operatorname{span}\{v_j - v_{N-j};\ 1 \leqslant j \leqslant [\tfrac{1}{2}(N-1)]\},$$

where $v_j = (1, \chi^j, \chi^{2j}, \dots, \chi^{(N-1)j})^{\mathrm{T}}$ with $\chi = \mathrm{e}^{2\mathrm{i}\pi/N}$, $j = 0, 1, 2, \dots, N-1$. Due to the proof of lemma 2.1, we have

$$\det \Delta_{0,0}(\tau, \lambda) = \prod_{j=0}^{[N/2]} p_j(\tau, \lambda) \quad \text{and} \quad \det \Delta_{0,1}(\tau, \lambda) = \prod_{j=1}^{[(N-1)/2]} p_j(\tau, \lambda),$$

where $p_j(\tau, \lambda)$ is defined as (2.5). Using a similar argument as in the proof of theorem 3.3, we can choose positive constants b, c and δ so that the only possible eigenvalue $u + \mathrm{i}v$ of $A(\tau_{k,s})$ with $(u, v) \in \partial\Omega$ is $\mathrm{i}\beta_{k,s}$, where $\Omega = (0, b) \times (\beta_{k,s} - c, \beta_{k,s} + c)$, and if $(\tau, \beta) \in [\tau_{k,s} - \delta, \tau_{k,s} + \delta] \times [\beta_{k,s} - c, \beta_{k,s} + c]$, then $\mathrm{i}\beta$ is an eigenvalue of $A(\tau)$ if and only if $\tau = \tau_{k,s}$ and $\beta = \beta_{k,s}$. By lemma 2.2, the above b, c and δ can be chosen so that, for the analytic function $p_k(\tau, \lambda)$, we find that $p_k(\tau_{k,s} - \delta, \lambda)$ has no zero in $\bar{\Omega}$, while $p_k(\tau_{k,s} + \delta, \lambda)$ has exactly one zero in $\bar{\Omega}$ and this zero is simple and is in the interior of $\bar{\Omega}$. Thus,

$$\deg_B(p_k(\tau_{k,s} - \delta, \cdot), \Omega) = 0 \quad \text{and} \quad \deg_B(p_k(\tau_{k,s} + \delta, \cdot), \Omega) = 1.$$

Hence,

$$\begin{aligned} c_0(0, \tau_{k,s}, \beta_{k,s}) &= \deg_B(\det \Delta_{0,0}(\tau_{k,s} - \delta, \cdot), \Omega) \\ &\quad - \deg_B(\det \Delta_{0,0}(\tau_{k,s} + \delta, \cdot), \Omega) \\ &= \sum_{j=0}^{[N/2]} \deg_B(p_j(\tau_{k,s} - \delta, \cdot), \Omega) - \sum_{j=0}^{[N/2]} \deg_B(p_j(\tau_{k,s} + \delta, \cdot), \Omega) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} c_1(0, \tau_{k,s}, \beta_{k,s}) &= \deg_B(\det \Delta_{0,1}(\tau_{k,s} - \delta, \cdot), \Omega) \\ &\quad - \deg_B(\det \Delta_{0,1}(\tau_{k,s} + \delta, \cdot), \Omega) \\ &= \sum_{j=1}^{[(N-1)/2]} \deg_B(p_j(\tau_{k,s} - \delta, \cdot), \Omega) - \sum_{j=1}^{[(N-1)/2]} \deg_B(p_j(\tau_{k,s} + \delta, \cdot), \Omega) \\ &= -1. \end{aligned}$$

Let S_j, $j = 0, 1$, denote the closure in $[0, \infty) \times C(\mathbb{R}; \mathbb{R}^N) \times [0, \infty)$ of the set of all triples $(\tau, z, \beta) \notin M^*$ such that

$$x(t) := z\left(\frac{\beta}{2\pi} t\right)$$

is a $2\pi/\beta$-periodic solution of (1.2) with

$$x_i(t) = x_{N+2-i}\left(t - \frac{2\pi}{\beta}\frac{j}{2}\right) \quad \text{for } t \in \mathbb{R} \text{ and } i (\operatorname{mod} N).$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

