PROBABILISTIC ASPECTS OF CRITICAL GROWTH-FRAGMENTATION EQUATIONS

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Abstract

The self-similar growth-fragmentation equation describes the evolution of a medium in which particles grow and divide as time proceeds, with the growth and splitting of each particle depending only upon its size. The critical case of the equation, in which the growth and division rates balance one another, was considered in Doumic and Escobedo (2015) for the homogeneous case where the rates do not depend on the particle size. Here, we study the general self-similar case, using a probabilistic approach based on Lévy processes and positive self-similar Markov processes which also permits us to analyse quite general splitting rates. Whereas existence and uniqueness of the solution are rather easy to establish in the homogeneous case, the equation in the nonhomogeneous case has some surprising features. In particular, using the fact that certain self-similar Markov processes can enter $(0, \infty)$ continuously from either 0 or ∞, we exhibit unexpected spontaneous generation of mass in the solutions.

Keywords: Growth-fragmentation equation; self-similarity; self-similar Markov process; branching process

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1. Introduction

The growth-fragmentation equation is a linear differential equation intended to describe the evolution of a medium in which particles grow and split as time passes. It is frequently expressed in terms of the concentration of particles with size $x > 0$ at time t, say $u(t, x)$, as follows:

$$\partial_t u(t, x) + \partial_x(\tau(x)u(t, x)) + B(x)u(t, x) = \int_x^\infty k(x, y)B(y)u(t, y)\,\mathrm{d}y. \tag{1.1}$$

Here $\tau(x)$ is the speed of growth of a particle with size x, $B(x)$ is the rate at which a particle of size x splits, and $k(y, x) = k(x - y, x)$ is twice the probability density that a particle with size x splits into two particles with sizes y and $y - x$ (the factor 2 is due to the symmetry of the splitting events). This type of equation has a variety of applications in mathematical modelling, notably in biology where particles should be thought of as cells, and has motivated several works in recent years; see, for example, [11], which also contains a summary of some recent literature.

We are interested here in the situation $\tau(x) = cx^{\alpha+1}$, $B(x) = x^\alpha$ for some $\alpha \in \mathbb{R}$, and k of the form $k(y, x) = x^{-1}k_0(y/x)$; for these parameters, (1.1) possesses a useful self-similarity property. This is referred to as the *critical* case by Doumic and Escobedo [11], who studied in

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depth the situation when, additionally, $\alpha = 0$. For our purposes, it will be more convenient to write the equation in weak form, as follows. For $x > 0$ and $y \in [\frac{1}{2}, 1)$, we write $(y \mid x)$ for the pair $\{yx, (1-y)x\}$, which we view as the dislocation of a mass x into two smaller masses, and then, for every function $f\colon (0, \infty) \to \mathbb{R}$, we set

$$f(y \mid x) := f(yx) + f((1-y)x).$$

Consider test functions $f \in \mathcal{C}_c^\infty(0, \infty)$, that is, f is infinitely differentiable with compact support. For a measure μ on $(0, \infty)$, we write $\langle \mu, f \rangle := \int_{(0,\infty)} f(x)\mu(\mathrm{d}x)$. By integrating (1.1), we obtain

$$\partial_t \langle \mu_t, f \rangle = \langle \mu_t, \mathcal{L} f \rangle, \tag{1.2}$$

where $\mu_t(\mathrm{d}x) = u(t, x)\,\mathrm{d}x$ and the operator $\mathcal{L}$ has the form

$$\mathcal{L} f(x) = x^\alpha \biggl(cxf'(x) + \int_{[1/2,1)} (f(y \mid x) - f(x)) K(\mathrm{d}y) \biggr), \tag{1.3}$$

where

$$K(\mathrm{d}y) := k_0(y)\,\mathrm{d}y = k_0(1-y)\,\mathrm{d}y, \qquad y \in [\tfrac{1}{2}, 1),$$

is referred to as the *dislocation measure*. The advantage of this formulation is that we do not require absolute continuity of the solution μ_t or the dislocation measure K. More generally, one might also consider nonbinary dislocation measures, but we refrain from doing so in this work in order to simplify the presentation.

In this article, we study (1.2) for operators of the form

$$\mathcal{L}_\alpha f(x) := x^\alpha \biggl(ax^2 f''(x) + bxf'(x) + \int_{[1/2,1)} (f(y \mid x) - f(x) + xf'(x)(1-y)) K(\mathrm{d}y) \biggr), \tag{1.4}$$

where $a \geq 0$ and $b \in \mathbb{R}$, and we now assume only that the measure K satisfies the weaker requirement

$$\int_{[1/2,1)} (1-y)^2 K(\mathrm{d}y) < \infty. \tag{1.5}$$

Our notion of a *solution* of (1.2) is a collection of locally finite measures $(\mu_t)_{t \geq 0}$ on $(0, \infty)$ such that, for every $f \in \mathcal{C}_c^\infty(0, \infty)$ and $t \geq 0$, there holds the identity

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \int_0^t \langle \mu_s, \mathcal{L} f \rangle \,\mathrm{d}s.$$

(This requires implicitly that $s \mapsto \langle \mu_s, \mathcal{L} f \rangle$ be a well-defined, locally integrable function, and in particular that the family $(\mu_t)_{t \geq 0}$ is then vaguely continuous.)

We offer a comparison between the original operator (1.3) and our operator (1.4). Besides the appearance of a second-order derivative, there is a new term $xf'(x)(1-y)$ in the integral in (1.4). The latter should be interpreted as an additional growth term which, in some sense, balances the accumulation of small dislocations. We stress that (1.5) is the necessary and sufficient condition for (1.4) to be well defined, and that when the measure K is finite (or at least fulfils $\int_{[1/2,1)} (1-y) K(\mathrm{d}y) < \infty$), every operator of the form (1.3) can also be expressed in the form (1.4). Our motivation for considering this more general setting stems from the recent work [3], in which a new class of growth-fragmentation stochastic processes is constructed such that, loosely speaking, the strong rates of dislocation that would instantaneously shatter the entire mass can be somehow compensated by an intense growth; the dislocation measure associated with such a fragmentation process need only satisfy (1.5).

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In short, the purpose of this work is to demonstrate the usefulness of some probabilistic methods for the study of these critical growth-fragmentation equations. More precisely, we shall see that solutions to (1.2) for $\mathcal{L} = \mathcal{L}_\alpha$ can be related to the one-dimensional distributions of certain self-similar Markov processes, and this will enable us to reveal some rather unexpected features of the former. Although, in the homogeneous case $\alpha = 0$, we establish existence and uniqueness of the solution in full generality, this feature is lost for $\alpha \neq 0$. In particular, we shall see that, under a fairly general assumption on the parameters of the model, the critical growth-fragmentation equation permits spontaneous generation, i.e. there exist nondegenerate solutions starting from the null initial condition.

We need some notation before describing more precisely our main results. We first introduce the function $\kappa\colon [0,\infty) \to (-\infty,\infty]$ which plays a major role in our approach and is given by

$$\kappa(q) := aq^2 + (b-a)q + \int_{[1/2,1)} (y^q + (1-y)^q - 1 + q(1-y))K(\mathrm{d}y), \qquad q \geq 0. \tag{1.6}$$

There are two principal 'Malthusian hypotheses' which we will require when $\alpha \neq 0$:

(M_+) $\inf_{q\geq 0} \kappa(q) < 0$;

(M_-) there exist $0 \leq \omega_- < \omega_+$ and $\varepsilon > 0$ such that $\kappa(\omega_-) = \kappa(\omega_+) = 0$ and $\kappa(\omega_- - \varepsilon) < \infty$.

We now summarise our main results, deferring their proofs to the body of the article.

- For $\alpha = 0$, (1.2) with operator (1.4) and initial condition $\mu_0 = \delta_1$ has a unique solution.
- For $\alpha < 0$, suppose that (M_+) holds. Then (1.2) with operator (1.4) has a solution with initial condition $\mu_0 = \delta_1$. There exists further a nondegenerate solution started from $\mu_0 = 0$; in particular, uniqueness fails.
- For $\alpha > 0$, if (M_+) holds then (1.2) with operator (1.4) has a solution with initial condition $\mu_0 = \delta_1$. If (M_-) holds then there also exists a nondegenerate solution started from $\mu_0 = 0$; again, uniqueness fails.

We shall also observe that, under essentially the converse assumption to (M_+), namely that $\inf_{q\geq 0} \kappa(q) > 0$, the particle system that corresponds to the stochastic version of the model may explode in finite time almost surely. This is a strong indication that (1.2) should have no global solution in the latter case.

The rest of this article is organised as follows. In the next section we provide brief preliminaries on the function κ and the use of the Mellin transform in the study of growth-fragmentation equations. Section 3 is devoted to the homogeneous case $\alpha = 0$, and then the general self-similar case $\alpha \neq 0$ is presented in Section 4. In Section 5 we investigate a stochastic model related to the growth-fragmentation equation, to demonstrate that explosion may occur when the Malthusian hypothesis fails. Finally, in Section 6 we briefly discuss another interpretation of the growth-fragmentation equation in terms of branching particle systems and many-to-one formulas, placing the results of Sections 3 and 4 in context.

2. The Mellin transform and the growth-fragmentation equation

We observe first that, for any $\alpha \in \mathbb{R}$, the operator $\mathcal{L}_\alpha$ fulfils a self-similarity property. Specifically, for every $c > 0$, if we denote by $\varphi_c(x) = cx$ the dilation function with factor c then, for a generic $f \in \mathcal{C}_c^\infty(0,\infty)$, there holds the identity

$$\mathcal{L}_\alpha(f \circ \varphi_c) = c^{-\alpha}(\mathcal{L}_\alpha f) \circ \varphi_c. \tag{2.1}$$

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As a consequence, if $(\mu_t)_{t\geq 0}$ is a solution to (1.2) for all $f \in \mathcal{C}_c^\infty(0,\infty)$ with initial condition $\mu_0 = \delta_1$, and if $\tilde{\mu}_t$ denotes the image of μ_t by the dilation φ_c, then $(\tilde{\mu}_{c^\alpha t})_{t\geq 0}$ is a solution to (1.2) for all $f \in \mathcal{C}_c^\infty(0,\infty)$ with initial condition $\tilde{\mu}_0 = \delta_c$. For the sake of simplicity, we shall therefore focus on the growth-fragmentation equation with initial condition $\mu_0 = \delta_1$, since this does not induce any loss of generality.

Recall that the function κ has been introduced in (1.6); its domain is clarified by the following result.

Lemma 2.1. (i) *For every $q \geq 0$, $\kappa(q)$ is well defined with values in $(-\infty, \infty]$. The function κ is convex, and we define* $\operatorname{dom}\kappa = \{q \geq 0 \colon \kappa(q) < \infty\}$.

(ii) *For $q \geq 0$, $\kappa(q) < \infty$ if and only if $\int_{[1/2,1)} (1-y)^q K(\mathrm{d}y) < \infty$; in particular,* $[2,\infty) \subseteq \operatorname{dom}\kappa$.

(iii) *For every function f in $\mathcal{C}_c^\infty(0,\infty)$, $\mathcal{L}_\alpha f$ is a continuous function on $(0,\infty)$ and is identically 0 in some neighbourhood of 0. Furthermore, $\mathcal{L}_\alpha f(x) = \mathrm{o}(x^{q+\alpha})$ as $x \to \infty$ for every $q \in \operatorname{dom}\kappa$, and, thus, in particular for $q = 2$.*

Proof. (i–ii) First, the integral $\int_{[1/2,1)} (y^q - 1 + q(1-y))K(\mathrm{d}y)$ converges absolutely thanks to (1.5), since $y^q - 1 + q(1-y) = \mathrm{O}((1-y)^2)$. It follows that $\kappa(q)$ is then well defined with values in $(-\infty,\infty)$ if and only if $\int_{[1/2,1)} (1-y)^q K(\mathrm{d}y) < \infty$, and otherwise $\kappa(q) = \infty$.

