Economics and Philosophy, **17** (2001) 235–243

THE NORMAL FORM IS NOT SUFFICIENT

Antonio Quesada
Universidad de Murcia

1. INTRODUCTION

The relationship between extensive and normal form analyses in non-cooperative game theory seems to be dominated, at least traditionally, by the so-called 'sufficiency of the normal form principle', according to which all that is necessary to analyse and 'solve' an extensive game is already in its normal form representation. The traditional defence of the sufficiency principle, that Myerson (1991, p. 50) attributes to von Neumann and Morgenstern, holds that, with respect to extensive games, it can be assumed without loss of generality, that players formulate simultaneously and independently their strategic plans at the beginning of the game – a situation which, it is claimed, is exactly described by the normal representation of an extensive game.

Accordingly, as Kohlberg and Mertens (1986, p. 1012) point out, in order to prove the insufficiency of the normal form one would need two extensive games with the same normal form whose 'reasonable equilibria' are completely different. Abreu and Pearce (1984, pp. 172–73) devised the required example by postulating that, if G' comes from extensive game G by replacing a subgame of G with its solution, the solution of G' must be that of G restricted to G'. The weakness of the example is that their requirement is strong: Example 7.7 in Peleg and Tijs (1996, p. 32) proves that subgame perfectness does not satisfy it.

This note exhibits another example of two extensive games with the same normal form representation whose 'reasonable equilibria' (according to a certain principle of 'cautious' behaviour) are *completely* different. The example has nonetheless two shortcomings: the 'reasonable

My thanks to the referees, whose comments and suggestions have substantially improved this article. Imperfections and remaining errors are exclusively mine.

https://doi.org/10.1017/S0266267101000256 Published online by Cambridge University Press

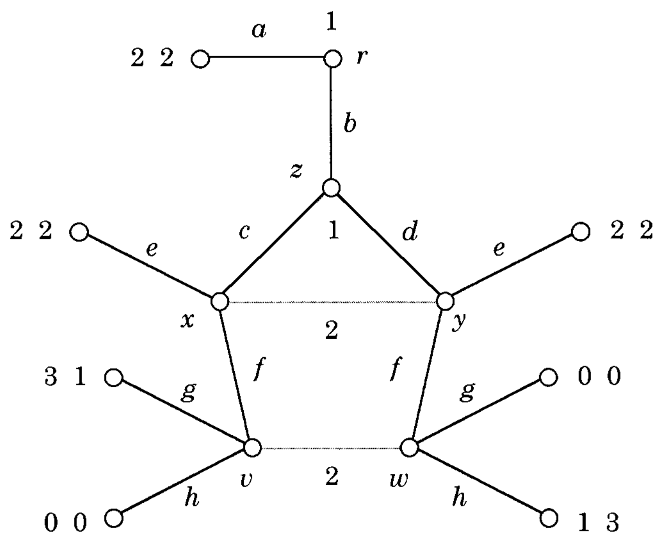

FIGURE 1

equilibria' are not robust to small payoff perturbations and the payoffs associated with these equilibria are the same in both games.

2. OBJECTION TO THE SUFFICIENCY PRINCIPLE: FORMAL VERSION

Consider extensive games G_1 of Figure 1 and G_2 of Figure 2, both having the same normal form (and the same reduced normal form) representation. With $G \in \{G_1, G_2\}$, i designates an arbitrary player of G, j his opponent, k the player assigned to the root, n the player not assigned to the root and H_i player i's set of information sets. A behaviour strategy (or, simply, 'strategy') for i is an assignment to each $h \in H_i$ of a probability distribution over the actions at h. Denoting by Σ_i player i's strategy set and identifying $\sigma_i \in \Sigma_i$ such that $\sigma_i(s) = \sigma_i(s') = 1$ with pure strategy (s, s'), if $\sigma_i \in \Sigma_i, h \in H_i$ and s is an action available at h, σ_i^h is the probability distribution that σ_i assigns to h and $\sigma_i(s)$ the probability that σ_i^h attributes to s.

