J. Aust. Math. Soc. **109** (2020), 93–111
doi:10.1017/S1446788719000375

ALGEBRAIC CUNTZ–KRIEGER ALGEBRAS

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(Received 4 June 2018; accepted 10 August 2019; first published online 23 September 2019)

Communicated by L. O. Clark

Abstract

We show that a directed graph E is a finite graph with no sinks if and only if, for each commutative unital ring R, the Leavitt path algebra $L_R(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra if and only if the C^*-algebra $C^*(E)$ is unital and $\mathrm{rank}(K_0(C^*(E))) = \mathrm{rank}(K_1(C^*(E)))$. Let k be a field and $k^\times$ be the group of units of k. When $\mathrm{rank}(k^\times) < \infty$, we show that the Leavitt path algebra $L_k(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra if and only if $L_k(E)$ is unital and $\mathrm{rank}(K_1(L_k(E))) = (\mathrm{rank}(k^\times) + 1)\mathrm{rank}(K_0(L_k(E)))$. We also show that any unital k-algebra which is Morita equivalent or stably isomorphic to an algebraic Cuntz–Krieger algebra, is isomorphic to an algebraic Cuntz–Krieger algebra. As a consequence, corners of algebraic Cuntz–Krieger algebras are algebraic Cuntz–Krieger algebras.

2010 *Mathematics subject classification*: primary 16W99; secondary 16E20, 16D90.

Keywords and phrases: Leavitt path algebra, Cuntz–Krieger algebra, stably isomorphic, Morita equivalence.

1. Introduction

Cuntz–Krieger algebras, introduced and first investigated by Cuntz and Krieger [11] in 1980, are a prominent class of C^*-algebras arising from dynamical systems. The Cuntz–Krieger algebra $\mathcal{O}_A$ was originally associated to a finite square $\{0, 1\}$ matrix A [11], but it can also be viewed as the graph C^*-algebra of a finite directed graph with no sinks and no sources [22]. Graph C^*-algebras and their generalizations have been intensively investigated by analysts for more than two decades (see [19] for an overview of the subject).

The algebraic Cuntz–Krieger algebras arose as specific examples of fractional skew monoid rings [5]. Leavitt path algebras are the algebraic version of graph C^*-algebras. Leavitt path algebras $L_k(E)$ are quotients of path algebras associated to an extended graph $\widehat{E}$ and a field k, modulo additional relations. An algebraic Cuntz–Krieger algebra $C\mathcal{K}_k(E)$ is a Leavitt path algebra $L_k(E)$ of a finite graph E with no sinks and no sources.

This research was in part supported by a grant from IPM (no. 95170419).

Initially, Leavitt path algebras were introduced by Ara *et al.* in [6] and by Abrams and Aranda Pino in [1]. Tomforde in [21] generalizes the construction of Leavitt path algebras by replacing the field k with a commutative unital ring R. Leavitt path algebras are also a generalization of the algebras constructed by Leavitt in [16] to produce rings without the invariant basis number property (i.e., $R_R^m \cong R_R^n$ as left R-modules with $m \neq n$). Leavitt path algebras include many well-known algebras such as matrix algebras $\mathbb{M}_n(R)$ for $n \geq 1$, the Laurent polynomial ring $R[x, x^{-1}]$, or the Leavitt algebras $L(1, n)$ for $n \geq 2$.

Many times in the literature, the conditions characterizing some analytic properties of the graph C^*-algebra turned out to be exactly the same conditions characterizing the corresponding algebraic version of these properties. In this sense the Leavitt path algebra theory, being more recent, has benefited from the inspiration that the graph C^*-algebra world provided. This is the case once more for the topic discussed in the current paper: the analytic results were given in [9] for the Cuntz–Krieger algebras, and we give here the algebraic analogue for algebraic Cuntz–Krieger algebras. Some of the ideas in this paper are contained in [9]. Theorem 3.12 of [9] says that the graph C^*-algebra $C^*(E)$ is isomorphic to a Cuntz–Krieger algebra if and only if $C^*(E)$ is unital and $\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E)))$. In the algebraic setting we prove a slightly different result (Theorem 3.10). Let k be a field such that $\operatorname{rank}(k^\times) < \infty$. Then the Leavitt path algebra $L_k(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra if and only if $L_k(E)$ is unital and $\operatorname{rank}(K_1(L_k(E))) = (\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E)))$. We also show that the assumption $\operatorname{rank}(k^\times) < \infty$ is necessary.

The paper is organized as follows. In Section 2 we give all the background information, definitions and basic properties of Leavitt path algebras that we need in this paper.

In Section 3 we give a characterization of algebraic Cuntz–Krieger algebras. In the first step of this process, we provide a class of operations on graphs that preserve isomorphism of associated Leavitt path algebras. With this useful result, we show that algebraic Cuntz–Krieger algebras are Leavitt path algebras of finite graphs with no sinks. Finally, in Corollary 3.9 and Theorem 3.10 we derive further conditions for $L_R(E)$ to be isomorphic to an algebraic Cuntz–Krieger R-algebra when R is a commutative unital ring or a field.

In Section 4 we first show that the corner $P_X L_R(E) P_X$, where X is a finite subset of E^0, is isomorphic to a Leavitt path algebra $L_R(E(T))$. Also we show that if $L_R(E)$ is an algebraic Cuntz–Krieger algebra, then $L_R(E(T))$ is an algebraic Cuntz–Krieger algebra. Finally, after proving similar results for $\mathbb{M}_n(L_R(E))$ and $\mathbb{M}_\infty(L_R(E))$, we show that if A is an algebraic Cuntz–Krieger R-algebra then, for each positive integer n, $\mathbb{M}_n(A)$ is isomorphic to an algebraic Cuntz–Krieger algebra.

In the last section, we show that if a unital k-algebra A is Morita equivalent or stably isomorphic to an algebraic Cuntz–Krieger algebra, then it is isomorphic to an algebraic Cuntz–Krieger algebra. As a consequence, A is an algebraic Cuntz–Krieger algebra if and only if the full $n \times n$ matrix algebra over A is an algebraic Cuntz–Krieger algebra. Also we show that if A is an algebraic Cuntz–Krieger algebra, then the corners eAe and $e'\mathbb{M}_\infty(A)e'$ are isomorphic to algebraic Cuntz–Krieger algebras.

2. Preliminaries

A directed *graph* $E = (E^0, E^1, r_E, s_E)$ consists of two sets E^0 and E^1 together with maps $r_E, s_E : E^1 \to E^0$, identifying the range and source of each edge. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If a vertex v emits no edges, that is, if $s_E^{-1}(v)$ is empty, then v is called a *sink*. A vertex v is called a *regular vertex* if $s_E^{-1}(v)$ is a finite nonempty set. The set of regular vertices is denoted by E^0_{reg}. We let $E^0_{\mathrm{sing}} := E^0 \backslash E^0_{\mathrm{reg}}$ and refer to an element of E^0_{sing} as a singular vertex.

A finite path μ in a graph E is a finite sequence of edges $\mu = e_1 \cdots e_n$ such that $r_E(e_i) = s_E(e_{i+1})$ for $i = 1, \ldots, n-1$. In this case, $n = l(\mu)$ is the length of μ. We view the elements of E^0 as paths of length 0. For any $n \in \mathbb{N}$ the set of paths of length n is denoted by E^n. Also, Path(E) stands for the set of all finite paths, that is, $\mathrm{Path}(E) = \bigcup_{n=0}^{\infty} E^n$. We denote by μ^0 the set of the vertices of the path μ, that is, the set $\{s(e_1), r(e_1), \ldots, r(e_n)\}$.

A path $\mu = e_1 \cdots e_n$ is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle of length one is called a *loop*. An *exit* for a path $\mu = e_1 \cdots e_n$ is an edge e such that $s_E(e) = s_E(e_i)$ for some i and $e \neq e_i$.

A right-infinite path μ in a graph E is an infinite sequence of edges $\mu = e_1e_2e_3 \ldots$ such that $r_E(e_i) = s_E(e_{i+1})$ for each i. A left-infinite path μ in a graph E is an infinite sequence of edges $\mu = \cdots e_{-3}e_{-2}e_{-1}$ such that $r_E(e_i) = s_E(e_{i+1})$ for each i. A bi-infinite path μ in a graph E is an infinite sequence of edges $\mu = \cdots e_{-3}e_{-2}e_{-1}e_0e_1e_2e_3 \cdots$ such that $r_E(e_i) = s_E(e_{i+1})$ for each i. We denote by E^∞ the set of all (right-, left-, bi-)infinite paths in E.