(iii) The first assertions are straightforward, and so we check only the last assertion. Take $q \in \operatorname{dom}\kappa$ and recall from above that $\int_{[1/2,1)} (1-y)^q K(\mathrm{d}y) < \infty$. This entails that $K([\frac{1}{2}, 1-\varepsilon)) = \mathrm{o}(\varepsilon^{-q})$ as $\varepsilon \to 0+$. Since f has compact support in $(0,\infty)$, then, for sufficiently large x, we have $\mathcal{L}_0 f(x) = \int_{[1/2,1)} f(x(1-y))K(\mathrm{d}y)$, and we easily conclude that $\mathcal{L}_0 f(x) = \mathrm{o}(x^q)$. □

Doumic and Escobedo [11] studied certain growth-fragmentation equations with homogeneous operators given by (1.3) for $\alpha = 0$, and observed that the Mellin transform plays an important role. In this direction, it is useful to introduce the notation $h_q \colon (0,\infty) \to (0,\infty)$, $h_q(x) = x^q$, for the power function with exponent q, and recall that the Mellin transform of a measurable function $f \colon (0,\infty) \to \mathbb{R}$ is defined for $z \in \mathbb{C}$ by

$$\mathcal{M}f(z) := \int_0^\infty f(x) x^{z-1}\,\mathrm{d}x$$

whenever the integral on the right-hand side converges. It follows from Lemma 2.1 that the Mellin transform of $\mathcal{L}_\alpha f$ is well defined for all $z < -2-\alpha$, or more generally for all z such that $-z-\alpha \in (\operatorname{dom}\kappa)^\circ$.

The role of κ in this study stems from the following lemma, which is easily checked from elementary properties of the Mellin transform; see [10, Section 12.3] and [11, Section 1.1].

Lemma 2.2. (i) *Let $q \in \operatorname{dom}\kappa$. Then*

$$\mathcal{L}_\alpha h_q(x) = \kappa(q) h_{q+\alpha}(x), \qquad x > 0.$$

In particular, for $\alpha = 0$, h_q is an eigenfunction for $\mathcal{L}_0$ with eigenvalue $\kappa(q)$.

(ii) *For every q such that $q - \alpha \in (\operatorname{dom}\kappa)^\circ$ and every $f \in \mathcal{C}_c^\infty(0,\infty)$, there holds the identity*

$$\mathcal{M}(\mathcal{L}_\alpha f)(-q) = \kappa(q-\alpha)\mathcal{M}f(-q+\alpha).$$

3. The homogeneous case

Throughout this section, we assume that $\alpha = 0$, and refer to this case as *homogeneous*. Recall that, when $a = 0$ and the dislocation measure K fulfils the stronger condition,

$$\int_{[1/2,1)} (1-y)K(\mathrm{d}y) < \infty. \tag{3.1}$$

Then we can express the operator $\mathcal{L}_0$ in the simpler form

$$\mathcal{L}_{c,K} f(x) := cxf'(x) + \int_{[1/2,1)} (f(y \mid x) - f(x))K(\mathrm{d}y),$$

with $c = b + \int_{[1/2,1)}(1-y)K(\mathrm{d}y)$. This situation was considered in depth by Doumic and Escobedo [11], and most of the results of this section should be viewed as extensions of those in [11] to the case when either $a > 0$, or K fulfils (1.5) but not (3.1). Furthermore, the case $c \le 0$ was considered by Haas [13] using the same method that we employ below.

3.1. Main results

The key observation in Lemma 2.2(i), that power functions h_q are eigenfunctions of the operator $\mathcal{L}_0$, underlies the analysis of the homogeneous case. Specifically, if we knew that $(\mu_t)_{t\ge 0}$ solves (1.2) with $f = h_q$ for $q \ge 2$, the Mellin transform of μ_t, $M_t(z) = \langle \mu_t, h_{z-1}\rangle$, would solve the linear equation

$$\partial M_t(q+1) = \kappa(q)M_t(q+1). \tag{3.2}$$

Focusing for simplicity on the initial condition $\mu_0 = \delta_1$, so that $M_0(q) = 1$, we would find that

$$M_t(q+1) = \exp(t\kappa(q)). \tag{3.3}$$

In order to check that (3.3) is indeed the Mellin transform of a positive measure, we define, for every $\omega \in \operatorname{dom}\kappa$, a new function by shifting κ:

$$\Phi_\omega(q) := \kappa(\omega+q) - \kappa(\omega), \qquad q \ge 0.$$

This is a smooth, convex function, and it has a simple probabilistic interpretation, which will play a major role throughout. In this direction, recall first that a *Lévy process* is a stochastic process issuing from the origin with stationary and independent increments and càdlàg paths. It is further called *spectrally negative* if all its jumps are negative. If $\xi := (\xi(t))_{t\ge 0}$ is a spectrally negative Lévy process with law $\mathbb{P}$ then, for all $t \ge 0$, the *Laplace exponent* Φ, given by

$$\mathbb{E}[\exp(q\xi(1))] = \exp(\Phi(q)),$$

is well defined (and finite) for all $q \ge 0$, and satisfies the classical Lévy–Khintchin formula

$$\Phi(q) = \mathrm{a}q + \frac{1}{2}\sigma^2 q^2 + \int_{(-\infty,0)} (\mathrm{e}^{qx} - 1 - qx\mathbf{1}_{\{|x|\le 1\}})\Upsilon(\mathrm{d}x), \qquad q \ge 0,$$

where $\mathrm{a} \in \mathbb{R}$, $\sigma \ge 0$, and Υ is a measure (the *Lévy measure*) on $(-\infty, 0)$ such that $\int_{(-\infty,0)}(1 \wedge x^2)\Upsilon(\mathrm{d}x) < \infty$. Conversely, any such function Φ whose parameters satisfy the constraints above is in fact the Laplace exponent of a spectrally negative Lévy process; see [21, Theorem 8.1].

Lemma 3.1. *Let $\omega \in \operatorname{dom} \kappa$. Then the following assertions hold.*

(i) *The function Φ_ω is the Laplace exponent of a spectrally negative Lévy process, which we will call $\xi_\omega = (\xi_\omega(t))_{t\geq 0}$.*

(ii) *For every $t \geq 0$, there exists a unique probability measure $\rho_t^{[\omega]}$ on $(0,\infty)$ with Mellin transform given by*

$$\mathcal{M}\rho_t^{[\omega]}(q+1) := \int_{(0,\infty)} x^q \rho_t^{[\omega]}(\mathrm{d}x) = \exp(t\Phi_\omega(q)), \qquad q \geq 0. \tag{3.4}$$

The family of measures has the representation $\rho_t^{[\omega]} = \mathbb{P}(\exp(\xi_\omega(t)) \in \cdot)$ for $t \geq 0$.

Proof. We prove both parts simultaneously, and focus first on the case $\omega = 2$, where we write $\Phi := \Phi_2$. We can express Φ in the form

$$\Phi(q) = aq^2 + b'q + \int_{[1/2,1)} (y^q - 1 + q(1-y))y^2 K(\mathrm{d}y) + \int_{[1/2,1)} ((1-y)^q - 1)(1-y)^2 K(\mathrm{d}y)$$

with $b' = 3a + b + \int_{[1/2,1)}(1-y)(1-y^2)K(\mathrm{d}y)$.

Denote by $\Lambda(\mathrm{d}x)$ the image of $y^2 K(\mathrm{d}y)$ by the map $y \mapsto x = \ln y$, and denote by $\Pi(\mathrm{d}x)$ the image of $(1-y)^2 K(\mathrm{d}y)$ by the map $y \mapsto x = \ln(1-y)$. Then, thanks to (1.5), Λ is a measure on $[-\ln 2, 0)$ with $\int x^2 \Lambda(\mathrm{d}x) < \infty$, and Π is a finite measure on $(-\infty, -\ln 2]$, and there hold the identities

$$\int_{[1/2,1)} (y^q - 1 + q(1-y))y^2 K(\mathrm{d}y) = \int_{[-\ln 2,0)} (\mathrm{e}^{qx} - 1 + q(1-\mathrm{e}^x))\Lambda(\mathrm{d}x)$$

and

$$\int_{[1/2,1)} ((1-y)^q - 1)(1-y)^2 K(\mathrm{d}y) = \int_{(-\infty,-\ln 2]} (\mathrm{e}^{qx} - 1)\Pi(\mathrm{d}x).$$

This shows that Φ is given by a Lévy–Khintchin formula, and, therefore, Φ can be viewed as the Laplace exponent of a spectrally negative Lévy process $\xi = (\xi(t))_{t\geq 0}$ (see Chapter VI of [1] for background), i.e.

$$\mathbb{E}[\exp(q\xi(t))] = \exp(t\Phi(q)), \qquad t, q \geq 0.$$

We conclude that (3.4) does indeed determine a probability measure ρ_t which arises as the distribution of $\exp(\xi(t))$.

Finally, if $\omega \neq 2$, we observe that the function Φ_ω can be written as

$$\Phi_\omega(q) = \Phi(q + \omega - 2) - \Phi(\omega - 2),$$

which implies that Φ_ω is given by an Esscher transform of Φ, and hence is also the Laplace exponent of a spectrally negative Lévy process; see [16, Theorem 3.9] or [21, Theorem 33.1]. □

Remark 3.1. More generally, if ζ denotes a random time having an exponential distribution, say with parameter $\mathtt{k} \geq 0$, which is, further, independent of ξ, then the process

$$\xi_\dagger(t) = \begin{cases} \xi(t) & \text{if } t < \zeta, \\ -\infty & \text{if } t \geq \zeta, \end{cases}$$

is referred to as a killed Lévy process. Note that if we set $\Phi_\dagger(q)) = \mathtt{k} + \Phi(q)$ then

$$\mathbb{E}[\exp(q\xi_\dagger(t))] = \exp(t\Phi_\dagger(q)),$$

with the convention that $\exp(q\xi_\dagger(t)) = 0$ for $t \geq \zeta$. So Lemma 3.1 shows that whenever $\kappa(\omega) \leq 0$, the function $q \mapsto \kappa(\omega + q)$ can be viewed as the Laplace exponent of a spectrally negative Lévy process killed at an independent exponential time with parameter $-\kappa(\omega)$.

Recall from Lemma 2.1(ii) that $2 \in \operatorname{dom} \kappa$ always; we will write ρ_t for $\rho_t^{[2]}$. Since ρ_t is guaranteed to exist, this collection of measures will play a particular role in the case $\alpha = 0$. We stress that in the cases $\alpha < 0$ and $\alpha > 0$, we will need to choose different values of ω, and the notation ρ_t will then refer to a different distribution.

We point out the following property of the probability measures $\rho_t^{[\omega]}$, which essentially rephrases Kolmogorov's forward equation.

Corollary 3.1. *The family of probability measures* $(\rho_t^{[\omega]})_{t\geq 0}$ *defined in Lemma 3.1 depends continuously on the parameter t for the topology of weak convergence. Furthermore, for every* $g \in \mathcal{C}_c^\infty(0,\infty)$*, the function* $t \mapsto \langle \rho_t^{[\omega]}, g\rangle$ *is differentiable, and* $\partial_t \langle \rho_t^{[\omega]}, g\rangle = \langle \rho_t^{[\omega]}, \mathcal{A}g\rangle$*, where*

$$\mathcal{A}g(x) := x^{-\omega}\mathcal{L}_0(h_\omega g)(x) - \kappa(\omega)g(x), \qquad x > 0.$$

Proof. Recall that every Lévy processes fulfils the Feller property and in particular, for every function $\varphi \in \mathcal{C}_0(\mathbb{R})$, the map $t \mapsto \mathbb{E}[\varphi(\xi_\omega(t))]$ is continuous. Taking $\varphi(x) = g(\mathrm{e}^x)$ with $g \in \mathcal{C}_0(0,\infty)$ yields the weak continuity of the map $t \mapsto \rho_t^{[\omega]}$.

