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A Note on Chirally Cosmetic Surgery on Cable Knots

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Abstract. We show that a (p,q)-cable of a non-trivial knot K does not admit chirally cosmetic surgeries for $q \neq 2$, or $q = 2$ with additional assumptions. In particular, we show that a (p,q)-cable of a non-trivial knot K does not admit chirally cosmetic surgeries as long as the set of JSJ pieces of the knot exterior does not contain the $(2,r)$-torus exterior for any r. We also show that an iterated torus knot other than the $(2,p)$-torus knot does not admit chirally cosmetic surgery.

1 Introduction

Let $S^3_K(m/n)$ be the Dehn surgery along a knot K in S^3 of slope m/n. Two Dehn surgeries $S^3_K(m/n)$ and $S^3_K(m'/n')$ along different slopes are *purely cosmetic* (resp. *chirally cosmetic*) if $S^3_K(m/n) \cong S^3_K(m'/n')$ (resp. $S^3_K(m/n) \cong -S^3_K(m'/n')$). Here for an oriented 3-manifold M, we denote by $-M$ the same 3-manifold with opposite orientation, and $M \cong N$ means that they are orientation preservingly homeomorphic.

It is expected that a non-trivial knot K in S^3 does not have purely cosmetic surgeries (cosmetic surgery conjecture [Ki, Problem 1.81 (A)]), whereas there are two families of chirally cosmetic surgeries on non-trivial knots;

(a) For an amphicheiral knot K, $S^3_K(m/n) \cong -S^3(-m/n)$.

(b) For a $(2,r)$-torus knot K, we have $S^3_K(\frac{2r^2(2m+1)}{r(2m+1)+1}) \cong -S^3_K(\frac{2r^2(2m+1)}{r(2m+1)-1})$ for any $m \in \mathbb{Z}$ ([Ro], see also Appendix of [IIS]).

Since no other examples of chirally cosmetic surgery of knots in S^3 are currently known, one encounters a natural question.

Question 1 *Is chirally cosmetic surgery of knots in S^3 either (a) or (b) ?*

At first glance, this may sound too optimistic, since there are several unexpected phenomenon or clever constructions that negate naive conjectures on Dehn surgeries. Moreover, when we extend our attention to knots in general 3-manifolds M, there are more examples of chirally cosmetic surgeries that are not generalizations of the above examples (a) and (b) [BHW, IJ].

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Nevertheless, we recently observed some results supporting the affirmative answer to this question [It, IIS]. In this note, we show the non-existence of chirally cosmetic surgery for cable knots under some technical assumptions.

Let $E(K) = S^3 \setminus N(K)$ be the knot exterior, where $N(K)$ denotes an open tubular neighborhood of K. There is a family of essential tori $\mathcal{T} = \{T_1, \dots, T_n\}$ (possibly empty) of $E(K)$ such that each component of $E(K) \setminus \mathcal{T} := E(K) \setminus (\bigcup_i T_i)$ is geometric (*i.e.,* either hyperbolic or Seifert fibered). Such a family of tori $\mathcal{T}$, called the *JSJ tori*, is unique up to isotopy when we take a minumum one. We call a connected component X of $E(K) \setminus \mathcal{T}$ a *JSJ piece* of $E(K)$.

Theorem 1.1 *Let $K_{p,q}$ be the (p,q)-cable of a non-trivial knot K. Assume that one of the following conditions is satisfied.*

(i) $q \neq 2$.
(ii) $q = 2$, $p \neq \pm 1$, *and the set of JSJ pieces of $E(K)$ does not contain the $(-p,2)$-torus knot exterior.*
(iii) $q = 2$, $p = \pm 1$, *and the set of JSJ pieces of $E(K)$ does not contain the $(r,2)$-torus knot exterior for any r.*
(iv) $q = 2$, $p = \pm 1$, *and $a_2(K) \neq 0$.*

Then $K_{p,q}$ does not admit chirally cosmetic surgeries.

Here, $a_2(K)$ is the coefficient of z^2 for the Conway polynomial $\nabla_K(z)$ of K. We remark that in our notation, the (p,q) cable $K_{p,q}$ of K is defined so that it has wrapping number q; $K_{p,q}$ intersects with $\{pt\} \times D^2 \subset S^1 \times D^2 \cong N(K)$ at q points.

