Proceedings of the Royal Society of Edinburgh, **133A**, 1389–1407, *2003*

On p-adic valuations of $L(1)$ of elliptic curves with CM by $\sqrt{-3}$

Derong Qiu*
Center for Advanced Study, Tsinghua University, Beijing 100084, People's Republic of China (derong@castu.tsinghua.edu.cn)

(MS received 12 February 2002; accepted 30 June 2003)

For positive rational integers λ, we study the Hecke L-series attached to elliptic curves $y^2 = x^3 - 2^4 3^3 D^\lambda$ over the quadratic field $\mathbb{Q}(\sqrt{-3})$ and obtain various bounds of $p(= 2, 3)$-adic valuations of their values at $s = 1$ according to the cases of D and λ. In particular, for the case of even λ, we obtain a criterion of reaching the bounds of 3-adic valuations. From this, combining with the work of Coates and Wiles and Rubin, we obtain some results about the conjecture of Birch and Swinnerton-Dyer of these curves.

1. Introduction

We consider the $p(= 2, 3)$-adic valuations of $L(1)$ of the Hecke L-series attached to elliptic curves

$$E_{D^\lambda} : y^2 = x^3 - 2^4 3^3 D^\lambda,$$

where D is a square-free integer in $\mathbb{Q}(\sqrt{-3})$ and $\lambda \in \mathbb{Z}$ is a positive rational integer not divisible by 6. The 3-power divisibility problems of $L(1)$ have been studied in [5, 6, 10] in some cases. In fact, Stephens [10] proved that if $D > 2$ is a cube-free rational integer, then $3^{1/2} D^{1/3} L(E_{D^2}/\mathbb{Q}, 1)/\Omega$ is a rational integer, divisible by 3 when $9 \mid D$. And recently (see [5, 6]) we proved that if D is a square-free integer in $\mathbb{Q}(\sqrt{-3})$ with n distinct prime factors, then $L(\bar{\psi}_{D^2}, 1)/\Omega$ is divisible by $3^{n/2-1}$, where ψ_{D^2} is the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to E_{D^2}.

In this paper, for E_{D^λ} over $\mathbb{Q}(\sqrt{-3})$, we further study the 3-adic valuation of $L(1)$ in the case of $\lambda = 2, 4$, and study the 2-adic valuation of $L(1)$ in the case of $\lambda = 3$. We obtain lower bounds of these valuations. In particular, for $\lambda = 2, 4$, we obtain a criterion of reaching the bound of 3-adic valuation. Then, by the results of Coates and Wiles [2] and Rubin [7], we can deduce some results about the conjecture of Birch and Swinnerton-Dyer of these curves. For example, if $D \equiv 1 \pmod 6$ is a prime integer of $\mathbb{Q}(\sqrt{-3})$, and the 3-adic valuation of the difference of its complex conjugate and 1 is equal to 1, then the Mordell–Weil group and Shafarevich–Tate group of E_{D^λ} over $\mathbb{Q}(\sqrt{-3})$ are finite. In particular, if D is a rational integer, the first part of the conjecture of Birch and Swinnerton-Dyer for E_{D^λ} over $\mathbb{Q}$ is shown to be true in some cases (see corollary 2.4 below).

*Present address: Department of Mathematics, Tsinghua University, Beijing 100084, People's Republic of China (dqiu@math.tsinghua.edu.cn).

2. Main results

Throughout this paper, let $\tau = \frac{1}{2}(-1+\sqrt{-3})$ be a primitive cubic root of unity and $O_K = \mathbb{Z}[\tau]$ the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{-3})$. We study the elliptic curves

$$E_{D^\lambda} : y^2 = x^3 - 2^4 3^3 D^\lambda, \quad \text{with } D = \pi_1 \cdots \pi_n,$$

where π_k are distinct prime elements in O_K $(k = 1,\dots,n)$, $\lambda = 1,\dots,5$. Let ψ_{D^λ} be the Hecke character (i.e. Grössencharacter) of K attached to the elliptic curve E_{D^λ} : $y^2 = x^3 - 2^4 3^3 D^\lambda$, and $L(\bar{\psi}_{D^\lambda}, s)$ the Hecke L-series of $\bar{\psi}_{D^\lambda}$ (the complex conjugate of ψ_{D^λ}). For the definition of such a Hecke L-series attached to an elliptic curve, see [9]. Also, we let Ω_0 denote the real period of the Weierstrass Σ-function satisfying the differential equation $\Sigma'(z)^2 = 4\Sigma(z)^3 - 1$ (as stated in lemma 2.1 below). Then, from [2,7], it can easily be seen that the values $\Omega_0^{-k} L(\bar{\psi}^k_{D^\lambda}, k)$ $(k = 1, 2, \dots)$ are all algebraic numbers. We will discuss the case $k = 1$.

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at p-adic valuation for any rational prime p, $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ be the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, and let v_p be the normalized p-adic additive valuation of $\bar{\mathbb{Q}}_p$ (i.e. $v_p(p) = 1$). Fix an isomorphic embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then $v_p(\alpha)$ is defined for any algebraic number α in $\bar{\mathbb{Q}}$. The value $v_p(\alpha)$ for $\alpha \in \bar{\mathbb{Q}}$ depends on the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, but this does not affect our discussion in this paper. We will discuss two cases: $p = 2$ and $p = 3$.

In order to state our main results, we first give a fundamental lemma to express $L(1)$ by the values of Weierstrass Σ-functions. To see this, let $S = \{\pi_1, \dots, \pi_n\}$. For any subset T of the set $\{1, \dots, n\}$, define

$$D_T = \prod_{k \in T} \pi_k, \qquad \hat{D}_T = \frac{D}{D_T}$$

and put $D_\emptyset = 1$ when $T = \emptyset$ (empty set). Let $\psi_{D_T^\lambda}$ be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D_T^\lambda} : y^2 = x^3 - 2^4 3^3 D_T^\lambda$, and let $L_S(\bar{\psi}_{D_T^\lambda}, s)$ be the Hecke L-series of $\bar{\psi}_{D_T^\lambda}$ (the complex conjugate of $\psi_{D_T^\lambda}$) with the Euler factors omitted at all primes in S. We have the following uniform formulae for special values $L_S(\bar{\psi}_{D_T^\lambda}, 1)$ of the above Hecke L-series at $s = 1$, which are expressed by the values of Weierstrass Σ-functions.

LEMMA 2.1. *Let $\lambda = 1, \dots, 5$, and let $D = \pi_1 \cdots \pi_n$, where $\pi_k \equiv 1 \pmod{6 \times 2^{\sigma(\lambda)}}$ are distinct prime elements of $\mathbb{Z}[\tau]$ $(k = 1, \dots, n)$. Then, for any factor D_T of D and the corresponding Hecke character $\psi_{D_T^\lambda}$ as defined above, we have*

$$\frac{D}{\Omega_0}\left(\frac{3 \times 4^{\sigma(\lambda)}}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda}, 1) = \frac{1}{2\sqrt{3}} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda \frac{1}{\Sigma(4^{\sigma(\lambda)} c \Omega_0 / D) - 1} + \frac{1}{3\sqrt{3}} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda,$$

where $\sigma(\lambda) = \frac{1}{2}(1 - (-1)^\lambda)$, $(\cdot)_6$ is the sextic residue symbol, $\mathcal{C}$ is any complete set of representatives of the relatively prime residue classes of O_K modulo D, $\Sigma(z)$ is *the Weierstrass Σ-function satisfying $\Sigma'(z)^2 = 4\Sigma(z)^3 - 1$ with period lattice $L_{\Omega_0} = \Omega_0 O_K$ (corresponding to the elliptic curve $y^2 = x^3 - \frac{1}{4}$) and $\Omega_0 = 3.059\,908\ldots$ is an absolute constant.*

2.1. The 3-adic valuation of $L(1)$ of E_{D^λ} in the even-λ case

Now we come to state our main results. The first one is about the 3-adic valuation of $L(1)$ of E_{D^λ} with even λ.

