Journal of the Inst. of Math. Jussieu (2010) **9**(3), 463–476 463
doi:10.1017/S1474748009000206

ENERGY FUNCTIONALS AND COMPLEX MONGE–AMPÈRE EQUATIONS

ZUOLIANG HOU AND QI LI

Mathematics Department, Columbia University, 2990 Broadway, New York, NY 10027, USA (hou@math.columbia.edu; liqi@math.columbia.edu)

(Received 6 May 2008; revised 13 April 2009; accepted 11 May 2009)

Abstract We introduce certain energy functionals to complex Monge–Ampère equations over bounded domains with inhomogeneous boundary conditions, and use these functionals to show the convergence of solutions to certain parabolic Monge–Ampère equations.

Keywords: energy functional; parabolic partial differential equation; complex Monge–Ampère equation

AMS 2010 *Mathematics subject classification:* Primary 35K60; 35K20

1. Introduction

It is generally hoped that the solution of a parabolic equation should converge to the solution of the corresponding elliptic equation. For example, it is well known that on a compact Riemannian manifold (M, g), the solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

will converge to the solution of $\Delta u = 0$. Another example is the Kähler–Ricci flow on compact Kähler manifold with positive first Chern class: Perelman showed that if M admits a Kähler–Einstein metric, then any solution of the Kähler–Ricci flow

$$\frac{\partial g}{\partial t} = -\operatorname{Ric}_g + g$$

with initial metric g_0 in $c_1(M)$ will converge to a Kähler–Einstein metric in the sense of Cheeger–Gromov (see [**12**] for more detail on this result).

In this paper, we will study the parabolic complex Monge–Ampère equation over a smooth bounded domain in $\mathbb{C}^n$ and its convergence property.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $\partial\Omega$. For $T > 0$, define $\mathcal{Q}_T = \Omega \times (0, T)$, $B = \Omega \times \{0\}$, $\Gamma = \partial\Omega \times \{0\}$ and $\Sigma_T = \partial\Omega \times (0, T)$. Let $\partial_p\mathcal{Q}_T = B \cup \Gamma \cup \Sigma_T$ be the parabolic boundary of $\mathcal{Q}_T$. We consider the following boundary-value problem:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \log\det(u_{\alpha\bar{\beta}}) &= f(t, z, u) && \text{in } \mathcal{Q}_T, \\ u &= \varphi && \text{on } \partial_p\mathcal{Q}_T, \end{aligned}\right\} \tag{1.1}$$

where $f \in \mathcal{C}^\infty(\mathbb{R} \times \bar{\Omega} \times \mathbb{R})$ and $\varphi \in \mathcal{C}^\infty(\partial_p \mathcal{Q}_T)$. We always assume that

$$\frac{\partial f}{\partial u} \leqslant 0. \tag{1.2}$$

The main results of this paper are the following theorems.

Theorem 1.1. *Suppose there exists a subsolution to (1.1), i.e. a spatial plurisubharmonic (psh) function* $\underline{u} \in \mathcal{C}^2(\bar{\mathcal{Q}}_T)$ *such that*

$$\left.\begin{aligned} \underline{u}_t - \log\det(\underline{u}_{\alpha\bar{\beta}}) \leqslant f(t, z, \underline{u}) \quad \text{in } \mathcal{Q}_T,\\ \underline{u} \leqslant \varphi \text{ on } B \quad \text{and} \quad \underline{u} = \varphi \text{ on } \Sigma_T \cup \Gamma. \end{aligned}\right\} \tag{1.3}$$

Then there exists a spatial psh solution $u \in \mathcal{C}^\infty(\mathcal{Q}_T) \cap \mathcal{C}^{2,1}(\bar{\mathcal{Q}}_T)^*$ *of (1.1) with* $\underline{u} \leqslant u$ *if the following compatibility condition is satisfied:* $\forall z \in \partial\Omega$,

$$\left.\begin{aligned} \varphi_t - \log\det(\varphi_{\alpha\bar{\beta}}) &= f(0, z, \varphi(z)),\\ \varphi_{tt} - (\log\det(\varphi_{\alpha\bar{\beta}}))_t &= f_t(0, z, \varphi(z)) + f_u(0, z, \varphi(z))\varphi_t. \end{aligned}\right\} \tag{1.4}$$

Theorem 1.2. *If both* φ *and* f *are independent of the time variable* t*, and* f *satisfies (1.2), then the solution* u *of (1.1) exists for* $T = +\infty$*, and as* t *approaches* $+\infty$, $u(\cdot, t)$ *approaches the unique solution* v *of the Dirichlet problem*

$$\left.\begin{aligned} \det(v_{\alpha\bar{\beta}}) &= \mathrm{e}^{-f(z,v)} \quad &&\text{in } \Omega,\\ v &= \varphi &&\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.5}$$

in $\mathcal{C}^\infty(\bar{\Omega})$.

Theorem 1.3. *Assume that* Ω *is strong pseudoconvex, then for any* $f \in \mathcal{C}^\infty(\bar{\Omega} \times \mathbb{R})$ *satisfying (1.2) and* $\varphi \in \mathcal{C}^\infty(\partial\Omega)$*, the solution of (1.1) exists for* $T = +\infty$*, and as* $t \to \infty$, $u(\cdot, t)$ *approaches the unique solution of (1.5) in* $\mathcal{C}^\infty(\bar{\Omega})$.