We mention that the non-existence of purely cosmetic surgery of cable knots are shown in [Ta]. Although there are many similarities we do not use this result. Indeed, a mild modification of the proof of Theorem 1.1 proves a non-existence of purely cosmetic surgery on cable knots.

In light of example (b) of chirally cosmetic surgery and Theorem 1.1, one can think that an iterated torus knot is a possible candidate for new chirally cosmetic surgeries. However, we show that iterated torus knots does not admit chirally cosmetic surgery.

Theorem 1.2 *An iterated torus knot that is not a $(2,p)$-torus knot does not admit chirally cosmetic surgeries.*

2 Dehn Surgery of Cable Knots

For a torus boundary component T of a 3-manifold X, a *slope* γ (on T) is an isotopy class of a non-trivial unoriented simple closed curve on T. We take an ordered basis (α, β) of $H_1(T;\mathbb{Z})$ to identify the set of slopes with $\mathbb{Q} \cup \{\infty = \frac{1}{0}\}$. We view γ as an oriented simple closed curve by taking one of its orientations. Then $[\gamma] = p\alpha + q\beta \in H_1(T;\mathbb{Z})$ for coprime integers p and q. We assign the slope γ a rational number $p/q \in \mathbb{Q} \cup \{\infty = \frac{1}{0}\}$ (note that p and q depend on a choice of orientation, whereas p/q does not).

In the case of a knot complement $E(K)$, we take the standard meridian-longitude pair $([\mu],[\lambda])$ as an ordered basis of $H_1(\partial E(K);\mathbb{Z})$. The m/n-surgery on K is the

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3-manifold $S^3_K(m/n)$ obtained from $E(K)$ by attaching the solid torus $S^1 \times D^2$ along $\partial E(K)$ so that the slope m/n bounds a disk in the attached solid torus.

The (p,q) torus knot $T_{p,q}$ is a slope $\frac{p}{q}$ curve on a boundary of the standardly embedded solid torus $S^1 \times D^2$ in S^3, with respect to the basis $([\{*\} \times \partial D^2], [S^1 \times \{*\}])$ of $H_1(\partial S^1 \times D^2; \mathbb{Z})$. Thus, in our convention, the (p,q)-torus knot $T_{p,q}$ is the closure of the q-braid $(\sigma_1 \cdots \sigma_{q-1})^p$. In the sequel, we will often view $T_{p,q}$ as a knot in the solid torus $S^1 \times D^2$.

The (p,q)-cable $K_{p,q}$ of the knot K is the image $f(T_{p,q})$ of the standard torus knot $T_{p,q} \subset S^1 \times D^2$, where $f : S^1 \times D^2 \to \overline{N(K)}$ is a homeomophism such that $f(S^1 \times \{*\}) = \lambda$, $f(\{*\} \times \partial D^2) = \mu$, and $\overline{N(K)}$ denotes the closure of $N(K)$. Since $K_{p,q} = K$ if $q = 1$, in the sequel, we always assume that $q > 1$.

By [Go], the Dehn surgery along a cable knot is described as follows:

$$S^3_{K_{p,q}}(m/n) = \begin{cases} S^3_K(p/q) \# L(q,p) & |npq - m| = 0, \\ S^3_K(m/nq^2) & |npq - m| = 1, \\ E(K) \cup_T P_{p,q,m,n} & |npq - m| > 1. \end{cases}$$

In the last case $P_{p,q,m,n}$ is a Seifert fibered space with base surface D^2 having two singular fibers, glued along the boundary $T := \partial E(K)$ of $E(K)$. Moreover, $P_{p,q,m,n}$ is a JSJ piece of $S^3_{K_{p,q}}(m/n)$.

In the following, we prove Theorem 1.1 by dividing arguments into the following four cases, according to $|npq - m|$ and $|n'pq - m|$.

Case 1: $|npq - m| = 0$ (Lemma 2.3)
Case 2: $|npq - m| = |n'pq - m| = 1$ (Lemma 2.4).
Case 3: $|npq - m| = 1$, $|n'pq - m| > 1$ (Lemma 2.5).
Case 4: $|npq - m|, |n'pq - m| > 1$ (Lemma 2.6).

In Case 4, we use additional assumptions (i)–(iv).

Before starting discussions, we review some known results on chirally cosmetic surgery that will be used in the argument.