Let $\lambda = 2, 4$. For $D = \pi_1 \cdots \pi_n$ as above, we define the functions $\delta_n^{(\lambda)}(D)$ (where $n = n(D)$ is the number of distinct prime factors of D in $\mathbb{Z}[\tau]$) inductively as follows,

$$\delta_1^{(\lambda)}(\pi) = \bar{\pi} - 1 \quad (n = 1,\ D = \pi), \quad \text{where } \bar{\pi} \text{ is the complex conjugate of } \pi,$$

$$\delta_n^{(\lambda)}(D) = \delta_n^{(\lambda)}(\pi_1 \cdots \pi_n) = \sum_{\emptyset \neq T \subsetneqq \{1,\ldots,n\}} (-1)^{n+1-t} \cdot \left(\prod_{\pi_k \mid \hat{D}_T} \left(\pi_k - \left(\frac{D_T}{\pi_k} \right)_6^{\lambda} \right) \right) \cdot \delta_t^{(\lambda)}(D_T) \quad (n \geqslant 2),$$

where T runs over all non-trivial subsets of $\{1, \ldots, n\}$ and $t = t(T) = \#T$ is the cardinal of T.

THEOREM 2.2. *For $\lambda = 2, 4$, let $D = \pi_1 \cdots \pi_n$, where $\pi_k \equiv 1 \pmod 6$ are distinct prime elements in $\mathbb{Z}[\tau]$ $(k = 1, \ldots, n)$, and let ψ_{D^λ} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D^\lambda} : y^2 = x^3 - 2^4 3^3 D^\lambda$. Then, for the 3-adic valuation of the algebraic number $L(\bar{\psi}_{D^\lambda}, 1)/\Omega_0$, we have*

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega_0) \geqslant \tfrac{1}{2}n - 1,$$

and the equality holds if and only if

$$v_3(\delta_n^{(\lambda)}(D)) = \tfrac{1}{2}(n + 1).$$

REMARK 2.3. The lower bound for the case $\lambda = 2$ was obtained in [5,6]. However, the condition for reaching the bound was not determined there. Here, for $\lambda = 2$ as well as the new case $\lambda = 4$, by a new method, we solve this problem when obtaining the lower bounds (see the proof in the following).

COROLLARY 2.4.

(i) *Let λ, D be as in theorem 2.2. If*

$$v_3(\delta_n^{(\lambda)}(D)) = \tfrac{1}{2}(n + 1),$$

then the Mordell–Weil group $E(K)$ and the Shafarevich–Tate group Sha(E/K) *are finite and there is a $u \in O_K[\frac{1}{6}]^\times$ such that*

$$\#(\mathrm{Sha}(E/K)) = u \cdot (\#E(K))^2 \cdot \frac{L(E/K, 1)}{\Omega\bar{\Omega}},$$

where the elliptic curve $E = E_{D^\lambda} : y^2 = x^3 - 2^4 3^3 D^\lambda$ and $\Omega \in \mathbb{C}^\times$ generates the period lattice of a minimal model of E/K.

(ii) *In particular, if D in (i) is a rational integer, and $v_3(\delta_n^{(\lambda)}(D)) = \frac{1}{2}(n+1)$, then the first part of the conjecture of Birch and Swinnerton-Dyer is true for the elliptic curve $E_{D^\lambda} : y^2 = x^3 - 2^4 3^3 D^\lambda$ over $\mathbb{Q}$, that is,*

$$\operatorname{rank}(E_{D^\lambda}(\mathbb{Q})) = \operatorname{ord}_{s=1}(L(E_{D^\lambda}/\mathbb{Q}, s)) = 0,$$

where $n = n(D)$ is the number of distinct prime factors of D in $\mathbb{Z}[\tau]$.

Proof of corollary 2.4. (i) If $v_3(\delta_n^{(\lambda)}(D)) = \frac{1}{2}(n+1)$, then, by theorem 2.2, we get

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega_0) = \tfrac{1}{2}n - 1.$$

In particular, $L(\bar{\psi}_{D^\lambda}, 1) \neq 0$, so, by [7, theorem 11.1], we obtain the result.

(ii) If D is a rational integer, then $L(\psi_{D^\lambda}, 1) = L(\bar{\psi}_{D^\lambda}, 1)$, which is equal to $L(E_{D^\lambda}/\mathbb{Q}, 1)$ up to a finite number of non-zero Euler factors at $s = 1$ (see [2]). So, if $v_3(\delta_n^{(\lambda)}(D)) = \frac{1}{2}(n+1)$, then, by theorem 2.2, we get $L(E_{D^\lambda}/\mathbb{Q}, 1) \neq 0$, and then, by [2], we obtain the result. □

Remark 2.5. There is much literature studying the 3-descent on elliptic curves $y^2 = x^3 + A$. Unfortunately, at present, we cannot obtain good enough results about the Selmer group of E_{D^λ} via 3-isogeny under the condition of our theorem 2.2. If this could be well established, then, by theorem 2.2, some further results about the full conjecture of Birch and Swinnerton-Dyer for E_{D^λ} may be obtained.

Proposition 2.6. *If $\pi_k \equiv 1 \pmod{18}$ for all $k = 1, \dots, n$, then, for ψ_{D^λ} in theorem 2.2, we have*

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega_0) \geqslant \tfrac{1}{2}(n-1).$$

2.2. The 2-adic valuation of $L(1)$ of E_{D^3}

Our second main result concerns the 2-adic valuation of the special values at $s = 1$ of the Hecke L-series attached to E_{D^λ} with $\lambda = 3$.

Theorem 2.7. *Let $D = \pi_1 \cdots \pi_n$, where $\pi_k \equiv 1 \pmod{12}$ are distinct prime elements of $\mathbb{Z}[\tau]$ ($k = 1, \dots, n$), and let ψ_{D^3} be the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_{D^3} : y^2 = x^3 - 2^4 3^3 D^3$. Then, for the 2-adic valuation of $L(\bar{\psi}_{D^3}, 1)/\Omega_0$, we have*

$$v_2(L(\bar{\psi}_{D^3}, 1)/\Omega_0) \geqslant n.$$

Remark 2.8.

(i) Our results in theorem 2.7 are consistent with the predictions of the conjecture of Birch and Swinnerton-Dyer in a certain sense. In fact, by the methods in [11], it is easy to verify that, under our hypothesis of D, the Tamagawa factor $c_v = 1, 2, 4$ for any finite place v satisfying $v \mid N_E$ and $v \nmid 6$, where N_E is the conductor of $E = E_{D^3}$ in theorem 2.7. Let $\Omega \in \mathbb{C}^\times$ be an O_K-generator of the period lattice of a minimal model of E. Consider the case $L(E/K, 1) \neq 0$. (The case $L(E/K, 1) = 0$ does not need to be considered since $v_2(L(E/K, 1)) = \infty$.) Then the conjecture of Birch and Swinnerton-Dyer [1,8] predicts that $L(E/K, 1)/\Omega\bar{\Omega}$ has the factor

$$\prod_{v \mid N(E) \text{ and } v \nmid 6} c_v = 2^{m_2} \quad \text{for certain exponent } m_2.$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Furthermore, when the number $n(D)$ of distinct prime factors of D becomes greater, $N(E)$ would have more prime factors v, so m_2 would become greater. This is consistent with our results of theorem 2.7, since we have (see, for example, [9])

$$L(E/K,1)=L(\psi,1)L(\bar{\psi},1), \quad \text{where } \psi=\psi_{D^3}.$$

(ii) For the families of elliptic curves E_{D^λ} in lemma 2.1 with $\lambda=1,5$, it can also be easily verified by the same methods in [11] that the Tamagawa factor $c_v=1$ for any finite place $v \mid N_E$ and $v \nmid 6$, where N_E is the conductor of $E=E_{D^\lambda}$. This means that the product of the Tamagawa factors

$$\prod_{v|N(E) \text{ and } v\nmid 6} c_v$$

does not increase. So we do not consider the problem of $p(=2,3)$-adic valuation for these cases.

