

MAXIMIZING THE p th MOMENT OF THE EXIT TIME OF PLANAR BROWNIAN MOTION FROM A GIVEN DOMAIN

MAHER BOUDABRA,* AND
GREG MARKOWSKY,**^{***} *Monash University*

Abstract

In this paper we address the question of finding the point which maximizes the p th moment of the exit time of planar Brownian motion from a given domain. We present a geometrical method for excluding parts of the domain from consideration which makes use of a coupling argument and the conformal invariance of Brownian motion. In many cases the maximizing point can be localized to a relatively small region. Several illustrative examples are presented.

Keywords: Planar Brownian motion; exit time

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1. Introduction

Let $Z_t := X_t + iY_t$ be a planar Brownian motion starting at a point a in a domain U . We will let $\tau_U = \tau_U(a)$ be the first time that Z_t exits U , and we will use the standard notation \mathbb{E}_a to denote expectation conditioned on $Z_0 = a$ a.s. The focus of this paper is the following optimization problem.

For a given domain in the plane and $0 < p < \infty$, find the point a which maximizes the quantity $\mathbb{E}_a[(\tau_U)^p]$.

We will refer to such a point as a p th center of U ; it is not generally unique, as the easy example of an infinite strip shows. For many domains, even simple ones such as an isosceles triangle, it is difficult to find any of the p th centers, but we will show how elementary coupling arguments and the conformal invariance of Brownian motion in many cases allows us to locate a small region in U which must contain all p th centers. In certain cases in which the domain in question has a high degree of symmetry, it will allow us to locate all p th centers.

Before describing our methods, we present a brief overview of some earlier works related to this problem. The case $p = 1$ is commonly referred to as the ‘torsion problem’ due to its connection with mechanics, and is naturally the most tractable. The function $h(a) = \mathbb{E}_a[\tau_U]$ satisfies $\Delta h = -2$, and therefore PDE techniques can be employed to great effect. Sperb [17, Chapter 6] gives a good account of this problem and methods for attacking it in special cases, such as when the domain in question is convex. Further results along the same lines, focusing in particular on convex domains, can be found in [9], [12], and [17].

Other interesting related problems have been tackled by PDE methods. For example, in the famous paper [2] (see also the related work [3]) eigenvalue techniques are used to demonstrate

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* Postal address: Clayton, Victoria, Australia.

** Email address: greg.markowsky@monash.edu

relationships between $\mathbb{E}_a[\tau_U]$ and geometric qualities of the domain, such as the size of the hyperbolic density and the inradius (the radius of the largest disk contained in the domain). The methods developed there have been extended by other authors in a number of different directions. For example, Méndez-Hernández [14] proved a number of related stochastic domination results concerning convex domains in \mathbb{R}^n and various types of symmetrizations. These results allow conclusions to be reached concerning the comparison of p th moments of the exit times from these domains. One striking consequence of the eigenvalue methods is the fact that over all domains with a given area, the disk maximizes the p th moment of the exit time of Brownian motion for all p . The recent work by Kim [11] contains a discussion and refinements of this result. We would also like to mention the interesting and very recent preprint [5], in which questions similar to ours are addressed; the authors there demonstrate bounds on the quantity $\lambda_1^p(U) \sup_{a \in U} \mathbb{E}_a[(\tau_U)^p]$, where $\lambda_1(U)$ denotes the first Dirichlet eigenvalue for the Laplacian in U , and prove the existence of extremal domains with regards to this quantity over various classes of convex domains.

Our results differ from those described above in the following ways. We have not employed PDE methods at all, choosing instead to work with an elementary coupling method. Perhaps as a consequence of this, convexity plays little role in our discussion, although a weaker concept called Δ -convexity (defined below) will be important. The type of coupling we will use is not entirely new, and has found a number of uses in related topics, for instance in investigations into the ‘hot spots’ conjecture such as [1], [4], and [16]. However, we believe that it has not yet been applied directly in the manner we use here. Furthermore, we restrict our attention to two dimensions, which allows conformal mappings to take prominence and to extend the standard notion of coupling. We present several methods for localizing the p -centers of a domain, and then consider a number of specific domains, showing in each case how our methods can be used to localize the p th centers of the domain. In what follows we assume p is a fixed positive number. However, in order to reduce the qualifications needed to state our results, for any planar domain U for which we are interested in maximizing the p th moment, we will assume that $\mathbb{E}_a[(\tau_U)^p] < \infty$ for all points $a \in U$; this would follow if $\mathbb{E}_a[(\tau_U)^p] < \infty$ for any $a \in U$, as is shown in [7].

2. Partial symmetry and convexity with respect to a line.

Definition 2.1. Let U be a domain of \mathbb{C} . We say that a line $\Delta: ax + by + c = 0$ is a partial symmetry axis for U if one of the two sets $U^+ := U \cap \{ax + by + c > 0\}$ or $U^- := U \cap \{ax + by + c < 0\}$ can be folded over Δ and fits into U , more precisely if $\sigma_\Delta(U^+)$ or $\sigma_\Delta(U^-)$ remains inside U , where σ_Δ denotes the symmetry over Δ . The subset among U^\pm that satisfies this property (i.e. the smaller side with respect to the symmetry) is called symmetric side of U over Δ . So, for instance, any line intersecting $\mathbb{D} = \{|z| < 1\}$ is a partial symmetry axis for \mathbb{D} but the line $y = 2x$ is not one for the square $\{|x| < 1, |y| < 1\}$, since the reflection over this line of the point $(1, 1)$ is the point $(1/5, 7/5)$, which is not in the closure of the square. Note that both of U^\pm are symmetric sides if and only if Δ is a symmetry axis for U .