A knot K is an *L-space knot* if a Dehn surgery on K yields an L-space. For an L-space knot K, its Alexander polynomial $\Delta_K(t)$, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$ hold, is of the form

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

for some $0 < n_1 < n_2 < \cdots < n_k = g(K)$ [OS1, Corollary 1.3]. From this property, we have the following proposition.

Proposition 2.1 *If K is an L-space knot that is not an unknot, then $a_2(K) \neq 0$.*

Proof The coefficient of z^2 of the Conway polynomial $\nabla_K(z)$ is given by

$$a_2(K) = \frac{1}{2}\Delta''_K(1) = \sum_{j=1}^{k}(-1)^{k-j} n_j^2 \neq 0. \qquad \blacksquare$$

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The relevance of L-space knots and (chirally) cosmetic surgery comes from the following result.

Theorem 2.2 [OS2, Theorem 1.6] *If* $S^3_K(r) \cong \pm S^3_K(r')$ *with* $rr' > 0$, *then* K *is an L-space knot.*

Then we turn to the proof of Theorem 1.1.

Lemma 2.3 *If* $|npq - m| = 0$, *then* $S^3_{K_{p,q}}(n/m) \ncong -S^3_{K_{p,q}}(m/n')$.

Proof $S^3_{K_{p,q}}(m/n) = S^3_K(p/q)\#L(q,p)$ is reducible, but $S^3_{K_{p,q}}(m/n')$ is irreducible whenever $n'pq - m \neq 0$ [Sc]. Hence, they are not homeomorphic. ∎

Lemma 2.4 *If* $|npq - m| = |n'pq - m| = 1$, *then* $S^3_{K_{p,q}}(m/n) \ncong -S^3_{K_{p,q}}(m/n')$.

Proof We can assume that $npq = m + 1$ and $n'pq = m - 1$, hence $(n - n')pq = 2$. Therefore, we have $(p,q) = (\pm 1, 2)$ and consequently $2n = m + 1$ and $2n' = m - 1$, or, $-2n = m + 1$ and $-2n' = m - 1$. We consider the former case $2n = m + 1$ and $2n' = m - 1$. The latter case is similar.

Since $S^3_{K_{p,q}}(m/n) = S^3_K(m/4n) = S^3_K(m/2m+2)$ and $S^3_{K_{p,q}}(m/n') = S^3_K(m/4n') = S^3_K(m/2m-2)$, we have a chirally cosmetic surgery on the knot K :

$$S^3_K(m/2m+2) \cong -S^3_K(m/2m-2).$$

Since $(m/2m+2)(m/2m-2) > 0$, *i.e.*, the sign of two surgery slopes are the same, by Theorem 2.2, K is an L-space knot. Hence, $a_2(K) \neq 0$ by Proposition 2.1.

On the other hand, by the surgery formula of Casson–Walker invariant λ [BL, Wa], we have

$$\lambda(S^3_K(m/2m+2)) = \frac{2m+2}{m}a_2(K) - \frac{1}{2}s(2m+2,m),$$
$$\lambda(-S^3_K(m/2m+2)) = -\frac{2m-2}{m}a_2(K) + \frac{1}{2}s(2m-2,m);$$

here, $s(a,b)$ denotes the Dedekind sum. Since the Dedekind sum has the properties

$$s(a,b) = s(a',b) \text{ if } a \equiv a' \pmod b, \quad s(-a,b) = -s(a,b),$$

$s(2m+2,m) + s(2m-2,m) = 0$. Since $\lambda(S^3_K(m/2m+2)) = \lambda(-S^3_K(m/2m+2))$, we have

$$8a_2(K) = s(2m+2,m) + s(2m-2,m) = 0.$$

This is a contradiction. ∎

Lemma 2.5 *If* $|npq - m| = 1$ *and* $|n'pq - m| > 1$, *then* $S^3_{K_{p,q}}(n/m) \ncong -S^3_{K_{p,q}}(m/n')$.

Proof Let k be the number of JSJ tori of $E(K)$, and let X_0 be the JSJ piece of $E(K)$ that contains $\partial E(K)$. When X_0 is hyperbolic, the simplicial volume of its exterior

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satisfies $\|E(K)\| \geq \|E(X_0)\| > 0$. Since the simplicial volume strictly decreases under Dehn fillings when X_0 is hyperbolic,

$$\|S^3_{K_{p,q}}(m/n)\| = \|S^3_K(m/4n)\| < \|E(K)\|$$

On the other hand,

$$\|S^3_{K_{p,q}}(m/n')\| = \|E(K) \cup_{T^2} P_{p,q,m,n'}\| = \|E(K)\|;$$

we conclude $S^3_{K_{p,q}}(m/n) \neq -S^3_{K_{p,q}}(m/n')$.