(iii) For the elliptic curves $E=E_{D^\lambda}$ in lemma 2.1 with $\lambda=2,4$, which were studied in [5,6] and theorem 2.2, it can be easily shown that the product of the Tamagawa factors

$$\prod_{v|N(E) \text{ and } v\nmid 6} c_v = 3^{m_3} \quad \text{for certain exponent } m_3.$$

Furthermore, m_3 varies depending on the number of distinct prime factors of D. This was exactly the problem of 3-adic valuation that we studied in [5,6] and theorem 2.2 in this paper for these cases.

3. Proofs of the main results

Proof of lemma 2.1. The case of $\lambda=2$ has been proved in [5,6], and the case of $\lambda=4$ can be similarly proved. Now we give a sketch of the proof for the odd-λ cases. So we let $\lambda=1,3,5$. By the method in [11], it can be verified that the conductor of $\psi_{D_T^\lambda}$ is $(\sqrt{-3}D_T)$ or $(3D_T)$ (as integral ideals of O_K). Then, by the results (especially proposition 5.5) in [3], as done in [5,6], we can get

$$\frac{3D}{\Omega_0}\cdot L_S(\bar{\psi}_{D_T^\lambda},1)=\sum_{c\in\mathcal{C}} E_1^*\left(\frac{\psi_{D_T^\lambda}(12c+D)\Omega_0}{3D},\Omega_0 O_K\right),$$

where $\mathcal{C}$ is as in lemma 2.1, a complete set of representatives of $(O_K/(D))^\times$ and $E_1^*(z,L)$ is the Eisenstein E^*-function (see [12]).

Since λ is odd by assumption, $\pi_k \equiv 1 \pmod{12}$ for all $k=1,\dots,n$. Hence $D=\pi_1\cdots\pi_n\equiv 1 \pmod{12}$, $D_T\equiv 1 \pmod{12}$ and $12c+D\equiv 1 \pmod{12}$ for all $c\in\mathcal{C}$. Since $(c,D)=1$, we have $(12c+D,D_T)=1$. Thus, by the definition [9] of

$\psi_{D_T^\lambda}$ and the law of sextic reciprocity [4], we get

$$\begin{aligned}\psi_{D_T^\lambda}(12c+D) &= \overline{\left(\frac{D_T}{12c+D}\right)_6^\lambda}\cdot(12c+D)\\ &= \overline{\left(\frac{12c+D}{D_T}\right)_6^\lambda}\cdot(12c+D)\\ &= \overline{\left(\frac{12c}{D_T}\right)_6^\lambda}\cdot(12c+D) \quad \text{for all } c\in\mathcal{C}.\end{aligned}$$

Since $E_1^*(wz,wL)=w^{-1}E_1^*(z,L)$, we have

$$\frac{D}{\Omega_0}\left(\frac{12}{D_T}\right)_6^{6-\lambda}\cdot L_S(\bar{\psi}_{D_T^\lambda},1)=\frac{1}{3}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_6^\lambda\cdot E_1^*\left(\frac{4c\Omega_0}{D}+\tfrac{1}{3}\Omega_0,\Omega_0 O_K\right).$$

Then, by the results and a similar calculation in [5,6], we can obtain

$$\frac{D}{\Omega_0}\left(\frac{12}{D_T}\right)_6^{6-\lambda}L_S(\bar{\psi}_{D_T^\lambda},1)=\frac{1}{2\sqrt{3}}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_6^\lambda\frac{1}{\Sigma(4c\Omega_0/D)-1}+\frac{1}{3\sqrt{3}}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_6^\lambda.$$

This proves lemma 2.1. □

3.1. Proofs of theorem 2.2 and proposition 2.6

LEMMA 3.1. *Let λ, D be as in theorem 2.2. Then we have*

$$v_3(\delta_n^{(\lambda)}(D))\geqslant\tfrac{1}{2}(n+1).$$

Proof. If $n=1$, then, by definition, $\delta_1^{(\lambda)}(\pi_1)=\bar{\pi}_1-1$. Since $\pi_1\equiv 1 \pmod 6$, we also have $\bar{\pi}_1\equiv 1 \pmod 6$, so

$$v_3(\delta_1^{(\lambda)}(\pi_1))=v_3(\bar{\pi}_1-1)\geqslant 1=\tfrac{1}{2}(1+1).$$

Assume that the conclusion is true for $1,2,\dots,n-1$ and consider the case n, $D=\pi_1\cdots\pi_n$. For every non-trivial subset T of $\{1,\dots,n\}$, since

$$\left(\frac{D_T}{\pi_k}\right)_6^\lambda=1,\ \tau \text{ or } \tau^2 \quad \text{for any } \pi_k\mid\hat{D}_T,$$

we have

$$v_3\left(\pi_k-\left(\frac{D_T}{\pi_k}\right)_6^\lambda\right)\geqslant\tfrac{1}{2}.$$

Hence

$$\begin{aligned}v_3\Biggl(\Biggl(\prod_{\pi_k\mid\hat{D}_T}&\left(\pi_k-\left(\frac{D_T}{\pi_k}\right)_6^\lambda\right)\Biggr)\cdot\delta_t^{(\lambda)}(D_T)\Biggr)\\ &=\sum_{\pi_k\mid\hat{D}_T}v_3\left(\left(\pi_k-\left(\frac{D_T}{\pi_k}\right)_6^\lambda\right)\right)+v_3(\delta_t^{(\lambda)}(D_T))\\ &\geqslant\tfrac{1}{2}(n-t)+\tfrac{1}{2}(t+1)=\tfrac{1}{2}(n+1).\end{aligned}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Therefore, by definition and properties of valuation, we get $v_3(\delta_n^{(\lambda)}(D)) \geqslant \frac{1}{2}(n+1)$. This proves lemma 3.1. □

Proof of theorem 2.2. For $\lambda = 2, 4$ and each subset T of $\{1, \dots, n\}$, multiply the two sides of the corresponding formula of lemma 2.1 by $2^{n-t(T)}$ (where $t(T) = \#T$ is the cardinal of T). Adding them up, we obtain

$$\begin{aligned}\sum_T 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda}, 1) &= \frac{1}{2\sqrt{3}} \sum_T 2^{n-t(T)} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda \frac{1}{\varSigma(c\Omega_0/D) - 1} + \frac{1}{3\sqrt{3}} \sum_T 2^{n-t(T)} \sum_{c\in\mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda \\ &= S_\lambda^*(D) + \frac{1}{3\sqrt{3}} \sum_T 2^{n-t(T)} \sum_{c\in\mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda,\end{aligned}$$

where

$$S_\lambda^*(D) = \frac{1}{2\sqrt{3}} \sum_{c\in\mathcal{C}} \frac{1}{\varSigma(c\Omega_0/D) - 1} \sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_6^\lambda.$$

It could be easily verified that

$$\sum_T 2^{n-t(T)} \sum_{c\in\mathcal{C}} \left(\frac{c}{D_T}\right)_6^\lambda = 2^{n-t(\emptyset)} \sum_{c\in\mathcal{C}} \left(\frac{c}{D_\emptyset}\right)_6^\lambda = 2^n \cdot \#\mathcal{C}.$$

Thus

$$\sum_T 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda}, 1) = S_\lambda^*(D) + \frac{2^n}{3\sqrt{3}} \cdot \#\mathcal{C}. \tag{3.1}$$