Furthermore, it is well known that the domain of the infinitesimal generator of a Lévy process contains $\mathcal{C}_c^\infty(\mathbb{R})$ (see, e.g. Theorem 31.5 of [21]), and it follows similarly that, for $g \in \mathcal{C}_c^\infty(0,\infty)$, the map $t \mapsto \langle \rho_t^{[\omega]}, g\rangle$ is differentiable. To compute the derivative, that is, to find the infinitesimal generator, take $q \geq 0$ and recall that $h_q(x) = x^q$ for $x > 0$; then simply observe from (3.4) that

$$\partial_t\langle \rho_t, h_q\rangle = \Phi_\omega(q)\exp(t\Phi_\omega(q)) = \langle \rho_t^{[\omega]}, \Phi_\omega(q)h_q\rangle = \langle \rho_t^{[\omega]}, \kappa(q+\omega)h_q - \kappa(\omega)h_q\rangle.$$

Using Lemma 2.2(i), we can express

$$\kappa(q+\omega)h_q = h_{-\omega}\kappa(q+\omega)h_{q+\omega} = h_{-\omega}\mathcal{L}_0(h_\omega h_q),$$

which shows that $\partial_t\langle \rho_t^{[\omega]}, h_q\rangle = \langle \rho_t^{[\omega]}, \mathcal{A}h_q\rangle$ for all $q \geq 0$. That the same holds when h_q is replaced by a function $g \in \mathcal{C}_c^\infty$ now follows from standard arguments, using linear combinations of h_q. □

We would now like to invoke Lemma 3.1 to invert the Mellin transform (3.3), observing that (using (3.4) with $\rho = \rho^{[2]}$)

$$\exp(t\kappa(q)) = \exp(t\kappa(2))\langle \rho_t, x^{q-2}\rangle,$$

and conclude (3.5) (below). However, $h_q \notin \mathcal{C}_c^\infty(0,\infty)$ and we cannot directly apply this simple argument. Nonetheless we claim the following.

Theorem 3.1. *Equation (1.2), for* $f \in \mathcal{C}_c^\infty(0,\infty)$ *and with* $\mathcal{L} = \mathcal{L}_0$*, has a unique solution started from* $\mu_0 = \delta_1$*, given by*

$$\mu_t(\mathrm{d}x) = \exp(t\kappa(2))x^{-2}\rho_t(\mathrm{d}x), \qquad t \geq 0, \tag{3.5}$$

where ρ_t *is the probability measure on* $(0,\infty)$ *defined by (3.4) for* $\omega = 2$.

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Remark 3.2. In particular, the unique solution in Theorem 3.1 fulfils $\langle \mu_t, h_q \rangle = \exp(t\kappa(q))$, as we expected from (3.3). From a probabilistic perspective, this does not come as a surprise. In [3], a homogeneous growth-fragmentation stochastic process $\boldsymbol{Z}(t) = (Z_1(t), Z_2(t), \dots)$ was constructed whose evolution is, informally speaking, governed by the stochastic growth-fragmentation dynamics described in the introduction. Using a spine technique, it may be shown (we omit the proof) that the solution $(\mu_t)_{t\geq 0}$ has the representation $\langle \mu_t, f \rangle = \mathbb{E}[\sum_{i=1}^{\infty} f(Z_i(t))]$ for any f for which the right-hand side is finite; and in [3, Theorem 1] it was proved that $\mathbb{E}[\sum_{i=1}^{\infty} Z_i^q(t)] = \exp(t\kappa(q))$ for all $q \geq 2$. We offer a more detailed discussion of the spine technique in Section 6.

Proof of Theorem 3.1. It is straightforward to check that $\mu_t(\mathrm{d}x) = \exp(t\kappa(2))x^{-2}\rho_t(\mathrm{d}x)$ is indeed a solution. Specifically, we deduce from (3.4) that

$$\langle \mu_t, h_q \rangle = \exp(t\kappa(2)) \exp(t\Phi(q-2)) = \exp(t\kappa(q)).$$

We thus see that $(\mu_t)_{t\geq 0}$ solves (1.2) with $\mathcal{L} = \mathcal{L}_0$ and $f = h_q$ for every $q \geq 0$, and it follows from classical properties of the Mellin transform that this entails that $(\mu_t)_{t\geq 0}$ solves (1.2) more generally for all $f \in \mathcal{C}_c^{\infty}(0, \infty)$.

Conversely, given a solution $(\mu_t)_{t\geq 0}$ to (1.2) with $\mu_0 = \delta_1$, set

$$\tilde{\rho}_t(\mathrm{d}x) = \exp(-t\kappa(2))x^2\mu_t(\mathrm{d}x).$$

Take $g \in \mathcal{C}_c^{\infty}(0, \infty)$ and define $f(x) = x^2 g(x)$ for $x > 0$, so $f \in \mathcal{C}_c^{\infty}(0, \infty)$. Then we have $\langle \tilde{\rho}_t, g \rangle = \exp(-t\kappa(2))\langle \mu_t, f \rangle$ and

$$\partial_t \langle \tilde{\rho}_t, g \rangle = -\kappa(2)\langle \tilde{\rho}_t, g \rangle + \exp(-t\kappa(2))\langle \mu_t, \mathcal{L}_0 f \rangle,$$

that is,

$$\partial_t \langle \tilde{\rho}_t, g \rangle = \langle \tilde{\rho}_t, \mathcal{A} g \rangle, \tag{3.6}$$

with $\mathcal{A}g(x) = x^{-2}\mathcal{L}_0 f(x) - \kappa(2)g(x)$, as in the notation of Corollary 3.1. We can thus interpret (3.6) as Kolmogorov's forward equation for the infinitesimal generator of the Feller process $(\exp(\xi(t)))_{t\geq 0}$. This will in turn enable us to identify $\tilde{\rho}_t = \rho_t$.

To be precise, examining the proof of [12, Proposition 4.9.18], we see that (3.6) for all $g \in \mathcal{C}_c^{\infty}(0, \infty)$ has at most one solution (in the sense of a vaguely right-continuous collection of measures $(\tilde{\rho}_t)_{t\geq 0}$) so long as the image of $\mathcal{C}_c^{\infty}(0, \infty)$ by $\lambda - \mathcal{A}$ is separating (see [12, p. 112]) for each $\lambda > 0$. Since $\mathcal{A}$ is the generator of a Feller process and $\mathcal{C}_c^{\infty}(0, \infty)$ is a core (cf. Theorem 31.5 of [21]), we know that the image of $\mathcal{C}_c^{\infty}(0, \infty)$ by $\lambda - \mathcal{A}$ is a dense subset of $\mathcal{C}_0$, and this implies that it is separating. If $(\tilde{\rho}_t)_{t\geq 0}$ is a collection of measures solving (3.6) then, for any $g \in \mathcal{C}_c^{\infty}$, the function $t \mapsto \langle \tilde{\rho}_t, g \rangle$ is right continuous. Hence, the solution of (3.6) restricted to $\mathcal{C}_c^{\infty}$ is unique, and this transfers to (1.2). □

3.2. Some properties of solutions

We next present some properties of the solution identified in Theorem 3.1, by means of translating known results on Lévy processes.

We first point out that, depending on whether (3.1) holds and $a = 0$, the support of the solution μ_t is bounded or not. Specifically, if $a = 0$ and (3.1) holds, we set

$$d := b + \int_{[1/2,1)} (1-y)K(\mathrm{d}y);$$

otherwise, $d = \infty$. It is easy to verify that $d = \lim_{q\to\infty} q^{-1}\kappa(q)$.

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Corollary 3.2. *If* $a = 0$ *and (3.1) holds, then, for every* $t > 0$, e^{dt} *is the supremum of the support of* μ_t, *i.e. we have, for every* $\varepsilon > 0$,

$$\mu_t((\mathrm{e}^{td}, \infty)) = 0 \quad \textit{and} \quad \mu_t((\mathrm{e}^{td} - \varepsilon, \mathrm{e}^{td}]) > 0.$$

Proof. The spectrally negative Lévy process $\xi = \xi_2$, arising in Lemma 3.1, has bounded variation with drift coefficient d exactly when the conditions of the result hold, and it is then well known that td is the maximum of the support of the distribution of $\xi(t)$. Therefore, we have $\rho_t((\mathrm{e}^{td}, \infty)) = 0$ and $\rho_t((\mathrm{e}^{td} - \varepsilon, \mathrm{e}^{td}]) > 0$, and our claim follows from Theorem 3.1. □

In the case when the assumptions of Corollary 3.2 are not fulfilled, we have the following large deviations estimates for the tail $\bar{\mu}_t(x) := \mu_t((x, \infty))$ of μ_t. Recall that κ is a convex function, and observe that $\lim_{q\to+\infty} \kappa'(q) = +\infty$ when either $a > 0$ or (3.1) fails. Thus, for every sufficiently large r, the equation $\kappa'(q) = r$ has a unique solution, which we denote by $\theta(r)$, and the Legendre–Fenchel transform of κ is given by

$$\kappa^*(r) := \sup_{q>0}\{rq - \kappa(q)\} = r\theta(r) - \kappa(\theta(r)).$$

Corollary 3.3. *Suppose that* $a > 0$ *or (3.1) fails. Then, for every sufficiently large* $r > 0$ *, we have*

$$\lim_{t\to\infty} t^{-1} \ln \bar{\mu}_t(\mathrm{e}^{tr}) = -\kappa^*(r).$$

Proof. This follows easily from the identity $\langle \mu_t, h_q \rangle = \exp(t\kappa(q))$ by adapting the classical arguments of Cramér and Chernoff; see, for instance, Theorem 1 of [7]. □

The estimate of Corollary 3.3 can easily be reinforced by using the local central limit theorem. Here is a typical example (compare with Theorem 1.3 of [11]).

Corollary 3.4. *Suppose that* $a > 0$ *or (3.1) fails, and further that* $\kappa'(q) < 0$ *for some* q. *Then* $\theta(0)$ *is well defined,* $0 < \kappa''(\theta(0)) < \infty$, *and, for every* $f \in \mathcal{C}_c$, *we have*

$$\langle \mu_t, f \rangle \sim \frac{\mathrm{e}^{t\kappa(\theta(0))}}{\sqrt{2\pi t\kappa''(\theta(0))}} \int_0^\infty f(x)x^{-(\theta(0)+1)}\,\mathrm{d}x.$$

Proof. The first assertions about the existence of $\theta(0)$ and $\kappa''(\theta(0))$ are immediate from the convexity of κ and the fact that $\lim_{+\infty} \kappa = +\infty$.

The function $\tilde{\Phi}(q) := \kappa(q + \theta(0)) - \kappa(\theta(0)) = \Phi(q + \theta(0)) - \Phi(\theta(0))$ is the Laplace exponent of a spectrally negative Lévy process $(\tilde{\xi}(t))_{t\geq 0}$ which is centred and has finite variance $\kappa''(\theta(0))$. Furthermore, we see from the Esscher transform and Theorem 3.1 that

$$\mu_t(\mathrm{d}x) = \mathrm{e}^{t\kappa(\theta(0))}x^{-\theta(0)}\mathbb{P}(\exp(\tilde{\xi}(t)) \in \mathrm{d}x).$$

Our claim then follows readily from the local central limit theorem for the Lévy process. □

Corollary 3.5. *If* $a > 0$ *or the absolutely continuous component of* $K(\mathrm{d}y)$ *has an infinite total mass, then* μ_t *is absolutely continuous for every* $t > 0$.