The parabolic complex Monge–Ampère equation on a complex manifold has been studied extensively by many authors because of its close connection with Kähler–Ricci flow: see [**5**, **6**, **11**]. On the other hand, the elliptic complex Monge–Ampère equations on both bounded domains and complex manifolds were developed in [**2**, **4**, **8**, **18**]. We will follow the treatment in [**4**] and [**7**] to study the Dirichlet boundary-value problem (1.1).

Our proof of Theorem 1.1 is based on an *a priori* estimate, an approach similar to real parabolic Monge–Ampère flow and real parabolic Hessian flow studied in [**16**] and [**17**]. More precisely, we have the following proposition.

Proposition 1.4. *Under the condition in Theorem 1.1, if* $u \in \mathcal{C}^{4,1}(\bar{\mathcal{Q}}_T)$ *solves equation (1.1), then there exist constants* C, λ *and* Λ *depends on* n, Ω, φ, f *and* $\underline{u}$*, such that*

$$|u| + |\nabla u| + |\nabla^2 u| \leqslant C \quad \text{in } \mathcal{Q} \tag{1.6}$$

and

$$\lambda|\xi|^2 \leqslant u_{\alpha\bar{\beta}}\xi_\alpha\xi_{\bar{\beta}} \leqslant \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{C}^n. \tag{1.7}$$

* $\mathcal{C}^{m,n}(\bar{\mathcal{Q}}_T)$ means m times and n times differentiable in the space direction and the time direction, respectively; and the same for the $\mathcal{C}^{m,n}$-norm.

We will prove this proposition in § 2. If for now we assume it to be true, we can finish the proof of Theorem 1.1 as follows. Since equation (1.1) is strictly parabolic, the short time solution always exists under the compatibility condition. Since the function $\log\det(\cdot)$ is concave on the space of the positive Hermitian matrix, so by the work of Krylov [**9**], Proposition 1.4 implies that for any $\varepsilon \in (0, T)$, there exist C_ε and γ such that

$$\|u\|_{\mathcal{C}^{2+\gamma,1+\gamma/2}(\Omega\times(\varepsilon,T))} \leqslant C_\varepsilon$$

with constant C_ε and γ depends on $1/\varepsilon$. Then a standard bootstrap argument implies that for any $l \in \mathbb{N}$, there exists $C(l, \varepsilon)$ such that

$$\|u\|_{\mathcal{C}^{l+\gamma,1+\gamma/2}(\Omega\times(\varepsilon,T))} \leqslant C(l,\varepsilon).$$

Hence u is smooth on $\Omega \times (\varepsilon, T)$, and the solution can always be extended to a long time solution. Besides, since ε is arbitrary, u is smooth on $\Omega \times (0, T)$.

Motivated by the energy functionals, especially the Mabuchi energy, in the study of the Kähler–Ricci flow, we introduce an energy functional to the complex Monge–Ampère problem over bounded smooth domain. Given $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, denote

$$\mathcal{P}(\Omega,\varphi) = \{u \in \mathcal{C}^2(\bar{\Omega}) \mid u \text{ is psh and } u = \varphi \text{ on } \partial\Omega\}. \tag{1.8}$$

We will show that there is a well-defined functional F^0 whose variation is given by

$$\delta F^0(u) = \int_\Omega \delta u \det(u_{\alpha\bar{\beta}}). \tag{1.9}$$

We will study the basic property of this functional in § 3. Then, in § 4, we will use this functional and follow an idea in [**11**] to prove Theorem 1.2.

Theorem 1.3 follows from Theorem 1.2, because a subsolution can be constructed for any φ and f, using the fact that the domain Ω is strong pseudoconvex.

Remark 1.5. Similar energy functionals have been studied in [**1**, **14**–**17**] for the real Monge–Ampère equation and the real Hessian equation with homogeneous boundary condition $\varphi = 0$, and the convergence for the solution of the real Hessian equation was also proved in [**15**]. Our construction of the energy functionals and the proof of the convergence also work for these cases. Li [**10**] and Błocki [**3**] studied the Dirichlet boundary-value problems for the complex k-Hessian equations over bounded complex domains. Similar energy functionals can also be constructed for the parabolic complex k-Hessian equations and can be used for the proof of the convergence.

Remark 1.6. Since equation (1.1) is nonlinear, even with the compatibility condition (1.4) we can only show that the solution is smooth for $t > 0$. On the other hand, using approximation, it is possible to show that a solution still exits without the compatibility condition, but then the solution will not be in $\mathcal{C}^{2,1}(\bar{\mathcal{Q}})$.

2. An *a priori* $\mathcal{C}^2$ estimate

In this section, we will prove Proposition 1.4.

Step 0. C^0-estimate.

Since u is spatial psh and $u \geqslant \underline{u}$, so

$$\underline{u} \leqslant u \leqslant \sup_{\Sigma_T} \underline{u},$$

i.e.

$$\|u\|_{\mathcal{C}^0(\mathcal{Q}_T)} \leqslant M_0. \tag{2.1}$$

Step 1. $|u_t| \leqslant C_1$ in $\bar{\mathcal{Q}}_T$.