Now we prove

$$v_3(S_\lambda^*(D)) > \tfrac{1}{2}n - 1. \tag{3.2}$$

In fact, since $(c/\pi_k)_6^\lambda = 1$, τ or τ^2, and $v_3(1-\tau) = \frac{1}{2}$, we have

$$v_3\left(2 + \left(\frac{c}{\pi_k}\right)_6^\lambda\right) \geqslant \tfrac{1}{2}$$

(we always write $v_3(0) = +\infty$). Thus

$$v_3\left(\sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_6^\lambda\right) = v_3\left(\prod_{k=1}^n \left(2 + \left(\frac{c}{\pi_k}\right)_6^\lambda\right)\right) = \sum_{k=1}^n v_3\left(2 + \left(\frac{c}{\pi_k}\right)_6^\lambda\right) \geqslant \tfrac{1}{2}n,$$

that is,

$$v_3\left(\sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_6^\lambda\right) \geqslant \tfrac{1}{2}n \quad \text{for all } c \in \mathcal{C}. \tag{3.3}$$

Now for the Weierstrass $\varSigma$-function $\varSigma(z, L_{\Omega_0})$ in lemma 2.1. By [10, p.128, lemmas 1 and 2], it could be easily verified that

$$v_3\left(\varSigma\left(\frac{c\Omega_0}{D}, L_{\Omega_0}\right) - 1\right) = \tfrac{1}{3} \quad \text{for all } c \in \mathcal{C}. \tag{3.4}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Therefore, by properties of valuation, we obtain

$$\begin{aligned} v_3(S_\lambda^*(D)) &\geqslant v_3\left(\frac{1}{2\sqrt{3}}\right) + v_3\left(\frac{1}{\Sigma(c\Omega_0/D)-1}\right) + \min_{c\in\mathcal{C}} v_3\left(\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_6^\lambda\right) \\ &\geqslant \tfrac{1}{2}n - \tfrac{5}{6} \\ &> \tfrac{1}{2}n - 1. \end{aligned}$$

This proves (3.2).

Also, for $\mathcal{C}$ in (3.1) above, by assumption, we have $\#\mathcal{C} = \prod_{k=1}^n (\pi_k\bar{\pi}_k - 1)$, and then

$$v_3\left(\frac{2^n}{3\sqrt{3}}\cdot\#\mathcal{C}\right) = v_3\left(\frac{2^n}{3\sqrt{3}}\right) + \sum_{k=1}^n v_3(\pi_k\bar{\pi}_k - 1) \geqslant n - \tfrac{3}{2} \geqslant \tfrac{1}{2}n - 1, \quad n \geqslant 1.$$

It is obvious that $L_S(\bar{\psi}_{D_T^\lambda}, 1) = L(\bar{\psi}_{D^\lambda}, 1)$ when $T = \{1, \dots, n\}$. And when $T = \emptyset$ we have

$$L_S(\bar{\psi}_{D_\emptyset^\lambda}, 1) = L_S(\bar{\psi}_1, 1) = L(\bar{\psi}_1, 1)\prod_{k=1}^n\left(1 - \frac{1}{\pi_k}\right).$$

From [10], we know that

$$L(\bar{\psi}_1, 1) = L(\psi_1, 1) = L_1(1) = \tfrac{1}{9}\sqrt{3}\Omega_0,$$

where ψ_1 is the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_1 : y^2 = x^3 - 2^4 3^3$. Thus

$$L_S(\bar{\psi}_1, 1) = \tfrac{1}{9}\sqrt{3}\Omega_0\prod_{k=1}^n\left(1 - \frac{1}{\pi_k}\right).$$

Now we come to prove the following assertion, which is the core of our proof.

ASSERTION 3.2. Let

$$\Sigma_n^{(\lambda)}(D) = \frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda}, 1) + (-1)^n\cdot\tfrac{1}{9}(2^n\sqrt{3})\cdot\delta_n^{(\lambda)}(D).$$

Then

$$v_3(\Sigma_n^{(\lambda)}(D)) > \tfrac{1}{2}n - 1.$$

In fact, if $n = 1$, then $D = \pi_1$. So, by (3.1),

$$\begin{aligned} 2^{1-1}\frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda}, 1) &+ 2^{1-0}\frac{\pi_1}{\Omega_0}\left(\frac{3}{D_\emptyset}\right)_6^{6-\lambda} L_{\pi_1}(\bar{\psi}_{D_\emptyset^\lambda}, 1) \\ &= S_\lambda^*(\pi_1) + \tfrac{1}{9}(2\sqrt{3})(\pi_1\bar{\pi}_1 - 1). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda}, 1) &= S_\lambda^*(\pi_1) + \tfrac{1}{9}(2\sqrt{3})(\pi_1\bar{\pi}_1 - 1) - \frac{2\pi_1}{\Omega_0}\cdot\tfrac{1}{9}\sqrt{3}\Omega_0\left(1 - \frac{1}{\pi_1}\right) \\ &= S_\lambda^*(\pi_1) + \tfrac{1}{9}(2\sqrt{3})(\pi_1\bar{\pi}_1 - \pi_1) \\ &= S_\lambda^*(\pi_1) + \tfrac{1}{9}(2\sqrt{3})(\pi_1 - 1)(\bar{\pi}_1 - 1) + \tfrac{1}{9}(2\sqrt{3})\delta_1^{(\lambda)}(\pi_1), \end{aligned}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

since, by definition, $\delta_1^{(\lambda)}(\pi_1) = \bar{\pi}_1 - 1$. Thus

$$\begin{aligned}\Sigma_1^{(\lambda)}(\pi_1) &= \frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda}, 1) + (-1)^1 \cdot \tfrac{1}{9}(2^1\sqrt{3})\delta_1^{(\lambda)}(\pi_1) \\ &= \frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda}, 1) - \tfrac{1}{9}(2\sqrt{3})\delta_1^{(\lambda)}(\pi_1) \\ &= S_\lambda^*(\pi_1) + \tfrac{1}{9}(2\sqrt{3})(\pi_1 - 1)(\bar{\pi}_1 - 1).\end{aligned}$$

From (3.2), $v_3(S_\lambda^*(\pi_1)) > \frac{1}{2} - 1$, and obviously

$$v_3(\tfrac{1}{9}(2\sqrt{3})(\pi_1 - 1)(\bar{\pi}_1 - 1)) \geqslant -\tfrac{3}{2} + 1 + 1 = \tfrac{1}{2} > \tfrac{1}{2} - 1.$$

Therefore, $v_3(\Sigma_1^{(\lambda)}(\pi_1)) > \frac{1}{2} - 1$.

Assume that our assertion is true for $1, \dots, n-1$ and consider the case n ($n \geqslant 2$), $D = \pi_1 \cdots \pi_n$. For any non-trivial subset T of $\{1, \dots, n\}$, put $t = t(T) = \#T$. Then $0 < t < n$ and, by the inductive assumption, we have

$$\Sigma_t^{(\lambda)}(D_T) = \frac{D_T}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L(\bar{\psi}_{D_T^\lambda}, 1) + (-1)^t \cdot \tfrac{1}{9}(2^t\sqrt{3}) \cdot \delta_t^{(\lambda)}(D_T)$$

and

$$v_3(\Sigma_t^{(\lambda)}(D_T)) > \tfrac{1}{2}t - 1.$$

Therefore,

$$\frac{D_T}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L(\bar{\psi}_{D_T^\lambda}, 1) = \Sigma_t^{(\lambda)}(D_T) - (-1)^t \cdot \tfrac{1}{9}(2^t\sqrt{3}) \cdot \delta_t^{(\lambda)}(D_T),$$