DEFINITION 2.1. Strategy $\sigma_k \in \Sigma_k$ is a best reply (b.r.) to $\sigma_n \in \Sigma_n$ if, after assigning to each action s of n probability $\sigma_n(s)$, σ_k yields k maximum expected payoff. Strategy $\sigma_n \in \Sigma_n$ is a best reply to $\sigma_k \in \Sigma_k$ if, after assigning probability 1 to the action leading from k's first to k's second information set h and probability $\sigma_k(s)$ to each action s at h: (i) σ_n yields n maximum expected payoff; and (ii) after assigning probability 1 to the action leading from n's first to his second information set h^*, $\sigma_n^{h^*}$ yields n maximum payoff.

https://doi.org/10.1017/S0266267101000256 Published online by Cambridge University Press

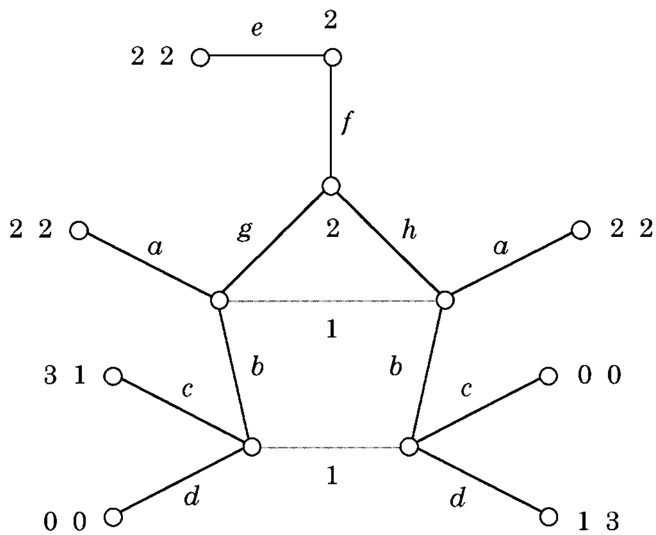

FIGURE 2

For k, σ_k is a b.r. to σ_n if it maximizes k's payoff given σ_n. For n, σ_n is a b.r. to σ_k if it is a sequentially rational best response to σ_k at n's two information sets, in the sense that: (i) σ_n maximizes n's payoff assuming that n's first information set is reached and given what σ_k prescribes for k's second information set h; and (ii) for n's second information set h^*, $\sigma_n^{h^*}$ maximizes n's payoff assuming that h^* is reached and given σ_k^h.

DEFINITION 2.2. With $H_j = \{h, h^*\}, \sigma_i \in \Sigma_i$ is a cautious best reply (c.b.r.) to $\sigma_j \in \Sigma_j$ if, for every $\lambda \in (0,1)$ there are $\varepsilon \in (0,\lambda)$ and $\delta \in (0,\lambda)$ such that, for all $\beta_j \in \Sigma_j, \sigma_i$ is a b.r. to $\tau_j \in \Sigma_j$ with $\tau_j^h = (1-\varepsilon)\sigma_j^h + \varepsilon\beta_j^h$ and $\tau_j^{h^*} = (1-\delta)\sigma_j^{h^*} + \delta\beta_j^{h^*}$.

That σ_i is c.b.r. to σ_j means that it is b.r. to σ_j and to every τ_j which, being 'sufficiently close' to σ_j, attributes positive probability to every choice in each of j's information sets; see Section 3 for a justification of this concept. Though one of the reviewers notes that the definition of c.b.r. is very strong, the fact is that the assumptions adopted only operate when some c.b.r. exists.

LEMMA 2.3. In G_1, if $\sigma_1 \in \Sigma_1$ is c.b.r. to $\sigma_2 \in \Sigma_2$ and $\sigma_1(b) > 0$ then $\sigma_1 = (b, c)$.