Theorem 2.1. *Let S be the symmetric side of U over a partial symmetry axis Δ . Then, for any $a \in S$ we can find Brownian motions Z_t starting at a and \tilde{Z}_t starting at $\sigma_\Delta(a)$ defined on the same probability space such that $\tau_U \leq \tilde{\tau}_U$ a.s. (where $\tilde{\tau}_U$ is the exit time from U of \tilde{Z}). Furthermore, if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$ then $P(\tau_U < \tilde{\tau}_U) > 0$. In particular, $\mathbb{E}_a[\tau_U^p] \leq \mathbb{E}_{\sigma_\Delta(a)}[\tilde{\tau}_U^p]$ (with strict inequality if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$).*

Proof. This follows from a coupling argument. Let Z_t start at $a \in S$, and let H_Δ be its hitting time of the line Δ . Form the process \tilde{Z}_t by the rule

$$\tilde{Z}_t = \begin{cases} \sigma_\Delta(Z_t) & \text{if } t < H_\Delta, \\ Z_t & \text{if } t \geq H_\Delta. \end{cases}$$

It follows from the strong Markov property and the reflection invariance of Brownian motion that \tilde{Z}_t is a Brownian motion. Clearly $\tau_U = \tilde{\tau}_U$ on the set $\{\tau_U \geq H_\Delta\}$, and our conditions on S imply $\tau_U \leq \tilde{\tau}_U$ on the set $\{\tau_U < H_\Delta\}$. Furthermore, if $\sigma_\Delta(S)$ is strictly contained in $U \setminus (S \cup \Delta)$, then Z_t has some positive probability of leaving U before \tilde{Z}_t does; this is implied for instance by [6, Theorem I.6.6]. The result follows. \square

This theorem allows us in essence to exclude the symmetric side of any partial symmetry axis for U in our search for p th centers. The only exception to this rule is when $\sigma_\Delta(S) = U \setminus (S \cup \Delta)$, i.e. when Δ is a symmetry axis of U . However, in most cases a symmetry axis will contain all p th centers. To see why this is so, we need another definition.

Definition 2.2. Let Δ be a symmetry axis of U . We say that U is Δ -convex if

$$tz + (1 - t)\sigma_\Delta(z) \in U \quad \text{for all } (z, t) \in U \times [0, 1].$$

In other words, for every $z \in U$, the segment joining z and $\sigma_\Delta(z)$ remains inside U .

It is clear that any convex U is Δ -convex for any symmetry axis Δ , and a less trivial example can be given by $\{-f(x) < y < f(x)\}$, where f is a positive continuous function on the real line, which is Δ -convex with $\Delta = \mathbb{R}$. A domain which is not Δ -convex with respect to a symmetry axis can be given by

$$W_\varepsilon = \{-\varepsilon < y < \varepsilon, |x| < 1\} \cup \left\{ |z - 1| < \frac{1}{2} \right\} \cup \left\{ |z + 1| < \frac{1}{2} \right\}$$

with $\varepsilon < 1/2$; this has the real and imaginary axes as symmetry axes but is not Δ -convex with respect to the imaginary axis (though it is with respect to the real line). As will be seen below, this domain also shows why Δ -convexity is required in the following proposition.

Proposition 2.1. *Suppose U is Δ -convex with respect to a symmetry axis Δ . Then all p th centers of U lie on Δ .*

Proof. Let $a \in U \setminus \Delta$, and let

$$\hat{a} = \frac{1}{2}a + \frac{1}{2}\sigma_\Delta(a)$$

be the orthogonal projection of a onto Δ . Let L be the line parallel to Δ which passes through the point

$$\frac{1}{2}a + \frac{1}{2}\hat{a}.$$

Speaking informally, this is the line halfway between a and Δ . Δ -convexity implies that L is a partial symmetry axis of U , and if S is the component of $U \setminus L$ containing a then $\sigma_L(S)$ is strictly contained in $U \setminus (S \cup L)$. It therefore follows from Theorem 2.1 that $\mathbb{E}_a[\tau_U^p] < \mathbb{E}_{\hat{a}}[\tau_U^p]$. The result follows. \square

Note that this proposition completely solves our problem in the case that our domain is Δ -convex with respect to two or more non-parallel symmetry axes, since all p th centers must lie at their point of intersection, and we have also incidentally proved the purely geometrical fact that all such symmetry axes must coincide at a unique point; more on this in the final section.

Thus, for instance, all p -centers of any regular polygon, a circle, an ellipse, a rhombus, and any number of other easily constructed examples must lie at their natural centers. To see an example of a domain with intersecting symmetry axes but where the point of intersection is not a p th center, let us return to the domains W_ε described immediately before this proposition. Proposition 2.1 implies that all p th centers lie on the real line, but it is easy to see that if we make ε sufficiently small then 0, the intersection point of the two symmetry axes, will not be a p th center (clearly $\tau_{W_\varepsilon}(1) \geq \tau_{\{|z-1|<1/2\}}(1)$, so that $\mathbb{E}_1[\tau_{W_\varepsilon}^p]$ always remains greater than a positive constant, but $\tau_{W_\varepsilon}(0)$ decreases monotonically to 0 a.s. as $\varepsilon \searrow 0$, so that $\mathbb{E}_1[\tau_{W_\varepsilon}^p] \searrow 0$).

Let us now look at an example that shows the use of the results proved up to this point, but also their limitations. Suppose U is the isosceles right-angled triangle with vertices at $-1, 1$, and i . The imaginary axis is an axis of symmetry, and U is Δ -convex with respect to this axis, so all p th centers must lie on the imaginary axis. The line $\{y = 1/2\}$ is a partial symmetry axis for U , with $U \cap \{y > 1/2\}$ the symmetric side, so all p -centers must lie on $\{x = 0, y \leq 1/2\}$. Now common sense tells us that the p th centers cannot be too close to the real axis as well, because this is a boundary component, but there is no good partial symmetry axis to apply to conclude that rigorously. The way out of this difficulty is to extend our method of reflection to curves more general than straight lines. For this, we will need to utilize the conformal invariance of Brownian motion, via the following famous theorem of Lévy (see [6] or [15]).