A path $\mu = e_1e_2e_3 \cdots$ is called vertex-simple if the sequence $s(e_1), r(e_1), r(e_2), \ldots$ contains no repeated vertices. A graph E is called path-finite if E^∞ contains no vertex-simple paths. A graph E is called row-finite if, for each $v \in E^0$, $s_E^{-1}(v)$ is a finite set. For each $e \in E^1$, we call e^* a *ghost edge*. We let $r_E(e^*)$ denote $s_E(e)$, and we let $s_E(e^*)$ denote $r_E(e)$.

Definition 2.1. Let E be a graph. The *graph C^*-algebra $C^*(E)$* is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ together with partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$ which satisfy the following conditions.

(1) (The ‘CK-1 relations’) for all $e \in E^1$, $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$.

(2) (The ‘CK-2 relations’) for every regular vertex $v \in E^0$,

$$p_v = \sum_{\{e \in E^1 \mid \, s_E(e) = v\}} s_e s_e^*.$$

Definition 2.2. Let E be an arbitrary graph and R be a commutative ring with unit. The *Leavitt path algebra $L_R(E)$ with coefficients in R* is the universal R-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions.

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(1) $s_E(e)e = e = er_E(e)$ for all $e \in E^1$.
(2) $r_E(e)e^* = e^* = e^* s_E(e)$ for all $e \in E^1$.
(3) (The 'CK-1 relations') for all $e, f \in E^1$, $e^*e = r_E(e)$ and $e^*f = 0$ if $e \neq f$.
(4) (The 'CK-2 relations') for every regular vertex $v \in E^0$,

$$v = \sum_{\{e \in E^1 \mid s_E(e)=v\}} ee^*.$$

Another definition for $L_R(E)$ can be given using the extended graph $\widehat{E}$. This graph has the same set of vertices E^0 and the same set of edges E^1 together with the so-called ghost edges e^* for each $e \in E^1$, whose directions are opposite to those of the corresponding $e \in E^1$. $L_R(E)$ can be defined as the usual path algebra $R\widehat{E}$ with coefficients in R subject to Cuntz–Krieger relations (3) and (4) above.

Definition 2.3. Let E be a graph and $\mathcal{A}$ be an R-algebra with involution $*$. A *Cuntz–Krieger E-family in $\mathcal{A}$* is a collection $\Sigma = (S_\mu)_{\mu \in E^0 \cup E^1} \subset \mathcal{A}$ which satisfies the following relations.

(1) For all $v, w \in E^0$, $S_v S_v = S_v$ and $S_v S_w = 0$ if $v \neq w$.
(2) $S_v^* = S_v$ for all $v \in E^0$.
(3) $S_{s(e)} S_e = S_e = S_e S_{r(e)}$ for all $e \in E^1$.
(4) For all $e, f \in E^1$, $S_e^* S_e = S_{r(e)}$ and $S_e^* S_f = 0$ if $e \neq f$.
(5) For every regular vertex $v \in E^0$,

$$S_v = \sum_{\{e \in E^1,\ s(e)=v\}} S_e S_e^*.$$

Let A be an R-algebra with a Cuntz–Krieger E-family; thus by the universal homomorphism property of $L_R(E)$, there is a unique R-algebra homomorphism from $L_R(E)$ to A mapping the generators of $L_R(E)$ to their appropriate counterparts in A. We will refer to this property as the universal homomorphism property of $L_R(E)$.

Definition 2.4. Let E be a finite graph with no sinks and no sources and R a commutative ring with unit. The Leavitt path algebra $L_R(E)$ is called their algebraic Cuntz–Krieger algebra, which is denoted by $\mathcal{CK}_R(E)$.

If E has a finite number of vertices, then $L_R(E)$ is unital with $\sum_{v \in E^0} v = 1_{L_R(E)}$; otherwise, $L_R(E)$ is a ring with a set of local units (that is, a set of elements X such that, for every finite collection $a_1, \ldots, a_n \in L_R(E)$, there exists $x \in X$ such that $a_i x = a_i = x a_i$) consisting of sums of distinct vertices of the graph.

If $\mu = e_1 \cdots e_n$ is a path in E, we write μ^* for the element $e_n^* \cdots e_1^*$ of $L_R(E)$. With this notation it can be shown that the Leavitt path algebra $L_R(E)$ can be viewed as

$$L_R(E) = \operatorname{span}_R\{\alpha\beta^* : \alpha, \beta \in \operatorname{Path}(E) \text{ and } r(\alpha) = r(\beta)\}$$

and $rv \neq 0$ for all $v \in E^0$ and all $r \in R \setminus \{0\}$ (see [21, Proposition 3.4]). Also $L_R(E)$ is a $*$-algebra with linear anti-multiplicative involution defined by $(\sum_{i=1}^n r_i \alpha_i \beta_i^*)^* = \sum_{i=1}^n r_i \beta_i \alpha_i^*$.

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Let G be a group. A ring $A = \bigoplus_{g\in G} A_g$ is called a G-graded ring, if each A_g is an additive subgroup of A and $A_gA_{g'} \subseteq A_{g+g'}$ for all $g, g' \in G$. A G-graded ring $A = \bigoplus_{g\in G} A_g$ is called a strongly graded ring if $A_gA_{g'} = A_{g+g'}$ for all $g, g' \in G$. Let $\Phi : A \to B$ be a ring homomorphism between G-graded rings. Φ is a graded ring homomorphism if $\Phi(A_g) \subseteq B_g$, for all $g \in G$. Leavitt path algebras can be viewed as graded algebras. Let G be a group with the identity element e and $w : E^1 \to G$ be a weight map. Also let $w(\alpha^*) = w(\alpha)^{-1}$ and $w(v) = e$, for each $\alpha \in E^1$ and $v \in E^0$. Thus the path algebra $R\widehat{E}$ of the extended graph $\widehat{E}$ is a G-graded R-algebra, and since Cuntz–Krieger relations are homogeneous, $L_R(E)$ is a G-graded R-algebra. The natural grading given to a Leavitt path algebra is a $\mathbb{Z}$-grading by setting $w(\alpha) = 1$, $w(\alpha^*) = -1$ and $w(v) = 0$, for each $\alpha \in E^1$ and $v \in E^0$. In this case the Leavitt path algebra can be decomposed as a direct sum of homogeneous components $L_R(E) = \bigoplus_{n\in\mathbb{Z}} L_R(E)_n$ satisfying $L_R(E)_nL_R(E)_m \subseteq L_R(E)_{n+m}$. Actually,

$$L_R(E)_n = \operatorname{span}_R\{pq^* : p, q \in \operatorname{Path}(E), l(p) - l(q) = n\}.$$

Every element $x \in L_R(E)_n$ is a homogeneous element of degree n.

An ideal I is graded if it inherits the grading of $L_R(E)$, that is, if $I = \bigoplus_{n\in\mathbb{Z}}(I \cap L_R(E)_n)$. Tomforde in [21] (see also [7, Theorem 3.5]) proved the graded uniqueness theorem. Let E be a graph and let $L_R(E)$ be the associated Leavitt path algebra with the usual $\mathbb{Z}$-grading. If A is a $\mathbb{Z}$-graded ring, and $\pi : L_R(E) \to A$ is a graded ring homomorphism with $\pi(rv) \neq 0$ for all $v \in E^0$ and $r \in R\backslash\{0\}$, then π is injective.

We define a relation $\geq$ on E^0 by setting $v \geq w$ if there exists a path μ in E from v to w, that is, $v = s(\mu)$ and $w = r(\mu)$. A subset X of E^0 is called *hereditary* if, for each $v \in X$, $v \geq w$ implies that $w \in X$. For any subset $X \subseteq E^0$, the smallest hereditary subset of E^0 containing X is denoted by $H_E(X)$. A subset $H \subseteq E^0$ is called *saturated* if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies that $v \in H$. An ideal I of $L_R(E)$ is called basic if $rv \in I$ for $r \in R \setminus \{0\}$ implies that $v \in I$. Tomforde [21, Theorem 7.9] proved that the map $H \longrightarrow I_H$ defines a lattice isomorphism between the saturated hereditary subsets of E^0 and the graded basic ideals of $L_R(E)$, where I_H is a two-sided ideal in $L_R(E)$ generated by a saturated hereditary subset H of E^0.