When X_0 is Seifert fibered, $S^3_{K_{p,q}}(m/n) = S^3_K(m/nq^2)$ has at most k essential tori, whereas $-S^3_{K_{p,q}}(m/n')$ contains $(k+1)$ essential tori, so they are not homeomorphic. ∎

To treat Case 4, we give a more precise description of the Seifert fibered piece $P_{p,q,m,n}$ and how $E(K)$ is attached to $P_{p,q,m,n}$.

In the following, we use Hatcher's notation $M(g,b;\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_n}{\beta_n})$ for Seifert fibered manifold [Ha]. For a compact oriented surface with genus g and b boundary components B, let $B' := B \setminus (D_1 \cup \cdots \cup D_n)$ where $D_1,\dots,D_n \subset \operatorname{Int}(B)$ are disjoint disks. Let $\pi : M' \to B'$ be the circle bundle over B' with orientable total space. By taking a cross section $\sigma : B' \to M$, we identify the total space M' with $\sigma(B') \times S^1 = B' \times S^1$. For each torus boundary component T of M', we have a canonical ordered basis given by $([c_T] := [B' \times \{*\} \cap T], [h] := [\{*\} \times S^1])$, which we call a *section-regular fiber basis*. Then $M(g,b;\frac{\alpha_1}{\beta_1},\dots,\frac{\alpha_n}{\beta_n})$ is a 3-manifold obtained by attaching n tori along each torus boundary $T_i := \partial D_i' \times S^1$ so that the slope $\frac{\alpha_i}{\beta_i}$ bounds a disk.

Let $C = C_{p,q} := (S^1 \times D^2) \setminus N(T_{p,q})$ be the cable space, the complement of the regular neighborhood of the (p,q) torus knot $T_{p,q}$ in a solid torus $S^1 \times D^2$. We fix integers s, r so that $pr + qs = 1$. With a suitable choice of section, the cable space $C_{p,q}$ is identified with $M(0,2;\frac{r}{q})$.

Besides a section-regular fiber basis, the boundaries of $C_{p,q}$ has another natural ordered basis. Let $\partial_1 C := \partial N(T_{p,q})$. By viewing $T_{p,q}$ as usual (p,q) torus knot lying in $S^1 \times D^2 \subset S^3$, we have the standard meridian-longitude basis (μ,λ) of $H_1(\partial_1 C;\mathbb{Z})$. In terms of the meridian-longitude basis, the section-regular fiber basis $([c_1],[h])$ is written by

$$[c_1] = -[\mu], \quad [h] = pq[\mu] + [\lambda] \in H_1(\partial_1 C;\mathbb{Z}).$$

Since $m[\mu] + n[\lambda] = (npq - m)[c_1] + n[h]$, we have an identification

$$P_{p,q,m,n} = M\Big(0,1;\frac{r}{q},\frac{n}{npq-m}\Big).$$

For $\partial_2 C := \partial(S^1 \times D^2)$, we have a natural basis $([M] = [\{*\} \times \partial D^1], [L] = [S^1 \times \{*\}])$ of $H_1(\partial_2 C;\mathbb{Z})$ which we call *outer torus basis*. In terms of the section-regular fiber basis $([c_2],[h])$, the outer torus basis $([M],[L])$ is written by

$$[M] = q[c_2] - r[h],\ [L] = p[c_2] + s[h] \in H_1(\partial_2 C;\mathbb{Z}).$$

By the definition of cabling, the exterior $E(K)$ is glued to $P_{p,q,m,n}$ by the homeomorphism $\varphi : \partial E(K) \to \partial P_{p,q,m,n}$ such that $\varphi(\mu_K) = [M]$ and $\varphi(\lambda_K) = [L]$.