Let $G = u_t(2M_0 - u)^{-1}$. If G attains its minimum on $\bar{\mathcal{Q}}_T$ at the parabolic boundary, then $u_t \geqslant -C_1$ where C_1 depends on M_0 and $\underline{u}_t$ on Σ. Otherwise, at the point where G attains the minimum,

$$\left.\begin{aligned} G_t &\leqslant 0, \quad \text{i.e. } u_{tt} + (2M_0 - u)^{-1}u_t^2 \leqslant 0,\\ G_\alpha &= 0, \quad \text{i.e. } u_{t\alpha} + (2M_0 - u)^{-1}u_t u_\alpha = 0,\\ G_{\bar\beta} &= 0, \quad \text{i.e. } u_{t\bar\beta} + (2M_0 - u)^{-1}u_t u_{\bar\beta} = 0,\end{aligned}\right\} \tag{2.2}$$

and the matrix $G_{\alpha\bar\beta}$ is non-negative, i.e.

$$u_{t\alpha\bar\beta} + (2M_0 - u)^{-1}u_t u_{\alpha\bar\beta} \geqslant 0. \tag{2.3}$$

Hence

$$0 \leqslant u^{\alpha\bar\beta}(u_{t\alpha\bar\beta} + (2M_0 - u)^{-1}u_t u_{\alpha\bar\beta}) = u^{\alpha\bar\beta}u_{t\alpha\bar\beta} + n(2M_0 - u)^{-1}u_t, \tag{2.4}$$

where $(u^{\alpha\bar\beta})$ is the inverse matrix for $(u_{\alpha\bar\beta})$, i.e.

$$u^{\alpha\bar\beta}u_{\gamma\bar\beta} = \delta^\alpha{}_\gamma.$$

Differentiating (1.1) in t, we get

$$u_{tt} - u^{\alpha\bar\beta}u_{t\alpha\bar\beta} = f_t + f_u u_t, \tag{2.5}$$

so

$$\begin{aligned} (2M_0 - u)^{-1}u_t^2 &\leqslant -u_{tt}\\ &= -u^{\alpha\bar\beta}u_{t\alpha\bar\beta} - f_t - f_u u_t\\ &\leqslant n(2M_0 - u)^{-1}u_t - f_u u_t - f_t,\end{aligned}$$

hence

$$u_t^2 - (n - (2M_0 - u)f_u)u_t + f_t(2M_0 - u) \leqslant 0.$$

Therefore, at point p, we get

$$u_t \geqslant -C_1, \tag{2.6}$$

where C_1 depends on M_0 and f.

https://doi.org/10.1017/S1474748009000206 Published online by Cambridge University Press

Similarly, by considering the function $u_t(2M_0+u)^{-1}$ we can show that

$$u_t \leqslant C_1. \tag{2.7}$$

Step 2. $|\nabla u| \leqslant M_1$.

Extend $\underline{u}|_{\Sigma}$ to a spatial harmonic function h, then

$$\underline{u} \leqslant u \leqslant h \text{ in } \mathcal{Q}_T \quad \text{and} \quad \underline{u} = u = h \text{ on } \Sigma_T. \tag{2.8}$$

So

$$|\nabla u|_{\Sigma_T} \leqslant M_1. \tag{2.9}$$

Let L be the linear differential operator defined by

$$Lv = \frac{\partial v}{\partial t} - u^{\alpha\bar{\beta}} v_{\alpha\bar{\beta}} - f_u v. \tag{2.10}$$

Then

$$\begin{aligned} L(\nabla u + \mathrm{e}^{\lambda|z|^2}) &= L(\nabla u) + L\mathrm{e}^{\lambda|z|^2} \\ &\leqslant \nabla f - \mathrm{e}^{\lambda|z|^2}\Big(\lambda \sum u^{\alpha\bar{\alpha}} - f_u\Big). \end{aligned} \tag{2.11}$$

Notice that both u and $\dot{u}$ are bounded and that

$$\det(u_{\alpha\bar{\beta}}) = \mathrm{e}^{\dot{u}-f},$$

so

$$0 < c_0 \leqslant \det(u_{\alpha\bar{\beta}}) \leqslant c_1, \tag{2.12}$$

where c_0 and c_1 depends on M_0 and f. Therefore,

$$\sum u^{\alpha\bar{\alpha}} \geqslant n c_1^{-1/n}. \tag{2.13}$$

Hence after taking λ large enough, we can get

$$L(\nabla u + \mathrm{e}^{\lambda|z|^2}) \leqslant 0,$$

thus

$$|\nabla u| \leqslant \sup_{\partial_p \mathcal{Q}_T} |\nabla u| + C_2 \leqslant M_1. \tag{2.14}$$

https://doi.org/10.1017/S1474748009000206 Published online by Cambridge University Press

Step 3. $|\nabla^2 u| \leqslant M_2$ on Σ.

At point $(p,t) \in \Sigma$, we choose coordinates $z_1, \dots, z_n$ for Ω, such that $z_1 = \cdots = z_n = 0$ at p and the positive x_n-axis is the interior normal direction of $\partial\Omega$ at p. We set $s_1 = y_1$, $s_2 = x_1$, $\dots$, $s_{2n-1} = y_n$, $s_{2n} = x_n$ and $s' = (s_1, \dots, s_{2n-1})$. We also assume that near p, $\partial\Omega$ is represented as a graph

$$x_n = \rho(s') = \frac{1}{2} \sum_{j,k<2n} B_{jk} s_j s_k + O(|s'|^3). \tag{2.15}$$

Since $(u - \underline{u})(s', \rho(s'), t) = 0$, we have for $j,k < 2n$,

$$(u - \underline{u})_{s_j s_k}(p,t) = -(u - \underline{u})_{x_n}(p,t) B_{jk}, \tag{2.16}$$

hence

$$|u_{s_j s_k}(p,t)| \leqslant C_3, \tag{2.17}$$

where C_3 depends on $\partial\Omega$, $\underline{u}$ and M_1.