Then, as discussed in the proof of theorem 3.3, lemma 2.3, together with the test of equation (2.10), implies that S_j must have a non-empty connected component E_j passing through $(\tau_{k,s}, 0, \beta_{k,s})$ and this component must be unbounded. Moreover, we can also conclude that the projection of $c_j(0, \tau_{k,s}, \beta_{k,s})$ on the τ-axis includes $[\tau_{k,s}, \infty)$ and period

$$\omega = \frac{2\pi}{\beta} \in \left(\frac{1}{s+1}, 2\right).$$

Namely, we have the following two conclusions.

THEOREM 3.4. *Assume that (H1)–(H5) hold. Then, for each $\tau > \tau_{k,s}$, $s \in \mathbb{N}_0$, system (1.2) has one standing wave satisfying $x_j(t) = x_{N+j-2}(t - \frac{1}{2}\omega)$ for $t \in \mathbb{R}$ and $j (\mathrm{mod}\, N)$, where ω is a period of x and satisfies that $1/(s+1) < \omega < 2$.*

THEOREM 3.5. *Assume that (H1)–(H5) hold. Then, for each $\tau > \tau_{k,s}$, $s \in \mathbb{N}_0$, system (1.2) has one mirror-reflecting wave satisfying $x_j(t) = x_{N+2-j}(t)$ for $t \in \mathbb{R}$ and $j (\mathrm{mod}\, N)$, with period $\omega \in (1/(s+1), 2)$.*

REMARK 3.6. Due to the D_N-symmetry, theorems 3.3–3.5 in fact imply the existence of N standing waves, N mirror-reflecting waves and two discrete waves for each $\tau > \tau_{k,s}$. It follows from (H1) and (H2) that $\gamma + 4\beta \sin^2(j\pi/N) > 1$ for all $j \in \{k, k+1, \dots, [\frac{1}{2}(N-1)]\}$. Thus, for $j \in \{k, k+1, \dots, [\frac{1}{2}(N-1)]\}$ and $s \in \mathbb{N}_0$, $\tau_{j,s}$ is a critical value of the delay, at each one of which $(2(N+1))$ branches of phase-locked oscillations, standing waves and mirror-reflecting waves may bifurcate simultaneously from the trivial solution. Note that, for $s \in \mathbb{N}_0$,

$$\tau_{k,s} > \tau_{k+1,s} > \tau_{k+2,s} > \cdots > \tau_{[(N-1)/2],s}.$$

Note also that

$$\tau_{k,0} < \tau_{k,1} < \tau_{k,2} < \cdots .$$

The above results establish the existence of (Ns) standing waves, (Ns) mirror-reflecting waves and $(2s)$ discrete waves.

REMARK 3.7. Theorems 3.3–3.5 mean that all $(2(N+1))$ branches of waves are *global*, i.e. all branches of waves exist for $\tau > \tau_{k,s}$.

REMARK 3.8. It should be mentioned that in the above theorems, ω is not necessarily the minimal period and several branches of waves may coincide at some values of τ. We will further discuss the coincidence of these waves in next section.

4. Coincidence of waves

In this section, we discuss whether several branches of waves may coincide at some values of τ.

THEOREM 4.1. *If N is an odd number, then a branch of non-trivial discrete waves and a branch of mirror-reflecting waves cannot coincide at any value of τ.*

Proof. If not, there exists a non-trivial ω-periodic solution x of (1.2) such that

$$x_i(t) = x_{i-1}\left(t \pm \frac{k\omega}{N}\right) \quad \text{for } i (\mathrm{mod}\, N) \text{ and } x_j(t) = x_l(t)$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

for some $j \neq k$. Without loss of generality, we assume that

$$x_i(t) = x_{i-1}\left(t + \frac{k\omega}{N}\right) \quad \text{and} \quad x_i(t) = x_{N+2-i}(t) \quad \text{for } i (\operatorname{mod} N) \text{ and } t \in \mathbb{R}.$$

