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SECOND-ORDER LIMIT LAWS FOR OCCUPATION TIMES OF FRACTIONAL BROWNIAN MOTION

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Abstract

We prove a second-order limit law for additive functionals of a d-dimensional fractional Brownian motion with Hurst index $H = 1/d$, using the method of moments and extending the Kallianpur–Robbins law, and then give a functional version of this result. That is, we generalize it to the convergence of the finite-dimensional distributions for corresponding stochastic processes.

Keywords: Fractional Brownian motion; short-range dependence; limit law; method of moments

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1. Introduction

Let $\{B^H(t) = (B^{H,1}(t), \dots, B^{H,d}(t)),\ t \geq 0\}$ be a d-dimensional fractional Brownian motion (FBM) with Hurst index H in (0, 1). If $Hd = 1$ then the local time of an FBM B^H does not exist; this is called the critical case. For example, two-dimensional Brownian motion ($H = \frac{1}{2}$ and $d = 2$) does not have local time at the origin. There is much work on limit theorems for two-dimensional Brownian motion. Kallianpur and Robbins [6] proved that, for any bounded integrable function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{1}{\log n} \int_0^n f(B^{1/2}(s))\, \mathrm{d}s \xrightarrow{\mathrm{D}} \frac{Z}{2\pi} \int_{\mathbb{R}^2} f(x)\, \mathrm{d}x, \qquad n \to \infty,$$

where '$\xrightarrow{\mathrm{D}}$' denotes convergence in law and Z is an exponentially distributed random variable with mean 1.

Kasahara and Kotani [5] gave a functional version of this result. They also proved the second-order result that, when $\int_{\mathbb{R}^2} f(x)\, \mathrm{d}x = 0$ and $n \to \infty$,

$$\frac{1}{\sqrt{n}} \int_0^{nt\mathrm{e}^{2nt}} f(B^{1/2}(s))\, \mathrm{d}s$$

is weakly M_1-convergent to $\sqrt{\langle f \rangle}\, W(\ell(M^{-1}(t)))$, where

$$\langle f \rangle = -\frac{4}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) f(y) \log|x - y|\, \mathrm{d}x\, \mathrm{d}y,$$

$W(t)$ is a one-dimensional Brownian motion, $\ell(t)$ is the local time at 0 of another one-dimensional Brownian motion $b(t)$ that is independent of $W(t)$, and $M(t) = \max_{0 \leq s \leq t} b(s)$.

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Kasahara and Kotani also pointed out that

$$\frac{1}{\sqrt{n}}\int_0^{e^{nt}} f(B^{1/2}(s))\,ds \xrightarrow{\text{F.D.}} \sqrt{\langle f\rangle}W(\ell(M^{-1}(t))), \qquad n\to\infty, \tag{1.1}$$

where '$\xrightarrow{\text{F.D.}}$' denotes the convergence of finite-dimensional distributions.

The above results were extended to Markov processes (see [2] and [3] and references therein). Subsequently, Kôno [7] extended the Kallianpur–Robbins law to the FBM case:

$$\frac{1}{\log t}\int_0^{t} f(B^{H}(s))\,ds \xrightarrow{\text{D}} \bigg(\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)\,dx\bigg)Z, \qquad t\to\infty.$$

Kasahara and Kosugi [4] obtained the corresponding functional version:

$$\frac{1}{n}\int_0^{e^{nt}} f(B^{H}(s))\,ds \xrightarrow{\text{F.D.}} \bigg(\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)\,dx\bigg)Z(t), \qquad n\to\infty, \tag{1.2}$$

where $Z(t)=\ell(M^{-1}(t))$. The reason for using the normalizing factor $1/n$ instead of $1/\log n$ was pointed out in Remark 1.1 of [4]. However, the corresponding second-order limit law ($\int_{\mathbb{R}^d} f(x)\,dx=0$) was left open.

In this paper we prove second-order limit laws for the above result in [4]. The following two theorems are the main results of this paper. One is the limit theorem for random variables. The other is the convergence of finite-dimensional distributions for stochastic processes.

Theorem 1.1. *Suppose that $Hd=1$, f is bounded and integrable with $\int_{\mathbb{R}^d} f(x)\,dx=0$ and $\int_{\mathbb{R}^d}|f(x)|\,|x|^{\beta}\,dx<\infty$ for some positive constant $\beta>0$. Then, for any $t>0$,*

$$\frac{1}{\sqrt{n}}\int_0^{e^{nt}} f(B^{H}(s))\,ds \xrightarrow{\text{D}} C_{f,d}\sqrt{Z(t)}\eta, \qquad n\to\infty,$$

where

$$C_{f,d}=\bigg(\frac{d\Gamma(d/2)}{\pi^{d/2}(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,dx\bigg)^{1/2},$$

$\widehat{f}$ is the Fourier transform of f, and η is a standard normal random variable independent of $Z(t)$.

Theorem 1.2. *Suppose that $Hd=1$, f is bounded and integrable with $\int_{\mathbb{R}^d} f(x)\,dx=0$ and $\int_{\mathbb{R}^d}|f(x)|\,|x|^{\beta}\,dx<\infty$ for some positive constant $\beta>0$. Then*

$$\frac{1}{\sqrt{n}}\int_0^{e^{nt}} f(B^{H}(s))\,ds \xrightarrow{\text{F.D.}} C_{f,d}W(\ell(M^{-1}(t))) \qquad n\to\infty,$$

where $W(t)$ is a one-dimensional Brownian motion independent of $B^H(\cdot)$ and $\ell(M^{-1}(t))$.

Remark 1.1. Since the process $\ell(M^{-1}(t))$ is not in $C((0,\infty])$, we could not use the Skorokhod J_1-topology. For properties of the process $Z(t)=\ell(M^{-1}(t))$, see [2] and references therein. So far, we still have no idea how to show the weak M_1-convergence of the result in Theorem 1.2 (this has been proved for two-dimensional Brownian motion).

Remark 1.2. Since the function f in the above two theorems is bounded, the constant β in Theorems 1.1 and 1.2 can always be assumed to be less than or equal to 1. Also, the $Hd < 1$ case has been considered in [1, 8].

It is known that FBM with Hurst index not equal to $\frac{1}{2}$ is neither a Markov process nor a semimartingale. Therefore, the methods that have been applied for two-dimensional Brownian motion and Markov processes cannot be used here to prove Theorems 1.1 and 1.2. In proving limit theorems for (additive) functionals of FBM, one often uses the method of moments. Another possible candidate is the Malliavin calculus.

In order to prove Theorems 1.1 and 1.2, we shall utilize the method of moments combined with a chaining argument. The chaining argument was first developed in [8] to prove the central limit theorem for an additive functional of a d-dimensional FBM with Hurst index $H \in (1/(d+2), 1/d)$. It has been proved to be very powerful when obtaining the asymptotic behaviour of moments. To some extent, it reveals the essential difference between the first- and second-order limit laws. However, the situation here is very different from that in [8]: we consider FBMs in the critical case and use a completely different normalizing factor. So, many key techniques applied in [1] and [8] do not work here. For example, we cannot use the scaling technique. Moreover, to use the chaining argument in [8], major modifications and new ideas are required.

The main difficulty when applying the method of moments comes from the convergence of even moments. The techniques used in [1] and [8] fail here. To overcome this difficulty, we first estimate the covariance between two increments of FBM with Hurst index $H < \frac{1}{2}$ and then show that, under certain conditions, these covariances do not contribute to the limit of even moments (see Lemma 2.3 and Step 3 in the proof of Proposition 3.2 for more details). Roughly speaking, the short-range dependence for FBM with Hurst index $H < \frac{1}{2}$ and proper normalizing factor give us a kind of weak independence. This leads to connections with the Brownian motion case in [4].

After some preliminaries in Section 2, Sections 3 and 4 are devoted to the proof of Theorems 1.1 and 1.2, respectively. Throughout the paper, if not mentioned otherwise, c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. Further, $x \cdot y$ denotes the usual inner product in $\mathbb{R}^d$.

