

A FUNCTIONAL LIMIT THEOREM FOR GENERAL SHOT NOISE PROCESSES

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Abstract

By a general shot noise process we mean a shot noise process in which the counting process of shots is arbitrary locally finite. Assuming that the counting process of shots satisfies a functional limit theorem in the Skorokhod space with a locally Hölder continuous Gaussian limit process, and that the response function is regularly varying at infinity, we prove that the corresponding general shot noise process satisfies a similar functional limit theorem with a different limit process and different normalization and centering functions. For instance, if the limit process for the counting process of shots is a Brownian motion, then the limit process for the general shot noise process is a Riemann–Liouville process. We specialize our result for five particular counting processes. Also, we investigate Hölder continuity of the limit processes for general shot noise processes.

Keywords: Hölder continuity; shot noise process; weak convergence in the Skorokhod space

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1. Introduction and main result

Let $(S_k)_{k \in \mathbb{N}_0}$ be a not necessarily monotone sequence of nonnegative random variables. Define the counting process $(N(t))_{t \geq 0}$ by

$$N(t) := \sum_{k \geq 0} \mathbf{1}_{\{S_k \leq t\}}, \quad t \geq 0,$$

where $\mathbf{1}_A = 1$ if the event A holds and 0 otherwise. Throughout the paper we always assume that $N(t) < \infty$ almost surely (a.s.) for $t \geq 0$.

Denote by $D := D[0, \infty)$ the Skorokhod space of right-continuous real-valued functions which are defined on $[0, \infty)$ and have finite limits from the left at each positive point. For a function $h \in D$, the random process $X := (X(t))_{t \geq 0}$, which is the main object of our investigation, is given by

$$X(t) := \sum_{k \geq 0} h(t - S_k) \mathbf{1}_{\{S_k \leq t\}} = \int_{[0, t]} h(t - y) dN(y), \quad t \geq 0.$$

We call X a *general shot noise process*, for no assumptions are imposed apart from $N(t) < \infty$ a.s. Plainly, $X \in D$ a.s.

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Denote by $Y_1 := (Y_1(t))_{t \geq 0}$, $Y_2 := (Y_2(t))_{t \geq 0}$, ... independent and identically distributed (i.i.d.) random processes with paths in D . Assume that, for $k \in \mathbb{N}_0$, Y_{k+1} is independent of (S_0, \dots, S_k) . In particular, the case of complete independence of $(Y_j)_{j \in \mathbb{N}}$ and $(S_k)_{k \in \mathbb{N}_0}$ is not excluded. Set

$$Y(t) := \sum_{k \geq 0} Y_{k+1}(t - S_k) \mathbf{1}_{\{S_k \leq t\}}, \quad t \geq 0$$

and call $Y := (Y(t))_{t \geq 0}$ a random process with immigration at random times. The interpretation is that associated with the k th immigrant arriving at the system at time S_{k-1} is the process Y_k which defines a model-dependent ‘characteristic’ of the k th immigrant. For instance, $Y_k(t - S_{k-1})$ may be the fitness of the k th immigrant at time t . The value of $Y(t)$ is then given by the sum of ‘characteristics’ of all immigrants arriving at the system up to and including time t . Assume that the function $g(t) := E[Y_1(t)]$ is finite for all $t \geq 0$, not identically 0, and that $g \in D$. To investigate weak convergence of the process Y , properly normalized and centered, it is natural to use the decomposition

$$Y(t) = \sum_{k \geq 0} (Y_{k+1}(t - S_k) - g(t - S_k)) \mathbf{1}_{\{S_k \leq t\}} + \sum_{k \geq 0} g(t - S_k) \mathbf{1}_{\{S_k \leq t\}}, \quad t \geq 0. \quad (1)$$

For fixed $t > 0$, while the first summand is the terminal value of a martingale, the second is the value at time t of a general shot noise process. The summands should be treated separately, for each of these requires a specific approach. Weak convergence of the first summand in (1) is investigated in [11].

In the present paper we aim to prove a functional limit theorem for a general shot noise process X under natural assumptions. Besides being of independent interest, our findings pave the way towards controlling the asymptotic behavior of the second summand in (1). These, taken together with the results from [11], should eventually lead to understanding of the asymptotics of processes Y .

A rich source of random processes Y with immigration at random times are queueing systems and various branching processes with or without immigration. For example, particular instances of random variables $Y(t)$ are given by the number of the k th-generation individuals ($k \geq 2$) with positions $\leq t$ in a branching random walk; the number of customers served up to and including time t ; or the number of busy servers at time t in a GEN/G/ ∞ queueing system, where GEN means that the arrival of customers is regulated by a general point process. Nowadays, rather popular objects of research are queueing systems in which the input process is more complicated than the renewal process, for instance, a Cox process (also known as a doubly stochastic Poisson process) [8] or a Hawkes process [10, 28], and branching processes with immigration governed by a process which is more general than the renewal process, for instance, an inhomogeneous Poisson process [33] or a Cox process [7]. Note that some authors have investigated the processes X , Y , or the like from a purely mathematical viewpoint. An incomplete list of relevant publications includes [9, 30, 31, 32] and the recent article [29]. On the other hand, we stress that the results obtained in the aforementioned papers do not overlap with ours.