Theorem 2.2. *If f is a holomorphic function, then $f(Z_t)$ is a time-changed Brownian motion. More precisely, $f(Z_{\kappa^{-1}(t)})$ is a Brownian motion where*

$$\kappa(t) := \int_0^t |f'(Z_s)|^2 ds \quad \text{for } t \geq 0.$$

This allows us to extend Theorem 2.1 as follows.

Proposition 2.2. *Suppose U is a domain with an axis of symmetry Δ , and suppose f is a conformal map defined on U with the property that $|f'(z)| \geq |f'(\sigma_\Delta(z))|$ for all $z \in A$, where A is one component of $U \setminus \Delta$ and σ_Δ is the symmetry over Δ . Then, for any $a \in A$, we can find Brownian motions Z_t starting at $f(a)$ and \tilde{Z}_t starting at $f(\sigma_\Delta(a))$ defined on the same probability space such that $\tau_{f(U)}(f(a)) \geq \tilde{\tau}_{f(U)}(f(\sigma_\Delta(a)))$ a.s. (where $\tilde{\tau}_U$ is the exit time from U of \tilde{Z}). In particular,*

$$\mathbb{E}_{f(a)}[\tau_{f(U)}^p] \geq \mathbb{E}_{f(\sigma_\Delta(a))}[\tau_{f(U)}^p].$$

If there is any point in A at which $|f'(z)| > |f'(\sigma_\Delta(z))|$ then

$$\mathbb{E}_{f(a)}[\tau_{f(U)}^p] > \mathbb{E}_{f(\sigma_\Delta(a))}[\tau_{f(U)}^p]$$

(for this statement we recall the assumption that $\mathbb{E}_w[\tau_{f(U)}^p] < \infty$ for any $w \in f(U)$).

Proof. Let Z_t be a Brownian motion starting at a , and let \tilde{Z}_t be defined as in Theorem 2.1. According to Theorem 2.2, the processes $f(Z_t)$ and $f(\tilde{Z}_t)$ are time-changed Brownian motions, and the time changes are given by $\kappa^{-1}(s)$ and $\tilde{\kappa}^{-1}(s)$, respectively, where

$$\left. \begin{aligned} \kappa(t) &= \int_0^t |f'(Z_t)|^2 dt \\ \tilde{\kappa}(t) &= \int_0^t |f'(\tilde{Z}_t)|^2 dt \end{aligned} \right\} t < \tau_U.$$

Now our assumptions imply $|f'(Z_t)| \geq |f'(\tilde{Z}_t)|$ a.s. for all $t < \tau_U$, and thus $\kappa(t) \geq \tilde{\kappa}(t)$ a.s. for all $t < \tau_U$. It follows from this that $\tau \geq \tilde{\tau}$ a.s., where τ and $\tilde{\tau}$ are the exit times from $f(U)$ of the Brownian motions $f(Z_{\kappa^{-1}(s)})$ and $f(\tilde{Z}_{\tilde{\kappa}^{-1}(s)})$, which begin at $f(a)$ and $f(\sigma_\Delta(a))$, respectively. The result follows. \square

We can obtain a corollary that will be useful for the isosceles triangle and in other cases by taking

$$f(z) = z_0 + R \left(\frac{z+i}{z-i} \right) \quad \text{for } z_0 \in \mathbb{C} \text{ and } R > 0,$$

which takes the real axis to the circle $C = \{|z - z_0| = R\}$, the upper half-plane to the outside of C , and the lower half-plane to the inside. We have

$$|f'(z)| = \frac{2R}{|z-i|^2},$$

and it is easy to check that $|f'(z)| > |f'(\bar{z})|$ for all z in the upper half-plane. Applying Proposition 2.2 and working through the implications yields the following.

Corollary 2.1. *Let $C = \{|z - z_0| = R\}$ be a circle in \mathbb{C} , with inside \mathcal{I} and outside \mathcal{O} . If U is a domain such that $\sigma_C(U \cap \mathcal{I}) \subseteq (U \cap \mathcal{O})$, then no p th center of U lies in \mathcal{I} .*

Note that here σ_C denotes reflection over the circular arc C , that is,

$$\sigma_C(z) = z_0 + \frac{R^2}{z - z_0}.$$

We remark further that the singularity that f has at i does not cause a problem in this result, because $\mathbb{E}_0(Z_t = i \text{ for some } t \geq 0) = 0$, so a.s. a Brownian motion starting at 0 will not hit the singularity in any case. The compact set $\{B_t : 0 \leq t \leq \tau_U\}$ is therefore bounded away from i a.s., and the result goes through.

Let us now apply this corollary to the isosceles right-angled triangle U shown in Figure 1. If C is the circle passing through -1 and 1 which intersects the real axis at angles of $\pi/8$, then the reflection of the set $\mathcal{A} = \mathcal{I} \cap U$ will be the region \mathcal{B} in the upper half-plane bounded by C and the circle which passes through -1 and 1 and intersects the real axis at angles of $\pi/4$; this can be seen by noting that the transformation σ_C preserves angles and also preserves the class of circles on the Riemann sphere (which includes lines, interpreted as circles through ∞).

As this region lies within U , we conclude that no p th centers lie within \mathcal{A} . A bit of Euclidean geometry shows that C intersects the imaginary axis at $\csc(\pi/8) - \cot(\pi/8) \approx 0.20$, and coupled with our observations above we see that all p th centers must lie on the imaginary axis between the points $0.2i$ and $0.5i$. In fact, the upper bound of $0.5i$ can be improved by using the angle bisector of the angles at 1 or -1 ; see Example 3.2. The reader may also have observed that the reflected circular domain does not do a good job of filling the triangle, and therefore it stands to reason that the lower bound may also be improved; more on this in the final section.