A right-infinite path $\tau = e_1e_2\cdots$ in E is called periodic if there exist integers $j, k \geq 1$, such that $e_{n+k} = e_n$ for every $n \geq j$. In this case, it is clear that the path $\rho = e_j \cdots e_{j+k-1}$ is closed. Take j and k such that $j + k$ is the smallest possible value which satisfies the condition $e_{n+k} = e_n$ for every $n \geq j$, and consider the paths $\alpha = e_1 \cdots e_{j-1}$ and $\lambda = e_j \cdots e_{j+k-1}$. The pair (α, λ) is called the seed of τ. Of course α may have zero length. In any case, λ is a closed path, which is called the period of τ. A right-infinite path τ which is periodic and whose period is a closed path without exits (which means that it has to be a cycle without exits) is called an infinite discrete essentially aperiodic trail. For any infinite discrete essentially aperiodic trail which is parameterized by the seed (α, λ_α) of the trail (that is, $\alpha \in \operatorname{Path}(E)$ is its essential head and $r(\alpha)$ is visited by the cycle without exits λ_α), the path α is called a distinguished path. In the case $l(\alpha) = 0$, α is called a distinguished vertex. For any distinguished path α, $\alpha\lambda_\alpha\alpha^*$ is denoted by ω_α.

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To finish this section we introduce a generalized uniqueness theorem which we will use later.

Theorem 2.5 [13, Theorem 5.2]. *Let E be a graph, R be a commutative ring with unit and $\mathcal{A}$ be an R-algebra. Consider $\Phi : L_R(E) \to \mathcal{A}$ a ring homomorphism. Then the following conditions are equivalent.*

(i) *Φ is injective.*
(ii) *The restriction of Φ to $M_R(E)$ is injective.*
(iii) *Both the following conditions are satisfied:*

   (a) *$\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$;*
   (b) *for every distinguished path α, the $*$ R-algebra $\langle\Phi(\omega_\alpha)\rangle$ generated by $\Phi(\omega_\alpha)$ is $*$-isomorphic to $R[x, x^{-1}]$, that is, $\langle\Phi(\omega_\alpha)\rangle \cong R[x, x^{-1}]$.*

3. Characterization of algebraic Cuntz–Krieger algebras

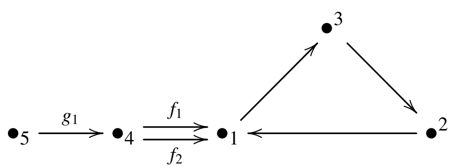
In this section we give a characterization of algebraic Cuntz–Krieger algebras.

Definition 3.1. [9, Definition 3.6] Let E be a graph, H be a hereditary subset of E^0 and $F(H) = \{\alpha | \alpha = e_1 e_2 \cdots e_n \in \mathrm{Path}(E), s_E(e_n) \notin H, r_E(e_n) \in H\}$. Let $\overline{F}(H)$ be another copy of $F(H)$; for each $\alpha \in F(H)$, the copy of α in $\overline{F}(H)$ is denoted by $\overline{\alpha}$. Define a graph $E(H)$ as follows

$$E(H)^0 = H \cup F(H),$$
$$E(H)^1 = s_E^{-1}(H) \cup \overline{F}(H).$$

$s_{E(H)}(e) = s_E(e)$ and $r_{E(H)}(e) = r_E(e)$ for each $e \in s_E^{-1}(H)$. $s_{E(H)}(\overline{\alpha}) = \alpha$ and $r_{E(H)}(\overline{\alpha}) = r_E(\alpha)$ for each $\overline{\alpha} \in \overline{F}(H)$.

Example 3.2. Consider the graph E given by



Let $H = \{1, 2, 3\}$. Thus $F(H) = \{f_1, f_2, g_1 f_1, g_1 f_2\}$ and $E(H)$ is the graph

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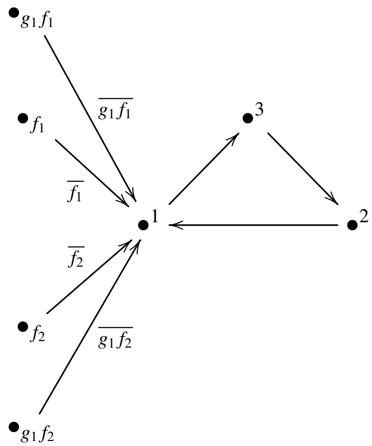

Theorem 3.3. *Let R be a commutative unital ring, E be a graph and H be a hereditary subset of E^0. Suppose that $(E^0\backslash H, r_E^{-1}(E^0\backslash H), r_E, s_E)$ is a finite acyclic graph, $v \geq H$ for all $v \in E^0\backslash H$ and the set $s_E^{-1}(E^0\backslash H) \cap r_E^{-1}(H)$ is finite. Then $L_R(E) \cong L_R(E(H))$.*

Proof. Let $\{e, v | e \in E^1, v \in E^0\}$ be a universal Cuntz–Krieger E-family. For $v \in E(H)^0$ define

$$Q_v = \begin{cases} v & \text{if } v \in H, \\ \alpha\alpha^* & \text{if } v = \alpha \in F(H), \end{cases}$$

and for $e \in E(H)^1$ define

$$T_e = \begin{cases} e & \text{if } e \in s_E^{-1}(H), \\ \alpha & \text{if } e = \overline{\alpha} \in \overline{F}(H). \end{cases}$$

The same argument as in [9, proof of the Theorem 3.8] (see also [17, Lemma 3.7]) shows that $\{T_e, Q_v | e \in E(H)^1, v \in E(H)^0\}$ is a Cuntz–Krieger $E(H)$-family in $L_R(E)$. Let $\{t_e, q_v | e \in E(H)^1, v \in E(H)^0\}$ be a universal Cuntz–Krieger $E(H)$-family. By the universal homomorphism property of $L_R(E(H))$ there exists a $*$-homomorphism $\Psi : L_R(E(H)) \to L_R(E)$ with $\Psi(q_v) = Q_v$ for each $v \in E(H)^0$ and $\Psi(t_e) = T_e$ for each $e \in E(H)^1$. Since $s_E^{-1}(E^0\backslash H) \cap r_E^{-1}(H)$ is finite, the same argument as in [9, proof of the Theorem 3.8] shows that Ψ is an epimorphism. Now let α be a distinguished path in $E(H)$ and $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$, where λ_α is a cycle without exits that starts and ends at $r_{E(H)}(\alpha)$. The cycles in $E(H)$ come from cycles in E all lying in the subgraph $(H, s_E^{-1}(H), s_E, r_E)$. Hence λ_α is a cycle without exits in E that starts and ends at $r_{E(H)}(\alpha)$. If α is a distinguished vertex, then $\omega_\alpha = \lambda_\alpha$ and $\Psi(\omega_\alpha) = \omega_\alpha$. If $l(\alpha) \neq 0$, then $\alpha = \mu\overline{\beta}$ for some $\mu = e_1 \cdots e_t \in \text{Path}(E)\backslash F(H)$ and $\beta \in F(H)$. Therefore $\Psi(\omega_\alpha) = \mu\beta\lambda_\alpha\beta^*\mu^*$ and so $\langle\Psi(\omega_\alpha)\rangle \cong R[x, x^{-1}]$. Now let $v \in E(H)^0$ and $r \in R\backslash\{0\}$. If $v \in H$, then $\Psi(rq_v) = rv$ and by [21, Proposition 3.4], $rv \neq 0$. Now assume that $v = \alpha$ for some $\alpha \in F(H)$. Hence $\Psi(rq_v) = r\alpha\alpha^*$ and by [21, Proposition 4.9], $r\alpha\alpha^* \neq 0$. Thus by the generalized

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uniqueness theorem, Ψ is injective. Therefore Ψ is an isomorphism and the result follows. □

Definition 3.4. [9, Definitions 3.2, 3.3 and 3.9] Let E be a graph and n be a positive integer.

(i) For any vertex $v_0 \in E^0$ define a graph $E(v_0, n)$ as follows

$$E(v_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\},$$
$$E(v_0, n)^1 = E^1 \cup \{e_1, e_2, \ldots, e_n\}.$$

$s_{E(v_0,n)}(e) = s_E(e)$ and $r_{E(v_0,n)}(e) = r_E(e)$ for each $e \in E^1$. $r_{E(v_0,n)}(e_i) = v_{i-1}$ and $s_{E(v_0,n)}(e_i) = v_i$ for each i.