Then

$$x_{[N/2]+1}(t) = x_{[N/2]+2}(t) = x_{[N/2]+1}\left(t + \frac{k\omega}{N}\right),$$

which implies that $k\omega/N$ is also a period of x. Thus, $x_i(t) = y(t)$ for all $i = 1, 2, \ldots, N$, where $y(t)$ satisfies the scalar equation

$$y'(t) = -\tau y(t) + \tau h(y(t-1)). \tag{4.1}$$

Namely, x is a synchronous periodic solution of (1.2). However, under assumption (H2), it is easy to see that system (4.1) takes $y = 0$ as its global attractor and hence has no non-constant periodic solutions, which is a contradiction. This completes the proof. □

THEOREM 4.2. *If N is an odd number, then a branch of non-trivial discrete waves and a branch of standing waves cannot coincide at any value of τ.*

Proof. If not, there exists a non-trivial ω-periodic solution x of (1.2) such that

$$x_i(t) = x_{i-1}\left(t \pm \frac{k\omega}{N}\right) \quad \text{for } i (\operatorname{mod} N)$$

and $x_j(t) = x_l(t + \frac{1}{2}\omega)$ for some $j \neq k$. Without loss of generality, we assume that $x_i(t) = x_{i-1}(t + k\omega/N)$ and $x_i(t) = x_{N+2-i}(t + \frac{1}{2}\omega)$ for $i (\operatorname{mod} N)$ and $t \in \mathbb{R}$. Then

$$x_{[N/2]+2}(t) = x_{[N/2]+1}\left(t + \frac{k\omega}{N}\right) = x_{[N/2]+1}(t + \tfrac{1}{2}\omega),$$

which implies that $(\frac{1}{2}\omega - k\omega/N)$ is also a period of x. Hence, for all $i (\operatorname{mod} N)$,

$$\begin{aligned} x_{[N/2]+i+2}(t) &= x_{[N/2]+2}\left(t + \frac{k\omega}{N}\right) = x_{[N/2]+i+1}\left(t + \frac{k\omega}{N} - \omega\right) \\ &= x_{[N/2]+i+1}(t - \tfrac{1}{2}\omega) = x_{[N/2]+i}(t). \end{aligned}$$

Namely, $x_{[N/2]+i+2}(t) = x_{[N/2]+i}(t)$ for all $i (\operatorname{mod} N)$. In view of the fact that N is an odd number, it is not difficult to verify that $x_i(t) = y(t)$ for all $i = 1, 2, \ldots, N$, where $y(t)$ satisfies the scalar equation (4.1). Using a similar argument to that above, we can obtain a contradiction and complete the proof. □

THEOREM 4.3. *If N is an odd number, then a branch of non-trivial discrete waves of the form $x_i(t) = x_{i-1}(t - k\omega/N)$ and a branch of discrete waves of the form $x_i(t) = x_{i-1}(t + k\omega/N)$ for $i (\operatorname{mod} N)$ and $t \in \mathbb{R}$ cannot coincide at any value of τ.*

Proof. If not, there exists a non-trivial ω-periodic solution x of (1.2) such that

$$x_i(t) = x_{i-1}\left(t - \frac{k\omega}{N}\right) \quad \text{and} \quad x_i(t) = x_{i-1}\left(t + \frac{k\omega}{N}\right) \quad \text{for } i (\operatorname{mod} N) \text{ and } t \in \mathbb{R}.$$

https://doi.org/10.1017/S0308210505000521 Published online by Cambridge University Press

Then $2k\omega/N$ is also a period of x. Thus,

$$x_{i+2}(t) = x_{i+1}\left(t + \frac{k\omega}{N}\right) = x_i\left(t + \frac{2k\omega}{N}\right) = x_i(t) \quad \text{for all } i (\text{mod } N) \text{ and } t \in \mathbb{R}.$$

In view of the fact that N is an odd number, it is not difficult to verify that $x_i(t) = y(t)$ for all $i = 1, 2, \ldots, N$, where $y(t)$ satisfies the scalar equation (4.1). Using a similar argument to that above, we can obtain a contradiction and complete the proof. □