PROOF. Let σ_1 be c.b.r. to σ_2 in G_1, with $\sigma_1(b) > 0$. To prove that $\sigma_1(c) = 1$, suppose $\sigma_1(d) > 0$. Since taking a yields payoff 2 and $\sigma_1(b) > 0$, for σ_1 to be b.r. to σ_2, it must be that $\sigma_2(e) = 1$. Yet, σ_1 with

https://doi.org/10.1017/S0266267101000256 Published online by Cambridge University Press

$\sigma_1(b) > 0 < \sigma_1(d)$ is not c.b.r. to σ_2, because (b, d) is weakly dominated by a. To prove that $\sigma_1(b) = 1$, suppose $\sigma_1(a) > 0$. Then, as σ_1 is c.b.r. to σ_2, there are small enough $\varepsilon > 0 < \delta$ such that, for *all* $\beta_2 \in \Sigma_2, \sigma_1$ is b.r. to τ_2 with $\tau_2(e) = (1-\varepsilon)\sigma_2(e) + \varepsilon\beta_2(e)$ and $\tau_2(g) = (1-\delta)\sigma_2(g) + \delta\beta_2(g)$. But, owing to $0 < \sigma_1(b) < 1$, a must yield player 1 the same expected payoff as (b, c). That is, for any such $\tau_2, 2 = 2\tau_2(e) + 3(1-\tau_2(e))\tau_2(g)$. Fix $\beta_2(e) < 1$, so $2/3 = \tau_2(g) = (1-\delta)\sigma_2(g) + \delta\beta_2(g)$ and $\beta_2(g)$ must be constant given $\beta_2(e)$. Thus, not for all $\beta_2 \in \Sigma_2$ is σ_1 b.r. to τ_2 with $\tau_2(e) = (1-\varepsilon)\sigma_2(e) + \varepsilon\beta_2(e)$ and $\tau_2(g) = (1-\delta)\sigma_2(g) + \delta\beta_2(g)$.■

LEMMA 2.4. In G_1, if $\sigma_2 \in \Sigma_2$ is c.b.r. to (b, c) then $\sigma_2 = (e, g)$.

PROOF. Showing that (e, g) is the only c.b.r. to (b, c) amounts to showing that (e, g) is the only c.b.r. to taking c with probability 1 in the subgame G^* of G_1 with root z. At 2's first information set in G^*, e is the only best reply to c and, if 2's second information set is reached, g is the only best reply to c. Given that e and g are still best replies in G^* when d is chosen with sufficiently small probability, (e, g) is a c.b.r. to (b, c).■

LEMMA 2.5. In G_1, if $\sigma_1 \in \Sigma_1$ is c.b.r. to (e, g) then $\sigma_1 = (b, c)$.

PROOF. Each of player 1's strategies is b.r. to (e, g). To see that only (b, c) is c.b.r. to (e, g), observe that being c.b.r. to (e, g) is tantamount to being b.r., for every sufficiently small but positive ε and δ, to each τ_2 such that $\tau_2(e) = 1-\varepsilon$ and $\tau_2(g) = 1-\delta$. The only strategy of player 1 satisfying this requirement is (b, c).■

A pair $(C_i, P_i) \subseteq (\Sigma_j, \Sigma_i)$ is associated with each player i of G. While C_i consists of the conjectures that i may 'reasonably' entertain concerning j's strategy choice, P_i consists of those strategies that i may 'reasonably' choose to play. Define $B_i(C_i) := \{\sigma_i \in \Sigma_i : \sigma_i$ is c.b.r. to some member of $C_i\}$ and set $P := P_k \times P_n$.

A1. For $i \in \{k, n\}$, if $B_i(C_i) \neq \varnothing$ then $\varnothing \neq P_i \subseteq B_i(C_i)$.

By A1, as long as the set $B_i(C_i)$ of i's strategies that are cautious best replies to some conjecture in C_i is non-empty, player i can only 'reasonably' choose to play members of $B_i(C_i)$. In other words, players are assumed to play c.b.r. to conjectures (if some such c.b.r. exists; in this respect, A1 is, like A2 and A3, a 'cautious' assumption).

A2. If $P_n \neq \varnothing$ then $\varnothing \neq C_k \subseteq P_n$.

By A2, the first player's 'reasonable' conjectures come from the second player's 'reasonable' strategy choices, provided some such choice exists.

Combined with A1, A2 in a sense expresses the fact that k believes that n plays some c.b.r. to conjectures.

A3. If s is the action at the root preceding n's information sets and $R_k := \{\sigma_k \in \Sigma_k : \sigma_k(s) > 0$ and σ_k is a c.b.r to some $\sigma_n \in \Sigma_n\} \neq \emptyset$ then $\emptyset \neq C_n \subseteq R_k$.