(ii) For each edge $e_0 \in E^1$ define a graph $E(e_0, n)$ as follows:

$$E(e_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\},$$
$$E(e_0, n)^1 = \{e_1, e_2, \ldots, e_{n+1}\} \cup E^1 \backslash \{e_0\}.$$

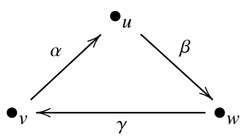
$s_{E(e_0,n)}(e) = s_E(e)$ and $r_{E(e_0,n)}(e) = r_E(e)$ for each $e \in E^1 \backslash \{e_0\}$. $r_{E(e_0,n)}(e_i) = v_{i-1}$ for each $2 \leq i \leq n+1$, $s_{E(e_0,n)}(e_i) = v_i$ for each $1 \leq i \leq n$, $r_{E(e_0,n)}(e_1) = r_E(e_0)$ and $s_{E(e_0,n)}(e_{n+1}) = s_E(e_0)$.

(iii) For any vertex $v_0 \in E^0$ define a graph $E'(v_0, n)$ as follows:

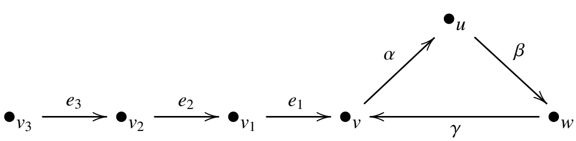
$$E'(v_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\},$$
$$E'(v_0, n)^1 = E^1 \cup \{e_1, e_2, \ldots, e_n\}.$$

$s_{E'(v_0,n)}(e) = s_E(e)$ and $r_{E'(v_0,n)}(e) = r_E(e)$ for each $e \in E^1$. $r_{E'(v_0,n)}(e_i) = v_0$ and $s_{E'(v_0,n)}(e_i) = v_i$ for each i.

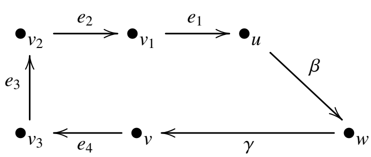
Example 3.5. Consider the graph E given by



Thus $E(v, 3)$ is the graph



$E(\alpha, 3)$ is the graph



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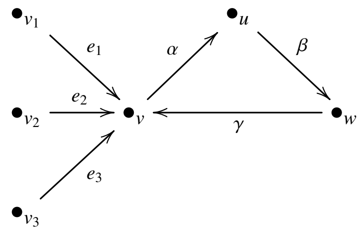
and $E'(v,3)$ is the graph



Corollary 3.6. *Let R be a commutative unital ring, E be a graph, $v_0 \in E^0$ be a vertex and n be a positive integer. Then $L_R(E(v_0,n)) \cong L_R(E'(v_0,n))$.*

Proof. An argument similar to that in [9, proof of the Corollary 3.10] shows that the result follows from Theorem 3.3. □

Proposition 3.7. *Let R be a commutative unital ring, E be a graph, $e_0 \in E^1$ be an edge and n be a positive integer. Then $L_R(E(r_E(e_0),n)) \cong L_R(E(e_0,n))$.*

Proof. Let $r_E(e_0) = v_0$ and $\{e, v | e \in E(e_0,n)^1, v \in E(e_0,n)^0\}$ be a universal Cuntz–Krieger $E(e_0,n)$-family. For $v \in E(v_0,n)^0$ define $Q_v = v$, and for $e \in E(v_0,n)^1$ define

$$T_e = \begin{cases} e & \text{if } e \neq e_0, \\ e_{n+1}e_n \cdots e_1 & \text{if } e = e_0. \end{cases}$$

The same argument as in [9, proof of the Proposition 3.5] shows that $\{T_e, Q_v | e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$ is a Cuntz–Krieger $E(v_0,n)$-family in $L_R(E(e_0,n))$. Let $\{t_e, q_v | e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$ be a universal Cuntz–Krieger $E(v_0,n)$-family. By the universal homomorphism property of $L_R(E(v_0,n))$ there exists a $*$-homomorphism $\Psi : L_R(E(v_0,n)) \to L_R(E(e_0,n))$ that $\Psi(q_v) = Q_v$ for each $v \in E(v_0,n)^0$ and $\Psi(t_e) = T_e$ for each $e \in E(v_0,n)^1$. The same argument as in [9, proof of the Proposition 3.5] shows that Ψ is an epimorphism.

Now let α be a distinguished path in $E(v_0,n)$ and $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$, where λ_α is a cycle without exits that starts and ends at $r_{E(v_0,n)}(\alpha)$. Suppose $\lambda_\alpha = f_1 f_2 \cdots f_m$. If $s_{E(v_0,n)}(f_i) \neq s_{E(v_0,n)}(e_0)$ for each i, then λ_α is a cycle without exits in $E(e_0,n)$. Thus $\Psi(t_{\lambda_\alpha}) = \lambda_\alpha$ and so $\langle \Psi(\omega_\alpha) \rangle \cong R[x,x^{-1}]$. Now assume that $s_{E(v_0,n)}(f_i) = s_{E(v_0,n)}(e_0)$, for some i. Since λ_α is a cycle without exits, $\lambda_\alpha = e_0 f_{i+1} f_{i+2} \cdots f_{i-1}$. $\Psi(t_{\lambda_\alpha}) = e_{n+1}e_n \cdots e_1 f_{i+1} f_{i+2} \cdots f_{i-1}$ and $e_{n+1}e_n \cdots e_1 f_{i+1} f_{i+2} \cdots f_{i-1}$ is a cycle without exits in $E(e_0,n)$. Thus $\langle \Psi(\omega_\alpha) \rangle \cong R[x,x^{-1}]$. Also, for each $v \in E(v_0,n)^0$ and $r \in R \backslash \{0\}$, $\Psi(rq_v) = rv \neq 0$. It follows from the generalized uniqueness theorem that Ψ is injective. Therefore Ψ is an isomorphism and the result follows. □

When E is a row-finite graph with no sinks and k is a field, Proposition 3.1 of [2] shows that, there exists a row-finite graph G with no sinks and no sources such that the Leavitt path algebras $L_k(E)$ and $L_k(G)$ are Morita equivalent. Also when E is a finite graph with no sinks and at least two vertices, Proposition 13 of [14] shows that, there exists a finite graph G with no sinks and no sources such that the Leavitt path algebras $L_k(E)$ and $L_k(G)$ are graded Morita equivalent (see also [18, Corollary 3.18]). The following corollary improves these known (graded) Morita equivalences to isomorphisms.

Corollary 3.8. *Let R be a commutative unital ring and E be a finite graph with no sinks. Then there exists a finite graph G with no sinks and no sources such that the Leavitt path algebras $L_R(E)$ and $L_R(G)$ are isomorphic.*