We will follow the construction of the barrier function by Guan [**7**] to estimate $|u_{x_n s_j}|$. For $\delta > 0$, define $\mathcal{Q}_\delta(p,t) = (\Omega \cap B_\delta(p)) \times (0,t)$.

Lemma 2.1. *Define the functions*

$$d(z) = \operatorname{dist}(z, \partial\Omega) \tag{2.18}$$

and

$$v = (u - \underline{u}) + a(h - \underline{u}) - N d^2. \tag{2.19}$$

Then for N sufficiently large and a, δ sufficiently small,

$$\left.\begin{aligned} Lv &\geqslant \epsilon\Big(1 + \sum u^{\alpha\bar\alpha}\Big) && \text{in } \mathcal{Q}_\delta(p,t), \\ v &\geqslant 0 && \text{on } \partial(B_\delta(p) \cap \Omega) \times (0,t), \\ v(z,0) &\geqslant c_3|z| && \text{for } z \in B_\delta(p) \cap \Omega, \end{aligned}\right\} \tag{2.20}$$

where ϵ depends on the uniform lower bound of the eigenvalues of $\{\underline{u}_{\alpha\bar\beta}\}$, and c_3 depends on M_1.

Proof. See the proof of Lemma 2.1 in [**7**]. □

For $j < 2n$, consider the operator

$$T_j = \frac{\partial}{\partial s_j} + \rho_{s_j} \frac{\partial}{\partial x_n}.$$

Then

$$\left.\begin{aligned} T_j(u - \underline{u}) &= 0 && \text{on } (\partial\Omega \cap B_\delta(p)) \times (0,t), \\ |T_j(u - \underline{u})| &\leqslant M_1 && \text{on } (\Omega \cap \partial B_\delta(p)) \times (0,t), \\ |T_j(u - \underline{u})(z,0)| &\leqslant C_4|z| && \text{for } z \in B_\delta(p). \end{aligned}\right\} \tag{2.21}$$

https://doi.org/10.1017/S1474748009000206 Published online by Cambridge University Press

So by Lemma 2.1 we may choose C_5 independent of u and $A \gg B \gg 1$ such that

$$\left.\begin{aligned} L(Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u})) \geqslant 0 \quad &\text{in } \mathcal{Q}_\delta(p,t), \\ Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) \geqslant 0 \quad &\text{on } \partial_p \mathcal{Q}_\delta(p,t). \end{aligned}\right\} \tag{2.22}$$

Hence, by the comparison principle,

$$Av + B|z|^2 - C_5(u_{y_n} - \underline{u}_{y_n})^2 \pm T_j(u - \underline{u}) \geqslant 0 \quad \text{in } \mathcal{Q}_\delta(p,t),$$

and at (p,t), therefore,

$$|u_{x_n y_j}| \leqslant M_2. \tag{2.23}$$

To estimate $|u_{x_n x_n}|$, we will follow the simplification in [**13**]. For $(p,t) \in \Sigma$, define

$$\lambda(p,t) = \min\{u_{\xi\bar{\xi}} \mid \text{complex vector } \xi \in T_p \partial\Omega \text{ and } |\xi| = 1\}.$$

Claim. $\lambda(p,t) \geqslant c_4 > 0$ where c_4 is independent of u.

Let us assume that $\lambda(p,t)$ attains the minimum at (z_0, t_0) with $\xi \in T_{z_o}\partial\Omega$. We may assume that

$$\lambda(z_0,t_0) < \tfrac{1}{2}\underline{u}_{\xi\bar{\xi}}(z_0,t_0).$$

Take a unitary frame $e_1, \dots, e_n$ around z_0, such that $e_1(z_0) = \xi$, and $\operatorname{Re} e_n = \gamma$ is the interior normal of $\partial\Omega$ along $\partial\Omega$. Let r be the function which defines Ω, then

$$(u - \underline{u})_{1\bar{1}}(z,t) = -r_{1\bar{1}}(z)(u - \underline{u})_\gamma(z,t), \quad z \in \partial\Omega.$$

Since $u_{1\bar{1}}(z_0,t_0) < \underline{u}_{1\bar{1}}(z_0,t_0)/2$, we have

$$-r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0) \leqslant -\tfrac{1}{2}\underline{u}_{1\bar{1}}(z_0,t_0).$$

Hence

$$r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t) \geqslant \tfrac{1}{2}\underline{u}_{1\bar{1}}(z_0,t) \geqslant c_5 > 0.$$

Since both ∇u and $\nabla \underline{u}$ are bounded, we get

$$r_{1\bar{1}}(z_0) \geqslant c_6 > 0.$$

Hence for any $z \in B_\delta(z_0) \cap \Omega$, where δ is a sufficiently small positive number, we have

$$r_{1\bar{1}}(z) \geqslant \tfrac{1}{2}c_6.$$

So by $u_{1\bar{1}}(z,t) \geqslant u_{1\bar{1}}(z_0,t_0)$, we get

$$\underline{u}_{1\bar{1}}(z,t) - r_{1\bar{1}}(z)(u - \underline{u})_\gamma(z,t) \geqslant \underline{u}_{1\bar{1}}(z_0,t_0) - r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0).$$