and then

$$\begin{aligned}&2^{n-t(T)}\frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda}, 1) \\ &\quad = 2^{n-t(T)}\frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} \cdot L(\bar{\psi}_{D_T^\lambda}, 1) \cdot \prod_{\pi_k \mid \hat{D}_T}\left(1 - \left(\frac{D_T}{\pi_k}\right)_6^\lambda \cdot \frac{1}{\pi_k}\right) \\ &\quad = 2^{n-t(T)}\left(\frac{D_T}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} \cdot L(\bar{\psi}_{D_T^\lambda}, 1)\right) \cdot \prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^\lambda\right) \\ &\quad = 2^{n-t(T)} \cdot (\Sigma_t^{(\lambda)}(D_T) - (-1)^t \cdot \tfrac{1}{9}(2^t\sqrt{3}) \cdot \delta_t^{(\lambda)}(D_T)) \cdot \prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^\lambda\right) \\ &\quad = 2^{n-t(T)} \cdot \Sigma_t^{(\lambda)}(D_T) \prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^\lambda\right) \\ &\qquad - (-1)^t \cdot \tfrac{1}{9}(2^n\sqrt{3})\delta_t^{(\lambda)}(D_T) \prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^\lambda\right).\end{aligned}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Since, from (3.1),

$$\sum_{T} 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1) = S_\lambda^*(D) + \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1),$$

we have

$$\begin{aligned}
&\frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda},1)\\
&\quad= \sum_{T} 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1) - 2^{n-t(\emptyset)} \frac{D}{\Omega_0}\left(\frac{3}{D_\emptyset}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_\emptyset^\lambda},1)\\
&\qquad - \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1)\\
&\quad= S_\lambda^*(D) + \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1) - 2^n \cdot \frac{D}{\Omega_0} \cdot \tfrac{1}{9}\sqrt{3}\Omega_0 \cdot \prod_{k=1}^{n}\left(1 - \frac{1}{\pi_k}\right)\\
&\qquad - \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} \Biggl(2^{n-t(T)} \cdot \Sigma_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\\
&\qquad\qquad - (-1)^t \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\Biggr)\\
&\quad= S_\lambda^*(D) + \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1) - \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k - 1)\\
&\qquad - \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} \left(2^{n-t(T)} \cdot \Sigma_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right)\\
&\qquad + \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} \left((-1)^t \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right)\\
&\quad= \Sigma_n' + \tfrac{1}{9}(2^n\sqrt{3}) \cdot \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} (-1)^t \cdot \left(\prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right) \cdot \delta_t^{(\lambda)}(D_T),
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_n' = {}& S_\lambda^*(D) + \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1) - \tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k - 1)\\
&- \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} \left(2^{n-t(T)} \cdot \Sigma_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k\mid\hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right).
\end{aligned}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Note that $(-1)^t = (-1)^{n+1} \cdot (-1)^{n+1-t}$, so

$$\begin{aligned}
&\frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda}, 1) \\
&\quad = \Sigma_n' + \tfrac{1}{9}(2^n\sqrt{3}) \cdot (-1)^{n+1} \\
&\qquad \cdot \sum_{\emptyset \neq T \subsetneqq \{1,\dots,n\}} (-1)^{n+1-t} \cdot \left(\prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right) \cdot \delta_t^{(\lambda)}(D_T) \\
&\quad = \Sigma_n' + (-1)^{n+1} \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D) \\
&\quad = \Sigma_n' - (-1)^{n} \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D).
\end{aligned}$$

Therefore,

$$\Sigma_n^{(\lambda)}(D) = \frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda}, 1) + (-1)^n \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D) = \Sigma_n'.$$

From (3.2), we know that $v_3(S_\lambda^*(D)) > \frac{1}{2}n - 1$. Also, we have

$$v_3\left(\tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1)\right) \geqslant -\tfrac{3}{2} + n > \tfrac{1}{2}n - 1 \quad \text{for } n \geqslant 2,$$

$$v_3\left(\tfrac{1}{9}(2^n\sqrt{3}) \cdot \prod_{k=1}^{n}(\pi_k - 1)\right) \geqslant -\tfrac{3}{2} + n > \tfrac{1}{2}n - 1 \quad \text{for } n \geqslant 2$$

and

$$\begin{aligned}
&v_3\left(2^{n-t(T)} \cdot \Sigma_t^{(\lambda)}(D_T) \cdot \prod_{\pi_k \mid \hat{D}_T}\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right) \\
&\qquad = v_3(\Sigma_t^{(\lambda)}(D_T)) + \sum_{\pi_k \mid \hat{D}_T} v_3\left(\left(\pi_k - \left(\frac{D_T}{\pi_k}\right)_6^{\lambda}\right)\right) \\
&\qquad > \tfrac{1}{2}t - 1 + \tfrac{1}{2}(n-t) \\
&\qquad = \tfrac{1}{2}n - 1.
\end{aligned}$$

Therefore, by properties of valuation, we get

$$v_3(\Sigma_n^{(\lambda)}(D)) = v_3(\Sigma_n') > \tfrac{1}{2}n - 1 \quad (n \geqslant 2).$$

This completes the proof of our assertion by induction.

Now, by the above assertion,

$$\frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda}, 1) = \Sigma_n^{(\lambda)}(D) - (-1)^n \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D)$$

and

$$v_3(\Sigma_n^{(\lambda)}(D)) > \tfrac{1}{2}n - 1.$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

From lemma 3.1, we know that $v_3(\delta_n^{(\lambda)}(D)) \geqslant \frac{1}{2}(n+1)$, and so

$$v_3((-1)^n \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D)) \geqslant -\tfrac{3}{2} + \tfrac{1}{2}(n+1) = \tfrac{1}{2}n - 1.$$

Therefore, by the properties of valuation, we get

$$v_3(L(\bar{\psi}_{D^\lambda}, 1)/\Omega_0) = v_3\left(\frac{D}{\Omega_0}\left(\frac{3}{D}\right)_6^{6-\lambda} L(\bar{\psi}_{D^\lambda}, 1)\right) \geqslant \tfrac{1}{2}n - 1,$$

and the equality holds if and only if

$$v_3((-1)^n \cdot \tfrac{1}{9}(2^n\sqrt{3}) \cdot \delta_n^{(\lambda)}(D)) = \tfrac{1}{2}n - 1,$$

which is equivalent to

$$v_3(\delta_n^{(\lambda)}(D)) = \tfrac{1}{2}(n+1).$$

This completes the proof of theorem 2.2. □

Proof of proposition 2.6. Since $\pi_k \equiv 1 \pmod{18}$, we have $\pi_k\bar{\pi}_k \equiv 1 \pmod{18}$, and then $v_3(\pi_k\bar{\pi}_k - 1) \geqslant 2$, so

$$v_3(\#\mathcal{C}) = v_3(\#(O_K/(D))^\times) = v_3\left(\prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1)\right) = \sum_{k=1}^{n} v_3(\pi_k\bar{\pi}_k - 1) \geqslant 2n.$$

Obviously, we also have $\pi_k \equiv 1 \pmod 6$, and thus $D = \pi_1 \cdots \pi_n$ completely satisfies the condition of D in theorem 2.2. Therefore, from the proof of theorem 2.2, we also have (see (3.1) above)

$$\sum_T 2^{n-t(T)} \frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda}, 1) = S_\lambda^*(D) + \frac{2^n}{3\sqrt{3}} \cdot \#\mathcal{C}$$

and

$$S_\lambda^*(D) = \frac{1}{2\sqrt{3}} \sum_{c\in\mathcal{C}} \frac{1}{\varSigma(c\Omega_0/D) - 1} \sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_6^\lambda,$$

where T runs over all the subset of $\{1, \dots, n\}$.