Proof. Using Sato [21, Theorem 27.7 and Lemma 27.1], it follows from the assumptions of the statement that the one-dimensional distributions of the Lévy process $\xi(t)$ are absolutely continuous for every $t > 0$. Our claim follows from the representation in Theorem 3.1. □

4. The self-similar case

We now turn our attention to the growth-fragmentation equation (1.2) for $\mathcal{L} = \mathcal{L}_\alpha$ given by (1.4) and $\alpha \neq 0$. We first point out that the function κ is nonincreasing if and only if $a = 0$, the dislocation measure K fulfils (3.1), and

$$b + \int_{[1/2,1)} (1-y)K(\mathrm{d}y) \leq 0.$$

In this case the operator $\mathcal{L}_\alpha$ can be expressed in the form (1.3) with $c \leq 0$, and (1.2) is then a pure fragmentation equation as studied in [13]. To avoid duplication of existing literature, this case will be implicitly excluded hereafter.

Recall the notation $h_q(x) = x^q$ for $x > 0$. In the self-similar case, power functions are no longer eigenfunctions of the operator $\mathcal{L}_\alpha$; however, there is the simple relation

$$\mathcal{L}_\alpha h_q = \kappa(q) h_{q+\alpha}, \qquad q \in \operatorname{dom}\kappa; \tag{4.1}$$

see Lemma 2.2(i). Hence, if (1.2) applies to power functions, the linear equation (3.2) for the Mellin transform $M_t(z) = \langle \mu_t, h_{z-1} \rangle$ in the homogeneous case has to be replaced by the system

$$\partial_t M_t(1+q) = \kappa(q) M_t(1+q+\alpha). \tag{4.2}$$

We make the fundamental assumption that

$$\inf_{q \geq 0} \kappa(q) < 0, \tag{4.3}$$

which is implicitly enforced throughout this section. The role and the importance of (4.3) will become clear in the sequel. Recall that κ is a convex function on $\mathbb{R}$, and is ultimately increasing, since throughout Section 4 we are excluding the case when κ is nonincreasing. Hence, condition (4.3) ensures the existence of a unique $\omega_+ \in \mathbb{R}$ with

$$\kappa(\omega_+) = 0 \quad \text{and} \quad \kappa'(\omega_+) > 0. \tag{4.4}$$

We refer to ω_+ as the *Malthusian parameter*.

The sign of the scaling parameter α plays a crucial role, and we shall study the two cases separately, even though some ideas are similar.

4.1. The case $\alpha < 0$

We now focus on the case $\alpha < 0$. We start by observing that the existence of a Malthusian parameter enables us to view (4.2) as a closed system for an arithmetic sequence, and thus to solve it.

Lemma 4.1. *Consider a sequence of functions $M_\bullet(1+q)\colon [0,\infty) \to (0,\infty)$, where $q = \omega_+ - k\alpha$, $k = -1, 0, 1, \ldots$, and $M_0(1+q) = 1$. Suppose that (4.2) and (4.3) hold, and recall that ω_+ is the Malthusian parameter defined by (4.4). Then $M_t(1+\omega_+) = 1$ for all $t \geq 0$, and, for $k = 1, 2, \ldots$, we have*

$$M_t(1+\omega_+ - k\alpha) = 1 + \sum_{\ell=1}^{k} \frac{\kappa(\omega_+ - \alpha k) \cdots \kappa(\omega_+ - \alpha(k-\ell+1))}{\ell!} t^\ell.$$

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Proof. Equation (4.2) applied to the Malthusian exponent ω_+ implies that the function $t \mapsto M_t(1+\omega_+)$ is constant. We can then solve (4.2) for $q = \omega_+ - \alpha k$ and $k = 1, 2, \ldots$ by induction in order to obtain the given formula. □

In comparison with the homogeneous case, Lemma 4.1 is a much weaker result than (3.3), as we are not able to compute the whole Mellin transform of a solution, but merely its moments for orders forming an arithmetic sequence. There is hence an additional crucial issue: it does not suffice to find a family of measures having the desired moments, as one needs also to ensure that the moment problem is determining. It turns out that moment calculations which were performed in [6] for self-similar Markov processes enable us to solve the moment problem in Lemma 4.1, and check that this indeed yields a solution to (1.4). Similar calculations also lead to a rather surprising result, namely that the self-similar growth-fragmentation permits spontaneous generation!

Theorem 4.1. *Assume that (4.3) holds and that* $\alpha < 0$.

(i) *For every* $t \geq 0$, *there exists a unique measure* μ_t *on* $(0, \infty)$ *such that* $\langle \mu_t, h_{\omega_+} \rangle = 1$ *and, for every integer* $k \geq 1$,

$$\langle \mu_t, h_{\omega_+ - k\alpha} \rangle = 1 + \sum_{\ell=1}^{k} \frac{\kappa(\omega_+ - \alpha k) \cdots \kappa(\omega_+ - \alpha(k - \ell + 1))}{\ell!} t^{\ell}.$$

In particular, $\mu_0 = \delta_1$ *and the family* $(\mu_t)_{t \geq 0}$ *solves (1.2) for all* $f \in \mathcal{C}_c^{\infty}(0, \infty)$ *when* $\mathcal{L} = \mathcal{L}_\alpha$ *is given by (1.4).*

(ii) *For every* $t > 0$, *there exists a unique measure* γ_t *on* $(0, \infty)$ *such that* $\langle \gamma_t, h_{\omega_+} \rangle = 1$ *and*

$$\langle \gamma_t, h_{\omega_+ - k\alpha} \rangle = t^k \frac{\kappa(\omega_+ - \alpha) \cdots \kappa(\omega_+ - \alpha k)}{k!} \qquad \textit{for every integer } k \geq 1.$$

If we further set $\gamma_0 \equiv 0$ *then the family* $(\gamma_t)_{t \geq 0}$ *solves (1.2) for all* $f \in \mathcal{C}_c^{\infty}(0, \infty)$ *when* $\mathcal{L} = \mathcal{L}_\alpha$ *is given by (1.4).*

Proof. (i) Let us define $\Phi_+ := \Phi_{\omega_+} = \kappa(\cdot + \omega_+)$, which, as we saw in Lemma 3.1, is the Laplace exponent of the Lévy process $\xi_+ := \xi_{\omega_+}$. Observe that $\Phi'_+(0) = \kappa'(\omega_+) > 0$, so this Lévy process has a strictly positive and finite first moment.

Proposition 1 of [6] then ensures, for every $t > 0$, the existence of a unique probability measure ρ_t on $(0, \infty)$, such that, for every integer $k \geq 1$,

$$\langle \rho_t, h_{-\alpha k} \rangle = 1 + \sum_{\ell=1}^{k} \frac{\kappa(\omega_+ - \alpha k) \cdots \kappa(\omega_+ - \alpha(k - \ell + 1))}{\ell!} t^{\ell},$$

so that, in particular, $\rho_0 = \delta_1$. Thus, we may set $\mu_t(\mathrm{d}x) = x^{-\omega_+} \rho_t(\mathrm{d}x)$, and then $\langle \mu_t, h_{\omega_+ - k\alpha} \rangle$ is given as in the statement for every integer $k \geq 0$. That this determines μ_t derives from the uniqueness of ρ_t.

Now, using (4.1), we immediately check that $\langle \mu_t, h_{\omega_+ - k\alpha} \rangle$ satisfies (4.2). It then follows that $(\mu_t)_{t \geq 0}$ solves (1.2) for every $f \in \mathcal{C}_c^{\infty}(0, \infty)$ (recall that the probability measure ρ_t is determined by its entire moments). Finally, the map $t \mapsto \langle \rho_t, h_{-\alpha k} \rangle$ is continuous, and we deduce that $(\rho_t)_{t \geq 0}$ is vaguely continuous (using again the fact that ρ_t is determined by its moments $\langle \rho_t, h_{-\alpha k} \rangle$ for $k \in \mathbb{N}$). Hence, the same holds for $(\mu_t)_{t \geq 0}$.

(ii) Recall from above that $\Phi_+ = \kappa(\omega_+ + \cdot)$ is the Laplace exponent of a spectrally negative Lévy process which has strictly positive and finite first moments. Proposition 1 of [6] ensures, for every $t > 0$, the existence of a unique probability measure π_t on $(0, \infty)$ such that its moments are given by

$$\langle \pi_t, h_{-\alpha k} \rangle = t^k \frac{\Phi_+(-\alpha) \cdots \Phi_+(-\alpha k)}{k!} = t^k \frac{\kappa(\omega_+ - \alpha) \cdots \kappa(\omega_+ - \alpha k)}{k!}$$

for $k = 1, 2, \ldots$, and this determines π_t. It follows immediately that $(\pi_t)_{t \geq 0}$ is vaguely continuous (recall that $\pi_0 = 0$).

We then define, for $t > 0$,

$$\gamma_t(\mathrm{d}x) = x^{-\omega_+} \pi_t(\mathrm{d}x), \qquad x > 0,$$

so

$$\langle \gamma_t, h_{\omega_+ - k\alpha} \rangle = t^k \frac{\kappa(\omega_+ - \alpha) \cdots \kappa(\omega_+ - \alpha k)}{k!}.$$

Then (4.1) entails that, for every integer $k \geq 1$, there holds the identity

$$\begin{aligned} \partial_t \langle \gamma_t, h_{\omega_+ - k\alpha} \rangle &= t^{k-1} \frac{\kappa(\omega_+ - \alpha) \cdots \kappa(\omega_+ - \alpha k)}{(k-1)!} \\ &= \kappa(\omega_+ - \alpha k) \langle \gamma_t, h_{\omega_+ - (k-1)\alpha} \rangle \\ &= \langle \gamma_t, \mathcal{L} h_{\omega_+ - k\alpha} \rangle, \end{aligned}$$

and the conclusion follows just as in (i). □

Theorem 4.1(ii) entails that uniqueness of the solution fails when one only requires (1.4) to be fulfilled for all $f \in \mathcal{C}_c^\infty(0, \infty)$, which contrasts sharply with the results of Haas [13] for the pure-fragmentation equation. We conjecture that the solution μ_t given in Theorem 4.1(i) is minimal, in the sense that if $(\tilde{\mu}_t)_{t \geq 0}$ is another solution with the same initial condition $\tilde{\mu}_0 = \delta_1$, then $\mu_t \leq \tilde{\mu}_t$ for every $t > 0$. We also stress that uniqueness of the solution can be restored by requiring (1.2) to hold for the functions h_q with $q \geq \omega_+ + \alpha$; see Lemma 4.1 and Theorem 4.1(i).

We now present a different approach to Theorem 4.1. In the homogeneous case $\alpha = 0$, we saw in the preceding section that (1.4) bears a close relationship with certain exponential Lévy processes. It turns out that in the self-similar case with $\alpha < 0$, the vital connection is with positive self-similar Markov processes, and is made via the Lamperti transformation which associates these with the class of Lévy processes. We first provide some background in this area.

A *positive self-similar Markov process* (*pssMp*) with *self-similarity index* $\gamma \in \mathbb{R}$ is a standard Markov process $R = (R_t)_{t \geq 0}$ with associated filtration $(\mathcal{F}_t)_{t \geq 0}$ and probability laws $(\mathrm{P}_x)_{x \in (0,\infty)}$, on $[0, \infty]$, which has 0 and ∞ as absorbing states and which satisfies the *scaling property* that, for every $x, c > 0$,

$$\text{the law of } (c R_{tc^{-\alpha}})_{t \geq 0} \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

Here, we mean 'standard' in the sense of [8], which is to say that $(\mathcal{F}_t)_{t \geq 0}$ is a complete, right-continuous filtration, and R has càdlàg paths and is strong Markov and quasi-left-continuous.

In the seminal paper [17], Lamperti described a one-to-one correspondence between pssMps and (possibly killed) Lévy processes, which we now outline. It may be worth noting that we have presented a slightly different definition of a pssMp from Lamperti; for the connection, see [22, Section 0].