2. Preliminaries

Let $\{B^H(t) = (B^{H,1}(t), \dots, B^{H,d}(t)),\ t \geq 0\}$ denote a d-dimensional FBM with Hurst index H in (0, 1), defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. the components of B^H are independent centered Gaussian processes with covariance function

$$\mathbb{E}[B^{H,i}(t)B^{H,i}(s)] = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Our first lemma gives comparable upper and lower bounds for the increments of d-dimensional FBM on n adjacent intervals (for the proof, see, e.g. [1]).

Lemma 2.1. *Given $n \geq 1$, there exist two positive constants κ_1 and κ_2 depending only on n, H, and d, such that, for any $0 = s_0 < s_1 < \cdots < s_n$ and $x_i \in \mathbb{R}^d$, $1 \leq i \leq n$,*

$$\kappa_1 \sum_{i=1}^{n} |x_i|^2 (s_i - s_{i-1})^{2H} \leq \operatorname{var}\bigg(\sum_{i=1}^{n} x_i \cdot [B^H(s_i) - B^H(s_{i-1})]\bigg) \leq \kappa_2 \sum_{i=1}^{n} |x_i|^2 (s_i - s_{i-1})^{2H}.$$

The inequalities in Lemma 2.1 can be reformulated as

$$\kappa_1 \sum_{i=1}^{n} \bigg| \sum_{j=i}^{n} x_j \bigg|^2 (s_i - s_{i-1})^{2H} \le \operatorname{var}\bigg(\sum_{i=1}^{n} x_i \cdot B^H(s_i) \bigg) \le \kappa_2 \sum_{i=1}^{n} \bigg| \sum_{j=i}^{n} x_j \bigg|^2 (s_i - s_{i-1})^{2H}.$$

The next lemma (for the proof, see [9]) gives a formula for the moments of $\sqrt{Z(t)}\eta$, where $Z(t)$ is an exponential random variable with parameter t and η is a standard normal random variable independent of $Z(t)$.

Lemma 2.2. *For any $m \in \mathbb{N}$ and $t > 0$,*

$$\mathbb{E}[(\sqrt{Z(t)}\eta)^{2m}] = \frac{(2m)!\, t^m}{2^m}.$$

Remark 2.1. This result says that $\sqrt{Z(t)}\eta$ has the Laplace distribution.

We also need the following lemma; it plays a very important role in proving the convergence of finite-dimensional distributions.

Lemma 2.3. *For any $H < \frac{1}{2}$, $0 < t_1 < t_2 < t_3 < t_4 < \infty$, and $\Delta t_i = t_i - t_{i-1}$ for $i = 2, 3, 4$, $a := |\mathbb{E}(B^{H,1}(t_4) - B^{H,1}(t_3))(B^{H,1}(t_2) - B^{H,1}(t_1))|$ satisfies*

(i) $a \le 2H\bigg(\dfrac{\Delta t_2}{\Delta t_3}\bigg)^{1/2-H}\bigg(\dfrac{\Delta t_4}{\Delta t_3}\bigg)^{1/2-H}(\Delta t_2)^H(\Delta t_4)^H,$

(ii) $a \le 2\bigg(\dfrac{\Delta t_2 \wedge \Delta t_4}{\Delta t_2 \vee \Delta t_4}\bigg)^{H}(\Delta t_2)^H(\Delta t_4)^H.$

Proof. Kôno [7] noted (ii), so it suffices to prove (i). Since $H < \frac{1}{2}$ and $0 < t_1 < \cdots < t_4$, it is easy to see that

$$\begin{aligned} |a| &= (\Delta t_4 + \Delta t_3)^{2H} + (\Delta t_3 + \Delta t_2)^{2H} - (\Delta t_4 + \Delta t_3 + \Delta t_2)^{2H} - (\Delta t_3)^{2H} \\ &= (\Delta t_3)^{2H}[(1+u)^{2H} + (1+v)^{2H} - (1+u+v)^{2H} - 1], \end{aligned}$$

where $u = \Delta t_2/\Delta t_3$ and $v = \Delta t_4/\Delta t_3$. From Taylor's expansion and $H < \frac{1}{2}$, it follows that

$$[(1+u)^{2H} + (1+v)^{2H} - (1+u+v)^{2H} - 1] \le 2H \min\{u, v\}.$$

Hence, $|a| \le 2H(\Delta t_3)^{2H}\sqrt{uv} = 2H(uv)^{1/2-H}(\Delta t_2)^H(\Delta t_4)^H$, and (i) follows. □

3. Proof of Theorem 1.1

Since f is bounded, we only need to consider the convergence of the random variables

$$F_n = \frac{1}{\sqrt{n}} \int_1^{\mathrm{e}^{nt}} f(B^H(s))\,\mathrm{d}s.$$

For any $m \in \mathbb{N}$, let

$$I_m^n = \frac{m!}{n^{m/2}}\mathbb{E}\bigg[\int_{D_{m,1}} \bigg(\prod_{i=1}^{m} f\Big(B^H(s_i)\Big)\bigg)\,\mathrm{d}s\bigg],$$

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where $D_{m,1} = \{(s_1, \dots, s_m) \in D_m : s_i - s_{i-1} \geq n^{-m},\ i = 2, 3, \dots, m\}$ and $D_m = \{1 < s_1 < \cdots < s_m < \mathrm{e}^{nt}\}$. Then, recalling that f is bounded, we obtain

$$|\mathbb{E}[(F_n)^m] - I_m^n| \leq \frac{m!}{n^{m/2}} \sum_{j=1}^m \mathbb{E}\bigg[\int_{D_m \cap \{|s_j - s_{j-1}| < n^{-m}\}} \bigg(\prod_{i=1}^m |f(B^H(s_i))|\bigg)\,\mathrm{d}s\bigg]$$
$$\leq \|f\|_\infty \frac{mm!}{n^{3m/2}} \mathbb{E}\bigg[\int_{D_{m-1}} \bigg(\prod_{i=1}^{m-1} |f(B^H(s_i))|\bigg)\,\mathrm{d}s\bigg].$$

Thus, by (1.2), we obtain

$$|\mathbb{E}[(F_n)^m] - I_m^n| \leq \frac{c_1}{n^{1+m/2}}. \tag{3.1}$$

Taking Fourier transforms, we can write

$$I_m^n = \frac{m!}{[(2\pi)^d \sqrt{n}]^m} \int_{\mathbb{R}^{md}} \int_{D_{m,1}} \bigg(\prod_{i=1}^m \widehat{f}(x_i)\bigg) \exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{i=1}^m x_i \cdot B^H(s_i)\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}x.$$

Making the change of variables $y_i = \sum_{j=i}^m x_j$ for $i = 1, 2, \dots, m$ yields

$$I_m^n = \frac{m!}{[(2\pi)^d \sqrt{n}]^m} \int_{\mathbb{R}^{md}} \int_{D_{m,1}} \bigg(\prod_{i=1}^m \widehat{f}(y_i - y_{i+1})\bigg)$$
$$\times \exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{i=1}^m y_i \cdot [B^H(s_i) - B^H(s_{i-1})]\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}y.$$

Set $I_{m,0}^n = I_m^n$, and for $k = 1, \dots, m$, define

$$I_{m,k}^n = \frac{m!}{[(2\pi)^d \sqrt{n}]^m} \int_{\mathbb{R}^{md}} \int_{D_{m,1}} I_k \prod_{i=k+1}^m \widehat{f}(y_i - y_{i+1})$$
$$\times \exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{i=1}^m y_i \cdot [B^H(s_i) - B^H(s_{i-1})]\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}y,$$

where

$$I_k = \begin{cases} \displaystyle\prod_{j=1}^{(k-1)/2} |\widehat{f}(y_{2j})|^2 \widehat{f}(-y_{k+1}) & \text{if } k \text{ is odd}, \\ \displaystyle\prod_{j=1}^{k/2} |\widehat{f}(y_{2j})|^2 & \text{if } k \text{ is even}. \end{cases}$$

The following proposition, which is similar to Proposition 4.1 in [8], controls the difference between $I_{m,k-1}^n$ and $I_{m,k}^n$. Fix a positive constant λ strictly less than $\frac{1}{2}$.