Now we prove that

$$v_3(S_\lambda^*(D)) \geqslant \tfrac{1}{2}(n-1). \tag{3.5}$$

Since $\pi_k \equiv 1 \pmod{18}$ $(k = 1, \dots, n)$, we have

$$N(D_T) \equiv N(D) \equiv 1 \pmod{18}, \quad \left(\frac{\tau}{D_T}\right)_6^2 = \tau^{(N(D_T)-1)/3} = 1.$$

We also have

$$\#\mathcal{C} = \prod_{k=1}^{n}(N(\pi_k) - 1) \equiv 0 \pmod{18},$$

so we may choose the set $\mathcal{C}$ properly such that $\pm c, \pm\tau c, \pm\tau^2 c \in \mathcal{C}$ (when $c \in \mathcal{C}$). That is, when $c \in \mathcal{C}$, all its associated elements are in $\mathcal{C}$. Let $V = \{c \in \mathcal{C} : c \equiv 1 \pmod 3\}$. Then

$$\mathcal{C} = \bigcup_{\mu\in\{\pm 1, \pm\tau, \pm\tau^2\}} \mu V.$$

Obviously,

$$\left(\frac{-c}{D_T}\right)_6^2 = \left(\frac{c}{D_T}\right)_6^2,$$

$$\Sigma\left(\frac{-c\Omega_0}{D}, L_{\Omega_0}\right) = \Sigma\left(\frac{c\Omega_0}{D}, L_{\Omega_0}\right),$$

$$\Sigma(\tau z, L_{\Omega_0}) = \tau\Sigma(z, L_{\Omega_0}),$$

$$\Sigma(\tau^2 z, L_{\Omega_0}) = \tau^2\Sigma(z, L_{\Omega_0}).$$

Thus

$$\begin{aligned} S_\lambda^*(D) &= \frac{1}{\sqrt{3}}\sum_{c\in V}\frac{1}{\Sigma(c\Omega_0/D, L_{\Omega_0})-1}\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_6^\lambda \\ &\quad + \frac{1}{\sqrt{3}}\sum_{c\in V}\frac{1}{\Sigma(\tau c\Omega_0/D, L_{\Omega_0})-1}\sum_T 2^{n-t(T)}\left(\frac{\tau c}{D_T}\right)_6^\lambda \\ &\quad + \frac{1}{\sqrt{3}}\sum_{c\in V}\frac{1}{\Sigma(\tau^2 c\Omega_0/D, L_{\Omega_0})-1}\sum_T 2^{n-t(T)}\left(\frac{\tau^2 c}{D_T}\right)_6^\lambda \\ &= \frac{1}{\sqrt{3}}\sum_T 2^{n-t(T)}\left\{\sum_{c\in V}\frac{3}{\gamma(c)}\cdot\left(\frac{c}{D_T}\right)_6^\lambda\right\} \\ &= \sqrt{3}\sum_{c\in V}\frac{1}{\gamma(c)}\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_6^\lambda, \end{aligned}$$

where

$$\gamma(c) = \left(\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right)\left(\tau\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right)\left(\tau^2\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right).$$

Since

$$\tau\Sigma\left(\frac{c\Omega_0}{D}\right)-1 = \frac{(\Sigma(c\Omega_0/D)-\tau^2)}{\tau^2} = \frac{1}{\tau^2}\left(\left(\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right)+(1-\tau^2)\right),$$

$$\tau^2\Sigma\left(\frac{c\Omega_0}{D}\right)-1 = \frac{(\Sigma(c\Omega_0/D)-\tau)}{\tau} = \frac{1}{\tau}\left(\left(\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right)+(1-\tau)\right),$$

by (3.4) in the proof of theorem 2.2, we obtain

$$v_3\left(\tau\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right) = v_3\left(\tau^2\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right) = v_3\left(\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right) = \tfrac{1}{3}.$$

Thus

$$v_3(\gamma(c)) = 3v_3\left(\Sigma\left(\frac{c\Omega_0}{D}\right)-1\right) = 3\cdot\tfrac{1}{3} = 1.$$

Hence, by (3.3) in the proof of theorem 2.2 and the properties of valuation, we get

$$v_3(S_\lambda^*(D)) \geqslant \tfrac{1}{2}+v_3\left(\frac{1}{\gamma(c)}\right)+\min_{c\in V} v_3\left(\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_6^\lambda\right) \geqslant \tfrac{1}{2}-1+\tfrac{1}{2}n = \tfrac{1}{2}(n-1).$$

This proves (3.5).

Also by the above discussion, we have

$$v_3\left(\frac{2^n}{3\sqrt{3}}\cdot \#\mathcal{C}\right) = -\tfrac{3}{2} + v_3(\#\mathcal{C}) \geqslant 2n - \tfrac{3}{2} > \tfrac{1}{2}(n-1) \quad (n \geqslant 1).$$

So, by properties of valuation, we obtain

$$v_3\biggl(\sum_T 2^{n-t(T)}\frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1)\biggr) \geqslant \tfrac{1}{2}(n-1). \tag{3.6}$$

By definition, we know that $L_S(\bar{\psi}_{D_T^\lambda},1) = L(\bar{\psi}_{D^\lambda},1)$ when $T=\{1,\dots,n\}$. When $T=\emptyset$, by our assumption, we have that

$$\begin{aligned} v_3\biggl(2^{n-t(\emptyset)}\frac{D}{\Omega_0}\left(\frac{3}{D_\emptyset}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_\emptyset^\lambda},1)\biggr) &= v_3(L_S(\bar{\psi}_1,1)/\Omega_0) \\ &= v_3\biggl(\tfrac{1}{9}\sqrt{3}\cdot\prod_{k=1}^n\left(1-\frac{1}{\pi_k}\right)\biggr) \\ &\geqslant \tfrac{1}{2} - 2 + 2n \\ &= 2n - \tfrac{3}{2} \\ &> \tfrac{1}{2}(n-1) \quad (n\geqslant 1). \end{aligned} \tag{3.7}$$

Now we use an induction method on n to prove that

$$v_3(L(\bar{\psi}_{D^\lambda},1)/\Omega_0) \geqslant \tfrac{1}{2}(n-1).$$

If $n=1$, $D=\pi_1$ and $S=\{\pi_1\}$, then

$$v_3(L_S(\bar{\psi}_{D_\emptyset^\lambda},1)/\Omega_0) = v_3(L_S(\bar{\psi}_1,1)/\Omega_0) > \tfrac{1}{2}(1-1) = 0.$$

Also, by (3.6), for $n=1$, we have

$$v_3\biggl(2\frac{\pi_1}{\Omega_0}\left(\frac{3}{D_\emptyset}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_\emptyset^\lambda},1) + \frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda},1)\biggr) \geqslant \tfrac{1}{2}(1-1) = 0.$$

Therefore, by the properties of valuation, we obtain

$$v_3(L(\bar{\psi}_{\pi_1^\lambda},1)/\Omega_0) = v_3\biggl(\frac{\pi_1}{\Omega_0}\left(\frac{3}{\pi_1}\right)_6^{6-\lambda} L(\bar{\psi}_{\pi_1^\lambda},1)\biggr) \geqslant 0 = \tfrac{1}{2}(1-1).$$