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Let $S(t) = \int_0^t (R_u)^{-\gamma}\,\mathrm{d}u$. This process is continuous and strictly increasing until R reaches 0. Let $(T(s))_{s\geq 0}$ be its inverse, and define

$$\eta_s = \log R_{T(s)}, \qquad s \geq 0.$$

Then $\eta := (\eta_s)_{s\geq 0}$ is a (possibly killed) Lévy process started at position $\log x$, possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of x. The real-valued process η with probability laws $(\mathbb{P}_y)_{y\in\mathbb{R}}$ is called the *Lévy process associated with* R, or the *Lamperti transform of* R.

Equivalent definitions of S and T, in terms of η instead of R, are given by defining $T(s) = \int_0^s \exp(\gamma\eta_u)\,\mathrm{d}u$ and S as its inverse. Then

$$R_t = \exp(\eta_{S(t)})$$

for all $t \geq 0$, and this shows that the Lamperti transform is a bijection. A useful fact is that, as a consequence of the definitions we have just given, it holds that $\mathrm{d}t = \exp(-\gamma\eta_{S(t)})\,\mathrm{d}S(t)$.

Most of the literature on pssMps (including Lamperti's original paper) assumes that $\gamma > 0$, and much of it is also given for $\gamma = 1$. Indeed, it is easy to change the index of self-similarity. If R is a pssMp of index γ and corresponds to the Lévy process η, then, for any $\gamma' \in \mathbb{R}$, the process $R^{\gamma'} = ((R_t)^{\gamma'})_{t\geq 0}$ is a pssMp with index γ/γ', corresponding to the Lévy process $\gamma'\eta$. It is also useful to note that the time changes appearing in the Lamperti transformation are almost surely equal for R and $R^{\gamma'}$. We should point out that the case $\gamma = 0$ is special, since in this case the time change does not have any effect, and the pssMps of index 0 are just exponential Lévy processes.

Note that if the Lévy process process η is killed at time ζ then we define $R_t = 0$ for $t \geq T(\zeta)$ if $\gamma \geq 0$, and $R_t = +\infty$ for $t \geq T(\zeta)$ if $\gamma < 0$.

Recall Lemma 3.1, and define $\Phi_+ := \Phi_{\omega+} = \kappa(\cdot + \omega_+)$, which is the Laplace exponent of the spectrally negative Lévy process $\xi_+ := \xi_{\omega_+}$. Let us denote by X_+ the pssMp with index $-\alpha$ associated with ξ_+ by the Lamperti transformation. Note that, because ξ_+ has positive mean, the process X_+ never reaches the absorbing boundary 0 or $+\infty$. We define the measure ρ_t to be the distribution of $X_+(t)$ under P_1, that is, the probability measure on $(0,\infty)$ defined by

$$\langle \rho_t, f\rangle = \mathrm{E}_1[f(X_+(t))], \qquad f \in \mathcal{C}_0(0,\infty),$$

and give first the following analogue of Corollary 3.1.

Lemma 4.2. *The family of probability measures $(\rho_t)_{t\geq 0}$ depends continuously on the parameter t for the topology of weak convergence. Furthermore, for every $g \in \mathcal{C}_c^\infty(0,\infty)$, the function $t \mapsto \langle \rho_t, g\rangle$ is differentiable with derivative $\partial_t\langle\rho_t, g\rangle = \langle \rho_t, \mathcal{A}_+^{(\alpha)} g\rangle$, where*

$$\mathcal{A}_+^{(\alpha)} g(x) := x^{-\omega_+}\mathcal{L}_\alpha(h_{\omega_+} g)(x), \qquad x > 0.$$

Proof. The first assertion follows easily from the Feller property of self-similar Markov processes; see Theorem 2.1 of [17] and the remark on page 212. In order to establish the second assertion, we work with the Lévy process ξ_+. The exponential Lévy process $\exp(\xi_+(\cdot))$ is a Feller process in $(0,\infty)$, and the same calculation as in the proof of Corollary 3.1 shows that its infinitesimal generator $\mathcal{A}_+$ is given by

$$\mathcal{A}_+ g(x) = x^{-\omega_+}\mathcal{L}_0(h_{\omega_+} g)(x), \qquad g \in \mathcal{C}_c^\infty(0,\infty).$$

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According to Dynkin's formula (see, e.g. Proposition 4.1.7 of [12]), for every $g \in \mathcal{C}_c^\infty(0,\infty)$, the process

$$g(\exp(\xi_+(t))) - \int_0^t \mathcal{A}_+ g(\exp(\xi_+(s)))\,\mathrm{d}s$$

is a martingale. Recall that, by definition, X_+ arises as the transform of $\exp(\xi_+(\cdot))$ by the time substitution S, which is given as the inverse of the additive functional $\int_0^t h_\alpha^{-1}(\exp(\xi_+(s))\,\mathrm{d}s$, and we have the identity

$$g(X_+(t)) - \int_0^{S(t)} \mathcal{A}_+ g(\exp(\xi_+(s)))\,\mathrm{d}s = g(X_+(t)) - \int_0^t h_\alpha(X_+(s))\mathcal{A}_+ g(X_+(s))\,\mathrm{d}s.$$

A priori, the time substitution above changes a martingale into a local martingale. However, using Lemma 2.1(iii) and the fact that $\alpha < 0$, we see that $\mathcal{A}_+^{(\alpha)} g := h_\alpha \mathcal{A}_+ g$ is bounded, and it follows that the process

$$g(X_+(t)) - \int_0^t \mathcal{A}_+^{(\alpha)} g(X_+(s))\,\mathrm{d}s$$

is a true martingale. Taking expectations, we arrive at $\langle \rho_t, g\rangle - g(1) = \int_0^t \langle \rho_s, \mathcal{A}_+^{(\alpha)} g\rangle\,\mathrm{d}s$, and our claim follows. □

The connection with solutions of the growth-fragmentation equation is as follows.

Corollary 4.1. *Let*

$$\tilde{\mu}_t = h_{-\omega_+}\rho_t, \qquad t \geq 0.$$

Then $(\tilde{\mu}_t)_{t\geq 0}$ is equal to the solution $(\mu_t)_{t\geq 0}$ of the growth-fragmentation equation appearing in Theorem 4.1(i).

Proof. We deduce immediately from Lemma 4.2 that, for every $f \in \mathcal{C}_c^\infty$,

$$\partial_t\langle \tilde{\mu}_t, f\rangle = \partial_t\langle \rho_t, h_{\omega_+}^{-1} f\rangle = \langle \rho_t, h_{\omega_+}^{-1}\mathcal{L}_\alpha f\rangle = \langle \tilde{\mu}_t, \mathcal{L}_\alpha f\rangle,$$

that is, the family $(\tilde{\mu}_t)_{t\geq 0}$ solves (1.2) with $\mathcal{L} = \mathcal{L}_\alpha$. That μ_t coincides with the measure appearing in Theorem 4.1(i), and that the notation ρ_t for the distribution of $X_+(t)$ is consistent with that used in the proof of Theorem 4.1(i), follows from Proposition 1 of [6]. □

This approach could also be adapted to prove the existence of $(\gamma_t)_{t\geq 0}$ using the process $X_+(t)$ started from 0, and indeed this will be our method for the case $\alpha > 0$ in Section 4.2.

We conclude the section by offering some results on the asymptotic behaviour of the solution $(\mu_t)_{t\geq 0}$ given by the previous theorem.

Our first result in this direction indicates that, thanks to the self-similarity property (2.1) of (1.2), the solution starting from 0 given above can be used to describe the asymptotic behaviour of μ_t as $t \to \infty$.

Proposition 4.1. *For any $f \in \mathcal{C}_b(0,\infty)$,*

$$\int f(t^{-1/|\alpha|}z) z^{\omega_+}\mu_t(\mathrm{d}z) \to \int f(z) z^{\omega_+}\gamma_1(\mathrm{d}z).$$

Proof. Since $\kappa'(\omega_+) > 0$, it is possible to extend the definition of X_+ in order to allow it to start from $X_+(0) = 0$, such that it is a Feller process on the state space $[0,\infty)$; this is a consequence of [5, Theorem 1]. For $x \geq 0$, we will denote by P_x the law of the process with $X_+(0) = x$.

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It then follows from the scaling property that $\mathrm{E}_x[f(t^{1/\alpha}X_+(t))] = \mathrm{E}_{xt^{1/\alpha}}[f(X_+(1))]$, and then the convergence

$$\mathrm{E}_x[f(t^{1/\alpha}X_+(t))] \to \mathrm{E}_0[f(X_+(1))] \quad \text{as } t \to \infty$$

follows from the scaling property of X_+.

Finally, we know from [6], which we used in the proof of Theorem 4.1, that the measures $x^{\omega_+}\mu_t(\mathrm{d}x)$ and $x^{\omega_+}\gamma_t(\mathrm{d}x)$ are respectively equal to $\mathrm{P}_1(X_+(t) \in \mathrm{d}x)$ and $\mathrm{P}_0(X_+(t) \in \mathrm{d}x)$. Our claim follows immediately. □

We remark that the statement of the proposition can easily be extended to solutions based on (μ_t) whose initial value is a measure with compact support in $(0, \infty)$.

Suppose now that the equation $\kappa(q) = 0$ has two solutions, ω_- and ω_+, with $\omega_- < \omega_+$. Then we can say a little more. Let X_- be the $(-\alpha)$-pssMp associated with the Lévy process $\xi_- := (\xi_-(t))_{t\geq 0}$ having Laplace exponent $\Phi_- := \Phi_{\omega_-}$. Recall that we say the Lévy process ξ_- is *lattice* if, for some $r \in \mathbb{R}$, the support of $\xi_-(1)$ almost surely lies in $r\mathbb{Z}$; otherwise, we say that ξ_- is *nonlattice*. If we define the random variable

$$I = \int_0^\infty \mathrm{e}^{|\alpha|\xi_-(t)}\,\mathrm{d}t,$$

then it is known from [19, Lemma 4] that, so long as ξ_- is nonlattice,

$$\lim_{t\to\infty} t^{(\omega_+-\omega_-)/|\alpha|}\mathbb{P}_0(I > t) = C$$

for some $0 < C < \infty$. We obtain the following result from [14].

Proposition 4.2. *Let $f \in \mathcal{C}_0(0, \infty)$, and assume that ξ_- is nonlattice. Then*

$$\frac{\int x^{\omega_-} f(t^{-1}x^{|\alpha|})\mu_t(\mathrm{d}x)}{\mathbb{P}(I > t)} \to \int f(x)\nu(\mathrm{d}x) \quad \text{as } t \to \infty,$$

where ν is the distribution of the random variable $J_{(\omega_+-\omega_-)/|\alpha|}$ in Equation (13) of [14].

Proof. As remarked in the proof of Proposition 4.1, the equation

$$x^{\omega_+}\mu_t(\mathrm{d}x) = \mathrm{P}_1(X_+(t) \in \mathrm{d}x)$$

holds as an identity of probability measures, where X_+ is the $(-\alpha)$-pssMp corresponding to the Lévy process ξ_+ with Laplace exponent $\Phi_+ = \Phi_{\omega_+}$. We now wish to use the 'Esscher transform' for pssMps, which is essentially obtained by standard arguments from the Esscher transform of Lévy processes given in [16, Theorem 3.9] (see, for instance, the discussion around [9, Theorem 14] for an application in the context of pssMps). This allows us to perform a change of measure to switch from the process X_+, related to the Laplace exponent Φ_+, to the process X_-, related to the Laplace exponent $\Phi_- = \Phi_+(\cdot + \omega_- - \omega_+)$. Specifically, we have

$$x^{\omega_-}\mu_t(\mathrm{d}x) = x^{\omega_- - \omega_+}x^{\omega_+}\mu_t(\mathrm{d}x) = x^{\omega_- - \omega_+}\mathrm{P}_1(X_+(t) \in \mathrm{d}x) = \mathrm{P}_1(X_-(t) \in \mathrm{d}x)$$

for $x > 0$. The process ξ_- (which corresponds to X_-) is a Lévy process with nonmonotone paths and which satisfies the conditions of [14, Theorem 1.6]. Applying this theorem gives the result. □

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4.2. The case $\alpha > 0$

In the case $\alpha > 0$, (4.2) for the Mellin transform is unfortunately much less useful, for the following reasons. Firstly, the analogue of Lemma 4.1 would require us to assume that $\langle \mu_t, h_q \rangle < \infty$ for all sufficiently negative q. Roughly speaking, this would force the scarcity of small particles, and this phenomenon occurs only for a very restricted class of dislocation measures K (informally, dislocations should not generate too many small particles, and in particular the total intensity of dislocations must be finite). Secondly, even if one were able to get an expression for the (negative) moments $\langle \mu_t, h_{\omega_+ - k\alpha} \rangle$ with $k \in \mathbb{N}$, this moment problem would be in general indeterminate, and the arguments used in the preceding section would thus collapse.

