Bull. Aust. Math. Soc. **105** (2022), 197–201
doi:10.1017/S0004972721000678

A PROOF OF MERCA'S CONJECTURES ON SUMS OF ODD DIVISOR FUNCTIONS

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(Received 16 July 2021; accepted 22 July 2021; first published online 10 September 2021)

Abstract

Merca ['Congruence identities involving sums of odd divisors function', *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* **22**(2) (2021), 119–125] posed three conjectures on congruences for specific convolutions of a sum of odd divisor functions with a generating function for generalised m-gonal numbers. Extending Merca's work, we complete the proof of these conjectures.

2020 *Mathematics subject classification*: primary 11A25; secondary 11P81.

Keywords and phrases: divisor function, polygonal number, partition.

Euler's partition function $p(n)$ is the number of partitions of a nonnegative integer n and its generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}, \quad |q| < 1.$$

The properties of the function $p(n)$, such as its asymptotic behaviour and its parity, have been an object of study for a long time. For instance, Ballantine and Merca [1] recently made a conjecture on when

$$\sum_{ak+1 \text{ is a square}} p(n-k)$$

is odd, which was proved by Hong and Zhang [2].

The function $p(n)$ is linked to the divisor function

$$\sigma(n) := \sum_{d|n} d,$$

The authors are grateful for the generous support of the National Science Foundation (DMS 2002265 and 205118), the National Security Agency (H98230-21-1-0059), the Thomas Jefferson Fund at the University of Virginia and the Templeton World Charity Foundation.

whose generating function is given by

$$\sum_{n=1}^{\infty}\sigma(n)q^n = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

In particular, $p(n)$ and $\sigma(n)$ satisfy the following convolution identities, which differ only in the values of $p(0)$ and $\sigma(0)$:

$$\sum_{k=-\infty}^{\infty}(-1)^k p(n-P_5(k)) = \delta_{0,n}, \quad \text{with } p(0)=1,$$

$$\sum_{k=-\infty}^{\infty}(-1)^k \sigma(n-P_5(k)) = 0, \quad \text{with } \sigma(0) \text{ replaced by } n,$$

where δ_{ij} is the Kronecker delta and $P_m(k)$ is the kth generalised m-gonal number

$$P_m(k) := \Big(\frac{m}{2}-1\Big)k^2 - \Big(\frac{m}{2}-2\Big)k. \tag{1}$$

Motivated by these identities as well as the fact that the divisor functions $\sigma(n)$ and

$$\sigma_{\text{odd}}(n) := \sum_{\substack{d|n\\ d \text{ odd}}} d,$$

where $\sigma_{\text{odd}}(n) := 0$ for $n \le 0$, have the same parity, Merca [3] recently studied the relationship between $\sigma_{\text{odd}}(n)$ and the generalised m-gonal numbers. More specifically, he investigated for which positive integers m the following congruences hold for all $n \in \mathbb{Z}^+$:

$$\sum_{k=-\infty}^{\infty}\sigma_{\text{odd}}(n-P_m(k)) \equiv \begin{cases} n \pmod 2 & \text{if } n = P_m(j),\ j\in\mathbb{Z},\\ 0 \pmod 2 & \text{otherwise,}\end{cases} \tag{2}$$

$$\sum_{k=-\infty}^{\infty}\sigma_{\text{odd}}(n-P_5(k)) \equiv \begin{cases} n \pmod m & \text{if } n = P_5(j),\ j\in\mathbb{Z},\\ 0 \pmod m & \text{otherwise,}\end{cases} \tag{3}$$

$$\sum_{k=-\infty}^{\infty}(-1)^{P_3(-k)}\sigma_{\text{odd}}(n-P_5(k)) \equiv \begin{cases} (-1)^{P_3(-j)}\cdot n \pmod m & \text{if } n = P_5(j),\ j\in\mathbb{Z},\\ 0 \pmod m & \text{otherwise.}\end{cases} \tag{4}$$

In particular, Merca posed the following conjectures in [3].

Conjecture 1. The following are true.

(i) The congruence (2) holds for all $n \in \mathbb{Z}^+$ if and only if $m \in \{5, 6\}$.
(ii) The congruence (3) holds for all $n \in \mathbb{Z}^+$ if and only if $m \in \{2, 3, 6\}$.
(iii) The congruence (4) holds for all $n \in \mathbb{Z}^+$ if and only if $m \in \{2, 4\}$.