By A3, the second player's 'reasonable' conjectures come from the first player's 'reasonable' strategy choices compatible with the reaching of the second player's first information set, if some such choice exists. In a way, A3 expresses the fact that the second player believes (insofar as it can be consistently held) that the first player chooses to play some c.b.r.; this is probably the crucial premise to prove that, under A1–A3, the 'reasonable' ways of playing G_1 and G_2 are completely different.

PROPOSITION 2.6. If A1–A3 hold then, in game G_1 of Figure 1, $P = \{(b, c, e, g)\}$

PROOF. By Lemma 2.3, $\{\sigma_1 \in \Sigma_1 : \sigma_1(b) > 0$ and σ_1 is a c.b.r to some $\sigma_2 \in \Sigma_2\} = \{(b, c)\}$. Hence, by A3, $C_2 = \{(b, c)\}$. By Lemma 2.4, $B_2(C_2) = \{(e, g)\}$. Therefore, by A1, $P_2 = \{(e, g)\}$. Given this, by A2, $C_1 = \{(e, g)\}$. By Lemma 2.5, $B_1(C_1) = \{(b, c)\}$ and, by A1, $P_1 = \{(b, c)\}$.■

PROPOSITION 2.7. If A1–A3 hold then, in game G_2 of Figure 2, $P = \{(f, h, a, d)\}$.

PROOF. Rewrite the proof of Lemmas 2.3–2.5 and Proposition 2.6 by renaming player 1, player 2 and actions a, b, c, d, e and g as, respectively, 2, 1, e, f, h, g, a and d.■

In consequence, if A1–A3 determine the 'reasonable' way of playing G_1 and G_2, the 'reasonable' way of playing G_1 is completely different from that of playing G_2, in spite of the fact that G_1 and G_2 have the same normal form representation.

3. OBJECTION TO THE SUFFICIENCY PRINCIPLE: INFORMAL VERSION

A way of interpreting and justifying the results and assumptions in Section 2 starts with a set Ω of 'states of the world', whose members are descriptions of everything one may consider 'relevant' in the playing of G_1 and G_2 that is not already in the description of these games (for instance, whether players usually read *Economics and Philosophy* or not). It is supposed that, in every $\omega \in \Omega$, each player i forms (under the hypothesis that he is given the move) exactly *one* conjecture $c_i(\omega) \in \Sigma_j$.

This assumption is justified on the grounds that the description that ω represents is so 'rich' that i always finds a reason to regard some conjecture more or less 'justifiable' than some other (if, in the last instance, i chooses among conjectures by randomizing, the outcome of the randomization could itself be incorporated into ω). Sets C_i in Section 2 can be then defined as $C_i := \cup_{\omega \in \Omega} c_i(\omega)$. In a similar vein, each player i is assumed to choose to play in ω a unique strategy $\sigma_i(\omega) \in \Sigma_i$, so that sets P_i in Section 2 would correspond to $\cup_{\omega \in \Omega} \sigma_i(\omega)$.

The requirement that a player forms a conjecture concerning the opponent's possible way of playing stems from the idea that the way each player chooses what to play is the result of a reasoning process that produces an estimate of the opponent's strategy choice. At a more general level, the conjecture serves the purpose of justifying (providing a reason for) the way the player chooses to play. In addition, that conjecture can be supposed to be equally valid under the conditional assumptions that the first or the second information sets are reached. In fact, there is no reason why the first player of G_1 should not entertain the same conjecture seeing himself at the root or at his second information set, because the only change in passing from one information set to the other is that he takes action b. The same applies to the second player: every conjecture held by player 2 assuming the reaching of $\{x, y\}$ seems to be also valid assuming the reaching of $\{v, w\}$, since the only difference in the situation he faces is having taken f.

A1–A3 partially define a way of deciding what to conjecture and what to choose to play in each ω. By A1, $\sigma_i(\omega)$ must be a c.b.r. to $c_i(\omega)$, provided some such reply exists. Why demand being c.b.r. and not merely b.r.? The notion of c.b.r. is motivated by the presumption that a player is never fully confident about his conjecture: though i forms a precise conjecture concerning j's strategy choice, i is supposed to be aware of the possibility of being wrong, for which reason it would be 'safer' for him to select a strategy which is still a b.r. in the event that the conjecture proved wrong but 'not by much'.