Proposition 3.1. *For $k = 1, 2, \dots, m$, there exists a positive constant c, which depends only on λ, such that*

$$|I_{m,k-1}^n - I_{m,k}^n| \leq c n^{-\lambda}.$$

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Proof. Consider first the case that k is odd. Making the change of variables $u_1 = s_1$, $u_i = s_i - s_{i-1}$ for $2 \le i \le m$, and then applying Lemma 2.1, it follows that $|I^n_{m,k-1} - I^n_{m,k}|$ is less than a constant multiple of

$$n^{-m/2} \int_{\mathbb{R}^{md}} \int_{[n^{-m},\, \mathrm{e}^{nt}]^m} \bigg(\prod_{i=k+1}^{m} |\widehat{f}(y_i - y_{i+1})| \bigg) |\widehat{f}(y_k - y_{k+1}) - \widehat{f}(-y_{k+1})| \times \bigg(\prod_{j=1}^{(k-1)/2} |\widehat{f}(y_{2j})|^2 \bigg) \exp\bigg(-\frac{1}{2}\kappa_1 \sum_{i=1}^{m} |y_i|^2 u_i^{2H} \bigg) \,\mathrm{d}u \,\mathrm{d}y$$

with the convention $y_{m+1} = 0$. The assumptions on f imply that $|\widehat{f}(x)| \le c_\alpha(|x|^\alpha \wedge 1)$ for any $\alpha \in [0, \beta]$, so $|I^n_{m,k-1} - I^n_{m,k}|$ is less than a constant multiple of

$$n^{-m/2} \int_{\mathbb{R}^{md}} \int_{[n^{-m},\, \mathrm{e}^{nt}]^m} |y_k|^\alpha \prod_{j=(k+1)/2}^{\lfloor m/2 \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha) \times \bigg(\prod_{j=1}^{(k-1)/2} |\widehat{f}(y_{2j})|^2 \bigg) \exp\bigg(-\frac{1}{2}\kappa_1 \sum_{i=1}^{m} |y_i|^2 u_i^{2H} \bigg) \,\mathrm{d}u \,\mathrm{d}y.$$

Integrating with respect to the y_i and u_i with $i \le k-1$, $|I^n_{m,k-1} - I^n_{m,k}|$ is bounded above by

$$\frac{c_1}{n^{(m-k+1)/2}} \int_{\mathbb{R}^{(m-k+1)d}} \int_{[n^{-m},\, \mathrm{e}^{nt}]^{m-k+1}} |y_k|^\alpha \prod_{j=(k+1)/2}^{\lfloor m/2 \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha) \times \exp\bigg(-\frac{1}{2}\kappa_1 \sum_{i=k}^{m} |y_i|^2 u_i^{2H} \bigg) \,\mathrm{d}\overline{u} \,\mathrm{d}\overline{y},$$

where $\mathrm{d}\overline{u} = \mathrm{d}u_k \cdots \mathrm{d}u_m$ and $\mathrm{d}\overline{y} = \mathrm{d}y_k \cdots \mathrm{d}y_m$. After some calculus,

$$\begin{aligned} |I^n_{m,k-1} - I^n_{m,k}| &\le c_2 n^{-(m-k+1)/2 + (\lfloor (m-k+1)/2 \rfloor + 1)(mH\alpha) + (m-k-\lfloor (m-k+1)/2 \rfloor)} \\ &= c_2\, n^{m/2 - \lfloor m/2 \rfloor - 1 + (\lfloor (m-k+1)/2 \rfloor + 1)(mH\alpha)}. \end{aligned}$$

Choose α so small that $\frac{1}{2}m - \lfloor \frac{1}{2}m \rfloor - 1 + (\lfloor \frac{1}{2}(m-k+1) \rfloor + 1)(mH\alpha) = -\lambda$; then

$$|I^n_{m,k-1} - I^n_{m,k}| \le c_2 n^{-\lambda}. \tag{3.2}$$

Now let k be even. By Lemma 2.1, $|I^n_{m,k-1} - I^n_{m,k}|$ is less than a constant multiple of

$$n^{-m/2} \int_{\mathbb{R}^{md}} \int_{[n^{-m},\, \mathrm{e}^{nt}]^m} |\widehat{f}(-y_k)| |\widehat{f}(y_k - y_{k+1}) - \widehat{f}(y_k)| \bigg(\prod_{i=k+1}^{m} |\widehat{f}(y_i - y_{i+1})| \bigg) \times \bigg(\prod_{j=1}^{k/2-1} |\widehat{f}(y_{2j})|^2 \bigg) \exp\bigg(-\frac{1}{2}\kappa_1 \sum_{i=1}^{m} |y_i|^2 u_i^{2H} \bigg) \,\mathrm{d}u \,\mathrm{d}y.$$

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Using arguments similar to those for the case of odd k,

$$|I^n_{m,k-1} - I^n_{m,k}| \le \frac{c_3}{n^{m/2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m},\, \mathrm{e}^{nt}]^m} |y_k|^\alpha |y_{k+1}|^\alpha \prod_{j=k/2+1}^{\lfloor m/2 \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha)$$

$$\times \bigg(\prod_{j=1}^{k/2-1} |\widehat{f}(y_{2j})|^2 \bigg) \exp\bigg(-\frac{\kappa_1}{2} \sum_{i=1}^m |y_i|^2 u_i^{2H} \bigg) \, \mathrm{d}u \, \mathrm{d}y$$

$$\le c_4\, n^{-(m-(k-2))/2+(\lfloor (m-k)/2 \rfloor + 2)(mH\alpha) + (m-k-\lfloor (m-k)/2 \rfloor)}$$

$$= c_4\, n^{m/2 - \lfloor m/2 \rfloor - 1 + (\lfloor (m-k)/2 \rfloor + 2)(mH\alpha)}.$$

Choose α so small that $\frac{1}{2}m - \lfloor \frac{1}{2}m \rfloor - 1 + (\lfloor \frac{1}{2}(m-k) \rfloor + 2)(mH\alpha) = -\lambda$; then

$$|I^n_{m,k-1} - I^n_{m,k}| \le c_4 n^{-\lambda}. \tag{3.3}$$

Combining (3.2) and (3.3) gives the desired estimates. □

Proposition 3.2. *Under the assumptions of Theorem 1.1, for any $t > 0$,*

$$\frac{1}{\sqrt{n}} \int_1^{\mathrm{e}^{nt}} f(B^H(s)) \mathrm{d}s \xrightarrow{\mathrm{D}} C_{f,d} \sqrt{Z(t)}\, \eta, \qquad n \to \infty,$$

where

$$C_{f,d} = \bigg(\frac{d\Gamma(d/2)}{\pi^{d/2}(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-d} \, \mathrm{d}x \bigg)^{1/2},$$

$\widehat{f}$ is the Fourier transform of f, $Z(t)$ is an exponential random variable with parameter t, and η is a standard normal random variable independent of $Z(t)$.

Proof. We give the proof in several steps.