Finally, the following result can be useful when U is mapped to itself by an antiholomorphic function \bar{f} (this means that the conjugate of $\bar{f}, \bar{f}(z)$, is holomorphic). We will denote the derivative of this function (with respect to \bar{z}) by $\bar{f}'(z)$. An example of this is when U is an annulus, as will be explored in Section 3.

Proposition 2.3. *Let $\bar{f}: U \rightarrow U$ be antiholomorphic, and consider the two sets*

$$\begin{aligned} \Omega &:= \{z \in U \mid |\bar{f}'(z)| < 1\}, \\ \Lambda &:= \{z \in U \mid |\bar{f}'(z)| = 1\}. \end{aligned}$$

If $\bar{f}(U \setminus \Omega) \subset \Omega$ and $\bar{f}|_\Lambda = id_\Lambda$, then all p th centers are contained in Ω .

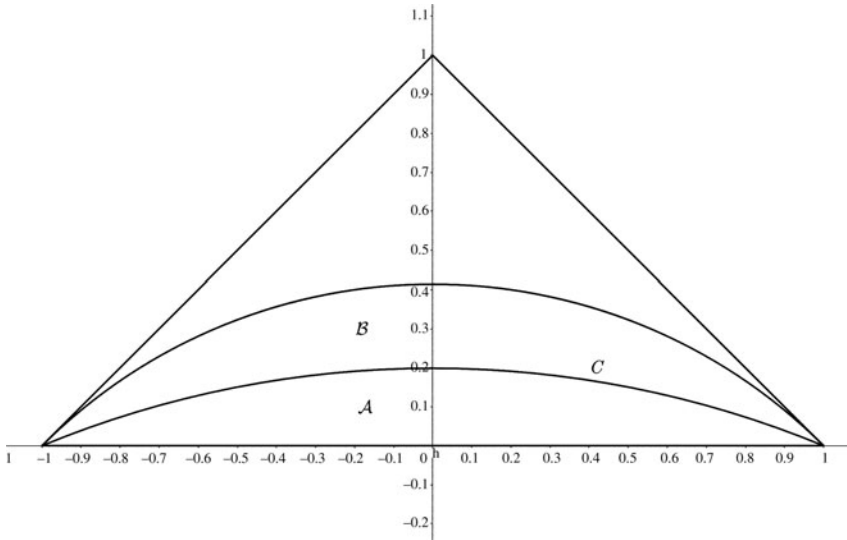


FIGURE 1: Reflection over a circle.

Proof. Let Z_t be a Brownian motion starting at $z \in U \setminus \Omega$ and let H_Λ be its hitting time of Λ , and consider W_t the Brownian motion derived from $f(Z_t)$, that is,

$$W_t := f(Z_{\kappa^{-1}(t)}),$$

where

$$\kappa(t) = \int_0^t |\tilde{f}'(Z_s)|^2 ds.$$

Now we are going to construct two Brownian motions \tilde{Z}_t and \tilde{W}_t starting respectively at z and $w := f(z)$, such that \tilde{W}_t leaves U before \tilde{Z}_t , as follows.

- (1) If $H_\Lambda < \tau_U^Z$, then run an independent Brownian motion, say B_t , starting at Z_{H_Λ} , and set

$$\begin{aligned} \tilde{Z}_t &:= Z_t \mathbf{1}_{\{t \leq H_\Lambda\}} + B_{t-H_\Lambda} \mathbf{1}_{\{H_\Lambda < t\}}, \\ \tilde{W}_t &:= W_t \mathbf{1}_{\{t \leq \kappa(H_\Lambda)\}} + B_{t-\kappa(H_\Lambda)} \mathbf{1}_{\{\kappa(H_\Lambda) < t\}}, \end{aligned}$$

where \tilde{W}_t and \tilde{Z}_t are well-defined Brownian motions as

$$\tilde{W}_{\kappa(H_\Lambda)} = W_{\kappa(H_\Lambda)} = f(Z_{H_\Lambda}) = Z_{H_\Lambda} = \tilde{Z}_{H_\Lambda}.$$

Now note that

$$\tau_U^{\tilde{Z}} = H_\Lambda + \inf\{t, B_t \notin U \mid B_0 = Z_{H_\Lambda}\} \leq \kappa(H_\Lambda) + \inf\{t, B_t \notin U \mid B_0 = Z_{H_\Lambda}\} = \tau_U^{\tilde{W}}.$$

- (2) If $H_\Lambda \geq \tau_U^Z$, then just set $\tilde{Z}_t = Z_t$ and $\tilde{W}_t = W_t$. Therefore

$$\tau_U^{\tilde{Z}} \leq \kappa(\tau_U^{\tilde{Z}}) = \tau_U^{\tilde{W}}.$$

In both cases we have

$$\tau_U^{\tilde{Z}} \stackrel{a.s.}{\leq} \tau_U^{\tilde{W}}.$$

Hence

$$\mathbb{E}_z((\tau_U^z)^p) \leq \mathbb{E}_w((\tau_U^w)^p),$$

which ends the proof. □

Remark 2.1. Reflection over the circle, obtained above as a corollary of Proposition 2.2, can just as easily be deduced as a corollary of Proposition 2.3.

3. Applications

In this section we work through a series of examples that show how our results may be applied.