The trembling-hand equilibrium refinement literature emerging from Selten (1975) is, in a way, based on cautious best replies. The standard interpretation in that literature is that i tries to protect himself from j's (potential) 'trembles', namely, mistakes in implementing what j is expected to implement. In contrast, the intended interpretation here is that i tries to protect himself from his own (potential) incompetence in the analysis of the game situation. For how can i, in trying to predict j's choice, be sure that he has taken into account every piece of relevant information, has excluded every piece of irrelevant information and, moreover, has used that information competently and without slips? It is therefore against doubts concerning the correctness of (or possible oversights in) his analysis of the game situation that i tries to protect himself by resorting to cautious best replies: by playing 'cautiously', *i* is covering himself against the risk of having conjectured wrongly. It is nonetheless to be noted that 'cautiousness' is not an unproblematic notion; see Cubitt and Sugden (1994) and Squires (1998).

In this respect, λ in Definition 2.2 is interpreted as a possible upper bound probability that *i* could ascribe to the event that he conjectures wrongly. Given λ, ε is then the maximum probability that *i* ascribes to the event that his conjecture σ_j^h relative to what *j* chooses to play in his first information set *h* is incorrect, while δ is the maximum probability that *i* ascribes to the event that his conjecture $\sigma_j^{h^*}$ relative to what *j* chooses to play in his second information set *h** is incorrect. In view of this, for any given λ, *i* must conclude that, in the extreme case, *j* may end choosing to play $\tau_j \in \Sigma_j$ with $\tau_j^h = (1-\varepsilon)\sigma_j^h + \varepsilon\beta_j^h$, where β_j is any member of Σ_j. Consequently, it seems more 'cautious' to choose a strategy that is not only a b.r. to σ_j^h but also to every $\tau_j^h = (1-\varepsilon)\sigma_j^h + \varepsilon\beta_j^h$, with $\beta_j \in \Sigma_j$. Similarly, since *i* must conclude that, in principle, anything could be chosen to be played at *h** with at most probability δ, it appears more 'cautious' to choose a strategy that is not only a b.r. to $\sigma_j^{h^*}$ but also to each $\tau_j^{h^*} = (1-\delta)\sigma_j^{h^*} + \delta\beta_j^{h^*}$.

The insufficiency result given by Propositions 2.6–2.7 is an immediate consequence of Lemmas 2.3–2.5. By Lemma 2.3, (*b*, *c*) is the only strategy in G_1 that reaches player 2's first information set and is a c.b.r. to some conjecture that player 1 may entertain. In consequence, all that is needed for player 2 to conjecture that 1 chooses to play (*b*, *c*) is to postulate that, in every ω, insofar as possible, 2's conjecture must be consistent with the reaching of his first information set and with the fact that 1 only plays a c.b.r. to his conjecture. This is what A3 postulates: roughly speaking, that 2 believes that 1 chooses to play a c.b.r. that gives 2 the move, as long as this can be consistently held (that is, as long as $\{\sigma_k \in \Sigma_k : \sigma_k(s) > 0$ and σ_k is a c.b.r to some $\sigma_n \in \Sigma_n\} \neq \varnothing$). This proviso is really the crux of the matter, because it establishes a priority in the definition of conjectures: a player should, first of all, try to 'rationalize' observed choices and, if they cannot be rationalized, resort in the last instance to the 'my opponent has made a mistake' explanation. Observe that a similar presumption operates in forward induction reasoning; see van Damme (1989).

Hence, by A3, since player 2 conjectures assuming that he is given the move (the premise behind Kreps and Wilson's (1982) sequential rationality), he must firstly try to make sense of player 1's presumed choice on the basis that it is consistent with playing a c.b.r. that reaches 2's first information set. And the virtue of the c.b.r. concept is that, thanks to Lemma 2.3, it singles out (*b*, *c*) as the unique strategy choice consistent with playing a c.b.r. and with the reaching of 2's first information set. By Lemma 2.4, (*e*, *g*) is the only c.b.r. to (*b*, *c*) so that, by

A1 (some c.b.r. to conjectures is played, whenever possible), 2 chooses to play (e, g).