Assume that our conclusion is true for $1,2,\dots,n-1$, and consider the case n, $D=\pi_1\cdots\pi_n$ and $S=\{\pi_1,\dots,\pi_n\}$. For any non-trivial subset T of $\{1,\dots,n\}$, put $t=t(T)=\#T$. Then $1\leqslant t<n$. So, by our induction assumption, we have

$$\begin{aligned} &v_3\biggl(2^{n-t(T)}\frac{D}{\Omega_0}\left(\frac{3}{D_T}\right)_6^{6-\lambda} L_S(\bar{\psi}_{D_T^\lambda},1)\biggr) \\ &\qquad = v_3(L(\bar{\psi}_{D_T^\lambda},1)/\Omega_0) + \sum_{\pi_k\mid\hat{D}_T} v_3\biggl(1-\left(\frac{D_T}{\pi_k}\right)_6^\lambda\frac{1}{\pi_k}\biggr) \\ &\qquad \geqslant \tfrac{1}{2}(t(T)-1) + \tfrac{1}{2}(n-t(T)) \\ &\qquad = \tfrac{1}{2}(n-1). \end{aligned} \tag{3.8}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Therefore, by (3.6)–(3.8) and the properties of valuation, we obtain that

$$\begin{aligned}
&v_3(L(\bar\psi_{D^\lambda},1)/\Omega_0)\\
&\quad= v_3\biggl(2^{n-n}\frac{D}{\Omega_0}\biggl(\frac{3}{D}\biggr)_6^{6-\lambda}L_S(\bar\psi_{D^\lambda},1)\biggr)\\
&\quad= v_3\biggl(\sum_T 2^{n-t(T)}\frac{D}{\Omega_0}\biggl(\frac{3}{D_T}\biggr)_6^{6-\lambda}L_S(\bar\psi_{D_T^\lambda},1)\\
&\qquad\qquad - \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} 2^{n-t(T)}\frac{D}{\Omega_0}\biggl(\frac{3}{D_T}\biggr)_6^{6-\lambda}L_S(\bar\psi_{D_T^\lambda},1)\\
&\qquad\qquad\qquad - 2^{n-t(\emptyset)}\frac{D}{\Omega_0}\biggl(\frac{3}{D_\emptyset}\biggr)_6^{6-\lambda}L_S(\bar\psi_{D_\emptyset^\lambda},1)\biggr)\\
&\quad\geqslant \tfrac{1}{2}(n-1).
\end{aligned}$$

This proves our conclusion by induction, and completes the proof of proposition 2.6. □

3.2. Proof of theorem 2.7

Proof of theorem 2.7. For $\lambda = 3$, sum the identity of lemma 2.1 over the 2^n subsets T of $\{1,\dots,n\}$ to obtain

$$\begin{aligned}
&\sum_T \frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3 L_S(\bar\psi_{D_T^3},1)\\
&\qquad= \frac{1}{2\sqrt{3}}\sum_T\sum_{c\in\mathcal{C}}\biggl(\frac{c}{D_T}\biggr)_6^3\frac{1}{\Sigma(4c\Omega_0/D)-1} + \frac{1}{3\sqrt{3}}\sum_T\sum_{c\in\mathcal{C}}\biggl(\frac{c}{D_T}\biggr)_6^3.
\end{aligned}$$

It can easily be verified that

$$\sum_T\sum_{c\in\mathcal{C}}\biggl(\frac{c}{D_T}\biggr)_6^3 = \sum_{c\in\mathcal{C}}\biggl(\frac{c}{D_\emptyset}\biggr)_6^3 = \#\mathcal{C}.$$

Hence

$$\sum_T \frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3 L_S(\bar\psi_{D_T^3},1) = S_3^*(D) + \frac{1}{3\sqrt{3}}\cdot\#\mathcal{C}, \tag{3.9}$$

where

$$S_3^*(D) = \frac{1}{2\sqrt{3}}\sum_{c\in\mathcal{C}}\frac{1}{\Sigma(4c\Omega_0/D)-1}\sum_T\biggl(\frac{c}{D_T}\biggr)_6^3.$$

Since $(c/\pi_k)_6^3 = \pm 1$, we have $v_2(1+(c/\pi_k)_6^3) \geqslant 1$ (we always write $v_2(0) = +\infty$). Thus

$$v_2\biggl(\sum_T\biggl(\frac{c}{D_T}\biggr)_6^3\biggr) = v_2\biggl(\prod_{k=1}^n\biggl(1+\biggl(\frac{c}{\pi_k}\biggr)_6^3\biggr)\biggr) = \sum_{k=1}^n v_2\biggl(1+\biggl(\frac{c}{\pi_k}\biggr)_6^3\biggr) \geqslant n,$$

that is,

$$v_2\biggl(\sum_T \biggl(\frac{c}{D_T}\biggr)_6^3\biggr) \geqslant n \quad \text{for all } c \in \mathcal{C}. \tag{3.10}$$

We also have

$$\#\mathcal{C} = \#(O_K/(D))^\times = \prod_{k=1}^{n}(\pi_k\bar{\pi}_k - 1).$$

Since $\pi_k \equiv 1 \pmod{12}$ for all $k = 1, \dots, n$, we get $v_2(\pi_k\bar{\pi}_k - 1) \geqslant 2$. Thus

$$v_2\biggl(\frac{\#\mathcal{C}}{3\sqrt{3}}\biggr) = v_2(\#\mathcal{C}) = \sum_{k=1}^{n} v_2(\pi_k\bar{\pi}_k - 1) \geqslant 2n. \tag{3.11}$$

For the Weierstrass Σ-function $\Sigma(z)$ in (3.9), by lemmas 1 and 2 in [10, p. 128], it can be easily verified that

$$v_2\biggl(\Sigma\biggl(\frac{4c\Omega_0}{D}\biggr) - 1\biggr) = 0 \quad \text{for all } c \in \mathcal{C}. \tag{3.12}$$

(See the calculation of the 3-adic valuation of $(\Sigma(c\Omega_0/D)-1)$ in [5] and [6, lemma 4].)

By assumption, we may choose $\mathcal{C}$ in such a way that $-c \in \mathcal{C}$ when $c \in \mathcal{C}$. Since $\Sigma(z)$ is even and $(-c/D_T)_6^3 = (c/D_T)_6^3$, by (3.10), (3.12) and the properties of valuation, we know that the first term in the right-hand side of (3.9) has 2-adic valuation greater than or equal to n, That is,

$$v_2(S_3^*(D)) \geqslant n. \tag{3.13}$$

Together with (3.11), we obtain

$$v_2\biggl(\sum_T \frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3 L_S(\bar{\psi}_{D_T^3}, 1)\biggr) \geqslant n. \tag{3.14}$$

It is obvious that $L_S(\bar{\psi}_{D_T^3}, 1) = L(\bar{\psi}_{D^3}, 1)$ when $T = \{1, \dots, n\}$. When $T = \emptyset$, we have

$$L_S(\bar{\psi}_{D_\emptyset^3}, 1) = L_S(\bar{\psi}_1, 1) = L(\bar{\psi}_1, 1)\prod_{k=1}^{n}\biggl(1 - \frac{1}{\pi_k}\biggr).$$

From [10], we know that

$$L(\bar{\psi}_1, 1) = L(\psi_1, 1) = L_1(1) = \tfrac{1}{9}\sqrt{3}\Omega_0,$$

where ψ_1 is the Hecke character of $\mathbb{Q}(\sqrt{-3})$ attached to the elliptic curve $E_1 : y^2 = x^3 - 2^4 3^3$. Thus

$$L_S(\bar{\psi}_{D_\emptyset^3}, 1) = L_S(\bar{\psi}_1, 1) = \tfrac{1}{9}\sqrt{3}\Omega_0\prod_{k=1}^{n}\biggl(1 - \frac{1}{\pi_k}\biggr).$$

So we get

$$\begin{aligned} v_2(L_S(\bar{\psi}_{D_\emptyset^3}, 1)/\Omega_0) &= v_2(\tfrac{1}{9}\sqrt{3}) + \sum_{k=1}^{n} v_2(\pi_k - 1) \\ &\geqslant 2n \quad (\text{since } \pi_k \equiv 1 \pmod{12}). \end{aligned} \tag{3.15}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Now we use induction on n to prove our assertion $v_2(L(\bar\psi_{D^3},1)/\Omega_0) \geqslant n$. If $n=1$, then $D=\pi_1$, $S=\{\pi_1\}$ and $L_S(\bar\psi_{D^3_\emptyset},1)=(\sqrt{3}/9)\Omega_0(1-1/\pi_1)$. By (3.15), we have

$$v_2\biggl(\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{D_\emptyset}\biggr)_6^3 L_S(\bar\psi_{D^3_\emptyset},1)\biggr)=v_2(L_S(\bar\psi_{D^3_\emptyset},1)/\Omega_0)\geqslant 2.$$