Example 3.1. Let U be the upper half-disk $\{|z| < 1, \text{Im}(z) > 0\}$. The imaginary axis is an axis of symmetry, and U is Δ -convex with respect to this axis, so all p th centers lie on the imaginary axis. The line

$$\Delta =: \left\{ \text{Im}(z) = \frac{1}{2} \right\}$$

is clearly a partial symmetry axis with symmetric part

$$U \cap \left\{ \text{Im}(z) > \frac{1}{2} \right\},$$

and U is Δ -convex as well. Thus all p -centers belong to the set

$$\left\{ \text{Re}(z) = 0, 0 \leq \text{Im}(z) \leq \frac{1}{2} \right\}.$$

Now, if we let C be the circle $\{|z + i| = \sqrt{2}\}$, then C passes through 1 and -1 , making an angle of $\pi/4$ at each point with the real axis. If we let $\mathcal{A} = U \cap \mathcal{I}$ as before, with \mathcal{I} the inside of C , then $\sigma_C(\mathcal{A}) = \mathcal{B}$, where $\mathcal{B} = U \cap \mathcal{O}$ and \mathcal{O} is the outside of C . By Corollary 2.1, no p th center lies in \mathcal{A} . Thus all p th centers lie on the line segment

$$\left\{ \text{Re}(z) = 0, \sqrt{2} - 1 \leq \text{Im}(z) \leq \frac{1}{2} \right\},$$

which is in bold in Figure 2.

Example 3.2. Now let U be an isosceles triangle with vertices at $-1, 1$, and Ni with $N > 0$. It will be convenient for us to index U by the angles at 1 and -1 , so if we let θ be this angle then $N = \tan \theta$. Proposition 2.1 tells us that all p th centers lie on the imaginary axis. We have seen already from the example discussed in connection with Proposition 2.2 that all p th centers must lie below $(M/2)i$, but we will now show how this can be improved. Let B be the angle bisector of one of the base angles of U . B is a partial symmetry axis of U , with symmetric side given by the component of $U \setminus B$ corresponding to the shorter side of the triangle. Thus, if $\theta > \pi/3$, then all p th centers must lie above B , while if $\theta < \pi/3$, then all p th centers must lie below B . Now let M be the perpendicular bisector of the edge connecting 1 to Mi (this is often referred to as the *mediator*). This is also a partial symmetry axis of U , and the symmetric side is the component of $U \setminus M$ which does not contain -1 . Thus, if $\theta > \pi/3$, then all p th centers must lie below M , while if $\theta < \pi/3$, then all p th centers must lie above M . Thus the intersections of

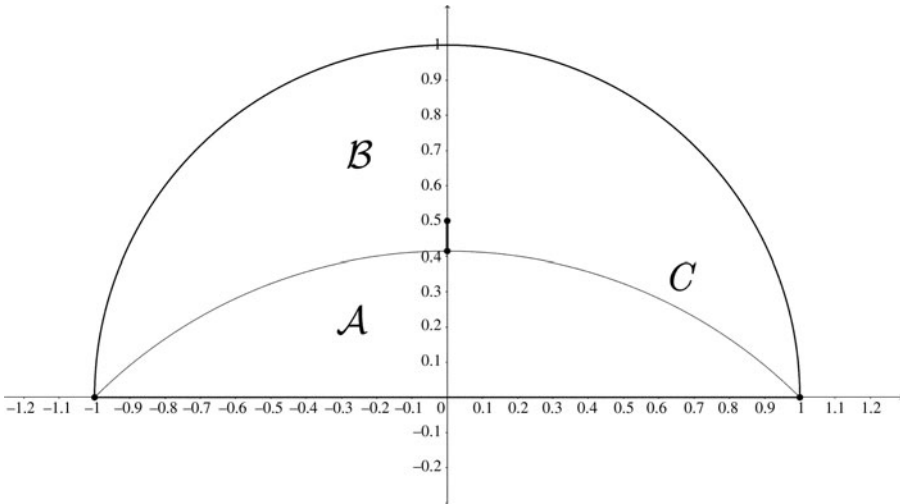


FIGURE 2: Reflection over a circle.

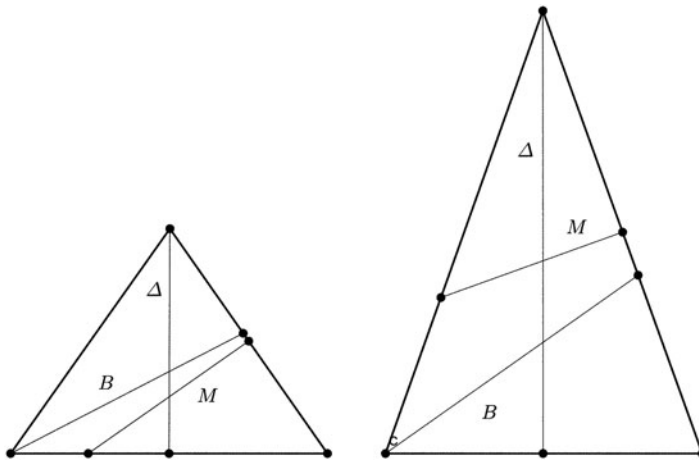


FIGURE 3: The angle bisector and mediator.

M and B with the imaginary axis provide upper and lower bounds for all p th centers, although which is the upper bound and which is the lower bound depends on θ . Figure 3 demonstrates this phenomenon (Δ denotes the imaginary axis).