Hence if we let

$$\Psi(z,t) = \frac{1}{r_{1\bar{1}}(z)}(r_{1\bar{1}}(z_0)(u - \underline{u})_\gamma(z_0,t_0) + \underline{u}_{1\bar{1}}(z,t) - \underline{u}_{1\bar{1}}(z_0,t_0)),$$

then Ψ is a function that depends on the geometry of $\partial\Omega$ and $\underline{u}$, and it satisfies

$$\begin{aligned}(u-\underline{u})_\gamma(z,t) &\leqslant \Psi(z,t) \quad \text{on } (\partial\Omega\cap B_\delta(z_0))\times(0,T),\\ (u-\underline{u})_\gamma(z_0,t_0) &= \Psi(z_0,t_0).\end{aligned}$$

Now take the coordinate system $z_1,\dots,z_n$ as before. Then

$$\left.\begin{aligned}(u-\underline{u})_{x_n}(z,t) &\leqslant \frac{1}{\gamma_n(z)}\Psi(z,t) \quad \text{on } (\partial\Omega\cap B_\delta(z_0))\times(0,T),\\ (u-\underline{u})_{x_n}(z_0,t_0) &= \frac{1}{\gamma_n(z_0)}\Psi(z_0,t_0),\end{aligned}\right\} \tag{2.24}$$

where γ_n depends on $\partial\Omega$. After taking C_6 independent of u and $A \gg B \gg 1$, we get

$$\begin{aligned}L\bigg(Av+B|z|^2-C_6(u_{y_n}-\underline{u}_{y_n})^2+\frac{\Psi(z,t)}{\gamma_n(z)}-(u-\underline{u})_{x_n}\bigg) &\geqslant 0 \quad \text{in } \mathcal{Q}_\delta(p,t),\\ Av+B|z|^2-C_6(u_{y_n}-\underline{u}_{y_n})^2+\frac{\Psi(z,t)}{\gamma_n(z)}-(u-\underline{u})_{x_n} &\geqslant 0 \quad \text{on } \partial_p\mathcal{Q}_\delta(p,t).\end{aligned}$$

So

$$Av+B|z|^2-C_6(u_{y_n}-\underline{u}_{y_n})^2+\frac{\Psi(z,t)}{\gamma_n(z)}-(u-\underline{u})_{x_n} \geqslant 0 \quad \text{in } \mathcal{Q}_\delta(p,t)$$

and

$$u_{x_nx_n}(z_0,t_0) \leqslant C_7.$$

Therefore, at (z_0,t_0), $u_{\alpha\bar\beta}$ is uniformly bounded, hence

$$u_{1\bar 1}(z_0,t_0) \geqslant c_4$$

with c_4 independent of u. Finally, from the equation

$$\det u_{\alpha\bar\beta} = \mathrm{e}^{\dot u - f}$$

we get

$$|u_{x_nx_n}| \leqslant M_2.$$

Step 4. $|\nabla^2 u| \leqslant M_2$ in $\mathcal{Q}$.

By the concavity of $\log\det$, we have

$$L(\nabla^2 u+\mathrm{e}^{\lambda|z|^2}) \leqslant O(1)-\mathrm{e}^{\lambda|z|^2}\Big(\lambda\sum u^{\alpha\bar\alpha}-f_u\Big).$$

So for λ large enough,

$$L(\nabla^2 u+\mathrm{e}^{\lambda|z|^2}) \leqslant 0$$

and

$$\sup|\nabla^2 u| \leqslant \sup_{\partial_p\mathcal{Q}_T}|\nabla^2 u| + C_8 \tag{2.25}$$

with C_8 depending on M_0, Ω and f.

3. The functionals I, J and F^0

Let us recall the definition of $\mathcal{P}(\Omega, \varphi)$ in (1.8):

$$\mathcal{P}(\Omega, \varphi) = \{u \in \mathcal{C}^2(\bar{\Omega}) \mid u \text{ is psh and } u = \varphi \text{ on } \partial\Omega\}.$$

Fixing $v \in \mathcal{P}$, for $u \in \mathcal{P}$, define

$$I_v(u) = -\int_\Omega (u - v)((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n). \tag{3.1}$$

Proposition 3.1. *There is a unique and well-defined functional J_v on $\mathcal{P}(\Omega, \varphi)$ such that*

$$\delta J_v(u) = -\int_\Omega \delta u((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n) \tag{3.2}$$

and $J_v(v) = 0$.

Proof. Notice that $\mathcal{P}$ is connected, so we can connect v to $u \in \mathcal{P}$ by a path u_t, $0 \leqslant t \leqslant 1$, such that $u_0 = v$ and $u_1 = u$. Define

$$J_v(u) = -\int_0^1 \int_\Omega \frac{\partial u_t}{\partial t}((\mathrm{i}\partial\bar{\partial}u_t)^n - (\mathrm{i}\partial\bar{\partial}v)^n)\,\mathrm{d}t. \tag{3.3}$$

We need to show that the integral in (3.3) is independent of the choice of path u_t. Let $\delta u_t = w_t$ be a variation of the path. Then

$$w_1 = w_0 = 0 \quad \text{and} \quad w_t = 0 \text{ on } \partial\Omega,$$

and

$$\delta \int_0^1 \int_\Omega \dot{u}((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n)\,\mathrm{d}t$$
$$= \int_0^1 \int_\Omega (\dot{w}((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n) + \dot{u}n\mathrm{i}\partial\bar{\partial}w(\mathrm{i}\partial\bar{\partial}u)^{n-1})\,\mathrm{d}t.$$