Lemma 2.6 *Assume the following hold.*

(i) *If $q = 2$ and $p \neq \pm 1$, then the set of JSJ pieces of $E(K)$ does not contain the $(-p, 2)$-torus knot exterior.*

(ii) *If $q = 2$, $p = \pm 1$, then either $a_2(K) = 0$, or, the set of JSJ pieces of $E(K)$ does not contain the $(r, 2)$-torus knot exterior for any r.*

Then for $|npq - m|, |n'pq - m| > 1$ with $n \neq n'$, $S^3_{K_{p,q}}(m/n) \not\cong -S^3_{K_{p,q}}(m/n')$.

Proof Assume, to the contrary that $S^3_{K_{p,q}}(m/n) \cong -S^3_{K_{p,q}}(m/n')$ so there is an orientation preserving homeomorphism $f : S^3_{K_{p,q}}(m/n) \to -S^3_{K_{p,q}}(m/n')$.

By isotopy, we assume that f induces homeomorphisms of JSJ pieces. By the assumption $|npq - m|, |n'pq - m| > 1$, $S^3_{K_{p,q}}(m/n)$, and $-S^3_{K_{p,q}}(m/n')$ have distinguished JSJ piece $P_{p,q,m,n}$ and $-P_{p,q,m,n'}$. Let $X_0 = P_{p,q,m,n}$ and $Y_0 = f(X_0)$.

Claim 2.7 $Y_0 \neq -P_{p,q,m,n'}$. ■

Proof Assume to the contrary that $Y_0 = -P_{p,q,m,n'}$, so f induces an orientation reversing homeomorphism $f_P = f|_{P_{p,q,m,n}} : P_{p,q,m,n} \to P_{p,q,m,n'}$. By uniqueness of Seifert fibration, f_P sends the regular fiber h of $P_{p,q,m,n}$ to the regular fiber h' of $P_{p,q,m,n}$. Since f is orientation reversing, it inverts the orientation of regular fiber; hence, we have $f([h]) = -[h']$.

On the other hand, f induces an orientation reversing homeomorphism $f|_{E(K)} : E(K) \to E(K)$ hence K is amphicheiral. In particular, we have $f([\mu_K]) = -[\mu_K]$, $f([\lambda_K]) = [\lambda_K]$.

As we have discussed, in $S^3_{K_{p,q}}(m/n)$, the outer torus basis $([M], [L])$ of $P_{p,q,m,n}$ are identified with $[\mu_K]$ and $[\lambda_K]$, respectively. Similarly, in $S^3_{K_{p,q}}(m/n')$, the outer torus basis $([M'], [L'])$ of $P_{p,q,m,n'}$ are identified with $[\mu_K]$ and $[\lambda_K]$, respectively. Therefore, $f([M]) = -[M']$ and $f([L]) = [L']$; hence, in terms of the section-regular fiber basis $([c_2], [h])$ and $([c_2'], [h'])$ of $P_{p,q,m,n}$ and $P_{p,q,m,n'}$, we have

$$f([M]) = f(q[c_2] - r[h]) = qf([c_2]) + r[h'] = -q[c_2'] + r[h'] = -[M'],$$
$$f([L]) = f(\,p[c_2] + s[h]) = pf([c_2]) - s[h'] = p[c_2'] + s[h'] = [L'].$$

The first equation shows $f([c_2]) = -[c_2']$, which contradicts the second equation. ■

Thus, $Y_0 = f(X_0) \cong P_{p,q,m,n}$ is a JSJ piece of $-E(K)$. Hence, there exists a JSJ piece X_1 of $E(K) = -(-E(K))$ that is homeomorphic to $-Y_0 \cong -P_{p,q,m,n}$.

The next claim, together with our assumption (i), shows that such a JSJ piece cannot be sent to $-P_{p,q,m,n'}$.

Claim 2.8 *Let X be a JSJ piece of $E(K)$ that is homeomorphic to $-P_{p,q,m,n}$. If $f(X) = -P_{p,q,m,n'}$, then $q = 2$ and X is homeomorphic to $(-p, 2)$-torus knot exterior. Moreover, $K_{p,q}$ is an L-space knot.*