Step 1. First we show tightness. Let $F_n = (1/\sqrt{n}) \int_1^{\mathrm{e}^{nt}} f(B^H(s)) \, \mathrm{d}s$. Using Fourier transforms,

$$\mathbb{E}[(F_n)^2] = \frac{2}{(2\pi)^{2d} n} \int_1^{\mathrm{e}^{nt}} \int_1^{s_2} \int_{\mathbb{R}^{2d}} \widehat{f}(x_1) \widehat{f}(x_2) \exp\bigg[-\frac{1}{2} \operatorname{var}\bigg(\sum_{i=1}^2 x_i \cdot B^H(s_i) \bigg) \bigg] \mathrm{d}s \, \mathrm{d}x.$$

Since $|\widehat{f}(x)| \le c_\alpha (|x|^\alpha \wedge 1)$ for all $x \in \mathbb{R}^d$ and any $\alpha \in [0, \beta]$, by Lemma 2.1, $\mathbb{E}[(F_n)^2]$ is bounded above by

$$\frac{c_1}{n} \int_1^{\mathrm{e}^{nt}} \int_1^{s_2} \int_{\mathbb{R}^{2d}} |\widehat{f}(x_2)| \exp\bigg(-\frac{\kappa_1 |x_2|^2 (s_2 - s_1)^{2H}}{2} - \frac{\kappa_1 |x_2 + x_1|^2 s_1^{2H}}{2} \bigg) \mathrm{d}s \, \mathrm{d}x$$

$$\le \frac{c_2}{n} \bigg(\int_1^{\mathrm{e}^{nt}} s_1^{-1} \, \mathrm{d}s_1 \bigg) \bigg(\int_{\mathbb{R}^d} |\widehat{f}(x_2)| \, |x_2|^{-d} \, \mathrm{d}x_2 \bigg)$$

$$\le c_3\, t.$$

Step 2. We show the convergence of moments when m is odd. Recall the estimate (3.1), which allows us to replace $\mathbb{E}[(F_n)^m]$ by I^n_m. By Proposition 3.1, we only need to show that

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$\lim_{n\to\infty} I^n_{m,m} = 0$, where

$$I^n_{m,m} = \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{D_{m,1}} \widehat{f}(y_m)\prod_{j=1}^{(m-1)/2}|\widehat{f}(y_{2j})|^2 \times \exp\biggl[-\frac{1}{2}\operatorname{var}\biggl(\sum_{i=1}^m y_i\cdot[B^H(s_i)-B^H(s_{i-1})]\biggr)\biggr]\,\mathrm{d}s\,\mathrm{d}y.$$

Make the change of variables $u_1 = s_1$, $u_i = s_i - s_{i-1}$ for $2\le i\le m$. By Lemma 2.1,

$$|I^n_{m,m}| \le \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{O_m}|\widehat{f}(y_m)|\prod_{j=1}^{(m-1)/2}|\widehat{f}(y_{2j})|^2\exp\biggl(-\frac{\kappa_1}{2}\sum_{i=1}^m|y_i|^2u_i^{2H}\biggr)\,\mathrm{d}u\,\mathrm{d}y,$$

where, much as for $D_{m,1}$ and D_m in the proof of Theorem 1.1,

$$O_m = \biggl\{(u_1,\dots,u_m)\colon 1<u_1,\ \sum\nolimits_{i=1}^m u_i<\mathrm{e}^{nt},\ n^{-m}<u_i<\mathrm{e}^{nt},\ i=2,\dots,m\biggr\}.$$

Since $O_m\subset[1,\mathrm{e}^{nt}]\times[n^{-m},\mathrm{e}^{nt}]^{m-1}$, $|I^n_{m,m}|$ is bounded above by

$$\begin{aligned}
&\frac{c_4}{n^{m/2}}\int_{\mathbb{R}^{md}}\int_{[1,\mathrm{e}^{nt}]\times[n^{-m},\mathrm{e}^{nt}]^{m-1}}|\widehat{f}(y_m)|\prod_{j=1}^{(m-1)/2}|\widehat{f}(y_{2j})|^2\exp\biggl(-\frac{\kappa_1}{2}\sum_{i=1}^m|y_i|^2\,u_i^{2H}\biggr)\,\mathrm{d}u\,\mathrm{d}y\\
&\quad\le\frac{c_5}{n^{m/2}}\biggl[\int_{\mathbb{R}^d}|\widehat{f}(y)|^2|y|^{-d}\mathrm{d}y\int_{n^{-m}}^{\mathrm{e}^{nt}}u^{-1}\,\mathrm{d}u\biggr]^{(m-1)/2}\int_{\mathbb{R}^d}|\widehat{f}(y)|\,|y|^{-d}\,\mathrm{d}y\\
&\quad\le\frac{c_6}{n^{1/2}}.
\end{aligned}$$

Combining these estimates gives $\lim_{n\to\infty}\mathbb{E}[(F_n)^m]=0$ when m is odd.

Step 3. Now we show the convergence of moments when m is even. Recall the estimate (3.1). By Proposition 3.1, it suffices to show that

$$\lim_{n\to\infty} I^n_{m,m} = \biggl[\frac{d\Gamma(d/2)}{\pi^{d/2}}(2\pi)^d\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\biggr]^{m/2}\mathbb{E}[(\sqrt{Z(t)}\eta)^m], \tag{3.4}$$

where

$$I^n_{m,m} = \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{D_{m,1}}\biggl(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\biggr) \tag{3.5}$$

$$\times\exp\biggl[-\frac{1}{2}\operatorname{var}\biggl(\sum_{i=1}^m y_i\cdot[B^H(s_i)-B^H(s_{i-1})]\biggr)\biggr]\,\mathrm{d}s\,\mathrm{d}y. \tag{3.6}$$

For $i=1,2,\dots,m$, set $\Delta s_i = s_i - s_{i-1}$ with the convention $s_0=0$. Define

$$\widehat{I}^n_{m,m} = \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{\widehat{O}_m}\biggl(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\biggr) \times\exp\biggl[-\frac{1}{2}\operatorname{var}\biggl(\sum_{i=1}^m y_i\cdot[B^H(s_i)-B^H(s_{i-1})]\biggr)\biggr]\,\mathrm{d}s\,\mathrm{d}y,$$

where $\widehat{O}_m = \{(s_1,s_2,\dots,s_m)\colon 1<\Delta s_1<\infty,\ n^{-m}<\Delta s_i<\mathrm{e}^{nt},\ i=2,\dots,m\}$.

Observe that $D_{m,1} \subseteq \widehat{O}_m$, so $I^n_{m,m} \le \widehat{I}^n_{m,m}$.

Define $\widetilde{I}^n_{m,m}$ much as $\widehat{I}^n_{m,m}$ above except that $\widehat{O}_m$ is replaced by

$$\widetilde{O}_m := \big\{(s_1, s_2, \dots, s_m) \colon n^2 < \Delta s_{2i-1} < \mathrm{e}^{nt},\ n^{-1} < \Delta s_{2i} < n,\ i = 1, 2, \dots, \tfrac{1}{2}m\big\}.$$

Then, by Lemma 2.1,

$$\lim_{n\to\infty}\sup I^n_{m,m} = \lim_{n\to\infty}\sup \widetilde{I}^n_{m,m}. \tag{3.7}$$

For any constant $\gamma > 1$, define

$$\widetilde{I}^{n,\gamma}_{m,m} = \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{\widetilde{O}_{m,\gamma}}\bigg(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\bigg) \times \exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{i=1}^{m} y_i\cdot[B^H(s_i) - B^H(s_{i-1})]\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}y, \tag{3.8}$$

where

$$\widetilde{O}_{m,\gamma} = \widetilde{O}_m \cap \bigg\{\frac{\Delta s_{2i-1}}{\Delta s_{2j-1}} > \gamma \text{ or } \frac{\Delta s_{2j-1}}{\Delta s_{2i-1}} > \gamma d \text{ for all } i, j = 1, 2, \dots, \frac{m}{2}\bigg\}.$$

By Lemma 2.1, $|\widetilde{I}^n_{m,m} - \widetilde{I}^{n,\gamma}_{m,m}|$ is less than a constant multiple of $(\log\gamma)/n$. Note that

$$\operatorname{var}\bigg(\sum_{i=1}^{m} y_i\cdot[B^H(s_i) - B^H(s_{i-1})]\bigg) = \sum_{i=1}^{m}|y_i|^2(\Delta s_i)^{2H} + 2\sum_{i<j}(y_i, y_j)a_{ij}, \tag{3.9}$$

where $a_{ij} = -\frac{1}{2}[(s_j - s_i)^{2H} + (s_{j-1} - s_{i-1})^{2H} - (s_j - s_{i-1})^{2H} - (s_{j-1} - s_i)^{2H}]$. Recall that $H = 1/d \le \frac{1}{2}$. By Lemma 2.3, the following estimates hold on the domain $\widetilde{O}_{m,\gamma}$:

- if both i and j are even then

$$|a_{ij}| \le c_7(1 - 2H)\frac{(\Delta s_i)^H(\Delta s_j)^H}{n^{(1-2H)}};$$

- if both i and j are odd then

$$|a_{ij}| \le c_7(1 - 2H)\bigg(\frac{\Delta s_i \wedge \Delta s_j}{\Delta s_i \vee \Delta s_j}\bigg)^H(\Delta s_i)^H(\Delta s_j)^H \le c_7(1 - 2H)\frac{(\Delta s_i)^H(\Delta s_j)^H}{\gamma^H};$$

- if just one of i and j is odd then

$$|a_{ij}| \le c_7(1 - 2H)\bigg(\frac{\Delta s_i \wedge \Delta s_j}{\Delta s_i \vee \Delta s_j}\bigg)^H(\Delta s_i)^H(\Delta s_j)^H \le c_7(1 - 2H)\frac{(\Delta s_i)^H(\Delta s_j)^H}{n^H}.$$

Let $\theta = (1 - 2H) \wedge H$. It follows from the Cauchy–Schwarz inequality and the estimates above that the variance at (3.9) satisfies

$$\bigg(1 - \frac{c_8}{\gamma^H} - \frac{c_9}{n^\theta}\bigg)\sum_{i=1}^{m}|y_i|^2(\Delta s_i)^{2H} \le \operatorname{var}\bigg(\sum_{i=1}^{m} y_i\cdot[B^H(s_i) - B^H(s_{i-1})]\bigg) \le \bigg(1 + \frac{c_8}{\gamma^H} + \frac{c_9}{n^\theta}\bigg)\sum_{i=1}^{m}|y_i|^2(\Delta s_i)^{2H}.$$

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Substituting the lower bound above in (3.8) yields

$$\begin{aligned}\widetilde{I}^{n,\gamma}_{m,m} &\le \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{\widetilde{O}_{m,\gamma}}\bigg(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\bigg)\\ &\qquad\qquad \times \exp\bigg[-\frac{1}{2}\bigg(1-\frac{c_8}{\gamma^H}-\frac{c_9}{n^\theta}\bigg)\sum_{i=1}^m |y_i|^2(\Delta s_i)^{2H}\bigg]\,\mathrm{d}s\,\mathrm{d}y\\ &\le \frac{m!}{[(2\pi)^d\sqrt{n}]^m}\int_{\mathbb{R}^{md}}\int_{\widetilde{O}_{m}}\bigg(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\bigg)\\ &\qquad\qquad \times \exp\bigg[-\frac{1}{2}\bigg(1-\frac{c_8}{\gamma^H}-\frac{c_9}{n^\theta}\bigg)\sum_{i=1}^m |y_i|^2(\Delta s_i)^{2H}\bigg]\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Evaluation via calculus shows that $\limsup_{n\to\infty}\widetilde{I}^{n,\gamma}_{m,m}$ is bounded above by

$$\frac{m!\,t^{m/2}}{(1-c_8/\gamma^H)^{md/2}}\bigg(\frac{1}{(2\pi)^{d/2}}\int_0^\infty \mathrm{e}^{-u^{2H}/2}\,\mathrm{d}u\bigg)^{m/2}\bigg(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\bigg)^{m/2}.$$

Trivially, $\limsup_{n\to\infty}\widetilde{I}^n_{m,m} \le \limsup_{n\to\infty}\widetilde{I}^{n,\gamma}_{m,m} + \limsup_{n\to\infty}|\widetilde{I}^n_{m,m}-\widetilde{I}^{n,\gamma}_{m,m}|$. Combining these inequalities with (3.7) gives

$$\limsup_{n\to\infty} I^n_{m,m} \le m!\,t^{m/2}\bigg[\frac{1}{(2\pi)^{d/2}}\int_0^\infty \mathrm{e}^{-u^{2H}/2}\,\mathrm{d}u\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\bigg]^{m/2}. \tag{3.10}$$

Recall the definition of $I^n_{m,m}$ in (3.6). Integrating with respect to y_{m-1} and then using the inequality

$$\int_{\mathbb{R}^d}\mathrm{e}^{-|x_1|^2u^{2H}/2-vx_1x_2}\,\mathrm{d}x_1 \ge \frac{(2\pi)^{d/2}}{u}, \tag{3.11}$$

leads to

$$\begin{aligned}I^n_{m,m} &\ge \frac{m!}{(2\pi)^{(m-1/2)d}n^{m/2}}\\ &\quad\times\int_{\mathbb{R}^{(m-1)d}}\int_{D_{m,1}}\bigg(\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\bigg)(\Delta s_{m-1})^{-1}\\ &\qquad\qquad\times\exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{i=1,\,i\ne m-1}^m y_i\cdot[B^H(s_i)-B^H(s_{i-1})]\bigg)\bigg]\\ &\qquad\qquad\times\,\mathrm{d}s\,\mathrm{d}y_1\cdots\mathrm{d}y_{m-2}\,\mathrm{d}y_m.\end{aligned}$$

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Repeating the above procedure for all the other y_i with i odd gives

$$I_{m,m}^n \geq \frac{m!}{(2\pi)^{3md/4} n^{m/2}} \times \int_{\mathbb{R}^{md/2}} \int_{D_{m,1}} \bigg(\prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2\bigg)\bigg(\prod_{j=1}^{m/2} (\Delta s_{2j-1})^{-1}\bigg) \times \exp\bigg[-\frac{1}{2}\operatorname{var}\bigg(\sum_{j=1}^{m/2} y_{2j}\cdot [B^H(s_{2j}) - B^H(s_{2j-1})]\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}\overline{y},$$

where $\mathrm{d}\overline{y} = \mathrm{d}y_2\,\mathrm{d}y_4\cdots \mathrm{d}y_m$.

Define

$$D_{m,2} = \bigg\{(s_1, s_2, \dots, s_m) \colon n^2 \leq \Delta s_{2j-1} \leq \frac{\mathrm{e}^{nt}}{m},\ n^{-1} < \Delta s_{2i} < n,\ j = 1, 2, \dots, \frac{m}{2}\bigg\}.$$

Note that $D_{m,2} \subseteq D_{m,1}$ when n is large enough. Applying Lemma 2.3,

$$\begin{aligned}\liminf_{n\to\infty} I_{m,m}^n &\geq \liminf_{n\to\infty} \frac{m!}{(2\pi)^{3md/4} n^{m/2}} \times \int_{\mathbb{R}^{md/2}} \int_{D_{m,2}} \bigg(\prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2\bigg)\bigg(\prod_{j=1}^{m/2} (\Delta s_{2j-1})^{-1}\bigg) \\ &\qquad \times \exp\bigg[-\frac{1}{2}\bigg(1 + \frac{c_{10}(1-2H)}{n^{1-2H}}\bigg)\sum_{j=1}^{m/2} |y_{2j}|^2 (\Delta s_{2j})^{2H}\bigg]\,\mathrm{d}s\,\mathrm{d}\overline{y} \\ &= m!\, t^{m/2}\bigg[\frac{1}{(2\pi)^{d/2}}\int_0^\infty \mathrm{e}^{-u^{2H}/2}\,\mathrm{d}u\bigg]^{m/2}\bigg[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-d}\,\mathrm{d}x\bigg]^{m/2}. \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12) gives

$$\begin{aligned}\lim_{n\to\infty} I_{m,m}^n &= m!\, t^{m/2}\bigg[\frac{1}{(2\pi)^{d/2}}\int_0^\infty \mathrm{e}^{-u^{2H}/2}\,\mathrm{d}u \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-d}\,\mathrm{d}x\bigg]^{m/2} \\ &= C_{f,d}^m \mathbb{E}(\sqrt{Z(t)}\eta)^m,\end{aligned}$$

where in the last equality we used Lemma 2.2 and the identity

$$\frac{2}{(2\pi)^{d/2}}\int_0^\infty \mathrm{e}^{-u^{2H}/2}\,\mathrm{d}u = \frac{d}{\pi^{d/2}}\Gamma\bigg(\frac{d}{2}\bigg).$$

So the statement follows. Using the method of moments completes the proof. □

Since f is bounded, the proof of Theorem 1.1 now follows easily from Proposition 3.2. □

Remark 3.1. Recall the convergence of finite-dimensional distributions for two-dimensional Brownian motion in (1.1). Theorem 1.1 implies that

$$\int_{\mathbb{R}^2} |\widehat{f}(x)|^2 |x|^{-2}\,\mathrm{d}x = -8\pi^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x) f(y) \log|x-y|\,\mathrm{d}x\,\mathrm{d}y$$

for all bounded functions f with compact support such that $\int_{\mathbb{R}^2} f(x)\,\mathrm{d}x = 0$.