The present work was preceded by several articles [16, 20, 21, 22, 27] in which weak convergence of renewal shot noise processes has been investigated. The latter is a particular case of processes X in which the input sequence $(S_k)_{k \in \mathbb{N}_0}$ is a standard random walk. Development of elements of the weak convergence theory for renewal shot noise processes was motivated by and effectively used for the asymptotic analysis of various characteristics of several random regenerative structures: the order of random permutations [12], the number of zero and

nonzero blocks of weak random compositions [2, 25], the number of collisions in coalescents with multiple collisions [13], the number of busy servers in a G/G/∞ queueing system [18], random processes with immigration at the epochs of a renewal process [23, 24]. Chapter 3 of [17] provides a survey of the results obtained in the aforementioned articles, pointers to relevant literature, and a detailed discussion of possible applications.

To formulate our main result we need additional notation. Denote by $W_\alpha := (W_\alpha(u))_{u \geq 0}$ a centered Gaussian process which is a.s. locally Hölder continuous with exponent $\alpha > 0$ and satisfies $W_\alpha(0) = 0$ a.s. In particular, for all $T > 0$, all $0 \leq x, y \leq T$, and some a.s. finite random variable M_T ,

$$|W_\alpha(x) - W_\alpha(y)| \leq M_T|x - y|^\alpha. \tag{2}$$

Define the random process $Y_{\alpha,\rho} := (Y_{\alpha,\rho}(u))_{u \geq 0}$ by

$$Y_{\alpha,\rho}(u) := \rho \int_0^u (u - y)^{\rho-1} W_\alpha(y) dy, \quad u > 0, \quad Y_{\alpha,\rho}(0) := \lim_{u \rightarrow +0} Y_{\alpha,\rho}(u), \tag{3}$$

when $\rho > 0$, and by

$$Y_{\alpha,\rho}(u) := u^\rho W_\alpha(u) + |\rho| \int_0^u (W_\alpha(u) - W_\alpha(u - y)) y^{\rho-1} dy, \quad u > 0, \tag{4}$$

$$Y_{\alpha,\rho}(0) := \lim_{u \rightarrow +0} Y_{\alpha,\rho}(u),$$

when $-\alpha < \rho < 0$. Also, put $Y_{\alpha,0} = W_\alpha$. Using (2), we conclude that $Y_{\alpha,\rho}(0) = 0$ a.s. whenever $\rho > -\alpha$.

Convergence of the integrals in (3) and (4) and a.s. continuity of the processes $Y_{\alpha,\rho}$ will be proved in Lemma 2.1 below. When W_α is a Brownian motion (so that $\alpha = 1/2 - \varepsilon$ for any $\varepsilon \in (0, 1/2)$), the process $Y_{\alpha,\rho}$ can be represented as a Skorokhod integral,

$$Y_{\alpha,\rho}(u) := \int_{[0,u]} (u - y)^\rho dW_\alpha(y), \quad u \geq 0.$$

The so defined process is called a *Riemann–Liouville process* or *fractionally integrated Brownian motion* with exponent ρ for $\rho > -1/2$. Since these processes appear several times in our presentation we reserve a special notation for them, R_ρ . When W_α is a more general Gaussian process satisfying the standing assumptions, the process $Y_{\alpha,\rho}$ may be called a *fractionally integrated Gaussian process*. Note that, for positive integer ρ , the process $Y_{\alpha,\rho}$ is up to a multiplicative constant an r -times integrated process W_α . This can be easily checked with the help of integration by parts.

Throughout the paper we assume that the spaces D and $D \times D$ are endowed with the J_1 -topology, and we denote weak convergence in these spaces by $\xrightarrow{J_1}$ and $\xrightarrow{J_1(D \times D)}$, respectively. Comprehensive information concerning the J_1 -topology can be found in [4, 26]. In what follows we use the notation $\mathbb{R}^+ := [0, \infty)$.