Merca showed the ‘if’ condition for each of these conjectures. Using his work, we obtain the following theorem.

THEOREM 2. *Merca's conjectures are true.*

PROOF. We begin by proving (ii). Merca showed that (3) holds if $m \in \{2, 3, 6\}$ [3, Theorem 3]. Hence, it suffices to show that if $m \notin \{2, 3, 6\}$, then there exists some $n \in \mathbb{Z}^+$ such that $n \neq P_5(j)$ for all $j \in \mathbb{Z}$ and

$$\sum_{k=-\infty}^{\infty} \sigma_{\mathrm{odd}}(n - P_5(k)) \not\equiv 0 \pmod{m}. \tag{5}$$

Since $\sigma_{\mathrm{odd}}(n - P_5(k)) = 0$ whenever $n - P_5(k) \le 0$, the sum in (5) is in fact finite and we easily compute $\sum_k \sigma_{\mathrm{odd}}(3 - P_5(k)) = 6$, where $3 \neq P_5(j)$ for $j \in \mathbb{Z}$. Thus, (5) holds unless $6 \equiv 0 \pmod{m}$. But this is the case only if $m \in \{2, 3, 6\}$.

Next, we prove (iii). Again, Merca proved that (4) holds if $m \in \{2, 4\}$ [3, Theorem 4]. Hence, it suffices to show that if $m \notin \{2, 4\}$, then there exists some $n \in \mathbb{Z}^+$ such that $n \neq P_5(j)$ for all $j \in \mathbb{Z}$ and

$$\sum_{k=-\infty}^{\infty} (-1)^{P_3(-k)} \sigma_{\mathrm{odd}}(n - P_5(k)) \not\equiv 0 \pmod{m}. \tag{6}$$

We compute $\sum_k (-1)^{P_3(-k)} \sigma_{\mathrm{odd}}(3 - P_5(k)) = 4$, where $3 \neq P_5(j)$ for $j \in \mathbb{Z}$, and so (6) holds unless $4 \equiv 0 \pmod{m}$. But this is the case only if $m \in \{2, 4\}$.

Finally, we prove (i). Since $\sigma_{\mathrm{odd}}(n)$ is odd if and only if n is a square or twice a square (see [3, page 3]),

$$\sum_{n=1}^{\infty} \sigma_{\mathrm{odd}}(n) q^n \equiv \sum_{n=1}^{\infty} q^{n^2} + \sum_{n=1}^{\infty} q^{2n^2} \pmod{2}. \tag{7}$$

The nth coefficient of

$$\Big(\sum_{\ell=1}^{\infty} \sigma_{\mathrm{odd}}(\ell) q^{\ell}\Big)\Big(\sum_{k=-\infty}^{\infty} q^{P_m(k)}\Big) = \sum_{\ell=1}^{\infty} \sum_{k=-\infty}^{\infty} \sigma_{\mathrm{odd}}(\ell) q^{\ell + P_m(k)} = \sum_{n=1}^{\infty} \Big(\sum_{k=-\infty}^{\infty} \sigma_{\mathrm{odd}}(n - P_m(k))\Big) q^n$$

is given by $\sum_{k=-\infty}^{\infty} \sigma_{\mathrm{odd}}(n - P_m(k))$. On the other hand, the nth coefficient of

$$\Big(\sum_{\ell=1}^{\infty} q^{\ell^2} + \sum_{\ell=1}^{\infty} q^{2\ell^2}\Big)\Big(\sum_{k=-\infty}^{\infty} q^{P_m(k)}\Big) = \sum_{\substack{\ell \ge 1 \\ k \in \mathbb{Z}}} (q^{\ell^2 + P_m(k)} + q^{2\ell^2 + P_m(k)})$$

is given by $a_m(n) + b_m(n)$, where

$$a_m(n) = |A_m(n)| := \#\{(\ell, k) \in \mathbb{Z}^+ \times \mathbb{Z} : \ \ell^2 + P_m(k) = n\},$$
$$b_m(n) = |B_m(n)| := \#\{(\ell, k) \in \mathbb{Z}^+ \times \mathbb{Z} : \ 2\ell^2 + P_m(k) = n\}.$$