It merely remains to add assumptions guaranteeing that, in every ω, player 1 can replicate the preceding conclusion. This is what A2 does: player 1 can only adopt as conjecture something that 2 can choose to play. As a result, by A2, player 1 conjectures that 2 chooses (e, g) to play and, by Lemma 2.5, the only c.b.r. to (e, g) is, precisely, (b, c). Thus, 2's conjecture is coherent with 1's choice and A1–A3, being consistent in G_1, confirm (b, c, e, g) as *the* 'reasonable' equilibrium.

The analysis of G_2 under A1–A3 proceeds along similar lines, for G_2 is obtained from G_1 by applying two re-labellings: for players, 1 becomes 2 and 2 becomes 1; for actions, a, b, c, d, e, f, g and h become, respectively, e, f, g, h, a, b, c and d. It is not hard to see that, if A1–A3 lead to (b, c, e, g) in G_1, they yield (f, h, a, d) in G_2. What causes the difference is that, in G_1, it is player 2 who, when given the move, is 'forced' to 'rationalize' 1's choice. As the rationalization relying on the notion of c.b.r. selects strategy (b, c), by giving the move to player 2, it is as if player 1 said 'I have played (b, c)' to player 2. And, given that the notion of c.b.r. makes (e, g) *the* right reply to (b, c) and vice versa, (b, c, e, g) emerges as *the* solution. The principle of cautiousness is therefore crucial in making the 'message' unambiguous. Contrariwise, in G_2 it is player 2 who unambiguously sends (again, thanks to the requirement that one must try to rationalize choices as cautious best replies) the message to 1 by giving him the move; now, the message says 'I have played (f, h)'. It is precisely the possibility of interpreting unambiguously 2's observed move (e was rejected) that forces 1 to give up what is his choice in G_1, namely, (b, c): in G_2, (b, c) is no longer a c.b.r. to the conjecture (f, h) that A3 (and the fact that 1 is this time the second player) compels player 1 to adopt.

In sum, it is the order of play and the possibility of exploiting it to 'send' unambiguous 'messages' to subsequent players (through, for instance, forward induction reasoning) that makes the extensive form matter in the analysis of G_1 and G_2 under A1–A3 (Amershi, Sadanand and Sadanand (1992) have combined forward induction with the order of play to produce a strong equilibrium refinement). This suggests that transformations of the extensive form that alter the order of play (when it is clearly defined, as in G_1 and G_2) may not be strategically inessential. So, granted that it can be assumed without loss of generality that players formulate simultaneously and independently their strategic plans at the beginning of the game, the point is, with what information do they formulate their plans. This paper claims that the order of play is information that *could* be decisive in deciding what to choose to play. And since the order of play is irretrievable from the normal form representation, it follows that this representation cannot always describe extensive games exactly.

REFERENCES

Abreu Dilip and David G. Pearce. 1984. 'On the inconsistency of certain axioms on solution concepts for non-cooperative games'. *Journal of Economic Theory*, 34:169–74

Amershi, Amin H., Asha Sadanand and Venkatraman Sadanand. 1992. 'Player importance and forward induction'. *Economics Letters*, 38:291–97

Cubitt, Robin and Robert Sugden. 1994. 'Rationally justifiable play and the theory of non-cooperative games'. *Economic Journal*, 104:798–803

Kohlberg, Elon and Jean-François Mertens. 1986. 'On the strategic stability of equilibria'. *Econometrica*, 54:1003–37

Kreps, David and Robert Wilson. 1982. 'Sequential equilibria'. *Econometrica*, 50:863–94

Myerson, Roger B. 1991. *Game Theory. Analysis of Conflict*. Harvard University Press

Peleg, Bezalel and Stef Tijs. 1996. 'The consistency principle for games in strategic form'. *International Journal of Game Theory*, 25:13–34

Selten, Reinhard. 1975. 'Reexamination of the perfectness concept for equilibrium points in extensive games'. *International Journal of Game Theory*, 4:25–55

Squires, David. 1998. 'Impossibility theorems for normal form games'. *Theory and Decision*, 44:67–81

van Damme, Eric. 1989. 'Stable equilibria and forward induction'. *Journal of Economic Theory*, 48:476–96