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Proof of Claim 2.8 Since $-X \cong P_{p,q,m,n}$ is a Seifert fibered space with disk base and two singular fibers that appears as a JSJ piece of the knot exterior $E(K)$, X is homeomorphic to the torus knot exterior $E(T_{P,Q})$ for some P, Q. We fix integers S, R so that $PR + QS = 1$. If $f(X) = -P_{p,q,m,n'}$,

$$-X \cong E(T_{P,Q}) \cong M\Big(0,1;\frac{R}{Q},\frac{S}{P}\Big) \cong M\Big(0,1;\frac{r}{q},\frac{n}{npq-m}\Big) \cong P_{p,q,m,n}$$
$$\cong M\Big(0,1;\frac{r}{q},\frac{n'}{n'pq-m}\Big) \cong P_{p,q,m,n'}$$

Thus, we can assume that we have $q = Q$, $r = R$, $P = npq - m = -(n'pq - m)$ and that there are integers i, j such that

$$\frac{n}{npq-m} + i = \frac{S}{P} \quad \text{and} \quad \frac{-n'}{npq-m} + j = \frac{S}{P}.$$

In particular, we have

$$\begin{cases} r(npq-m) + qn + qi(npq-m) = 1, \\ r(npq-m) - qn' + qj(npq-m) = 1. \end{cases} \tag{1}$$

Since $P = npq - m = -(n'pq - m)$, we have $(n + n')pq = 2m$. By (1), q and m are coprime hence we have $q = 2$. Consequently, we get $r = R = 1$, $q = Q = 2$, and

$$P = npq - m = 2np - (n + n')p = (n - n')p.$$

Then (1) is written as

$$\begin{cases} (n-n')p + 2n + 2i(n-n')p = 1, \\ (n-n')p - 2n' + 2j(n-n')p = 1. \end{cases}$$

So we have $(n - n')(1 + p + ip + jp) = 1$, hence $(n - n') = \pm 1$. If $(n - n') = 1$, we have $P = (n - n')p = p$, so $X \cong -P_{p,q,m,n} \cong -E(T_{P,2}) \cong E(T_{-p,2})$. Moreover, $n - n' = 1$ means that the signs of surgery slopes m/n and m/n' are the same; hence, $K_{p,q}$ is an L-space knot by Theorem 2.2.

If $(n - n') = -1$, we have $p(1 + i + j) = -2$, so $p = \pm 1$. Then we have $|npq - m| = |(n - n')p| = 1$, so it contradicts the assumption. ■

Thus, $Y_1 = f(X_1)$ is a JSJ piece of $-E(K)$. Hence, we have a JSJ piece X_2 of $E(K)$ that is homeomoprhic to $P_{p,q,m,n}$. The next claim, similar to Claim 2.8, together with the assumption (ii) shows that such a JSJ piece cannot be sent to $-P_{p,q,m,n'}$, either.

Claim 2.9 *Let X be a JSJ piece of $E(K)$ that is homeomorphic to $P_{p,q,m,n}$. If $f(X) = -P_{p,q,m,n'}$ then $q = 2$, $p = \pm 1$ and $a_2(K) = 0$. Moreover, X is homeomoprhic to $(2, \pm(n - n'))$-torus knot exterior.*

Proof of Claim 2.9 As in Claim 2.8, X is homeomorphic to the torus knot exterior $E(T_{P,Q})$ for some coprime P, Q, and we have

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$$X \cong E(T_{P,Q}) \cong M\Big(0,1;\frac{R}{Q},\frac{S}{P}\Big) \cong M\Big(0,1;\frac{r}{q},\frac{n}{npq-m}\Big) \cong P_{p,q,m,n}$$
$$\cong M\Big(0,1;-\frac{r}{q},-\frac{n'}{n'pq-m}\Big) \cong -P_{p,q,m,n'}.$$

Here, S, R are integers chosen so that $PR + QS = 1$. We can assume that $q = Q$, $r = R$, $P = npq - m$.

We have either $|npq - m| = |n'pq - m|$ or $|npq - m| = |q|$. In the latter case, we also have $|n'pq - m| = |q| = |n'pq - m|$, so in both cases, we always have $|npq - m| = |n'pq - m|$. Since $n \neq n'$, we have $npq - m = -n'pq + m$, so $(n + n')pq = 2m$.

On the other hand, there is an integer i such that $\frac{n}{npq-m} + i = \frac{S}{P}$, so we have $r(npq - m) + qn + qi(npq - m) = 1$. This implies that m and q are coprime so we have $q = Q = 2$. Consequently, $(n + n')p = m$; hence, $npq - m = 2np - (n + n')p = (n - n')p$.