On the other hand, if N is an even number, then we have some different properties. In fact, if at some value of τ, the system

$$\left.\begin{aligned} \tau^{-1}y'(t) &= -y(t) + h(y(t-1)) + 2g(y(t-1)) - 2g(z(t-1)),\\ \tau^{-1}z'(t) &= -z(t) + h(z(t-1)) + 2g(z(t-1)) - 2g(y(t-1)),\end{aligned}\right\} \tag{4.2}$$

has phase-locked $2k\omega/N$-periodic solutions $(y(t), z(t))^{\mathrm{T}}$ satisfying $y(t) = z(t + k\omega/N)$, then a branch of non-trivial discrete waves of system (1.2) may coincide with a branch of mirror-reflecting waves of system (1.2) such that $x_{2j-1}(t) = y(t)$ and $x_{2j}(t) = z(t)$ for all $j = 1, 2, \ldots, \frac{1}{2}N$. However, if for any $\tau > 0$, there exist no phase-locked $2k\omega/N$-periodic solutions $(y(t), z(t))^{\mathrm{T}}$ of system (4.2) satisfying $y(t) = z(t+k\omega/N)$, then it is easy to see that a branch of non-trivial discrete waves of system (1.2) and a branch of mirror-reflecting waves of system (1.2) do not coincide at any value of τ. Similarly, we can see that the coincidences between a branch of non-trivial discrete waves and a branch of standing waves, between a branch of non-trivial discrete waves of the form $x_i(t) = x_{i-1}(t - k\omega/N)$ and a branch of discrete waves of the form $x_i(t) = x_{i-1}(t + k\omega/N)$, are completely determined by the existence of phase-locked $2k\omega/N$-periodic solutions $(y(t), z(t))^{\mathrm{T}}$ of system (4.2) satisfying $y(t) = z(t + k\omega/N)$.

Because (4.1) has no non-trivial synchronous periodic solutions, we have the following result whatever the value of N.

Theorem 4.4. *A branch of non-trivial mirror-reflecting waves satisfying $x_i(t) = x_j(t)$ for some $i \neq j$ and a branch of mirror-reflecting waves satisfying $x_l(t) = x_m(t)$ for some $l \neq m$ cannot coincide at any value of τ if $(i, j) \neq (l, m)$.*

Unfortunately, the above arguments cannot be extended to rule out the possibility of the coincidence of a branch of non-trivial ω-periodic mirror-reflecting waves with $x_i(t) = x_j(t)$ for some $i \neq j$ and a branch of ω-periodic standing waves with $x_i(t) = x_j(t + \frac{1}{2}\omega)$ for some $i \neq j$. In fact, such a coincidence may occur at some value of τ where period doubling happens: $x_i(t) = x_i(t + \frac{1}{2}\omega)$, $i (\text{mod } N)$, $t \in \mathbb{R}$.

Acknowledgments

This work was partly supported by the Science Foundation of Hunan University, by National Natural Science Foundation of the People's Republic of China (Grants 10371034 and 10271044), and by the Key Project of the Chinese Ministry of Education (no. [2002]78).

References

1 J. C. Alexander. Bifurcation of zeros of parameterized functions. *J. Funct. Analysis* **29** (1978), 37–53.

2 J. C. Alexander and G. Auchmuty. Global bifurcations of phase-locked oscillators. *Arch. Ration. Mech. Analysis* **93** (1986), 253–270.

3 P. Anderson, O. Gross, T. Lomo and O. Sveen. Participation of inhibitory interneurons in the control of hippocampal cortical output. In *The interneuron* (ed. M. A. B. Brazier). (Berkeley: University of California Press, 1969).

4 J. Bélair, S. A. Campbell and P. van den Driessche. Frustration, stability and delay-induced oscillations in a neural network model. *SIAM J. Appl. Math.* **46** (1996), 245–255.