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4. Proof of Theorem 1.2

In this section we prove Theorem 1.2, the convergence of finite-dimensional distributions. Let $F_n(t) = (1/\sqrt{n}) \int_0^{e^{nt}} f(B^H(s))\,\mathrm{d}s$. We only need to prove that the finite-dimensional distributions of $F_n(t)$ converge to the corresponding distributions of $C_{f,d} W(\ell(M^{-1}(t)))$.

Fix a finite number of disjoint intervals $(a_i, b_i]$ with $i = 1, \dots, N$ and $b_i \le a_{i+1}$. Let $\boldsymbol{m} = (m_1, \dots, m_N)$ be a fixed multi-index with $m_i \in \mathbb{N}$ for $i = 1, \dots, N$. Set $|\boldsymbol{m}| = \sum_{i=1}^N m_i$ and $\boldsymbol{m}! = \prod_{i=1}^N m_i!$. We need to consider the sequence of random variables

$$G_n := \prod_{i=1}^N [F_n(b_i) - F_n(a_i)]^{m_i}$$

and compute $\lim_{n\to\infty} \mathbb{E}(G_n)$. This expectation can be expressed as

$$\mathbb{E}(G_n) = \boldsymbol{m}!\, n^{-|\boldsymbol{m}|/2} \mathbb{E}\bigg[\int_{D_{\boldsymbol{m}}} \prod_{i=1}^N \prod_{j=1}^{m_i} f(B^H(s_j^i))\,\mathrm{d}s\bigg],$$

where $D_{\boldsymbol{m}} = \{s \in \mathbb{R}^{|\boldsymbol{m}|} \colon e^{na_i} < s_1^i < \cdots < s_{m_i}^i < e^{nb_i},\ 1 \le i \le N\}$. Here and in the sequel we denote the coordinates of a point $s \in \mathbb{R}^{|\boldsymbol{m}|}$ by $s = (s_j^i)$, where $1 \le i \le N$ and $1 \le j \le m_i$. Define

$$D_{\boldsymbol{m},1} = \{s \in D_{\boldsymbol{m}} \colon s_j^i - s_{j-1}^i \ge n^{-|\boldsymbol{m}|},\ 1 \le j \le m_i,\ 1 \le i \le N\}, \tag{4.1}$$

with the convention that$s_0^i = s_{m_{i-1}}^{i-1}$ for $1 \le i \le N$.

For simplicity of notation, define

$$J_0 = \{(i, j) \colon 1 \le i \le N, 1 \le j \le m_i\}.$$

For any (i_1, j_1) and $(i_2, j_2) \in J_0$, we define the following dictionary ordering:

$$(i_1, j_1) \le (i_2, j_2)$$

if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 \le j_2$. For any (i, j) in J_0, under the above ordering, (i, j) is the $(\sum_{k=1}^{i-1} m_k + j)$th element in J_0 and we define $\#(i, j) = \sum_{k=1}^{i-1} m_k + j$.

Proposition 4.1. *Suppose that at least one of the exponents m_i is odd. Then*

$$\lim_{n\to\infty} \mathbb{E}(G_n) = 0.$$

Proof. Let $M_n = \boldsymbol{m}!\, n^{-|\boldsymbol{m}|/2} \mathbb{E}[\int_{D_{\boldsymbol{m},1}} \prod_{i=1}^N \prod_{j=1}^{m_i} f(B^H(s_j^i))\,\mathrm{d}s]$.

It is easy to see that $\lim_{n\to\infty}[\mathbb{E}(G_n) - M_n] = 0$, so it is enough to show that $\lim_{n\to\infty} M_n = 0$. By taking Fourier transforms, we see that M_n is equal to

$$\frac{\boldsymbol{m}!\, n^{-|\boldsymbol{m}|}/2}{(2\pi)^{|\boldsymbol{m}|d}} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\mathbf{m},1}} \bigg(\prod_{i=1}^N \prod_{j=1}^{m_i} \widehat{f}(y_j^i)\bigg) \exp\bigg[-\frac{1}{2} \operatorname{var}\bigg(\sum_{i=1}^N \sum_{j=1}^{m_i} y_j^i \cdot B^H(s_j^i)\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}y.$$

Making the change of variables $x_j^i = \sum_{(\ell,k)\ge(i,j)} y_k^\ell$ for $1 \le i \le N$ and $1 \le j \le m_i$,

$$M_n = \frac{\boldsymbol{m}!\, n^{-|\boldsymbol{m}|}/2}{(2\pi)^{|\boldsymbol{m}|d}} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \prod_{i=1}^N \prod_{j=1}^{m_i} \widehat{f}(x_j^i - x_{j+1}^i)$$
$$\times \exp\bigg[-\frac{1}{2} \operatorname{var}\bigg(\sum_{i=1}^N \sum_{j=1}^{m_i} x_j^i \cdot [B^H(s_j^i) - B^H(s_{j-1}^i)]\bigg)\bigg]\,\mathrm{d}s\,\mathrm{d}x.$$

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Applying Proposition 3.1, we obtain

$$\lim_{n\to\infty} M_n = \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \lim_{n\to\infty} n^{-|\boldsymbol{m}|/2} \times \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \Big(\prod_{(i,j)\in J_e} |\widehat{f}(x_j^i)|^2\Big) I_{|\boldsymbol{m}|} \times \exp\Big[-\frac{1}{2}\operatorname{var}\Big(\sum_{i=1}^N \sum_{j=1}^{m_i} x_j^i \cdot [B^H(s_j^i) - B^H(s_{j-1}^i)]\Big)\Big] \,\mathrm{d}s\,\mathrm{d}x,$$

where $J_e = \{(i,j) \in J_0 \colon \#(i,j) \text{ is even}\}$ and

$$I_{|\boldsymbol{m}|} = \begin{cases} \widehat{f}(x_{m_N}^N) & \text{if } |\boldsymbol{m}| \text{ is odd,} \\ 1 & \text{if } |\boldsymbol{m}| \text{ is even.} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} M_n = 0$ when $|\boldsymbol{m}|$ is odd. When $|\boldsymbol{m}|$ is even we show that $\lim_{n\to\infty} M_n = 0$ as follows:

$$\lim_{n\to\infty} M_n = \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \lim_{n\to\infty} n^{-|\boldsymbol{m}|/2} \times \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \Big(\prod_{(i,j)\in J_e} |\widehat{f}(x_j^i)|^2\Big) \times \exp\Big[-\frac{1}{2}\operatorname{var}\Big(\sum_{i=1}^N \sum_{j=1}^{m_i} x_j^i \cdot [B^H(s_j^i) - B^H(s_{j-1}^i)]\Big)\Big] \,\mathrm{d}s\,\mathrm{d}x.$$

Observe that the right-hand side of this equality is positive. By Lemma 2.1,

$$\begin{aligned} &\limsup_{n\to\infty} |M_n| \\ &\quad\le c_1 \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \Big(\prod_{(i,j)\in J_e} |\widehat{f}(x_j^i)|^2\Big) \times \exp\Big[-\frac{\kappa_1}{2} \sum_{i=1}^N \sum_{j=1}^{m_i} |x_j^i|^2 (s_j^i - s_{j-1}^i)^{2H}\Big] \,\mathrm{d}s\,\mathrm{d}x \\ &\quad:= c_1 \limsup_{n\to\infty} I_n. \end{aligned}$$