By comparing Seifert invariants, we have integers i, j such that

$$\begin{cases}(n-n')p + 2n + 2i(n-n')p = 1,\\ (n-n')p + 2n' + 2j(n-n')p = 1,\end{cases}$$

so we have $(n - n')(ip - jp + 1) = 0$. Consequently, we have $(i - j)p = -1$, so $p = \pm 1$. Thus $P = npq - m = (n - n')p = \pm(n - n')$.

Also, by $p = \pm 1$, we have $n + n' = \pm m$. This shows that $n + n' \equiv 0 \pmod m$. By the Casson–Walker invariant, we have

$$\lambda(S^3_{K_{\pm1,2}}(m/n)) = \frac{n}{m}a_2(K_{\pm1,2}) - \frac{1}{2}s(n,m)$$
$$= -\frac{n'}{m}a_2(K_{\pm1,2}) + \frac{1}{2}s(n',m) = \lambda(-S^3_{K_{\pm1,2}}(m/n'))$$

and

$$\frac{n+n'}{m}a_2(K_{\pm1,2}) = \frac{1}{2}\big(s(n,m) + s(n',m)\big) = 0.$$

Since $n + n' \neq 0$, because this implies $m = 0$, we have $a_2(K_{p,q}) = 0$. On the other hand, since $\Delta_{K_{p,q}}(t) = \Delta_K(t^q)\Delta_{T_{p,q}}(t)$, we have $a_2(K_{p,q}) = q^2 a_2(K) + a_2(T_{p,q})$. Thus, $a_2(K_{\pm1,2}) = 4a_2(K) = 0$. ■

Therefore, $Y_2 = f(X_2)$ appears as a JSJ piece of $-E(K)$; hence, we have a JSJ piece X_3 of $E(K)$ which is homeomoprhic to $-P_{p,q,m,n}$.

Then we repeat the argument; for each $i > 2$, we have a JSJ piece X_i that is homeomorphic to $-P_{p,q,m,n}$ (if i is odd) or $P_{p,q,m,n}$ (if i is even). Then by assumption (i) and Claim 2.8 (if i is odd) or by assumption (ii) and Claim 2.9 (if i is even), we see that $f(X_i) \neq -P_{p,q,m,n'}$. Hence, $Y_i := f(X_i)$ gives a new JSJ piece of $-E(K)$. This means that we find a new JSJ piece X_{i+1} in $E(K)$, homeomorphic to $-P_{p,q,m,n}$ (if i is even) or $P_{p,q,m,n}$ (if i is odd) (see Figure 1 for a schematic illustration). Thus, $E(K)$ contains infinitely many JSJ pieces, which is absurd.

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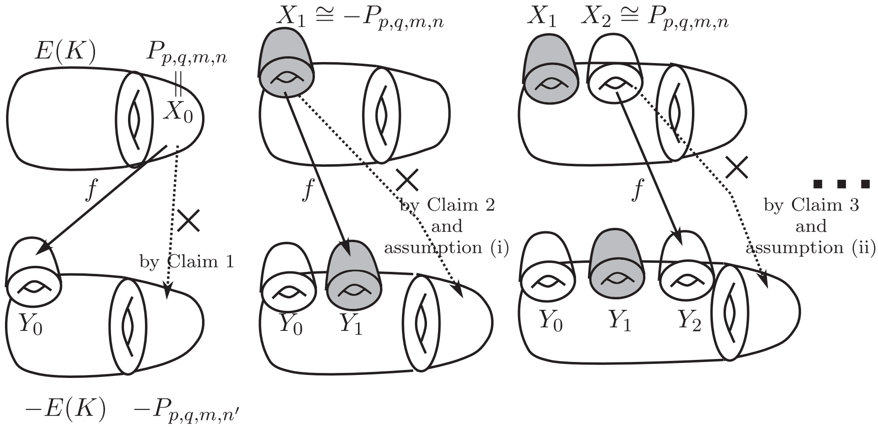

Figure 1: Proof of Lemma 4: $S^3_{K_{p,q}}(n/m) \ncong -S^3_{K_{p,q}}(m/n')$ imposes infinitely many JSJ pieces.