Proof. Let $E_0 = E$. Removing the sources of E_0, we get a subgraph E_1 of E_0. Removing the sources of E_1, we get a subgraph E_2 of E_1 (see [2, Definition 1.2]). Since E is a finite graph with no sinks, after finitely many repetitions, we get a subgraph $F = E_n$ of E that has no sinks and no sources. By induction we see that F^0 is a hereditary subset of E^0. We show that $(E^0\backslash F^0, r_E^{-1}(E^0\backslash F^0), r_E, s_E)$ is a finite acyclic graph such that, for any $v \in E^0\backslash F^0$, there exists a path from v to F^0 in E. Since E is a finite graph, $(E^0\backslash F^0, r_E^{-1}(E^0\backslash F^0), r_E, s_E)$ is finite. Let $e_1e_2\cdots e_r$ be a cycle in $(E^0\backslash F^0, r_E^{-1}(E^0\backslash F^0), r_E, s_E)$. Then $e_1e_2\cdots e_r$ is a cycle in E and so it is a cycle in E_1. Inductively, $e_1e_2\cdots e_r$ is a cycle in E_i for each $0 \leq i \leq n$. Since $s_E(e_1) \in E^0\backslash F^0$, $s_E(e_1)$ is a source in E_j for some $0 \leq j \leq n$, which is a contradiction. Therefore $(E^0\backslash F^0, r_E^{-1}(E^0\backslash F^0), r_E, s_E)$ is acyclic. Let $v \in E^0\backslash F^0$. Then there exists $0 \leq j \leq n$ such that v is a source in E_j. Since E has no sinks, there exists an edge $e_1 \in E^1$ such that $s_E(e_1) = v$ and $r_E(e_1) \in E^0_{j+1}$. If $j + 1 = n$, then e_1 is a path from v to F^0. Assume that $j + 1 < n$. If $r_E(e_1)$ is not a source in E_i for each $i \geq j + 1$, then $r_E(e_1) \in F^0$ and e_1 is a path from v to F^0. If $r_E(e_1)$ is a source in E_i for some $i \geq j + 1$, then there exists an edge $e_2 \in E^1$ such that $s_E(e_2) = r_E(e_1)$ and $r_E(e_2) \in E^0_{i+1}$. Continuing in this way, since E is a finite graph, we get a path from v to F^0. Thus by Theorem 3.3, $L_R(E) \cong L_R(E(F^0))$. By definition, $E(F^0)^0 = F^0 \cup F(F^0)$. Since $(E^0\backslash F^0, r_E^{-1}(E^0\backslash F^0), r_E, s_E)$ is a finite acyclic graph, $F(F^0) = \{\alpha | \alpha = e_1e_2\cdots e_n \in \text{Path}(E), s_E(e_n) \notin F^0, r_E(e_n) \in F^0\}$ is a finite set. Assume that $F(F^0) = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ for some positive integer p. $E(F^0)^1 = s_E^{-1}(F^0) \cup \{\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_p}\}$. Let $r_{E(F^0)}(\overline{\alpha_1}) = r_E(\alpha_1) = v_1 \in F^0$ and assume that $r^{-1}_{E(F^0)}(v_1) = r_F^{-1}(v_1) \cup \{\overline{\alpha_{1_1}}, \overline{\alpha_{1_2}}, \ldots, \overline{\alpha_{1_i}}\}$ for some $1 \leq i \leq p$, where $\overline{\alpha_{1_1}} = \overline{\alpha_1}$. Removing the vertices $\alpha_{1_1}, \alpha_{1_2}, \ldots, \alpha_{1_i}$ of $E(F^0)$, we get a graph G_1 such that $G'_1(v_1, i) = E(F^0)$. By Corollary 3.6, $L_R(E(F^0)) \cong L_R(G_1(v_1, i))$. F has no sources, then there exists an edge $e_1 \in F^1$ such that $r_F(e_1) = r_E(e_1) = v_1$, and so by Proposition 3.7, $L_R(G_1(v_1, i)) \cong L_R(G_1(e_1, i))$. Therefore $L_R(E(F^0)) \cong L_R(G_1(e_1, i))$. The above procedure shows that $G_1(e_1, i)$ is a finite graph with no sinks and with $p - i$ sources. Continuing in this way, after finitely many steps we get a finite graph G with no sinks and no sources that $L_R(E) \cong L_R(G)$. □

Corollary 3.9. *Let E be a graph. Then the following statements are equivalent.*

(1) *E is a finite graph with no sinks.*

(2) *For every commutative unital ring R, $L_R(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

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(3) *There exists a field k such that $L_k(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

(4) *E is a finite graph and, for every commutative unital ring R, $L_R(E)$ is strongly $\mathbb{Z}$-graded.*

(5) *$C^*(E)$ is unital and* $\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E)))$.

Proof. (1) ⇒ (2) follows from Corollary 3.8.

(2) ⇒ (3) is obvious.

For (3) ⇒ (1), suppose that there exist a field k and a finite graph E' with no sinks and no sources such that $L_k(E) \cong \mathcal{CK}_k(E')$. Hence $L_k(E)$ is unital and so E has a finite number of vertices. Then, by [20, Corollary 6.17], E has no singular vertices. Therefore E is a finite graph with no sinks and the result follows.

(1) ⇔ (4) follows from [15, Theorem 3.15].

(1) ⇔ (5) follows from [9, Theorem 3.12]. □

Let $(G, +)$ be an abelian group. A finite set of elements $\{g_1, g_2, \ldots, g_l\} \subseteq G$ is called linearly independent if whenever $\sum_{i=1}^{l} n_i g_i = 0$ for $n_1, \ldots, n_l \in \mathbb{Z}$, then $n_i = 0$ for each $1 \le i \le l$. Any two maximal linearly independent sets in G have the same cardinality. If there exits a maximal linearly independent set in G, the cardinality of this set is called the rank $\operatorname{rank}(G)$ of G, and if there is no maximal linearly independent set in G, the rank $\operatorname{rank}(G)$ of G is defined to be infinite.

We are now ready to prove the main result of this section.

Theorem 3.10. *Let k be a field such that* $\operatorname{rank}(k^\times) < \infty$ *and E be a graph. Then the following statements are equivalent.*

(1) *E is a finite graph with no sinks.*

(2) *$L_k(E)$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

(3) *E is a finite graph and $L_k(E)$ is strongly $\mathbb{Z}$-graded.*

(4) *$C^*(E)$ is unital and* $\operatorname{rank}(K_1(C^*(E))) = \operatorname{rank}(K_0(C^*(E)))$.

(5) *$L_k(E)$ is unital and* $\operatorname{rank}(K_1(L_k(E))) = (\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E)))$.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) follows from Corollary 3.9.

For (2) ⇒ (5), suppose that there exists a finite graph E' with no sinks and no sources such that $L_k(E) \cong \mathcal{CK}_k(E')$. Hence $L_k(E)$ is unital, and since $\operatorname{rank}(k^\times) < \infty$, by [12, Theorem 8.1] we have $|E'^0_{\text{sing}}| = (\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E'))) - \operatorname{rank}(K_1(L_k(E')))$. Thus $(\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E'))) = \operatorname{rank}(K_1(L_k(E')))$ and so $(\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E))) = \operatorname{rank}(K_1(L_k(E)))$.

For (5) ⇒ (1), suppose that $L_k(E)$ is unital and $\operatorname{rank}(K_1(L_k(E))) = (\operatorname{rank}(k^\times) + 1)$ $\operatorname{rank}(K_0(L_k(E)))$. Thus E^0 is a finite set, and by [12, Theorem 8.1] we have $|E^0_{\text{sing}}| = (\operatorname{rank}(k^\times) + 1)\operatorname{rank}(K_0(L_k(E))) - \operatorname{rank}(K_1(L_k(E)))$. Since $\operatorname{rank}(k^\times) < \infty$, $\operatorname{rank}(K_1(L_k(E))) < \infty$ by [12, Theorem 8.1]. Thus $|E^0_{\text{sing}}| = 0$ and so E has no singular vertices. Therefore E is a finite graph with no sinks and the result follows. □

The following example shows that the assumption $\operatorname{rank}(k^\times) < \infty$ in the Theorem 3.10 is necessary.

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Example 3.11. Let E be the graph $\bullet_1 \xrightarrow{\alpha} \bullet_2$ and $\mathbb{Q}$ be the field of rational numbers. $\mathrm{rank}(\mathbb{Q}^\times) = \infty$ and $K_1(\mathbb{Q}) \cong \mathbb{Q}^\times$. $L_{\mathbb{Q}}(E) \cong \mathbb{M}_2(\mathbb{Q})$, and so $K_1(L_{\mathbb{Q}}(E)) \cong K_1(\mathbb{Q}) \cong \mathbb{Q}^\times$. Thus $\mathrm{rank}(K_1(L_{\mathbb{Q}}(E))) = (\mathrm{rank}(\mathbb{Q}^\times) + 1)\mathrm{rank}(K_0(L_{\mathbb{Q}}(E))) = \infty$, $L_{\mathbb{Q}}(E)$ is unital and E is a finite graph but E has a sink.

4. Corners of Leavitt path algebras

In this section we show that there exists a graph $E(T)$ for the corner $P_X L_R(E) P_X$ of a Leavitt path algebra $L_R(E)$ associated to a finite vertex set X, such that $P_X L_R(E) P_X \cong L_R(E(T))$.

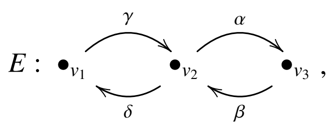
Let E be a graph. An acyclic subgraph T of E is called a directed forest in E if, for each $v \in T^0$, $|T^1 \cap r_E^{-1}(v)| \leqslant 1$. We denote by T^r the subset of T^0 consisting of those vertices v with $|T^1 \cap r_E^{-1}(v)| = 0$, and by T^l the subset of T^0 consisting of those vertices v with $|T^1 \cap s_E^{-1}(v)| = 0$.