Let m_ℓ be the first odd exponent of $\boldsymbol{m}$. If $s_{m_\ell}^\ell \ge \mathrm{e}^{nb_\ell} - nb_\ell$ then integrating with respect to the proper x_j^i and s_j^i gives

$$I_n \le \frac{c_2}{n} \sup_{s_{m_\ell-1}^\ell \in (\mathrm{e}^{na_\ell}, \mathrm{e}^{nb_\ell})} \int_{\mathbb{R}^d} \int_{Q_1} \frac{|\widehat{f}(x_1^{\ell+1})|^2}{s_{m_\ell}^\ell - s_{m_\ell-1}^\ell} \times \exp\big[-\tfrac{1}{2}\kappa_1 |x_1^{\ell+1}|^2 (s_1^{\ell+1} - s_{m_\ell}^\ell)^{2H}\big] \,\mathrm{d}s_1^{\ell+1}\,\mathrm{d}s_{m_\ell}^\ell\,\mathrm{d}x_1^{\ell+1},$$

where $Q_1 = \{e^{na_{\ell+1}} < s_1^{\ell+1} < e^{nb_{\ell+1}},\ e^{nb_\ell} - nb_\ell \le s_{m_\ell}^\ell < e^{nb_\ell},\ s_{m_\ell}^\ell - s_{m_{\ell-1}}^\ell \ge n^{-|\boldsymbol{m}|}\}$. Integrating with respect to $s_1^{\ell+1}$ and $x_1^{\ell+1}$ gives

$$I_n \le \frac{c_3}{n} \sup_{s_{m_\ell-1}^\ell \in (e^{na_\ell}, e^{nb_\ell})} \int_{(s_{m_\ell-1}^\ell + n^{-|\boldsymbol{m}|}) \vee (e^{nb_\ell} - nb_\ell)}^{e^{nb_\ell}} \frac{ds_{m_\ell}^\ell}{s_{m_\ell}^\ell - s_{m_\ell-1}^\ell} \le \frac{c_4}{n} \ln(1 + n^{|\boldsymbol{m}|+1} b_\ell).$$

If, on the other hand, $s_{m_\ell}^\ell \le e^{nb_\ell} - nb_\ell$, integrating with respect to the proper x_j^i and s_j^i gives

$$I_n \le \frac{c_5}{n} \sup_{s_{m_\ell-1}^\ell \in (e^{na_\ell}, e^{nb_\ell})} \int_{\mathbb{R}^d} \int_{Q_2} \frac{|\widehat{f}(x_1^{\ell+1})|^2}{s_{m_\ell}^\ell - s_{m_\ell-1}^\ell} \times \exp\bigl[-\tfrac{1}{2}\kappa_1 |x_1^{\ell+1}|^2 (s_1^{\ell+1} - s_{m_\ell}^\ell)^{2H}\bigr]\, ds_1^{\ell+1}\, ds_{m_\ell}^\ell\, dx_1^{\ell+1},$$

where $Q_2 = \{e^{na_{\ell+1}} < s_1^{\ell+1} < e^{nb_{\ell+1}},\ s_{m_{\ell-1}}^\ell + n^{-|\boldsymbol{m}|} \le s_{m_\ell}^\ell \le e^{nb_\ell} - nb_\ell\}$.

Recall from the proof of Proposition 3.1 that $|\widehat{f}(x)| \le c_\beta |x|^\beta$ for all $x \in \mathbb{R}^d$,

$$\begin{aligned} I_n &\le \frac{c_6}{n} \sup_{s_{m_\ell-1}^\ell \in (e^{na_\ell}, e^{nb_\ell})} \int_{\mathbb{R}^d} \int_{Q_2} \frac{|x_1^{\ell+1}|^{2\beta}}{s_{m_\ell}^\ell - s_{m_\ell-1}^\ell} \times \exp\bigl[-\tfrac{1}{2}\kappa_1 |x_1^{\ell+1}|^2 (s_1^{\ell+1} - s_{m_\ell}^\ell)^{2H}\bigr]\, ds_1^{\ell+1}\, ds_{m_\ell}^\ell\, dx_1^{\ell+1} \\ &\le \frac{c_7}{n} \sup_{s_{m_\ell-1}^\ell \in (e^{na_\ell}, e^{nb_\ell})} \int_{Q_2} \frac{1}{(s_{m_\ell}^\ell - s_{m_\ell-1}^\ell)(s_1^{\ell+1} - s_{m_\ell}^\ell)^{1+2H\beta}}\, ds_1^{\ell+1}\, ds_{m_\ell}^\ell \\ &\le \frac{c_8}{n^{1+2H\beta}} \sup_{s_{m_\ell-1}^\ell \in (e^{naell}, e^{nb_\ell})} \int_{s_{m_\ell-1}^\ell + n^{-|\boldsymbol{m}|}}^{e^{nb_\ell}} \frac{1}{s_{m_\ell}^\ell - s_{m_\ell-1}^\ell}\, ds_{m_\ell}^\ell \\ &\le \frac{c_9}{n^{2H\beta}}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} M_n = 0$ and, thus, $\lim_{n\to\infty} \mathbb{E}(G_n) = 0$. □

Consider now the convergence of moments when all exponents m_i are even.

Proposition 4.2. *Suppose that all exponents m_i are even. Then*

$$\lim_{n\to\infty} \mathbb{E}(G_n) = C_{f,d}^{|\boldsymbol{m}|} \mathbb{E}\biggl[\prod_{i=1}^N [W(\ell(M^{-1}(b_i))) - W(\ell(M^{-1}(a_i)))]^{m_i}\biggr].$$

Proof. Recall the domain $D_{\boldsymbol{m},1}$ in (4.1). Define

$$I_{\boldsymbol{m}}^n = \frac{\boldsymbol{m}!\, n^{-|\boldsymbol{m}|/2}}{(2\pi)^{|\boldsymbol{m}|d}} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \prod_{i=1}^N \prod_{k=1}^{m_i/2} |\widehat{f}(x_{2k}^i)|^2 \times \exp\biggl[-\frac{1}{2} \operatorname{var}\biggl(\sum_{i=1}^N \sum_{j=1}^{m_i} x_j^i \cdot [B^H(s_j^i) - B^H(s_{j-1}^i)]\biggr)\biggr]\, ds\, dx.$$

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The proof of Proposition 3.1 includes the result $\lim_{n\to\infty}[\mathbb{E}(G_n) - I^n_{\boldsymbol{m}}] = 0$. Make the change of variables $u^i_j = s^i_j - s^i_{j-1}$ for $1 \le j \le m_i$ and $1 \le i \le N$. Recall that $\theta = (1-2H) \wedge H$. Repeating the procedure for the proof of the inequality (3.10) gives

$$\begin{aligned}
&\limsup_{n\to\infty} I^n_{\boldsymbol{m}} \\
&\quad\le \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\qquad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} |\widehat{f}(x^i_{2k})|^2 \\
&\qquad\qquad \times \exp\biggl[-\frac{1}{2}\biggl(1 - \frac{c_1(1-2H)}{n^{\theta}}\biggr) \sum_{i=1}^{N} \sum_{j=1}^{m_i} |x^i_j|^2 (s^i_j - s^i_{j-1})^{2H}\biggr] \,\mathrm{d}s\,\mathrm{d}x \\
&\quad= \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\qquad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},2}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{|\widehat{f}(x^i_{2k})|^2}{u^i_{2k-1}} \\
&\qquad\qquad \times \exp\biggl[-\frac{1}{2}\biggl(1 - \frac{c_1(1-2H)}{n^{\theta}}\biggr) \sum_{i=1}^{N} \sum_{k=1}^{m_i/2} |x^i_{2k}|^2 (u^i_{2k})^{2H}\biggr] \,\mathrm{d}u\,\mathrm{d}\overline{x},
\end{aligned} \tag{4.2}$$

where $\mathrm{d}\overline{x} = \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \mathrm{d}x^i_{2k-1}$ and

$$D_{\boldsymbol{m},2} = \biggl\{ \mathrm{e}^{na_i} < \sum_{(\ell,k)\le(i,j)} u^{\ell}_k < \mathrm{e}^{nb_i}, u^i_j \ge n^{-|\boldsymbol{m}|},\ 1 \le i \le N,\ 1 \le j \le m_i \biggr\}.$$