3 Iterated Cables

For a sequence $(p_1, q_1), \dots, (p_N, q_N)$ of coprime integers with $q_i > 1$ and a knot K, we define an iterated cable $K_{(p_1,q_1),\dots,(p_N,q_N)}$ inductively by

$$K_{(p_1,q_1)} = K_{p_1,q_1}, \quad K_{((p_1,q_1),\dots,(p_N,q_N))} = (K_{((p_1,q_1),\dots,(p_{N-1},q_{N-1}))})_{(p_N,q_N)}.$$

When K is the unknot U, the iterated cable $U_{(p_1,q_1),\dots,(p_N,q_N)}$ is called the *iterated torus knot*.

We prove a theorem that is slightly more general than Theorem 1.2, by adding more arguments to Lemma 2.6.

Theorem 3.1 *Let K be a non-satellite knot. Then an iterated cable $K_{(p_1,q_1),\dots,(p_N,q_N)}$ for $N \geq 1$ that is not a $(2,p)$-torus knot does not admit chirally cosmetic surgery.*

Proof of Theorem 3.1 An iterated cable of torus knot is an iterated torus knot, so we can assume that K is either hyperbolic or unknot. We put $p = p_N$, $q = q_N$ and view the iterated cable $K_{(p_1,q_1),\dots,(p_N,q_N)}$ as $K^*_{(p,q)}$, the (p,q)-cable of the iterated cable $K^* = K_{(p_1,q_1),\dots,(p_{N-1},q_{N-1})}$.

The JSJ decomposition of $E(K^*)$ is given by

$$E(K^*) = E(T_{p_1,q_1}) \cup_{T_1} C_{p_2,q_2} \cup_{T_2} \cdots \cup_{T_{N-2}} C_{p_{N-1},q_{N-1}}$$

if K is unknot, and

$$E(K^*) = E(K) \cup_{T_0} C_{p_1,q_1} \cup_{T_1} C_{p_2,q_2} \cup_{T_2} \cdots \cup_{T_{N-2}} C_{p_{N-1},q_{N-1}}$$

otherwise (*i.e.*, K is hyperbolic). When K is hyperbolic, no JSJ piece of E_{K^*} is homeomorphic to the torus knot exterior so by Theorem 1.1, $K^*_{p,q}$ does not admit chirally cosmetic surgery. Thus, we assume that K^* is an iterated torus knot. Since the

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classification of chirally cosmetic surgery of torus knots are known [IIS, Ro], in the following, we assume that K is not a torus knot.

Assume, to the contrary that $S^3_{K^*_{p,q}}(m/n) \cong -S^3_{K^*_{p,q}}(m/n')$, so there is an orientation preserving homeomorphism $f : S^3_{K^*_{p,q}}(m/n) \to -S^3_{K^*_{p,q}}(m/n')$. By Lemmas 2.3, 2.4, and 2.5, $|npq - m|, |n'pq - m| > 1$; hence, $S^3_{K^*_{p,q}}(m/n) = E(K^*) \cup_T P_{p,q,m,n}, S^3_{K^*_{p,q}}(m/n') = E(K^*) \cup_T P_{p,q,m,n'}$.

By isotopy we assume that f induces homeomorphisms of JSJ pieces. By Claim 2.7 in the proof of Lemma 2.6, $f(P_{p,q,m,n})$ is a JSJ piece of $-E(K^*)$. Since the cable space C_{p_i,q_i} has two boundary components, whereas the boundary of $P_{p,q,m,n}$ is connected, we have $f(P_{p,q,m,n}) = -E(T_{p_1,q_1})$. Since $f(E(T_{p_1,q_1}))$ is a JSJ piece of $-S^3_{K^*_{p,q}}(m/n') = -E(K^*) \cup_T -P_{p,q,m,n'}$ other than $-E(T_{p_1,q_1})$, we have $f(E(T_{p_1,q_1})) = -P_{p,q,m,n'}$. This shows that f gives an orientation preserving homeomorphism

$$f : -f(P_{p,q,m,n}) = E(T_{p_1,q_1}) \longrightarrow -P_{p,q,m,n'}.$$

By Claim 2.8 in the proof of Lemma 2.6, we have $q_1 = 2, p_1 = -p$ and $K = K^*_{p,q}$ is an L-space knot. On the other hand, by [Hom] an iterated torus knot $K_{(-p,2),\ldots,(p_{N-1},q_{N-1}),(p,2)}$ is an L-space knot implies that $-p, p_2, p_3, \ldots, p$ has the same sign. This is a contradiction. ∎

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