Naturally they coincide at $\theta = \pi/3$. It can be checked that, regardless of θ , this gives a better upper bound than $N/2$, which was given by reflection over $\{\text{Im}(z) = N/2\}$. A bit of Euclidean geometry shows that the intersection of B with the imaginary axis is at the point $\tan(\theta/2)i$, and the intersection of M with the imaginary axis is at the point $(1/\tan \theta - 1/\sin 2\theta)i$. Furthermore, we always have as a lower bound the intersection of the imaginary axis and the circle passing through 1 and -1 , making an angle of $\theta/2$ with the real axis; this follows from Corollary 2.1

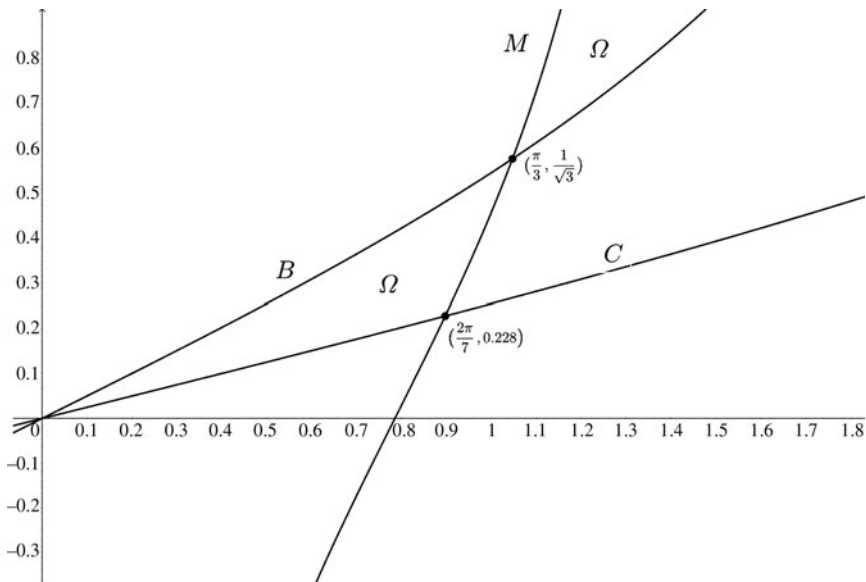


FIGURE 4: Upper and lower bounds for p th centers.

as above. This point is

$$\frac{1 - \cos(\theta/2)}{\sin(\theta/2)}.$$

Figure 4 shows these upper and lower bounds; all p th centers must lie in the regions labeled Ω .

Remark 3.1. We believe that a better lower bound can be achieved through numerical conformal mapping; more on this in the final section.

Example 3.3. Let $\mathcal{A}_{r,R}$ be the annulus $\{r < |z| < R\}$. Then all p -centers lie in $\{\sqrt{rR} < |z| < (R + r)/2\}$.

Proof. Consider $f(z) = rR/\bar{z}$, which maps $\mathcal{A}_{r,R}$ to itself. Under the same notation as in Proposition 2.3, we have

$$\Omega := \{\sqrt{rR} < |z| < R\},$$

$$\Delta := \{|z| = \sqrt{rR}\}$$

and we can check easily that f satisfies the requirements of Proposition 2.3). Therefore we can eliminate $U \setminus \Omega$ from consideration, and we obtain the lower bound \sqrt{rR} . In order to get the upper bound $(R + r)/2$ we can see, as illustrated by Figure 5, that the line Δ is a partial symmetry axis. The result follows. \square

Remark 3.2.

- Another way to get the same lower bound as above is to note that Proposition 2.2 extends to non-injective maps in suitable situations. We may use the map

$$f: \begin{matrix} \{\ln r < \operatorname{Re}(z) < \ln R\} & \longrightarrow & \mathcal{A}_{r,R} \\ z & \longmapsto & e^z \end{matrix}$$

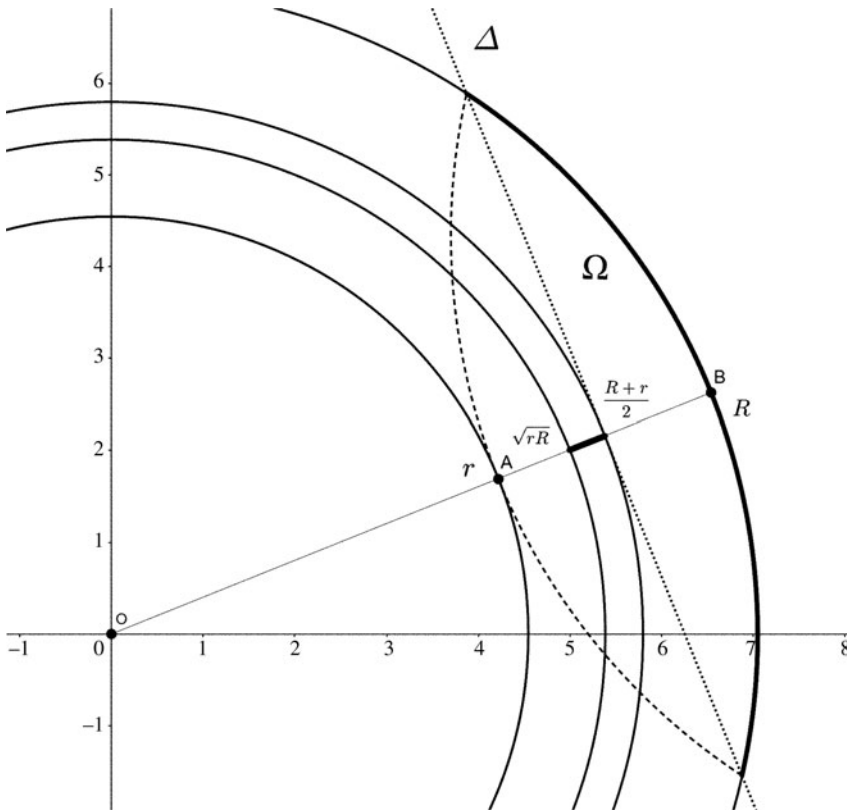


FIGURE 5: Bounds for the annulus.

and apply this extension of Proposition 2.2 with reflection axis $\{\text{Re}(z) = \ln R + \ln r/2\}$ in order to obtain the result.