Since $w_0 = w_1 = 0$, an integration by parts with respect to t gives

$$\begin{aligned}\int_0^1 \int_\Omega \dot{w}((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n)\,\mathrm{d}t &= -\int_0^1 \int_\Omega w\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{i}\partial\bar{\partial}u)^n\,\mathrm{d}t \\ &= -\int_0^1 \int_\Omega \mathrm{i}nw\partial\bar{\partial}\dot{u}(\mathrm{i}\partial\bar{\partial}u)^{n-1}\,\mathrm{d}t.\end{aligned}$$

Notice that both w and $\dot{u}$ vanish on $\partial\Omega$, so an integration by parts with respect to z gives

$$\begin{aligned}\int_\Omega \mathrm{i}nw\partial\bar{\partial}\dot{u}(\mathrm{i}\partial\bar{\partial}u)^{n-1} &= -\int_\Omega \mathrm{i}n\partial w \wedge \bar{\partial}\dot{u}(\mathrm{i}\partial\bar{\partial}u)^{n-1} \\ &= \int_\Omega \mathrm{i}n\dot{u}\partial\bar{\partial}w(\mathrm{i}\partial\bar{\partial}u)^{n-1}.\end{aligned}$$

So

$$\delta \int_0^1 \int_\Omega \dot{u}((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n)\,\mathrm{d}t = 0 \tag{3.4}$$

and the functional J is well defined. □

Using the J functional, we can define the F^0 functional as

$$F_v^0(u) = J_v(u) - \int_\Omega u(\mathrm{i}\partial\bar{\partial}v)^n. \tag{3.5}$$

Then, by Proposition 3.1, we have

$$\delta F_v^0(u) = -\int_\Omega \delta u(\mathrm{i}\partial\bar{\partial}u)^n. \tag{3.6}$$

Proposition 3.2. *The basic properties of I, J and F^0 are the following.*

(1) *For any $u \in \mathcal{P}(\Omega,\varphi)$, $I_v(u) \geqslant J_v(u) \geqslant 0$.*

(2) *F^0 is convex on $\mathcal{P}(\Omega,\varphi)$, i.e. $\forall u_0, u_1 \in \mathcal{P}$,*

$$F^0\left(\frac{u_0+u_1}{2}\right) \leqslant \frac{F^0(u_0)+F^0(u_1)}{2}. \tag{3.7}$$

(3) *F^0 satisfies the cocycle condition, i.e. $\forall u_1, u_2, u_3 \in \mathcal{P}(\Omega,\varphi)$,*

$$F_{u_1}^0(u_2) + F_{u_2}^0(u_3) = F_{u_1}^0(u_3). \tag{3.8}$$

Proof. Let $w = (u - v)$ and $u_t = v + tw = (1-t)v + tu$, then

$$\begin{aligned} I_v(u) &= -\int_\Omega w((\mathrm{i}\partial\bar{\partial}u)^n - (\mathrm{i}\partial\bar{\partial}v)^n) \\ &= -\int_\Omega w\left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{i}\partial\bar{\partial}u_t)^n\,\mathrm{d}t\right) \\ &= -\int_0^1 \int_\Omega \mathrm{i}nw\partial\bar{\partial}w(\mathrm{i}\partial\bar{\partial}u_t)^{n-1} \\ &= \int_0^1 \int_\Omega \mathrm{i}n\partial w \wedge \bar{\partial}w \wedge (\mathrm{i}\partial\bar{\partial}u_t)^{n-1} \geqslant 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} J_v(u) &= -\int_0^1 \int_\Omega w((\mathrm{i}\partial\bar{\partial}u_t)^n - (\mathrm{i}\partial\bar{\partial}v)^n)\,\mathrm{d}t \\ &= -\int_0^1 \int_\Omega w\left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{i}\partial\bar{\partial}u_s)^n\,\mathrm{d}s\right)\mathrm{d}t \\ &= -\int_0^1 \int_\Omega \int_0^t \mathrm{i}nw\partial\bar{\partial}w(\mathrm{i}\partial\bar{\partial}u_s)^{n-1}\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^1 \int_\Omega (1-s)\mathrm{i}n\partial w \wedge \bar{\partial}w \wedge (\mathrm{i}\partial\bar{\partial}u_s)^{n-1}\,\mathrm{d}s \geqslant 0. \end{aligned} \tag{3.10}$$

https://doi.org/10.1017/S1474748009000206 Published online by Cambridge University Press

Compare (3.9) and (3.10), it is easy to see that

$$I_v(u) \geqslant J_v(u) \geqslant 0.$$

To prove (3.7), let $u_t = (1-t)u_0 + tu_1$, then

$$F^0(u_{1/2}) - F^0(u_0) = -\int_0^{1/2}\int_\Omega (u_1 - u_0)(\mathrm{i}\partial\bar\partial u_t)^n \,\mathrm{d}t,$$
$$F^0(u_1) - F^0(u_{1/2}) = -\int_{1/2}^{1}\int_\Omega (u_1 - u_0)(\mathrm{i}\partial\bar\partial u_t)^n \,\mathrm{d}t.$$