Thus, due to (7),

$$\sum_{k=-\infty}^{\infty} \sigma_{\mathrm{odd}}(n - P_m(k)) \equiv a_m(n) + b_m(n) \pmod{2}.$$

https://doi.org/10.1017/S0004972721000678 Published online by Cambridge University Press

Suppose first that $m \geq 7$. Then we claim that $P_m(0) = 0, P_m(1) = 1$ and $P_m(k) > 3$ for all $k \notin \{0, 1\}$. From (1), it is clear that $P_m(0) = 0$ and $P_m(1) = 1$. To see that $P_m(k) > 3$ for all $k \notin \{0, 1\}$, note that since the leading term of $P_m(x)$ is positive and its minimum is at $(m-4)/(2m-4)$, where $0 < (m-4)/(2m-4) < 1$, we have $P_m(k) \geq P_m(2) = m \geq 7$ for $k \geq 2$ and $P_m(k) \geq P_m(-1) = m - 3 \geq 4$ for $k \leq -1$.

Now, let $n = 3$. Then the above shows that n is not a generalised m-gonal number for $m \geq 7$, and so for (2) to hold, we must have $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(3 - P_m(k)) \equiv 0 \pmod 2$. If $(\ell, k) \in A_m(3)$, then $\ell^2 = 3 - P_m(k)$, so that in particular $\ell^2 \leq 3$, which forces $\ell = 1$. But then we must have $P_m(k) = 2$, which we have seen to be impossible. Hence, $A_m(3)$ is empty and $a_m(3) \equiv 0 \pmod 2$. On the other hand, if $(\ell, k) \in B_m(3)$, we must again have $\ell = 1$. It follows that $P_m(k) = 1$, which is the case if and only if $k = 1$. Hence, $B_m(3) = \{(1, 1)\}$ and $b_m(3) \equiv 1 \pmod 2$. We conclude that $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(3 - P_m(k)) \equiv a_m(3) + b_m(3) \equiv 1 \not\equiv 0 \pmod 2$.

Merca showed that (2) holds for $m \in \{5, 6\}$ and, for $m \in \{1, 2\}$, the sum in (2) diverges; hence, it remains to consider $m \in \{3, 4\}$. Suppose first that $m = 3$ and note that $P_3(k) = \frac{1}{2}(k^2 + k)$. We have $3 = P_3(-3) = P_3(2)$, so for (2) to hold, we must have $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(3 - P_3(k)) \equiv 3 \equiv 1 \pmod 2$. If $(\ell, k) \in A_3(3)$, then $\ell = 1$ and $P_3(k) = 2$, which is impossible. Hence, $A_3(3)$ is empty. If $(\ell, k) \in B_3(3)$, then $\ell = 1$ and $P_3(k) = 1$, which is the case if and only if $k \in \{-2, 1\}$. It follows that $B_3(3) = \{(1, -2), (1, 1)\}$ and $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(3 - P_3(k)) \equiv a_3(3) + b_3(3) \equiv 0 \not\equiv 1 \pmod 2$.

Finally, suppose that $m = 4$ and note that $P_4(k) = k^2$. Since $4 = P_4(2)$, for (2) to hold we must have $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(4 - P_4(k)) \equiv 4 \equiv 0 \pmod 2$. If $(\ell, k) \in A_4(4)$, then either $\ell = 1$ and $P_4(k) = 3$, which is impossible, or $\ell = 2$ and $P_4(k) = 0$, which is the case if and only if $k = 0$. Thus, $A_4(4) = \{(2, 0)\}$. On the other hand, if $(\ell, k) \in B_4(4)$, then $\ell = 1$ and $P_3(k) = 2$, which is impossible. It follows that $B_4(4)$ is empty and $\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(4 - P_4(k)) \equiv a_4(4) + b_4(4) \equiv 1 \not\equiv 0 \pmod 2$. □

Acknowledgements

We would like to thank Ken Ono for suggesting this project and for several helpful conversations. We thank William Craig and Badri Pandey, as well as the referee, for their comments on the exposition in this note.

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https://doi.org/10.1017/S0004972721000678 Published online by Cambridge University Press

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