Definition 4.1 [10, Definition 3.1]. Let E be a graph, $X \subsetneqq E^0$ be a finite set and T be a row-finite, path-finite directed forest in E with $T^r = X$ and $T^0 = H_E(X)$. Define the T-corner, $E(T)$ of E as follows:

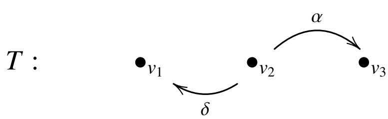
$$E(T)^0 = T^0 \backslash \{v | v \in T^0, \varnothing \neq s_E^{-1}(v) \subseteq T^1\},$$
$$E(T)^1 = \{e_u | e \in s_E^{-1}(T^0)\backslash T^1, u \in E(T)^0, r_E(e) \geq_T u\},$$
$$s_{E(T)}(e_u) = s_E(e), \quad r_{E(T)}(e_u) = u.$$

Let E be a graph, $X \subsetneqq E^0$ be a finite set. According to [10, Lemma 3.6] there is a forest T in E which satisfies the conditions of Definition 4.1 if and only if $H_E(X)$ is finite.

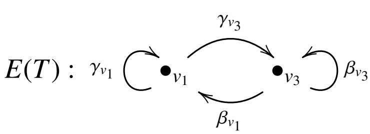
Example 4.2. Let



$X = \{v_2\}$ and



T is a row-finite, path-finite directed forest in E with $T^r = \{v_2\}$ and $T^0 = H_E(\{v_2\})$. Thus $E(T)$ is the following graph:



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Crisp in [10, Theorem 3.5] proved that $C^*(E(T)) \cong P_X C^*(E) P_X$, where $\sum_{v\in X} P_v$ converges strictly to a projection P_X in the multiplier algebra $M(C^*(E))$. In the following theorem we prove a similar result for Leavitt path algebras.

Theorem 4.3. *Let E be a graph, $X \subsetneq E^0$ be a finite set, T be a row-finite, path-finite directed forest in E with $T^r = X$ and $T^0 = H_E(X)$, $P_X = \sum_{x\in X} x$ and R be a commutative unital ring. Then $L_R(E(T)) \cong P_X L_R(E) P_X$. If in addition $L_R(E)$ is an algebraic Cuntz–Krieger algebra, then $L_R(E(T))$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

Proof. Let $\{e, v | e \in E^1, v \in E^0\}$ be the universal Cuntz–Krieger E-family. By [10, Lemma 2.1(i)], for any $v \in T^0$, there exists a unique path $\tau(v)$ in $\mathrm{Path}(T)$ such that $s_T(\tau(v)) \in T^r$ and $r_T(\tau(v)) = v$. For each $v \in T^0$, let $Q_v = \tau(v)\tau(v)^* - \sum_{e\in T^1\cap s^{-1}(v)} \tau(v)ee^*\tau(v)^*$. For each $e_u \in E(T)^1$, let $T_{e_u} = \tau(s(e))e\tau(r(e))^* Q_u$. An argument similar to that in [10, proof of the Proposition 3.8] shows that $\{T_{e_u}, Q_v | e_u \in E(T)^1, v \in E(T)^0\}$ is a Cuntz–Krieger $E(T)$-family in E. By the universal homomorphism property of $L_R(E(T))$ there exists a $*$-homomorphism $\Psi : L_R(E(T)) \to L_R(E)$ with $\Psi(v) = Q_v$ for each $v \in E(T)^0$ and $\Psi(e_u) = T_{e_u}$ for each $e_u \in E(T)^1$. Let $w : E^1 \to \mathbb{Z}$ be a weight map given by

$$w(e) = \begin{cases} l(\tau(r(e))) - l(\tau(s(e))) + 1 & \text{if } e \notin T^1; s(e), r(e) \in T^0, \\ 1 & \text{otherwise.} \end{cases}$$

Let $w(e^*) = -w(e)$ and $w(v) = 0$ for each $e \in E^1$ and $v \in E^0$. Thus $L_R(E)$ is a $\mathbb{Z}$-graded algebra. Also $L_R(E(T))$ is a $\mathbb{Z}$-graded algebra with the usual $\mathbb{Z}$-grading. We show that Ψ is a $\mathbb{Z}$-graded ring homomorphism. For each $v \in E(T)^0$, $\Psi(v) = Q_v = \tau(v)\tau(v)^* - \sum_{e\in T^1\cap s^{-1}(v)} \tau(v)ee^*\tau(v)^*$ and so the degree of $\Psi(v)$ is zero. For each $e_u \in E(T)^1$, $\Psi(e_u) = T_{e_u} = \tau(s(e))e\tau(r(e))^* Q_u$. $\tau(s(e)), \tau(r(e)) \in \mathrm{Path}(T)$ and so $\tau(s(e))e\tau(r(e))^* Q_u$ is homogeneous of degree $l(\tau(s(e))) + l(\tau(r(e))) - l(\tau(s(e))) + 1 - l(\tau(r(e))) = 1$. Thus Ψ is a $\mathbb{Z}$-graded ring homomorphism.

We now show that, for each $v \in E(T)^0$ and each $r \in R\backslash\{0\}$, $\Psi(rv) \neq 0$. Let $v \in E(T)^0$ and $r \in R\backslash\{0\}$. $\Psi(rv) = rQ_v = r(\tau(v)\tau(v)^* - \sum_{e\in T^1\cap s^{-1}(v)} \tau(v)ee^*\tau(v)^*)$. By Definition 4.1, either v is a sink in E or v emits an edge $f \in E^1\backslash T^1$. If v is a sink in E, then $Q_v = \tau(v)\tau(v)^*$ and, by [21, Proposition 4.9], for each $r \in R\backslash\{0\}$, $rQ_v \neq 0$. Now assume that v emits an edge $f \in E^1\backslash T^1$. If $rQ_v = 0$ for some $r \in R\backslash\{0\}$, then $r\tau(v)\tau(v)^* - r\sum_{e\in T^1\cap s^{-1}(v)} \tau(v)ee^*\tau(v)^* = 0$. Thus

$$\Big(r\tau(v)\tau(v)^* - r\sum_{e\in T^1\cap s^{-1}(v)} \tau(v)ee^*\tau(v)^*\Big)\tau(v)ff^*\tau(v)^* = 0.$$

Since $e \in T^1$ and $f \notin T^1$, $e^*f = 0$. Hence $r\tau(v)ff^*\tau(v)^* = 0$ and so $r\tau(v)f = 0$. This leads to a contradiction with [21, Proposition 4.9]. Thus $\Psi(rv) \neq 0$ for each $v \in E(T)^0$ and each $r \in R\backslash\{0\}$, and, by the graded uniqueness theorem, Ψ is injective. An argument similar to that in [10, proof of the Proposition 3.11] shows that $\Psi(L_R(E(T))) = P_X L_R(E) P_X$ and so $L_R(E(T)) \cong P_X L_R(E) P_X$.