Nonetheless, we have just seen from Lemma 4.2 that, for $\alpha < 0$, self-similar growth-fragmentation equations have a close connection with certain self-similar Markov processes, and using the intuition that we gained, we are able to offer a very similar set of results when $\alpha > 0$. Recall that $\kappa\colon [0, \infty) \to (-\infty, \infty]$ is a convex function, and that, since we are assuming (4.3) holds, we may pick $\omega > 0$ such that $\mathtt{k} := -\kappa(\omega) > 0$. Then the function

$$\Phi_\dagger(q) := \Phi_\omega(q) + \mathtt{k} = \kappa(\omega + q), \qquad q \geq 0,$$

is the Laplace exponent of a spectrally negative Lévy process, say $\xi_\dagger$, killed at an independent exponential time with parameter $\mathtt{k}$, and we denote by $X_\dagger$ the $(-\alpha)$-pssMp associated with $\xi_\dagger$ via the Lamperti transformation. Note that $X_\dagger$ hits the absorbing state $+\infty$ by a jump.

We write ρ_t for the sub-probability measure on $(0, \infty)$ induced by the distribution of $X_\dagger(t)$ and study its properties in the following result, which mirrors Lemma 4.2.

Lemma 4.3. *Suppose that (4.3) holds and $\alpha > 0$. Then the following assertions hold, in the notation above.*

(i) $\mathbb{E}[\sup_{t \geq 0} X_\dagger(t)^q] < \infty$ *for all* $0 \leq q < \omega_+ - \omega$.

(ii) $\mathbb{E}[\int_0^\infty X_\dagger(u)^p \, \mathrm{d}u] < \infty$ *for all* $0 < p - \alpha < \omega_+ - \omega$.

(iii) *The family $(\rho_t)_{t \geq 0}$ depends continuously on the parameter t for the topology of weak convergence. For every $g \in \mathcal{C}_c^\infty(0, \infty)$, the function $t \mapsto \langle \rho_t, g \rangle$ is differentiable with derivative $\partial_t \langle \rho_t, g \rangle = \langle \rho_t, \mathcal{A}_\dagger^{(\alpha)} g \rangle$, where*

$$\mathcal{A}_\dagger^{(\alpha)} g(x) := x^{-\omega} \mathcal{L}_\alpha(h_\omega g)(x), \qquad x > 0.$$

(iv) *$(\rho_t)_{t \geq 0}$ solves the above equation also for $g(x) = x^q$ with $0 < q < \omega_+ - \omega$.*

Proof. (i) From the very construction of $X_\dagger$, the overall supremum $\bar{X}_\dagger := \sup_{t \geq 0} X_\dagger(t)$ is given by $\bar{X}_\dagger = \exp(\bar{\xi}_\dagger)$, with $\bar{\xi}_\dagger := \sup_{t \geq 0} \xi_\dagger(t)$. We infer from Corollary VII.2 of [1] that $\bar{\xi}_\dagger$ has the exponential distribution with parameter $\omega_+ - \omega$ (which is the positive root to the equation $\Phi_\dagger(q) = 0$), and our claim follows.

(ii) We begin with the following calculation, using the discussion of pssMps in the preceding section. Recall that S is the time change appearing in the Lamperti transform relating $X_\dagger$ and $\xi_\dagger$, and that there is the identity $\mathrm{d}S(t) = \exp(\alpha \xi_\dagger(t)) \, \mathrm{d}t$. We therefore have

$$\begin{aligned}
\int_0^\infty X_\dagger(u)^p \, \mathrm{d}u &= \int_0^\infty \mathrm{e}^{p\xi_\dagger(S(u))} \, \mathrm{d}u \\
&= \int_0^\infty \mathrm{e}^{(p-\alpha)\xi_\dagger(S(u))} \, \mathrm{d}S(u) \\
&= \int_0^\infty \mathrm{e}^{(p-\alpha)\xi_\dagger(s)} \, \mathrm{d}s.
\end{aligned}$$

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But now we can consider the expectation:

$$\mathrm{E}_x\biggl[\int_0^\infty X_\dagger(u)^p\,\mathrm{d}u\biggr] = x^{p-\alpha}\mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{(p-\alpha)\xi_\dagger(s)}\,\mathrm{d}s\biggr]$$
$$= \begin{cases} \dfrac{x^{p-\alpha}}{\kappa(p-\alpha+\omega)} & \text{if } \kappa(p-\alpha+\omega)<0,\\ \infty & \text{otherwise.}\end{cases}$$

We complete the proof by recalling the definition of ω_+.

(iii) Just as in the proof of Lemma 4.2, the first assertion follows from the Feller property of self-similar Markov processes, and the process

$$N_t := g(X_\dagger(t)) - g(1) - \int_0^t \mathcal{A}_\dagger^{(\alpha)} g(X_\dagger(s))\,\mathrm{d}s \tag{4.5}$$

is a local martingale for every $g \in \mathcal{C}_c^\infty(0,\infty)$. We will show that

$$\mathbb{E}\Bigl[\sup_{s\le t}|N_s|\Bigr] < \infty, \qquad t\ge 0,$$

which implies that N is a true martingale; see [18, Theorem I.51].

Since g is bounded, certainly $\sup_{s\le t}|g(X_\dagger(s)) - g(x)|$ is in $L^1(\mathbb{P})$. In contrast to Lemma 4.2, the function $\mathcal{A}_\dagger^{(\alpha)} g$ may be unbounded for $\alpha>0$; however, we do know from Lemma 2.1(iii) that $\mathcal{A}_\dagger^{(\alpha)} g$ is 0 on some neighbourhood of 0 and, for any $q \in \operatorname{dom}\kappa$, fulfils $\mathcal{A}_\dagger^{(\alpha)} g = \mathrm{o}(x^{q+\alpha-\omega})$ as $x\to\infty$.

We let $\omega < q < \omega_+$ and keep it fixed for the rest of the proof. For some $K>0$, we then have

$$\mathbb{E}\biggl[\sup_{u\le t}\biggl|\int_0^u \mathcal{A}_\dagger^{(\alpha)} g(X_\dagger(s))\,\mathrm{d}s\biggr|\biggr] \le \mathbb{E}\biggl[\int_0^t |\mathcal{A}_\dagger^{(\alpha)} g(X_\dagger(s))|\,\mathrm{d}s\biggr]$$
$$\le t \sup_{[0,K]}|\mathcal{A}_\dagger^{(\alpha)} g| + \mathbb{E}\biggl[\int_0^t X_\dagger(s)^{q+\alpha-\omega}\mathbf{1}_{\{X_\dagger(s)>K\}}\,\mathrm{d}s\biggr].$$

Setting $p = q+\alpha-\omega$ in part (ii), we see that the right-hand side is finite.

We have thus shown that N is a true martingale, and

$$\mathbb{E}\biggl[\int_0^t |\mathcal{A}_\dagger^{(\alpha)} g(X_\dagger(s))|\,\mathrm{d}s\biggr] < \infty.$$

Taking expectations in (4.5) and applying Fubini's theorem, we obtain

$$\langle \rho_t, g\rangle - g(1) = \int_0^t \langle \rho_s, \mathcal{A}_\dagger^{(\alpha)} g\rangle\,\mathrm{d}s,$$

which completes the proof.

(iv) This part is proved by setting $g(x) = x^q$ in the previous part, as follows. Using the Markov property, we immediately see that the process

$$M_t = \mathrm{e}^{q\xi_\dagger(t)} - 1 - \kappa(\omega_- + q)\int_0^t \mathrm{e}^{q\xi_\dagger(s)}\,\mathrm{d}s, \qquad t\ge 0,$$

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is a martingale in the filtration of $\xi_\dagger$ for every $q \geq 0$. Applying the same reasoning with the time change as in Lemma 4.2, it follows that

$$N_t = X_\dagger(t)^q - 1 - \kappa(\omega + q) \int_0^t X_\dagger(s)^{q+\alpha}\,\mathrm{d}s, \qquad t \geq 0,$$

is a local martingale. (For our choice of g, we have $\mathcal{A}_\dagger^{(\alpha)} g(x) = \kappa(\omega + q)x^{q+\alpha}$, so this is consistent with the proof of part (iii).) We observe that

$$\sup_{t\geq 0}|N_t| \leq 1 + \sup_{t\geq 0} X_\dagger(t)^q - \kappa(\omega + q) \int_0^\infty X_\dagger(s)^{q+\alpha}\,\mathrm{d}s.$$

We now apply directly parts (i) and (ii) of this lemma in order to show that $\mathbb{E}[\sup_{t\geq 0}|N_t|] < \infty$. This is a sufficient criterion for N to be a uniformly integrable martingale (see [18, Theorem I.51]), and the remainder of the proof follows in the same way as in part (iii). □

We can then repeat the calculations which were made after the proof of Lemma 4.2, and arrive at the following result.

Corollary 4.2. *Suppose that (4.3) holds. Define $\mu_t = h_{-\omega}\rho_t$ for $t \geq 0$. Then the vaguely continuous family of measures $(\mu_t)_{t\geq 0}$ solves (1.2) with $\mathcal{L} = \mathcal{L}_\alpha$ both for $f \in \mathcal{C}_c^\infty(0, \infty)$ and for $f = h_q$ for any $\omega < q < \omega_+$.*

An interesting contrast with the case $\alpha < 0$ is that we do *not* show that μ_t solves (1.2) for all power functions.

We now give the basis of a solution to the growth-fragmentation equation starting from the zero measure; in this case, the solution should be interpreted not as spontaneous generation from infinitely small masses, but as starting from infinite mass and breaking apart instantaneously. Recall that $\kappa(q) = 0$ has at most two solutions. More precisely, we have already seen that there is always a unique solution ω_+ with $\kappa'(\omega_+) > 0$ (this is the Malthusian exponent defined by (4.4)). When a second solution, say ω_-, exists, then $\omega_- < \omega_+$ and $\kappa'(\omega_-) \in [-\infty, 0)$. We give the results under the assumption that

$$\text{the equation } \kappa(q) = 0 \text{ with } q \geq 0 \text{ has two solutions } \omega_- < \omega_+, \text{ and } \kappa'(\omega_-) > -\infty, \qquad (4.6)$$

which is thus stronger than (4.4). We write ξ_- for the spectrally negative Lévy process with Laplace exponent $\Phi_- := \Phi_{\omega_-} = \kappa(\cdot + \omega_-)$, and then X_- for the pssMp with index $-\alpha$ associated with ξ_- by Lamperti's transform.