Also, by (3.14), for the case $n=1$, we have

$$v_2\biggl(\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{D_\emptyset}\biggr)_6^3 L_S(\bar\psi_{D^3_\emptyset},1)+\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{\pi_1}\biggr)_6^3 L_S(\bar\psi_{\pi_1^3},1)\biggr)\geqslant 1.$$

Therefore, by the properties of valuation,

$$\begin{aligned}v_2(L(\bar\psi_{\pi_1^3},1)/\Omega_0)&=v_2\biggl(\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{\pi_1}\biggr)_6^3 L_S(\bar\psi_{\pi_1^3},1)\biggr)\\&=v_2\biggl(\biggl(\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{D_\emptyset}\biggr)_6^3 L_S(\bar\psi_{D^3_\emptyset},1)\\&\qquad\qquad+\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{\pi_1}\biggr)_6^3 L_S(\bar\psi_{\pi_1^3},1)\biggr)-\frac{\pi_1}{\Omega_0}\biggl(\frac{12}{D_\emptyset}\biggr)_6^3 L_S(\bar\psi_{D^3_\emptyset},1)\biggr)\\&\geqslant 1.\end{aligned}$$

Now assume our assertion is true for $1,2,\dots,n-1$ and consider $D=\pi_1\cdots\pi_n$. For any subset T of $\{1,\dots,n\}$, set $t=t(T)=\#T$. Then, by definition,

$$\frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3 L_S(\bar\psi_{D_T^3},1)=\frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3\cdot L(\bar\psi_{D_T^3},1)\cdot\prod_{\pi_k\mid\hat D_T}\biggl(1-\biggl(\frac{D_T}{\pi_k}\biggr)_6^3\frac{1}{\pi_k}\biggr).$$

Since $(D_T/\pi_k)_6^3=\pm1$ (for all $\pi_k\mid\hat D_T$), we have

$$1-\biggl(\frac{D_T}{\pi_k}\biggr)_6^3\frac{1}{\pi_k}=\frac{\pi_k-\mu}{\pi_k},\quad \mu\in\{1,-1\}.$$

Note that $\pi_k\equiv 1 \pmod{12}$. Therefore, $v_2(\pi_k-\mu)\geqslant 1$. Thus, when T is non-trivial (i.e. $1\leqslant t<n$), by our inductive assumption,

$$\begin{aligned}&v_2\biggl(\frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3 L_S(\bar\psi_{D_T^3},1)\biggr)\\&\quad=v_2\biggl(\frac{D}{\Omega_0}\biggl(\frac{12}{D_T}\biggr)_6^3\cdot L(\bar\psi_{D_T^3},1)\biggr)+v_2\biggl(\prod_{\pi_k\mid\hat D_T}\biggl(1-\biggl(\frac{D_T}{\pi_k}\biggr)_6^3\frac{1}{\pi_k}\biggr)\biggr)\\&\quad=v_2(L(\bar\psi_{D_T^3},1)/\Omega_0)+\sum_{\pi_k\mid\hat D_T}v_2\biggl(1-\biggl(\frac{D_T}{\pi_k}\biggr)_6^3\frac{1}{\pi_k}\biggr)\\&\quad\geqslant t+\#\{\pi_k:\pi_k\mid\hat D_T\}\\&\quad=t+(n-t)\\&\quad=n.\end{aligned}$$

https://doi.org/10.1017/S0308210500003000 Published online by Cambridge University Press

Furthermore, when $T=\emptyset$, by (3.15), we get

$$v_2\left(\frac{D}{\Omega_0}\left(\frac{12}{D_\emptyset}\right)_6^3 L_S(\bar{\psi}_{D_\emptyset^3},1)\right) = v_2(L_S(\bar{\psi}_{D_\emptyset^3},1)/\Omega_0) \geqslant 2n > n.$$

Therefore, by the properties of valuation, we obtain

$$\begin{aligned} v_2(L(\bar{\psi}_{D^3},1)/\Omega_0) &= v_2\left(\frac{D}{\Omega_0}\left(\frac{12}{D}\right)_6^3 L_S(\bar{\psi}_{D^3},1)\right) \\ &= v_2\Bigg(\sum_T \frac{D}{\Omega_0}\left(\frac{12}{D_T}\right)_6^3 L_S(\bar{\psi}_{D_T^3},1) \\ &\qquad - \sum_{\emptyset\neq T\subsetneqq\{1,\dots,n\}} \frac{D}{\Omega_0}\left(\frac{12}{D_T}\right)_6^3 L_S(\bar{\psi}_{D_T^3},1) - \frac{D}{\Omega_0}\left(\frac{12}{D_\emptyset}\right)_6^3 L_S(\bar{\psi}_{D_\emptyset^3},1)\Bigg) \\ &\geqslant n. \end{aligned}$$

This proves our conclusion by induction, and completes the proof of theorem 2.7. □

REMARK 3.3. If the inequalities (3.5) and (3.13) can be improved more explicitly such that they become as strict as (3.2), then similar criteria for reaching the bounds of valuations in proposition 2.6 and theorem 2.7, like the ones in theorem 2.2, can also be obtained.

REMARK 3.4. For any positive rational integer λ, by a simple change of variables, the elliptic curve $E_{D^\lambda}: y^2 = x^3 - 2^4 3^3 D^\lambda$ is always $\mathbb{Q}(\sqrt{-3})$-isomorphic to

$$E_{D^{\lambda_0}}: y^2 = x^3 - 2^4 3^3 D^{\lambda_0}, \quad \text{with } \lambda_0 = 0,1,\dots,5.$$

In particular, when $\lambda_0 = 0$,

$$E_{D^{\lambda_0}} = E_1 : y^2 = x^3 - 2^4 3^3,$$

which is the trivial case. Therefore, the study of the $p(=2,3)$-adic valuation of $L(1)$ of E_{D^λ} is reduced to the five essential cases $\lambda = 1,\dots,5$ that we consider in this paper.

Acknowledgments

The author thanks the referee for his helpful suggestion and encouragement.

References

1 B. J. Birch and H. P. F. Swinnerton-Dyer. Notes on elliptic curves. II. *J. Reine Angew. Math.* **218** (1965), 79–108.

2 J. Coates and A. Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.* **39** (1977), 223–251.

3 C. Goldstein and N. Schappacher. Séries d'Eisenstein et fonction L de courbes elliptiques à multiplication complexe. *J. Reine Angew. Math.* **327** (1981), 184–218.

4 F. Lemmermeyer. *Reciprocity laws* (Springer, 2000).

5 D.-R. Qiu. Mordell–Weil groups and related problems of elliptic curves. PhD thesis, Tsinghua University (2000).

6 D.-R. Qiu and X.-K. Zhang. Elliptic curves with CM by $\sqrt{-3}$- and 3-adic valuations of their L-series. *Manuscr. Math.* **108** (2002), 385–397.

7 K. Rubin. The 'main conjectures' of Iwasawa theory for imaginary quadratic fields. *Invent. Math.* **103** (1991), 25–68.

8 J. H. Silverman. *The arithmetic of elliptic curves*. Graduate Texts in Mathematics, vol. 106 (Springer, 1986).

9 J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*. Graduate Texts in Mathematics, vol. 151 (Springer, 1994).

10 N. M. Stephens. The diophantine equation $x^3 + y^3 = Dz^3$ and the conjectures of Birch and Swinnerton-Dyer. *J. Reine Angew. Math.* **231** (1968), 121–162.

11 J. Tate. Algorithm for determining the type of singular fiber in an elliptic pencil. In *Modular functions of one variable*, vol. IV. Lecture Notes in Mathematics, vol. 476 (Springer, 1975).

12 A. Weil. *Elliptic functions according to Eisenstein and Kronecker* (Springer, 1976).

(*Issued 19 December 2003*)