Now suppose that $L_R(E)$ is an algebraic Cuntz–Krieger algebra. Thus E is a finite graph with no sinks and no sources. Since E is a finite graph, any directed forest in E is finite and so $E(T)$ is a finite graph. Assume, on the contrary, that $v \in E(T)^0$

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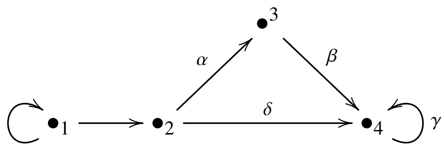
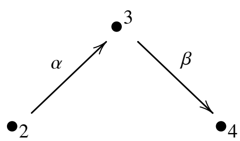
and $s_{E(T)}^{-1}(v) = \emptyset$. Since E has no sinks, $s_E^{-1}(v) \neq \emptyset$. Thus by the definition of $E(T)$, $s_E^{-1}(v) \nsubseteq T^1$. Therefore there exists $e \in E^1 \backslash T^1$ such that $s_E(e) = v$. If $r_E(e) \in E(T)^0$, then $e_{r_E(e)} \in E(T)^1$ and $s_{E(T)}(e_{r_E(e)}) = v$, which is a contradiction. Thus $r_E(e) \notin E(T)^0$. Since $T^0 = H_E(X)$ and $v \in T^0$, $r_E(e) \in T^0$. Also $s_E^{-1}(r_E(e)) \neq \emptyset$, then there exists $e_1 \in T^1$ with $s_E(e_1) = r_E(e)$. Let $r_E(e_1) = v_1$. If $v_1 \in E(T)^0$, then $s_{E(T)}(e_{v_1}) = v$, which is a contradiction. Thus $v_1 \in T^0 \backslash E(T)^0$. A similar argument shows that there exists $e_2 \in T^1$ with $s_E(e_2) = v_1$. Since T is an acyclic graph, by continuing in this way we get an infinite path $e_1 e_2 e_3 \cdots$ in T, which is a contradiction. Thus $E(T)$ is a finite graph with no sinks, and, by Corollary 3.9, $L_R(E(T))$ is isomorphic to an algebraic Cuntz–Krieger algebra. □

Let E be a finite graph with no sinks and no sources. In the proof of the above Theorem we show that $E(T)$ is a finite graph with no sinks. The following example shows that there exists a finite graph E with no sinks and no sources such that $E(T)$ has a source.

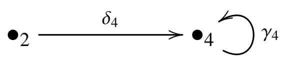
Example 4.4. Let E be the graph



$X = \{2\}$ and T be the directed forest

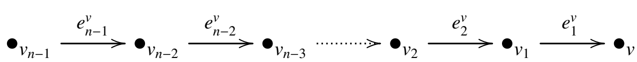


Thus $E(T)$ is the following graph:



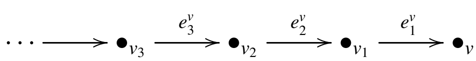
Definition 4.5 [3, Definitions 9.1 and 9.4]. Let E be a graph.

(1) Define $M_n E$ to be the graph formed from E by taking each $v \in E^0$ and attaching a head of length $n - 1$ of the form



to E.

(2) Define SE to be the graph formed from E by taking each $v \in E^0$ and attaching an infinite head of the form



to E. SE is called the stabilization of E.

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Let R be a ring. The ring of finitely supported, countably infinite square matrices with coefficients in R is denoted by $\mathbb{M}_\infty(R)$ [3, Definition 9.6]. Note that if R is an algebra (respectively, a $*$-algebra), then $\mathbb{M}_\infty(R)$ inherits an algebra (respectively, a $*$-algebra) structure. Algebras A and B are called stably isomorphic if $\mathbb{M}_\infty(A) \cong \mathbb{M}_\infty(B)$. Abrams and Tomforde in [3, Propositions 9.3 and 9.8] proved that, for any graph E and field k, $L_k(M_nE) \cong \mathbb{M}_n(L_k(E))$ and $L_k(SE) \cong \mathbb{M}_\infty(L_k(E))$. An argument similar to that in [3, proof of Propositions 9.3 and 9.8], with a commutative unital ring R in place of field k, shows that $L_R(M_nE) \cong \mathbb{M}_n(L_R(E))$ and $L_R(SE) \cong \mathbb{M}_\infty(L_R(E))$.

Corollary 4.6. *Let E be a graph and R be a commutative unital ring.*

(1) *Let X be a finite subset of $(SE)^0$ such that $H_{SE}(X)$ is a finite set and $e_X = \sum_{x\in X} x$. Then there exists a row-finite, path-finite directed forest T in SE such that $L_R(SE(T)) \cong e_X L_R(SE) e_X$. If in addition $L_R(SE)$ is an algebraic Cuntz–Krieger algebra, then $e_X L_R(SE) e_X$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

(2) *Let n be a positive integer, X be a finite subset of $(M_nE)^0$ such that $H_{M_nE}(X)$ is a finite set and $e_X = \sum_{x\in X} x$. Then there exists a row-finite, path-finite directed forest T in M_nE such that $L_R(M_nE(T)) \cong e_X L_R(M_nE) e_X$. If in addition $L_R(M_nE)$ is an algebraic Cuntz–Krieger algebra, then $e_X L_R(M_nE) e_X$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

Proof. Since $H_{SE}(X)\backslash X$ (respectively, $H_{M_nE}(X)\backslash X$) is a finite set, by [10, Lemma 3.6] there is a row-finite, path-finite directed forest T in SE (respectively, M_nE) which satisfies the conditions of Theorem 4.3. Thus the result follows from Theorem 4.3. □

An idempotent e of an algebra A is called full idempotent if $AeA = A$.

Remark 4.7. In Corollary 4.6, if in addition we assume that E is a graph with finitely many vertices and $E^0 \subseteq X$, then the smallest saturated subset of $(SE)^0$ (respectively, $(M_nE)^0$) containing X is $(SE)^0$ (respectively, $(M_nE)^0$), and so e_X is a full idempotent.

Proposition 4.8. *Let R be a commutative unital ring and A be an algebraic Cuntz–Krieger R-algebra. Then $\mathbb{M}_n(A)$ is isomorphic to an algebraic Cuntz–Krieger algebra for each positive integer n.*

Proof. Let E be a finite graph with no sinks and no sources such that $A = L_R(E)$, and let n be a positive integer. M_nE is a finite graph with no sinks and so, by Corollary 3.9, $L_R(M_nE)$ is isomorphic to an algebraic Cuntz–Krieger algebra. Thus $\mathbb{M}_n(L_R(E)) \cong L_R(M_nE)$ is isomorphic to an algebraic Cuntz–Krieger algebra. □

5. Algebras that are Morita equivalent to algebraic Cuntz–Krieger algebras

In this section we show that if a unital algebra A is stably isomorphic to an algebraic Cuntz–Krieger algebra, then A is isomorphic to an algebraic Cuntz–Krieger algebra. Also we show that if A is Morita equivalent to an algebraic Cuntz–Krieger algebra, then A is isomorphic to an algebraic Cuntz–Krieger algebra.

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Definition 5.1. Let R be a commutative unital ring, A be an R-algebra, $e^2 = e \in \mathbb{M}_n(A)$ and $f^2 = f \in \mathbb{M}_m(A)$. e is called Murray–von Neumann equivalent to f, denoted $e \sim f$, if there exist $x \in \mathbb{M}_{m,n}(A)$ and $y \in \mathbb{M}_{n,m}(A)$ such that $e = yx$ and $f = xy$.

Example 5.2. Let R be a commutative unital ring and E be a graph. Let $v \in E^0$ be a regular vertex. Thus, by the Cuntz–Krieger relations, we have $v \sim \sum_{e\in s_E^{-1}(v)} r_E(v)$.

For an idempotent $e \in A$ and a positive integer n, ne denotes the idempotent $M \in \mathbb{M}_n(A)$ such that $M = (m_{ij})$, $m_{ii} = e$ for each i and $m_{ij} = 0$ for each $i \neq j$.

The proof of the following lemma is similar to the proof of [9, Lemma 4.6], and we give the proof for the reader's convenience.

Lemma 5.3. *Let k be a field, E be a finite graph with no sinks and no sources such that every vertex of E is a base point of at least one cycle of length one, $\{v, e | v \in E^0, e \in E^1\}$ be a Cuntz–Krieger E-family and f be a full idempotent of $\mathbb{M}_n(L_k(E))$. Then there exists a set $\{m_v | v \in E^0, m_v \geq 1\}$ of positive integers such that $f \sim \sum_{v\in E^0} m_v v$.*

Proof. By [6, Theorem 3.5], there exists a set $\{n_v | v \in E^0, n_v \geq 0\}$ of nonnegative integers such that $f \sim \sum_{v\in E^0} n_v v$. Let H_0 be the smallest hereditary subset of E^0 which contains $S_0 = \{v | v \in E^0, n_v \neq 0\}$. By [9, Lemma 4.5], H_0 is saturated. Put $g = \sum_{v\in S_0} v \in I_{H_0}$. Since $f \sim \sum_{v\in E^0} n_v v$, the ideal generated by f is equal to the ideal generated by $e_{11} \otimes g$, where $\{e_{ij}\}_{i,j}$ is a system of matrix units. Thus $e_{11} \otimes g$ is a full idempotent in $\mathbb{M}_n(L_k(E))$ and so g is a full idempotent of $L_k(E)$. Thus $I_{H_0} = L_k(E)$ and hence $H_0 = E^0$. Thus, for each $w \in E^0$, there exists $v \in S_0$ such that $v \geq w$. Put $E^0 \backslash S_0 = \{w_0, w_1, \ldots, w_m\}$. Let $v \in S_0$ such that $v \geq w_0$. An argument similar to that in [9, proof of Lemma 4.3] shows that $v \sim w_0 + \sum_{u\in E^0} m_u(v, w_0)u$, where $m_u(v, w_0) \geq 0$ and $m_v(v, w_0) \geq |\{e \in E^1 | s_E(e)r_E(e) = v\}| \geq 1$. Hence, by using such equations for all $w_0, \ldots, w_m$, we obtain $f \sim \sum_{v\in E^0} n'_v v$, where $n'_v \geq 1$ for all $v \in E^0$. □