5 Y. Chen, Y. Huang and J. Wu. Desynchronization of large scale delayed neural networks. *Proc. Am. Math. Soc.* **128** (2000), 2365–2371.

6 M. A. Cohen and S. Grossberg. Absolute stability of global pattern formation and parallel memory storage by competitive neural networks. *IEEE Trans. Syst. Man Cybern.* **13** (1983), 815–826.

7 J. C. Eccles, M. Ito and J. Szenfagothai. *The cerebellum as neuronal machine* (Springer, 1967).

8 B. Fiedler. *Global bifurcation of periodic solutions with symmetry*. Lecture Notes in Mathematics, vol. 1309 (Springer, 1988).

9 J. Foss and J. Milton. Multistability in recurrent neural loops arising from delay. *J. Neurophysiol.* **84** (2000), 975–985.

10 J. Foss, F. Moss and J. Milton. Noise, multistability, and delayed recurrent loops. *Phys. Rev.* E **55** (1997), 4536–4543.

11 M. Golubitsky, I. Stewart and D. G. Schaeffer. *Singularities and groups in bifurcation theory* (Springer, 1988).

12 K. Gopalsamy and I. Leung. Delay-induced periodicity in a neural network of excitation and inhibition. *Physica* D **89** (1996), 395–426.

13 S. Guo and L. Huang. Hopf bifurcating periodic orbits in a ring of neurons with delays. *Physica* D **183** (2003), 19–44.

14 S. Guo and L. Huang. Stability analysis of a delayed Hopfield neural network. *Phys. Rev.* E **67** (2003), 061902.

15 S. Guo and L. Huang. Stability of nonlinear waves in a ring of neurons with delays. (Submitted.)

16 J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations* (Springer, 1993).

17 M. W. Hirsch. Convergent activation dynamics in continuous-time networks. *Neural Networks* **2** (1989), 331–349.

18 J. J. Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Natl Acad. Sci. USA* **81** (1984), 3088–3092.

19 L. Huang and J. Wu. Nonlinear waves in networks of neurons with delayed feedback: pattern formation and continuation. *SIAM J. Math. Analysis* **34** (2003), 836–860.

20 W. Krawcewicz and J. Wu. Theory and applications of Hopf bifurcations in symmetric functional differential equations. *Nonlin. Analysis* **35** (1999), 845–870.

21 A. Lasota and J. A. Yorke. Bounds for periodic solutions of differential equations in Banach spaces. *J. Diff. Eqns* **10** (1971), 83–91.

22 C. M. Marcus and R. M. Westervelt. Stability of analog neural networks with delay. *Phys. Rev.* A **39** (1989), 347–359.

23 L. Olien and J. Bélair. Bifurcations, stability and monotonicity propertities of a delayed neural network model. *Physica* D **102** (1997), 349–363.

24 H. G. Othmer and L. E. Scriven. Instability and dynamics pattern in cellular networks. *J. Theor. Biol.* **32** (1971), 507–537.

25 K. Pakdaman, C. Grotta-Ragazzo and C. P. Malta. Transient regime duration in continuous-time neural networks with delay. *Phys. Rev.* E **58** (1998), 3623–3627.

26 K. Pakdaman, C. Grotta-Ragazzo, C. P. Malta, O. Arino and J.-F. Vibert. Effect of delay on the boundary of the basin of attraction in a system of two neurons. *Neural Networks* **11** (1998), 509–519.

27 D. H. Sattinger. Bifurcation and symmetry breaking in applied mathematics. *Bull. Am. Math. Soc.* **3** (1980), 779–819.

28 J. Szenfagothai. The 'module-concept' in cerebral cortex architecture. *Brain Res.* **95** (1967), 475–496.

29 S. A. van Gils and T. Valkering. Hopf bifurcation and symmetry: standing and travelling waves in a circular chain. *Jpn. J. Appl. Math.* **3** (1986), 207–222.

30 J. Wu. Symmetric functional differential equations and neural networks with memory. *Trans. Am. Math. Soc.* **350** (1998), 4799–4838.

(*Issued 14 October 2005*)