Theorem 1.1. *Let $h \in D$ be an eventually nondecreasing function of bounded variation which is regularly varying at ∞ of index $\beta \geq 0$ (i.e., $\lim_{t \rightarrow \infty} (h(tx)/h(t)) = x^\beta$ for all $x > 0$). Assume that $\lim_{t \rightarrow \infty} h(t) = \infty$ when $\beta = 0$, and that*

$$\frac{N(t \cdot) - b(t \cdot)}{a(t)} \xrightarrow{J_1} W_\alpha(\cdot), \quad t \rightarrow \infty, \tag{5}$$

where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is regularly varying at ∞ of positive index, and $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function. Then

$$\frac{X(t \cdot) - \int_{[0, t]} h(t \cdot - y) db(y)}{a(t)h(t)} \xrightarrow{J_1} Y_{\alpha, \beta}(\cdot), \quad t \rightarrow \infty.$$

Remark 1.1. The proof given below reveals that the assumption $\lim_{t \rightarrow \infty} h(t) = \infty$ when $\beta = 0$ is not needed if h is nondecreasing on \mathbb{R}^+ rather than eventually nondecreasing.

Since b is nondecreasing and h is a locally bounded and almost everywhere continuous function (in view of $h \in D$), the integral $\int_{[0, t]} h(t - y) db(y)$ exists as a Riemann–Stieltjes integral.

The remainder of this article is organized as follows. In Section 2 we investigate local Hölder continuity of the limit processes in Theorem 1.1. In Section 3 we give five specializations of Theorem 1.1 for particular sequences $(S_k)_{k \in \mathbb{N}_0}$. Finally, we prove Theorem 1.1 in Section 4.

2. Hölder continuity of the limit processes

For the subsequent presentation, it is convenient to define the process W_α on the whole line. To this end, put $W_\alpha(x) = 0$ for $x < 0$. The right-hand side of (4) can then be given in an equivalent form,

$$Y_{\alpha, \rho}(u) = |\rho| \int_0^\infty (W_\alpha(u) - W_\alpha(u - y))y^{\rho-1} dy, \quad u > 0.$$

It is important for us that the formula (2) still holds true for negative x, y . More precisely, we claim that, for all $T > 0$, all $-\infty < x, y \leq T$, and the same random variable M_T as in (2),

$$|W_\alpha(x) - W_\alpha(y)| \leq M_T |x - y|^\alpha. \tag{6}$$

This inequality is trivially satisfied in the case $x \vee y \leq 0$, and follows from (2) in the case $x \wedge y \geq 0$. Assume now that $x \wedge y \leq 0 < x \vee y$. Then $|W_\alpha(x) - W_\alpha(y)| = |W_\alpha(x \vee y)| \leq M_T (x \vee y)^\alpha \leq M_T |x - y|^\alpha$. Here, the first inequality is a consequence of (2) with $y = 0$.

Lemma 2.1. *Let $\rho > -\alpha$. The following assertions hold:*

- (i) $|Y_{\alpha, \rho}(u)| < \infty$ a.s. for each fixed $u > 0$;
- (ii) the process $Y_{\alpha, \rho}$ is a.s. locally Hölder continuous with exponent $\min(1, \alpha + \rho)$ if $\alpha + \rho \neq 1$, and with arbitrary positive exponent less than 1 if $\alpha + \rho = 1$; more precisely, in the latter situation we have, for any $T^* > T$,

$$\sup_{0 \leq u \neq v \leq T} \frac{|Y_{\alpha, \rho}(u) - Y_{\alpha, \rho}(v)|}{|u - v| \log(T^* |u - v|^{-1})} < \infty \quad \text{a.s.}$$

Proof. The case $\rho = 0$ is trivial. Fix $T > 0$.

(i) Using (2) we obtain, for all $u \in [0, T]$,

$$\begin{aligned} |Y_{\alpha, \rho}(u)| &\leq \rho \int_0^u (u - y)^{\rho-1} |W_\alpha(y)| dy \leq M_T \rho \int_0^u (u - y)^{\rho-1} y^\alpha dy \\ &= M_T \rho B(\rho, \alpha + 1) u^{\rho+\alpha} < \infty \quad \text{a.s.} \end{aligned}$$

in the case $\rho > 0$, and

$$\begin{aligned} |Y_{\alpha,\rho}(u)| &\leq u^\rho |W_\alpha(u)| + |\rho| \int_0^u |W_\alpha(u) - W_\alpha(u-y)| y^{\rho-1} dy \\ &\leq M_T u^{\rho+\alpha} + |\rho| M_T (\rho + \alpha)^{-1} u^{\rho+\alpha} = M_T \alpha (\rho + \alpha)^{-1} u^{\rho+\alpha} < \infty \quad \text{a.s.} \end{aligned}$$

in the case $-\alpha < \rho < 0$.

(ii) By symmetry it is enough to investigate the case $0 \leq v < u \leq T$, and this is tacitly assumed throughout the proof.