Since

$$\begin{aligned}\int_0^{1/2}\int_\Omega (u_1 - u_0)(\mathrm{i}\partial\bar\partial u_t)^n \,\mathrm{d}t - \int_{1/2}^{1}\int_\Omega (u_1 - u_0)(\mathrm{i}\partial\bar\partial u_t)^n \,\mathrm{d}t \\ = \int_0^{1/2}\int_\Omega (u_1 - u_0)((\mathrm{i}\partial\bar\partial u_t)^n - (\mathrm{i}\partial\bar\partial u_{t+1/2})^n)\,\mathrm{d}t \\ = 2\int_0^{1/2}\int_\Omega (u_{t+1/2} - u_t)((\mathrm{i}\partial\bar\partial u_t)^n - (\mathrm{i}\partial\bar\partial u_{t+1/2})^n)\,\mathrm{d}t \geqslant 0,\end{aligned}$$

we have

$$F^0(u_1) - F^0(u_{1/2}) \geqslant F^0(u_{1/2}) - F^0(u_0).$$

The cocycle condition is a simple consequence of the variation formula (3.6). □

4. The convergence

In this section, let us assume that both f and φ are independent of t. For $u \in \mathcal{P}(\Omega, \varphi)$, define

$$F(u) = F^0(u) + \int_\Omega G(z,u)\,\mathrm{d}V, \tag{4.1}$$

where $\mathrm{d}V$ is the volume element in $\mathbb{C}^n$ and $G(z,s)$ is the function given by

$$G(z,s) = \int_0^s \mathrm{e}^{-f(z,t)}\,\mathrm{d}t.$$

Then the variation of F is

$$\delta F(u) = -\int_\Omega \delta u(\det(u_{\alpha\bar\beta}) - \mathrm{e}^{-f(z,u)})\,\mathrm{d}V. \tag{4.2}$$

Proof of Theorem 1.2. Since both φ and f are independent of t, by the Krylov–Evans theory and the uniform $\mathcal{C}^2(\bar\Omega)$ estimate in § 2, it follows that $u(\cdot\,,t)$ has uniform $\mathcal{C}^{2,\alpha}(\bar\Omega)$ estimate, and hence uniform $\mathcal{C}^\infty(\bar\Omega)$ estimate. Therefore, for any sequence $t_n \to \infty$ there is a subsequence $t_{n_j} \to \infty$ such that $u(\cdot\,, t_{n_j})$ converges to some function $\tilde u$ in $\mathcal{C}^\infty(\bar\Omega)$. If we can show that $\mathrm{e}^{\dot u}$ converges to 1 in measure, then $\tilde u$ solves the elliptic equation (1.5) in measure, hence $\tilde u = v$ by the uniqueness of the solution of (1.5), i.e. the sequential

https://doi.org/10.1017/S1474748009000206 Published online by Cambridge University Press

limit is independent of the sequence, therefore we get the full convergence of $u(\cdot,t)$ in $\mathcal{C}^\infty(\bar{\Omega})$. So we only need to show that

$$\mathrm{e}^{\dot{u}(\cdot,t)} \to 1 \quad \text{in the sense of measure.}$$

We will follow Phong and Sturm's proof of the convergence of the Kähler–Ricci flow in [**11**]. For any $t > 0$, the function $u(\cdot,t)$ is in $\mathcal{P}(\Omega,\varphi)$. So, by (4.2),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(u) &= -\int_\Omega \dot{u}(\det(u_{\alpha\bar{\beta}}) - \mathrm{e}^{-f(z,u)}) \\ &= -\int_\Omega (\log\det(u_{\alpha\bar{\beta}}) - (-f(z,u)))(\det(u_{\alpha\bar{\beta}}) - \mathrm{e}^{-f(z,u)}) \leqslant 0.\end{aligned}$$

Thus $F(u(\cdot,t))$ is monotonic decreasing as t approaches $+\infty$. On the other hand, $u(\cdot,t)$ is uniformly bounded in $\mathcal{C}^2(\bar{\Omega})$ by (1.6), so both $F^0(u(\cdot,t))$ and $f(z,u(\cdot,t))$ are uniformly bounded, hence $F(u)$ is bounded. Therefore,

$$\int_0^\infty \int_\Omega (\log\det(u_{\alpha\bar{\beta}}) + f(z,u))(\det(u_{\alpha\bar{\beta}}) - \mathrm{e}^{-f(z,u)})\,\mathrm{d}t < \infty. \tag{4.3}$$

Observe that by the mean value theorem, for $x, y \in \mathbb{R}$,

$$(x+y)(\mathrm{e}^x - \mathrm{e}^{-y}) = (x+y)^2\mathrm{e}^\eta \geqslant \mathrm{e}^{\min(x,-y)}(x+y)^2,$$

where η is between x and $-y$. Thus

$$(\log\det(u_{\alpha\bar{\beta}}) + f)(\det(u_{\alpha\bar{\beta}}) - \mathrm{e}^{-f}) \geqslant C_9(\log\det(u_{\alpha\bar{\beta}}) + f)^2 = C_9|\dot{u}|^2,$$

where C_9 is independent of t. Hence

$$\int_0^\infty \|\dot{u}\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant \infty. \tag{4.4}$$