- It should be mentioned that an explicit formula for the first moment can be obtained by Dynkin’s formula, and it is

$$\mathbb{E}_z(\tau_{\mathcal{A}_{r,R}}) = \frac{R^2 \ln(|z|/r) - r^2 \ln(|z|/R)}{\ln(R/r)} - |z|^2.$$

This can be shown to be maximal at

$$|z| = \sqrt{\frac{R^2 - r^2}{2 \ln(R/r)}}.$$

Our estimates are therefore not necessary for the first moments, but as an aside we obtain the inequality

$$\sqrt{Rr} < \sqrt{\frac{R^2 - r^2}{2 \ln(R/r)}} < \frac{R + r}{2}.$$

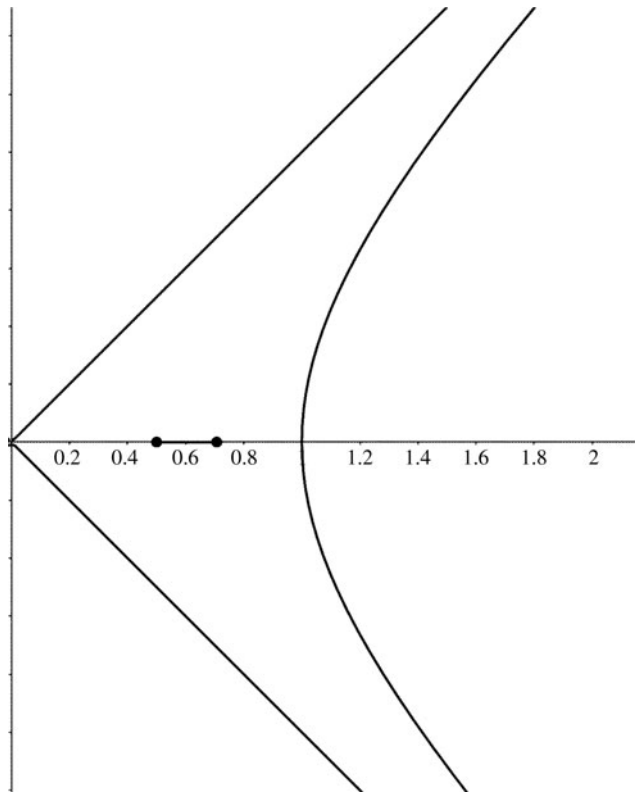


FIGURE 6: Bounds for the hyperbolic region.

Setting $a = R^2$, $b = r^2$, and squaring the inequalities gives the following:

$$\sqrt{ab} < \frac{a - b}{\ln(a/b)} < \frac{a + b + 2\sqrt{ab}}{4} \leq \frac{a + b}{2}.$$

The quantity

$$\frac{a - b}{\ln(a/b)}$$

is of great importance in the study of heat flow, and in that context it is known as the *logarithmic mean temperature difference*, or LMTD (see [10]). We have therefore given a new proof of the fundamental fact that the LMTD lies between the arithmetic and geometric means, and in fact have proved that the upper bound can be lowered to the arithmetic mean of the arithmetic and geometric means.

Example 3.4. Let \mathcal{H} be the region $\{|x| > |y|, x^2 - y^2 < 1\}$; this is the region bounded by the lines $y = \pm x$ and the hyperbola $x^2 - y^2 = 1$; see Figure 6.

It is perhaps not obvious for which p we have $\mathbb{E}_w[\tau_{\mathcal{H}}^p] < \infty$, but we can show that $\mathbb{E}_w[\tau_{\mathcal{H}}^p] < \infty$ for any $p > 0$ and $w \in \mathcal{H}$, as follows. \mathcal{H} is contained in the union of two infinite strips which are orthogonal. Any strip has all moments of its exit time finite: Brownian motion is rotation-invariant, so the moments are the same as for a horizontal strip, and these moments

in turn are the same as for a one-dimensional Brownian motion from a bounded interval, since that is what we obtain when we project the Brownian motion onto the imaginary axis; these moments are well known to be finite for all p , and in fact they can be calculated explicitly for integer p using the Hermite polynomials. We would like to conclude that the union of these two strips must then have finite p th moment, but easy examples show that it is not necessarily the case that the union of two domains with finite p th moment must itself have finite p th moment. A method does exist for reaching the desired conclusion, however, and it is contained in Theorem 3 and Lemmas 1 and 2 of [13]. It is straightforward to verify that our infinite strips satisfy the required conditions: their intersection is bounded, and boundary arcs intersect at non-zero angles. Therefore the exit time for their union has finite p th moments for all p , and thus so does \mathcal{H} . See [13] for details.

Now let us see how our methods can be used to localize the p th centers. The real axis is an axis of symmetry, but the domain is not Δ -convex, so we may not apply Proposition 2.1. However, all p -centers lie on the real axis, and we may prove this as follows. The map $f(z) = \sqrt{z}$ maps the strip $\{0 < \operatorname{Re}(z) < 1\}$ conformally onto \mathcal{H} . Any horizontal line can be used in Proposition 2.2, and we note that

$$|f'(z)| = \frac{1}{2\sqrt{|z|}}$$

is monotone decreasing in $|z|$ and therefore in $|\operatorname{Im}(z)|$. Thus all p th centers must lie on the image of the real axis under f , which is again the real axis. So we need only consider points on \mathbb{R} . The line $\{\operatorname{Re}(z) = 1/2\}$ is a partial symmetry axis, which gives a lower bound of $1/2$ for all p th centers. For an upper bound, note that $\{\operatorname{Re}(z) = 1/2\}$ is another axis of symmetry of $\{0 < \operatorname{Re}(z) < 1\}$, and the monotonicity of the derivative shows again via Proposition 2.2 that we only need to look in the region

$$f\left(\left\{0 < \operatorname{Re}(z) < \frac{1}{2}\right\}\right),$$

which is the region $\{x^2 - y^2 < 1/2\}$ inside U . This gives an upper bound of $1/\sqrt{2}$ on the real axis. Thus all p -centers lie on

$$\left\{\operatorname{Im}(z) = 0, \frac{1}{2} < \operatorname{Re}(z) < \frac{1}{\sqrt{2}}\right\}.$$

This set is in bold in Figure 6.