Proposition 5.4. *Let k be a field, E be a finite graph with no sinks and no sources, n be a positive integer and e be a full idempotent of $\mathbb{M}_n(L_k(E))$. Then there exists a finite graph F that has no sinks and no sources such that $L_k(F) \cong e\mathbb{M}_n(L_k(E))e$.*

Proof. First, we assume that every vertex of E is a base point of at least one cycle of length one. By [3, Proposition 9.3] and its proof, there exists an isomorphism $\Phi : \mathbb{M}_n(L_k(E)) \to L_k(M_nE)$ such that, for each $v \in E^0$, $K_0(\Phi)([e_{11} \otimes v]) = [v]$. Let e be a full idempotent of $\mathbb{M}_n(L_k(E))$; thus, by Lemma 5.3, $e \sim \sum_{v\in E^0} m_v v$ with $m_v \geq 1$ for all $v \in E^0$. Since $L_k(M_nE)$ is separative by [4, Theorem 6.3], $\Phi(e)$ is Murray–von Neumann equivalent to $e_X \in L_k(M_nE)$ such that X is a finite, hereditary subset of $(M_nE)^0$ with $E^0 \subseteq X$. By Corollary 4.6, $e_X L_k(M_nE)e_X \cong L_k(F)$ for some finite graph F with no sinks and no sources. Thus $e\mathbb{M}_n(L_k(E))e \cong \Phi(e)L_k(M_nE)\Phi(e) \cong e_X L_k(M_nE)e_X \cong L_k(F)$ and the result follows. Now let E be a finite graph with no sinks and no sources. Since M_nE is a finite graph with no sinks, we can apply [18, Theorem 3.1] to get a finite graph G with no sinks and no sources, and every vertex of G is a base point of at least one cycle of length one, such that $L_k(M_nE)$ and $L_k(G)$ are Morita equivalent. Therefore there exist a positive integer m and full idempotent f of $\mathbb{M}_m(L_k(G))$ such that $L_k(M_nE) \cong f\mathbb{M}_m(L_k(G))f$. Therefore $e\mathbb{M}_n(L_k(E))e \cong f'\mathbb{M}_m(L_k(G))f'$ for some full idempotent f' of $\mathbb{M}_m(L_k(G))$. Thus by the first case there exists a finite graph F that has no sinks and no sources such that $L_k(F) \cong f'\mathbb{M}_m(L_k(G))f' \cong e\mathbb{M}_n(L_k(E))e$, and the result follows. □

Corollary 5.5. *Let k be a field and A be a unital k-algebra which is Morita equivalent to an algebraic Cuntz–Krieger k-algebra. Then A is isomorphic to an algebraic Cuntz–Krieger algebra.*

Proof. Let E be a finite graph with no sinks and no sources such that A is Morita equivalent to the algebraic Cuntz–Krieger algebra $L_k(E)$. Therefore there exist a positive integer n and full idempotent e of $\mathbb{M}_n(L_k(E))$ such that $A \cong e\mathbb{M}_n(L_k(E))e$. Therefore, by Proposition 5.4, there exists a finite graph F that has no sinks and no sources such that $L_k(F) \cong e\mathbb{M}_n(L_k(E))e \cong A$. □

Corollary 5.6. *Let k be a field and A be a unital k-algebra which is stably isomorphic to an algebraic Cuntz–Krieger k-algebra. Then A is isomorphic to an algebraic Cuntz–Krieger algebra.*

Proof. By [3, Proposition 9.10], A is Morita equivalent to an algebraic Cuntz–Krieger k-algebra. Therefore the result follows by Corollary 5.5. □

Corollary 5.7. *Let k be a field and A be a k-algebra. Then the following statements are equivalent.*

(1) *A is an algebraic Cuntz–Krieger algebra.*
(2) *$\mathbb{M}_n(A)$ is isomorphic to an algebraic Cuntz–Krieger algebra for each $n \in \mathbb{N}$.*
(3) *$\mathbb{M}_n(A)$ is isomorphic to an algebraic Cuntz–Krieger algebra for some $n \in \mathbb{N}$.*

Proof. (1) ⇒ (2) follows from Proposition 4.8.

(2) ⇒ (3) is obvious.

(3) ⇒ (1) Assume that $\mathbb{M}_n(A)$ is isomorphic to an algebraic Cuntz–Krieger algebra for some $n \in \mathbb{N}$. Thus $\mathbb{M}_n(A)$ is unital and so A is a unital k-algebra. Since A is stably isomorphic to $\mathbb{M}_n(A)$, the result follows from Corollary 5.6. □

Corollary 5.8. *Let k be a field, A be an algebraic Cuntz–Krieger k-algebra and e be a nonzero idempotent of A. Then eAe is isomorphic to an algebraic Cuntz–Krieger algebra.*

Proof. Let $A = L_k(E)$ where E is a finite graph with no sinks and no sources. Let e be a nonzero idempotent of A and I be the ideal in $L_k(E)$ generated by e. As $eAe \subseteq I$ we have $eAe \subseteq eIe$ and so $eIe = eAe$. Since I is generated by an idempotent e, I is a graded ideal of $L_k(E)$ by [6, proof of Proposition 5.2 and Theorem 5.3]. Therefore there exists a hereditary saturated subset H of E^0 such that $I = I_H$. Let $E_H = (H, s_E^{-1}(H), r_E, s_E)$. E_H is a finite graph with no sinks, and, by [8, Lemma 2.4], $L_k(E_H)$ is Morita equivalent to I_H. Thus, by Corollaries 3.9 and 5.5, I_H is isomorphic to the algebraic Cuntz–Krieger algebra B. Hence $eAe = eI_He$ is isomorphic to fBf for some full idempotent f of B.

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By Proposition 5.4, there exists a finite graph F that has no sinks and no sources such that $L_k(F) \cong fBf$. Thus $eAe \cong L_k(F)$ and the result follows. □

COROLLARY 5.9. *Let k be a field, A be an algebraic Cuntz–Krieger k-algebra and e be a nonzero idempotent of $\mathbb{M}_\infty(A)$. Then $e\mathbb{M}_\infty(A)e$ is isomorphic to an algebraic Cuntz–Krieger algebra.*

PROOF. We use the same argument as in [9, proof of Corollary 4.10]. Let $A = L_k(E)$ where E is a finite graph with no sinks and no sources. By [6, Theorem 3.5], there exists a set $\{n_v | v \in E^0, n_v \geq 0\}$ of nonnegative integers such that $e \sim \sum_{v \in E^0} n_v v$. Let $X = \{v \in E^0 | n_v \neq 0\}$ and $f = \sum_{v \in X} v$. Thus f is a nonzero idempotent of $L_k(E)$, and, by Corollary 5.8, there exists a finite graph F that has no sinks and no sources such that $L_k(F) \cong fL_k(E)f$. By [6, Theorem 5.3], the ideal of $\mathbb{M}_\infty(A)$ generated by e and the ideal of $\mathbb{M}_\infty(A)$ generated by $e_{11} \otimes f$ coincide. Thus $fAf \otimes \mathbb{M}_\infty(k) \cong (e_{11} \otimes f)\mathbb{M}_\infty(A)(e_{11} \otimes f) \otimes \mathbb{M}_\infty(k) \cong e\mathbb{M}_\infty(A)e \otimes \mathbb{M}_\infty(k)$. Therefore $e\mathbb{M}_\infty(A)e$ is stably isomorphic to an algebraic Cuntz–Krieger algebra and the result follows from Corollary 5.6. □

Acknowledgement

The author would like to thank the referee for the careful reading of this paper and for making many helpful suggestions that improved the presentation of the paper.

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