Lemma 4.4. *Assume that (4.6) holds. Then there exists a càdlàg process $(\mathcal{X}(t))_{t>0}$ with values in $(0, \infty)$ and $\lim_{t\to 0+} \mathcal{X}(t) = \infty$ almost surely, such that*

- *for every $s > 0$, conditionally on $\mathcal{X}(s) = x$, the shifted process $(\mathcal{X}(s + t))_{t\geq 0}$ has the law P_x of the pssMp X_- started from x;*
- *for all $0 < \varepsilon < (\omega_+ - \omega_-)/\alpha$, there exists $c(\varepsilon) < \infty$ such that*

$$\mathbb{E}[\mathcal{X}(t)^{\alpha(1-\varepsilon)}] = c(\varepsilon)t^{\varepsilon-1}, \qquad t > 0.$$

Proof. Let Y denote the pssMp with self-similarity index α associated with the Lévy process $-\xi_-$, so Y has the same law as $1/X_-$. Because

$$\mathbb{E}[-\xi_-(1)] = -\Phi_-'(0+) := m \in (0, \infty),$$

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[5] shows that $0+$ is an entrance boundary for Y, i.e. there exists a càdlàg process $(\mathcal{Y}(t))_{t>0}$ with values in $(0, \infty)$ and $\lim_{t\to 0+} \mathcal{Y}(t) = 0$ almost surely, such that, for every $s > 0$, conditionally on $\mathcal{Y}(s) = y$, the shifted process $(\mathcal{Y}(s+t))_{t\geq 0}$ has the law Y started from y. Our first claim follows by setting $\mathcal{X}(t) = 1/\mathcal{Y}(t)$.

Furthermore, according to Theorem 1 of [5], there is the identity

$$\mathbb{E}[\mathcal{Y}^{\alpha(\varepsilon-1)}(t)] = \frac{1}{\alpha m}\mathbb{E}\left[I^{-1}\left(\frac{t}{I}\right)^{\varepsilon-1}\right],$$

where $I := \int_0^\infty \exp(\alpha\xi_-(s))\,\mathrm{d}s$. It thus follows that

$$\mathbb{E}[\mathcal{X}^{\alpha(1-\varepsilon)}(t)] = c(\varepsilon)t^{\varepsilon-1},$$

where $c(\varepsilon) = \mathbb{E}(I^\varepsilon)/(\alpha m) \in (0, \infty]$.

For every $0 < \varepsilon < (\omega_+ - \omega_-)/\alpha$, the Laplace exponent $q \mapsto \Phi_-(\alpha q)$ of $\alpha\xi_-$ satisfies $\Phi_-(\alpha\varepsilon) < 0$, and according to Lemma 3 of [20], this ensures that $\mathbb{E}(I^\varepsilon) < \infty$. □

We next further require that $\omega_- \in (\operatorname{dom}\kappa)^\circ$. This is only a little stronger than the condition $\kappa'(\omega_-) > -\infty$, which is necessary and sufficient for $\mathcal{X}$ to exist. We write $\mathcal{A}_-^{(\alpha)}$ for the operator $\mathcal{A}_\dagger^{(\alpha)}$ given in Lemma 4.3(iii) for $\omega = \omega_-$, and deduce the following.

Corollary 4.3. *Assume that (4.6) holds and that $\omega_- \in (\operatorname{dom}\kappa)^\circ$. For $t > 0$, write π_t for the distribution of $\mathcal{X}(t)$. Then, for every $f \in \mathcal{C}_c^\infty(0, \infty)$, we have*

$$\int_0^t |\langle \pi_s, \mathcal{A}_-^{(\alpha)} f\rangle|\,\mathrm{d}s < \infty,$$

and the identity

$$\langle \pi_t, f\rangle = \int_0^t \langle \pi_s, \mathcal{A}_-^{(\alpha)} f\rangle\,\mathrm{d}s.$$

Proof. Recall from Lemma 4.3(iii) that $h_{\omega_-}\mathcal{A}_-^{(\alpha)} f = \mathcal{L}_\alpha(h_{\omega_-} f)$, so Lemma 2.1(iii) and the assumption that $\omega_- - \alpha\varepsilon \in \operatorname{dom}\kappa$ for some $\varepsilon > 0$ entail

$$\mathcal{A}_-^{(\alpha)} f(x) = \mathrm{o}(x^{-\omega_- + \alpha + \omega_- - \alpha\varepsilon}) = \mathrm{o}(x^{\alpha(1-\varepsilon)}).$$

It now follows from the preceding lemma that $|\langle \pi_s, \mathcal{A}_-^{(\alpha)} f\rangle| \leq C(s^{\varepsilon-1} + 1)$, where C is a finite constant depending only on f and ε. Our first claim follows.

Recall then from the proof of Lemma 4.3(iii) that

$$f(X_-(t)) - \int_0^t \mathcal{A}_-^{(\alpha)} f(X_-(s))\,\mathrm{d}s, \qquad t \geq 0,$$

is a local martingale, and, thus, thanks to Lemma 4.4, so is

$$f(\mathcal{X}(t+s)) - f(\mathcal{X}(s)) - \int_0^t \mathcal{A}_-^{(\alpha)} f(\mathcal{X}(r+s))\,\mathrm{d}r, \qquad t \geq 0,$$

for all $s > 0$. Since

$$\mathbb{E}\left[\int_0^t |\mathcal{A}_-^{(\alpha)} f(\mathcal{X}(r+s))|\,\mathrm{d}r\right] = \int_s^{t+s} |\langle \pi_r, \mathcal{A}_-^{(\alpha)} f\rangle|\,\mathrm{d}r < \infty,$$

the above process is actually a true martingale, and taking expectations, we arrive at the identity

$$\langle \pi_t, f\rangle - \langle \pi_s, f\rangle = \int_s^t \langle \pi_r, \mathcal{A}_-^{(\alpha)} f\rangle\,\mathrm{d}r.$$

With the observation above, we can let $s \to 0+$, and since $\langle \pi_s, f \rangle \to 0$ thanks to Lemma 4.4, we conclude that $\langle \pi_t, f \rangle = \int_0^t \langle \pi_r, \mathcal{A}_-^{(\alpha)} f \rangle \, \mathrm{d}r$. □

We conclude this section with the following corollary, which demonstrates the existence of a solution to the growth-fragmentation equation started from zero mass.

Corollary 4.4. *Suppose that the hypotheses of Lemma 4.3 hold. Let* $\gamma_t(\mathrm{d}x) = x^{-\omega_-}\pi_t(\mathrm{d}x)$ *for* $t > 0$*, and set* $\gamma_0 \equiv 0$*. Then the family* $(\gamma_t)_{t\geq 0}$ *solves (1.2) for all* $f \in \mathcal{C}_c^\infty(0,\infty)$ *when* $\mathcal{L} = \mathcal{L}_\alpha$ *is given by (1.4).*

5. Explosion of the stochastic model

In this section we discuss the behaviour of the stochastic growth-fragmentation process in a simplified setting. In particular, we point out that, when the Malthusian hypotheses from Section 4 do not hold, this stochastic model may experience explosion, in the sense that some arbitrarily small compact sets contain infinitely many particles after a finite time. We will focus on the case where $\alpha < 0$, though similar arguments can be made for $\alpha > 0$.

Assume that the measure K is a probability measure on $[\frac{1}{2}, 1)$ which is not equal to $\delta_{\frac{1}{2}}$, and denote by Y a random variable with law K. Choose $c \in \mathbb{R}$ such that

$$\mathbb{E}[\log(1-Y)] + c < 0 < \mathbb{E}[\log Y] + c.$$

We now set up the stochastic model. Let $\mathcal{U} = \bigcup_{n\geq 0}\{L, R\}^n$, where $\{L, R\}^0 = \{\varnothing\}$. We view this as a binary tree, as follows. The root node $\varnothing$ gives rise to child nodes L and R; the former then has children LL and LR, while the latter has children RL and RR, and so on. We introduce the ancestry relationship '$\preceq$' by saying that, for individuals $u, u' \in \mathcal{U}$, $u \preceq u'$, if and only if there exists $u'' \in \mathcal{U}$ such that $uu'' = u'$; we also define the strict relation $u \prec u'$ to mean that $u \preceq u'$ but $u \neq u'$.

Let $\mathcal{V} = \{L, R\}^{\mathbb{N}}$. This is the set of infinite lines of descent, or rays, in $\mathcal{U}$. For instance, $LLRRLRLRRRRL \cdots \in \mathcal{V}$ traces a line of descent starting at individual $\varnothing$, and proceeding to L, then LL, then LLR, and so on. If $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we say (by a slight abuse of notation) that $u \prec v$ if and only if there exists $v' \in \mathcal{V}$ such that $uv' = v$.

To each $u \in \mathcal{U}$, we assign, independently of everything else, a *lifetime* T_u which has an exponential distribution of rate 1, and an *offspring distribution* Y_u which is distributed with law K. We then recursively assign positions ζ_u to the individuals in $\mathcal{U}$. The root is positioned at a given point $x \in \mathbb{R}$, that is, $\zeta_\varnothing = x$. Its descendents are positioned as follows:

$$\zeta_{uL} = \zeta_u + \log(1 - Y_u) + cT_u \quad \text{and} \quad \zeta_{uR} = \zeta_u + \log(Y_u) + cT_u, \qquad u \in \mathcal{U}.$$

This gives a model in which each individual lives an exponential time, dies, and (on average) scatters one child to the left and one to the right.

We now introduce a model in which individuals also move continuously, as follows. For $u \in \mathcal{U}$, define its birth time $a_u = \sum_{u' \prec u} T_{u'}$ and its death time $b_u = \sum_{u' \preceq u} T_{u'} = a_u + T_u$. Its position between those times is then given by $\xi_u(t) = \zeta_u + c(t - a_u)$ for $a_u \leq t < b_u$. By another abuse of notation, let us define also the positions of a ray: for $v \in \mathcal{V}$, let $\xi_v(t) = \xi_u(t)$, where $u \in \mathcal{U}$ is the unique individual with $a_u \leq t < b_u$ and $u \prec v$. We may now see the model as containing individuals which move to the right at constant rate c, until an exponential clock rings and the individual dies, scattering offspring to the left.

The model may also be viewed as a stochastic process $\mathcal{Y} = (\mathcal{Y}(t))_{t\geq 0}$, with

$$\mathcal{Y}(t) = \sum_{u \in \mathcal{U}} \delta_{\exp(\xi_u(t))} \mathbf{1}_{\{a_u \leq t < b_u\}},$$

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taking values in the space $\mathcal{N}$ of locally finite point measures and such that $\mathcal{Y}(0) = \delta_{e^x}$. With this perspective, it is an important and useful fact that the process has the branching property. Loosely speaking, this means that the behaviour of $(\mathcal{Y}(t+s))_{s\geq 0}$ is given by collecting the atoms of $\mathcal{Y}(t)$ and running from each one an independent copy of $\mathcal{Y}$; for a precise statement and proof, see, for instance, [3, Proposition 2].

The process $\mathcal{Y}$ we have just described is a stochastic model corresponding to the homogeneous fragmentation equation. In particular, if we define a collection of measures $(\mu_t)_{t\geq 0}$ via

$$\langle \mu_t, f\rangle = \mathbb{E}[\langle \mathcal{Y}(t), f\rangle] = \mathbb{E}\bigg[\sum_{u\in\mathcal{U}} f(\exp(\xi_u(t)))\mathbf{1}_{\{a_u\leq t<b_u\}}\bigg], \qquad f\in \mathcal{C}_c^\infty,$$

then we obtain a solution to (1.2) with $\alpha = 0$ and $\mathcal{L}$ given as in (1.3). This corresponds to the function κ satisfying, for $q\geq 0$,

$$\kappa(q) = cq + \int_{[1/2,1)} [y^q + (1-y)^q - 1]K(\mathrm{d}y) > q\bigg(c + \int_{[1/2,1)} \log y\, K(\mathrm{d}y)\bigg) > 0.$$

Here, the first inequality holds because it holds for the integrands, and the second inequality is by our assumption about c at the beginning of the section. Also, $\kappa(0) = 1$. Thus, $\inf_{q\geq 0}\kappa(q) > 0$, and so the Malthusian hypothesis (4.4), which was an important assumption in Section 4, is not satisfied for our model.