Define

$$\begin{aligned}
U_{\boldsymbol{m}} &= \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\quad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},1}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} |\widehat{f}(x^i_{2k})|^2 \\
&\qquad \times \exp\biggl[-\frac{1}{2}\biggl(1 - \frac{c_1(1-2H)}{n^{\theta}}\biggr) \sum_{i=1}^{N} \sum_{j=1}^{m_i} |x^i_j|^2 (s^i_j - s^i_{j-1})^{2H}\biggr] \,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

Then

$$U_{\boldsymbol{m}} \le C^{|\boldsymbol{m}|/2}_{f,d}\, \boldsymbol{m}! \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{\mathrm{d}\overline{u}}{u^i_{2k-1}},$$

where

$$O_{\boldsymbol{m}} = \biggl\{ \mathrm{e}^{na_i} < \sum_{(\ell,k)\le(i,j)} u^{\ell}_{2k-1} < \mathrm{e}^{nb_i},\ u^i_{2j-1} \ge n^{-|\boldsymbol{m}|},\ 1 \le i \le N,\ 1 \le j \le \frac{m_i}{2} \biggr\}$$

and $\mathrm{d}\overline{u} = \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \mathrm{d}u_{2k-1}^i$. On the other hand,

$$U_{\boldsymbol{m}} \geq \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},2}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} |\widehat{f}(x_{2k}^i)|^2 \times \exp\biggl(-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i} |x_j^i|^2 (u_j^i)^{2H}\biggr)\,\mathrm{d}u\,\mathrm{d}x$$

$$\geq \frac{\boldsymbol{m}!}{(2\pi)^{|\boldsymbol{m}|d}} \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{\mathbb{R}^{|\boldsymbol{m}|d}} \int_{D_{\boldsymbol{m},3}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} |\widehat{f}(x_{2k}^i)|^2 \times \exp\biggl(-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{m_i} |x_j^i|^2 (u_j^i)^{2H}\biggr)\,\mathrm{d}u\,\mathrm{d}x,$$

where $D_{\boldsymbol{m},3} = D_{\boldsymbol{m},2} \cap \{n^2 < u_{2k-1}^i,\ 1/n < u_{2k}^i < n,\ 1 \leq i \leq N,\ 1 \leq k \leq \frac{1}{2}m_i\}$. Defining $O_{\boldsymbol{m},1} = O_{\boldsymbol{m}} \cap \{\sum_{(\ell,k)\leq(i,j)} u_{2k-1}^{\ell} < \mathrm{e}^{nb_i} - |\boldsymbol{m}|n,\ u_{2j-1}^i \geq n^2,\ 1 \leq i \leq N,\ 1 \leq j \leq \frac{1}{2}m_i\}$ yields

$$U_{\boldsymbol{m}} \geq C_{f,d}^{|\boldsymbol{m}|/2}\boldsymbol{m}! \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m},1}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i}\,\mathrm{d}\overline{u}$$

$$= C_{f,d}^{|\boldsymbol{m}|/2}\boldsymbol{m}! \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i}\,\mathrm{d}\overline{u}.$$

Therefore,

$$U_{\boldsymbol{m}} = C_{f,d}^{|\boldsymbol{m}|/2}\boldsymbol{m}! \limsup_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i}\,\mathrm{d}\overline{u}. \tag{4.3}$$

Repeated application of (3.11) yields

$$\liminf_{n\to\infty} I_{\boldsymbol{m}}^n \geq \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},1}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \frac{|\widehat{f}(y_{2k}^i)|^2}{s_{2k-1}^i - s_{2k-2}^i} \times \exp\biggl[-\frac{1}{2}\operatorname{var}\biggl(\sum_{i=1}^{N}\sum_{k=1}^{m_i/2} y_{2k}^i \cdot [B^H(s_{2k}^i) - B^H(s_{2k-1}^i)]\biggr)\biggr]\,\mathrm{d}s\,\mathrm{d}y.$$

By Lemma 2.3,

$$\liminf_{n\to\infty} I_{\boldsymbol{m}}^n \geq \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},2}} \prod_{i=1}^{N}\prod_{k=1}^{m_i/2} \frac{|\widehat{f}(y_{2k}^i)|^2}{u_{2k-1}^i} \times \exp\biggl[-\frac{1}{2}\biggl(1 + \frac{c_2(1-2H)}{n^{1-2H}}\biggr)\sum_{i=1}^{N}\sum_{k=1}^{m_i/2} |y_{2k}^i|^2 (u_{2k}^i)^{2H}\biggr]\,\mathrm{d}u\,\mathrm{d}y \tag{4.4}$$

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$$
\begin{aligned}
&= \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\quad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},3}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{|\widehat{f}(y_{2k}^i)|^2}{u_{2k-1}^i} \\
&\qquad \times \exp\bigg[-\frac{1}{2}\bigg(1+\frac{c_2(1-2H)}{n^{1-2H}}\bigg) \sum_{i=1}^{N} \sum_{k=1}^{m_i/2} |y_{2k}^i|^2 (u_{2k}^i)^{2H}\bigg] \,\mathrm{d}u \,\mathrm{d}y \\
&\geq C_{f,d}^{|\boldsymbol{m}|/2} \boldsymbol{m}! \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m},1}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i} \,\mathrm{d}\overline{u} \\
&= C_{f,d}^{|\boldsymbol{m}|/2} \boldsymbol{m}! \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i} \,\mathrm{d}\overline{u}.
\end{aligned}
$$

Let

$$
\begin{aligned}
L_{\boldsymbol{m}} &= \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\quad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},3}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{|\widehat{f}(y_{2k}^i)|^2}{u_{2k-1}^i} \\
&\qquad \times \exp\bigg[-\frac{1}{2}\bigg(1+\frac{c_2(1-2H)}{n^{1-2H}}\bigg) \sum_{i=1}^{N} \sum_{k=1}^{m_i/2} |y_{2k}^i|^2 (u_{2k}^i)^{2H}\bigg] \,\mathrm{d}u \,\mathrm{d}y.
\end{aligned}
$$

Then

$$
\begin{aligned}
L_{\boldsymbol{m}} &\leq \frac{\boldsymbol{m}!}{(2\pi)^{3|\boldsymbol{m}|d/4}} \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \\
&\quad \times \int_{\mathbb{R}^{|\boldsymbol{m}|d/2}} \int_{D_{\boldsymbol{m},3}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{|\widehat{f}(y_{2k}^i)|^2}{u_{2k-1}^i} \exp\bigg[-\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{m_i/2} |y_{2k}^i|^2 (u_{2k}^i)^{2H}\bigg] \,\mathrm{d}u \,\mathrm{d}y \\
&= C_{f,d}^{|\boldsymbol{m}|/2} \boldsymbol{m}! \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m},1}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i} \,\mathrm{d}\overline{u} \\
&= C_{f,d}^{|\boldsymbol{m}|/2} \boldsymbol{m}! \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i} \,\mathrm{d}\overline{u}.
\end{aligned}
$$

Therefore,

$$
L_{\boldsymbol{m}} = C_{f,d}^{|\boldsymbol{m}|/2} \boldsymbol{m}! \liminf_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{1}{u_{2k-1}^i} \,\mathrm{d}\overline{u}. \tag{4.5}
$$

When $H = \frac{1}{2}$, inequalities (4.2) and (4.4) become identities. Recall the convergence of finite-dimensional distribution for two-dimensional Brownian motion in (1.1), Remark 3.1,

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(4.3), and (4.5). We can easily obtain

$$\boldsymbol{m}! \lim_{n\to\infty} n^{-|\boldsymbol{m}|/2} \int_{O_{\boldsymbol{m}}} \prod_{i=1}^{N} \prod_{k=1}^{m_i/2} \frac{\mathrm{d}\overline{u}}{u_{2k-1}^{i}} = \mathbb{E}\Bigg[\prod_{i=1}^{N} [W(\ell(M^{-1}(b_i))) - W(\ell(M^{-1}(a_i)))]^{m_i}\Bigg]. \tag{4.6}$$

Combining (4.2)–(4.6) gives the required result. □

The proof of Theorem 1.2 now follows easily from Propositions 4.1 and 4.2. □

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