Let

$$Y(t) = \int_\Omega |\dot{u}(\cdot,t)|^2 \det(u_{\alpha\bar{\beta}})\,\mathrm{d}V, \tag{4.5}$$

then

$$\dot{Y} = \int_\Omega (2\ddot{u}\dot{u} + \dot{u}^2 u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}})\det(u_{\alpha\bar{\beta}})\,\mathrm{d}V.$$

Differentiate (1.1) in t,

$$\ddot{u} - u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}} = f_u\dot{u}, \tag{4.6}$$

so

$$\begin{aligned}\dot{Y} &= \int_\Omega (2\dot{u}\dot{u}_{\alpha\bar{\beta}}u^{\alpha\bar{\beta}} + \dot{u}^2(2f_u + u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}}))\det(u_{\alpha\bar{\beta}})\,\mathrm{d}V \\ &= \int_\Omega (\dot{u}^2(2f_u + u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}}) - 2\dot{u}_\alpha\dot{u}_{\bar{\beta}}u^{\alpha\bar{\beta}})\det(u_{\alpha\bar{\beta}})\,\mathrm{d}V.\end{aligned}$$

Notice that both f_u and $u^{\alpha\bar{\beta}}\dot{u}_{\alpha\bar{\beta}}$ are uniformly bounded, so

$$\dot{Y} \leqslant C_{10} \int_{\Omega} \dot{u}^2 \det(u_{\alpha\bar{\beta}})\,\mathrm{d}V = C_{10} Y$$

and

$$Y(t) \leqslant Y(s)\mathrm{e}^{C_{10}(t-s)} \quad \text{for } t > s, \tag{4.7}$$

where C_{10} is independent of t. By (4.4), (4.7) and the uniform boundedness of $\det(u_{\alpha\bar{\beta}})$, we get

$$\lim_{t\to\infty} \|\dot{u}(\cdot\,,t)\|_{L^2(\Omega)} = 0.$$

Since Ω is bounded, the L^2 norm controls the L^1 norm, hence

$$\lim_{t\to\infty} \|\dot{u}(\cdot\,,t)\|_{L^1(\Omega)} = 0.$$

Notice that by the mean value theorem,

$$|\mathrm{e}^x - 1| < \mathrm{e}^{|x|}|x|,$$

so

$$\int_{\Omega} |\mathrm{e}^{\dot{u}} - 1|\,\mathrm{d}V \leqslant \mathrm{e}^{\sup|\dot{u}|} \int_{\Omega} |\dot{u}|\,\mathrm{d}V.$$

Hence $\mathrm{e}^{\dot{u}}$ converges to 1 in $L^1(\Omega)$ as t approaches $+\infty$, and this finishes the proof of Theorem 1.2. □

Acknowledgements. Both authors would like to thank Professor D. H. Phong for his constant encouragement and help. Both authors would also like to thank the referee for carefully reading the paper and for help in correcting a misstatement in Theorem 1.1.

References

1. I. J. Bakelman, Variational problems and elliptic Monge–Ampère equations, *J. Diff. Geom.* **18**(4) (1983), 669–699.
2. E. Bedford and B. A. Taylor, The Dirichlet problem for a complex Monge–Ampère equation, *Invent. Math.* **37**(1) (1976), 1–44.
3. Z. Błocki, Weak solutions to the complex Hessian equation, *Annales Inst. Fourier* **55**(5) (2005), 1735–1756.
4. L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, II, Complex Monge–Ampère, and uniformly elliptic, equations, *Commun. Pure Appl. Math.* **38**(2) (1985), 209–252.
5. H. D. Cao, Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, *Invent. Math.* **81**(2) (1985), 359–372.
6. X. X. Chen and G. Tian, Ricci flow on Kähler–Einstein surfaces, *Invent. Math.* **147**(3) (2002), 487–544.
7. B. Guan, The Dirichlet problem for complex Monge–Ampère equations and regularity of the pluri-complex Green function, *Commun. Analysis Geom.* **6**(4) (1998), 687–703.
8. S. Kołodziej, The complex Monge–Ampère equation, *Acta Math.* **180**(1) (1998), 69–117.
9. N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, *Izv. Akad. Nauk SSSR Ser. Mat.* **47**(1) (1983), 75–108.

10. S.-Y. Li, On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian, *Asian J. Math.* **8**(1) (2004), 87–106.
11. D. H. Phong and J. Sturm, On stability and the convergence of the Kähler–Ricci flow, *J. Diff. Geom.* **72**(1) (2006), 149–168.
12. G. Tian and X. Zhu, Convergence of Kähler–Ricci flow, *J. Am. Math. Soc.* **20**(3) (2007), 675–699 (electronic).
13. N. S. Trudinger, On the Dirichlet problem for Hessian equations, *Acta Math.* **175**(2) (1995), 151–164.
14. N. S. Trudinger and X.-J. Wang, Hessian measures, I, *Topol. Methods Nonlin. Analysis* **10**(2) (1997), 225–239 (dedicated to Olga Ladyzhenskaya).
15. N. S. Trudinger and X.-J. Wang, A Poincaré type inequality for Hessian integrals, *Calc. Var. Partial Diff. Eqns* **6**(4) (1998), 315–328.
16. K. Tso, On a real Monge–Ampère functional, *Invent. Math.* **101**(2) (1990), 425–448.
17. X. J. Wang, A class of fully nonlinear elliptic equations and related functionals, *Indiana Univ. Math. J.* **43**(1) (1994), 25–54.
18. S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Commun. Pure Appl. Math.* **31**(3) (1978), 339–411.