Example 3.5. Let \mathcal{C}_R be the crescent-like shape limited by the two circles

$$\left\{\left|z - \frac{1}{2}\right| = \frac{1}{2}\right\} \quad \text{and} \quad \left\{\left|z - \frac{R}{2}\right| = \frac{R}{2}\right\}$$

(see Figure 7). \mathcal{C}_R is the image of the region

$$\left\{\frac{1}{R} < \operatorname{Re}(z) < 1\right\}$$

under the conformal map $f(z) = 1/z$. Note that $|f'(z)| = 1/|z|^2$ is monotone decreasing in $|z|$, so by the same argument as in Example 3.4, all p th centers lie on the real axis. Furthermore,

$$\left\{\operatorname{Re}(z) = \frac{R+1}{2}\right\}$$

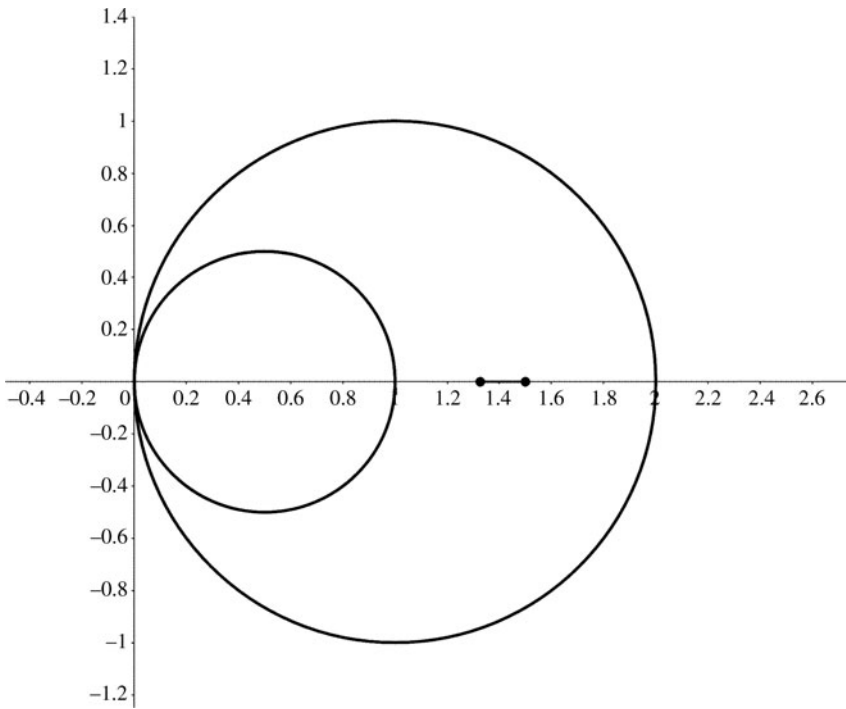


FIGURE 7: Bounds for the crescent region.

is a partial symmetry axis for \mathcal{C}_R , and this allows us to eliminate the region

$$\left\{ \operatorname{Re}(z) > \frac{R+1}{2} \right\}$$

from consideration. We may also use an axis of symmetry

$$\left\{ \operatorname{Re}(z) = \frac{1+1/R}{2} \right\}$$

to conclude via Proposition 2.2 that we can exclude the region

$$f\left(\left\{ \frac{1}{R} < \operatorname{Re}(z) < \frac{1+1/R}{2} \right\}\right),$$

which is the region

$$\mathcal{C}_R \cap \left\{ \left| z - \frac{R}{R+1} \right| < \frac{R}{R+1} \right\},$$

in the search for p th centers. We see that all p -centers lie on the interval

$$\left\{ \operatorname{Im}(z) = 0, \frac{2R}{R+1} < \operatorname{Re}(z) < \frac{R+1}{2} \right\},$$

which is in bold for $R = 2$ in Figure 7.

4. Concluding remarks

As remarked earlier, in Figure 1 the regions \mathcal{A} and \mathcal{B} do not fill all of U , and it is natural to search for a better bound by finding a conformal map that fills the entire domain. Let us consider the Schwarz–Christoffel transformation sending the unit disk to $U(\theta)$ given by (see [8, Chapter 2])

$$f(z) = A + C \int_0^z (1-w)^{\theta/\pi-1} (1+w)^{\theta/\pi-1} (1+iw)^{-2\theta/\pi} dw$$

for appropriate choices of constants A and C ; note that this is chosen so that 1 and -1 are mapped to the base angles, and i is mapped to the top angle. We have

$$|f'(z)| = |C| |1-z|^{\theta/\pi-1} |1+z|^{\theta/\pi-1} |1+iz|^{-2\theta/\pi}.$$

It can then be checked that $|f'(z)| > |f'(\bar{z})|$ whenever $\operatorname{Re}(z) > 0$, so Proposition 2.2 implies that no p th centers can be found in the image of $\mathbb{D} \cap \{\operatorname{Re}(z) < 0\}$. From this point a numerical method can be employed, and the resulting bound should improve the one we found, if desired.

As was mentioned in connection with Δ -convexity, there are some purely geometrical consequences of our results. In that context, the following may be proved.

Proposition 4.1. *Suppose a domain U is Δ -convex with respect to two parallel symmetry axes. Then we can find $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$ so that U is a rotation of the domain $\{a < \operatorname{Re}(z) < b\}$; in other words, U is all of \mathbb{C} , is a half-plane, or is an infinite strip.*

As a corollary of this, and of our probabilistic results above, we obtain the following.

Corollary 4.1. *Suppose U is a domain which is not all of \mathbb{C} , a half-plane, or an infinite strip. Then if there are multiple axes of symmetry to which U is Δ -convex, then they all meet at a unique point.*

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