We now give the model corresponding to the self-similar equation. To this end, we should first introduce the notion of a stopping line. We say that $S = (S_v)_{v\in\mathcal{V}}$ is a *stopping line* if

(i) for every $v\in\mathcal{V}$, S_v is a stopping time for the natural filtration of ξ_v;

(ii) if $u\in\mathcal{U}$ and $v, v'\in\mathcal{V}$ such that $u\prec v, v'$, then $\mathbb{P}(S_v = S_{v'} \mid a_u\leq S_v < b_u) = 1$.

Now, for $v\in\mathcal{V}$ and $\alpha\in\mathbb{R}$, let

$$T_v(s) = \int_0^s \mathrm{e}^{-\alpha\xi_v(r)}\,\mathrm{d}r,$$

and denote its inverse by S_v. Then $S(t) = (S_v(t))_{v\in\mathcal{V}}$ is a stopping line for every $t\geq 0$. If $u\in\mathcal{U}$ is an individual such that, for some $v\in\mathcal{V}$ with $u\prec v$, $a_u\leq S_v(t) < b_u$ holds, then we define $S_u(t)$ to be equal to $S_v(t)$; by property (ii) of the definition of a stopping line, this does not depend on the choice of v. We define

$$X_v(t) = \exp(\xi_v(S_v(t))), \qquad v\in\mathcal{V},\ t\geq 0,$$

and

$$\mathcal{X}(t) = \sum_{u\in\mathcal{U}} \delta_{\exp\xi_u(S_u(t))}\mathbf{1}_{\{a_u\leq S_u(t)<b_u\}}, \qquad t\geq 0, \tag{5.1}$$

where the sum is over only those u for which $S_u(t)$ is defined. The process $\mathcal{X}$ is called the *α-self-similar fragmentation process*. The stopping-line nature of S means that the process $\mathcal{X}$ retains the branching property; however, it is not clear that it should be locally finite, and indeed, our main result in this section is that $\mathcal{X}$ almost surely does *not* remain locally finite for all time.

Proposition 5.1. *For any $a > 0$, there exists some random time σ such that there are infinitely many individuals of $\mathcal{X}(\sigma)$ in the compact set* $[1, 1+a]$.

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Proof. We will study rays $p_k = L^k R^\infty$ which follow the left-hand offspring for k steps, and the right-hand offspring thereafter. Our first remark is that, if we define

$$\tau_0(v) = \inf\{t \geq 0 \colon X_v(t) = 0\}, \qquad v \in \mathcal{V},$$

then we have

$$\tau_0 := \tau_0(L^\infty) = T_{L^\infty}(\infty) = \int_0^\infty \mathrm{e}^{-\alpha \xi_{L^\infty}(t)}\, \mathrm{d}t.$$

Since ξ_{L^∞} is a Lévy process with negative mean (by our assumption on c), then $\tau_0 < \infty$ almost surely.

Consequently, for any $\eta > 0$, there exists some infinite set C such that, for each $k \in C$, $T_{p_k}(b_{L^k}) \in (\tau_0 - \eta, \tau_0)$ and $X_{p_k}(T_{p_k}(b_{L^k})) \to 0$ as $k \to \infty$. We define

$$L_1^+(v) = \sup\{t \geq 0 \colon X_v(t) \leq 1\}, \qquad v \in \mathcal{V},$$

which is the *last passage time* of the level 1 by the process X_v; then, for $k \in C$, we have

$$L_1^+(p_k) = T_{p_k}(b_{L^k}) + \tilde{L}_1^+(R^\infty),$$

where $\tilde{L}_1^+(R^\infty)$ is the last passage time of the level 1 by $\tilde{X}_{R^\infty}$, computed for an independent self-similar fragmentation process started at $X_{p_k}(b_{L^k})$.

We are therefore reduced to studying first passage times of the $(-\alpha)$-pssMp X_{R^∞} corresponding to a spectrally negative Lévy process ξ_{R^∞} started from a level $x < 0$ with Laplace exponent

$$\Phi_{R^\infty}(q) = cq + \int_{[1/2,1)} (y^q - 1) K(\mathrm{d}y).$$

We seek t such that, with positive probability (not depending on x), $\tilde{L}_1^+(X_{R^\infty}) \leq t$.

The first observation is that $\Phi'_{R^\infty}(0+) > 0$, which implies that the process X_{R^∞} can be extended to start at 0; it is then Feller on $[0, \infty)$. Furthermore, the pssMp drifts to $+\infty$ as $t \to \infty$. Hence, we pick $\varepsilon > 0$ and $t \geq 0$ such that $\mathbb{P}_0(L_1^+(X_{R^\infty}) \leq t) \geq 2\varepsilon$. By the Portmanteau theorem, $\liminf_{x \to 0} \mathbb{P}_x(L_1^+(X_{R^\infty}) \leq t) \geq 2\varepsilon$ also, and, therefore, for x sufficiently close to 0, $\mathbb{P}_x(L_1^+(X_{R^\infty}) \leq t) \geq \varepsilon$. Applying the Borel–Cantelli lemma to the paths referred to above, there are then infinitely many $k \in C$ such that

$$L_1^+(p_k) \leq T_{p_k}(b_{L^k}) + t \leq \tau_0 + t$$

with probability 1.

We therefore have infinitely many paths whose last passage time of 1 occurs in a (random) compact interval. In particular, there must exist some finite random time σ such that, for every $\delta > 0$, there are infinitely many paths which cross 1 for the last time in $(\sigma - \delta, \sigma)$. Since the particles are large, the time change is bounded, in that $S_{p_k}(L_1^+(p_k) + u) - S_{p_k}(L_1^+(p_k)) \leq u$ for all $u \geq 0$. Furthermore, the processes ξ_{p_k} can grow at most at rate c; thus, picking $\delta < c^{-1} \log(1 + a)$ ensures that, at time σ, all the selected particles are in $[1, 1 + a]$, which completes the proof. □

This result illustrates one example where the Malthusian hypothesis, under which we examined the growth-fragmentation equation, fails, and where the stochastic model $\mathcal{X}$ (whose mean measure could otherwise be expected to give a solution) does not remain locally finite. However, since uniqueness generally fails for the self-similar growth-fragmentation equation, it does not immediately imply that there is no global solution of (1.2).

The procedure of creating the growth-fragmentation process $\mathcal{Y}$ can be carried out under the general conditions on a, b, and K given in the main body of the paper (this is done in [3]) and the self-similar time change (5.1) can also be applied in this general context for any $\alpha \neq 0$ (see [4, Corollary 2]). However, it remains an open problem to determine necessary and sufficient conditions for the process $\mathcal{X}$ to be locally finite at all times, and to decide when global solutions of (1.2) exist.

6. Branching particle system and many-to-one formulas

In this concluding section we aim to clarify the connection between our work and the 'spine' or 'tagged fragment' approach to branching particle systems and fragmentation processes, with the hope of explaining the source of the solutions obtained in Theorem 3.1 and Corollaries 4.1 and 4.2. We refer the reader to the survey of Hardy and Harris [15] for results in the context of branching processes, Bertoin [2, Section 3.2.2] for the background on fragmentation processes, and Haas [13] for the use of the tagged fragment in solving the classical fragmentation equation. For the sake of simplicity, we focus on the case when the dislocation measure K is finite and the operator $\mathcal{L}$ has the form (1.3).

Let us assume that we can construct a system of branching particles in $(0, \infty)$, with the following dynamics: particles evolve independently of one other, each particle located at $x > 0$ grows at rate $cx^{\alpha+1}$, and a particle located at $x > 0$ is replaced by two particles located respectively at xy and $x(1-y)$ at rate $x^{\alpha}K(\mathrm{d}y)$. Let $\boldsymbol{Z}(t) = (Z_i(t))_{i\geq 1}$ denote the collection of particles in the system at time $t \geq 0$, starting at time 0 from a single particle located at 1. Informally, the verbal description of the dynamics of the particle system suggests that, for every test function $f \in \mathcal{C}_c^{\infty}(0, \infty)$, the functional $F(z) = \sum_i f(z_i)$ for $z = (z_i)_{i\geq 1}$ belongs to the domain of the infinitesimal generator $\mathcal{G}$ of the process $(\boldsymbol{Z}(t))_{t\geq 0}$, and that

$$\mathcal{G}F(z) = \sum_{i\geq 1} z_i^{\alpha}\biggl(cz_i f'(z_i) + \int_{[1/2,1)} (f(y \mid z_i) - f(z_i))K(\mathrm{d}y)\biggr).$$

Therefore, if we write μ_t for the intensity measure of $\boldsymbol{Z}(t)$, i.e.

$$\langle \mu_t, f\rangle = \mathbb{E}[F(\boldsymbol{Z}(t))] = \mathbb{E}\biggl[\sum_i f(Z_i(t))\biggr],$$

then Kolmogorov's forward equation entails that $(\mu_t)_{t\geq 0}$ solves the fragmentation equation (1.2).

The analysis of the system $(\boldsymbol{Z}(t))_{t\geq 0}$ can be significantly simplified by identifying a *spine* among the particles, and formulating questions about the entire system in terms of just the spine particle via a *many-to-one* formula. This proceeds roughly as follows. Suppose that we can identify a function $(t, z) \mapsto \varphi(t, z) > 0$ such that the process $M(t) = \sum_{i\geq 1} \varphi(t, Z_i(t))$ is a martingale. We introduce a new probability measure $\tilde{\mathbb{P}}$ by means of a martingale change of measure using M, simultaneously identifying one of the particles to be the spine; specifically, we identify particle i of $\boldsymbol{Z}(t)$ as the spine with $\tilde{\mathbb{P}}$-probability proportional to $\varphi(t, Z_i(t))$.

At each time $0 \leq s \leq t$, the spine particle has a unique ancestor, and we define the random variable $W(s)$ to be the position of this ancestor. We now aim to identify the law of certain functionals of $\boldsymbol{Z}$ in terms of the law of W. In particular, if we define ρ_t to be the law of $W(t)$, we obtain

$$\langle \mu_t, f\rangle = \mathbb{E}\biggl[\sum_{i\geq 1} f(Z_i(t))\biggr] = \tilde{\mathbb{E}}\biggl[\frac{f(W(t))}{\varphi(t, W(t))}\biggr] = \biggl\langle \rho_t, \frac{f}{\varphi(t, \cdot)}\biggr\rangle,$$

which is known as a many-to-one formula.

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The spine method for solving (1.2) can be summarised as follows. We first use the dynamics of the branching particle system and the effect of the martingale change of measure to identify the process W. The one-dimensional distributions of W give the collection of measures $(\rho_t)_{t\geq 0}$, and then the many-to-one formula gives us an explicit description of $(\mu_t)_{t\geq 0}$.

The method we have sketched can be made rigorous in the homogeneous case $\alpha = 0$, even in the more general situation when the dislocation measure K is infinite and fulfils (1.5). More precisely, one can take $\varphi(t, z) = \exp(-\kappa(\omega)t)z^{\omega}$ for any $\omega \in \operatorname{dom}\kappa$, and then the process $W(t)$ is the exponential of a Lévy process with no positive jumps and Laplace exponent $\Phi_{\omega} = \kappa(\omega + \cdot) - \kappa(\omega)$; this is the justification for Remark 3.2. We stress, however, that the general self-similar case $\alpha \neq 0$ is far less simple. In particular, it is not clear whether the branching particle system can indeed be constructed, since as noted in the previous section, explosion may occur. A fairly general class of growth-fragmentation processes was introduced recently in [4] by means of a Crump-Mode-Jagers process; however, although it is expected to be related to growth-fragmentation equations as described above, so far no many-to-one formula is known to make the connection rigorous.

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