Assume first that $-\alpha < \rho < 0$. Appealing to (6) and (2), we conclude that, for $v > 0$,

$$\begin{aligned} |\rho|^{-1} |Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)| &= \left| \int_0^\infty (W_\alpha(u) - W_\alpha(u-y) - W_\alpha(v) + W_\alpha(v-y)) y^{\rho-1} dy \right| \\ &\leq \int_0^{u-v} |W_\alpha(u) - W_\alpha(u-y)| y^{\rho-1} dy \\ &\quad + \int_0^{u-v} |W_\alpha(v) - W_\alpha(v-y)| y^{\rho-1} dy \\ &\quad + \int_{u-v}^\infty |W_\alpha(u) - W_\alpha(v)| y^{\rho-1} dy \\ &\quad + \int_{u-v}^\infty |W_\alpha(u-y) - W_\alpha(v-y)| y^{\rho-1} dy \\ &\leq 2M_T \left(\int_0^{u-v} y^{\rho-1+\alpha} dy + (u-v)^\alpha \int_{u-v}^\infty y^{\rho-1} dy \right) \\ &= 2M_T \alpha (|\rho|(\rho + \alpha))^{-1} (u-v)^{\rho+\alpha} \quad \text{a.s.} \end{aligned}$$

We already know from the proof of (i) that a similar inequality holds when $v = 0$. Thus, the claim of (ii) has been proved in the case $-\alpha < \rho < 0$.

Assume now that $\rho \geq 1$. We infer with the help of (2) that

$$\begin{aligned} |Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)| &= \left| \rho \int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1}) W_\alpha(y) dy + \rho \int_v^u (u-y)^{\rho-1} W_\alpha(y) dy \right| \\ &\leq \rho \int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1}) |W_\alpha(y)| dy + \rho \int_v^u (u-y)^{\rho-1} |W_\alpha(y)| dy \\ &\leq \rho M_T \int_0^v ((u-y)^{\rho-1} - (v-y)^{\rho-1}) y^\alpha dy + \rho M_T \int_v^u (u-y)^{\rho-1} y^\alpha dy \\ &= \rho M_T \int_0^u (u-y)^{\rho-1} y^\alpha dy - \rho M_T \int_0^v (v-y)^{\rho-1} y^\alpha dy \\ &= \rho M_T B(\rho, \alpha + 1) (u^{\rho+\alpha} - v^{\rho+\alpha}) \\ &\leq \rho (\rho + \alpha) M_T B(\rho, \alpha + 1) T^{\rho+\alpha-1} (u-v) \quad \text{a.s.,} \end{aligned}$$

where the last inequality follows from the mean value theorem for differentiable functions.

It remains to investigate the case $0 < \rho < 1$. We shall use the following decomposition:

$$\begin{aligned} |Y_{\alpha,\rho}(u) - Y_{\alpha,\rho}(v)| &= \left| \rho \int_0^u (u-y)^{\rho-1} W_\alpha(y) dy - \rho \int_0^v (v-y)^{\rho-1} W_\alpha(y) dy \right| \\ &= \left| \rho \int_0^v (W_\alpha(v) - W_\alpha(v-y))(y^{\rho-1} - (y+u-v)^{\rho-1}) dy \right. \\ &\quad \left. - \rho \int_0^{u-v} (W_\alpha(v) - W_\alpha(u-y))y^{\rho-1} dy + W_\alpha(v)(u^\rho - v^\rho) \right| \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \rho \int_0^v |W_\alpha(v) - W_\alpha(v-y)|(y^{\rho-1} - (y+u-v)^{\rho-1}) dy, \\ I_2 &:= \rho \int_0^{u-v} |W_\alpha(v) - W_\alpha(u-y)|y^{\rho-1} dy, \\ I_3 &:= |W_\alpha(v)|(u^\rho - v^\rho). \end{aligned}$$

The summand I_1 can be estimated as

$$\begin{aligned} I_1 &\leq \rho M_T \int_0^v y^\alpha (y^{\rho-1} - (y+u-v)^{\rho-1}) dy \\ &= \rho M_T (u-v)^{\alpha+\rho} \int_0^{v/(u-v)} t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) dt. \end{aligned}$$

Using the inequality $x^{\rho-1} - (x+1)^{\rho-1} \leq (1-\rho)x^{\rho-2}$ for $x > 0$ gives

$$\begin{aligned} I_1 &\leq \rho(1-\rho)M_T(u-v)^{\alpha+\rho} \int_0^{v/(u-v)} t^{\alpha+\rho-2} dt \\ &= \rho(1-\rho)(\alpha+\rho-1)^{-1} M_T v^{\alpha+\rho-1} (u-v) \\ &\leq \rho(1-\rho)(\alpha+\rho-1)^{-1} M_T T^{\alpha+\rho-1} (u-v) \end{aligned}$$

in the case $\alpha + \rho > 1$, and

$$\begin{aligned} I_1 &\leq \rho M_T (u-v)^{\alpha+\rho} \left(\int_0^1 t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) dt + \int_1^\infty t^\alpha (t^{\rho-1} - (t+1)^{\rho-1}) dt \right) \\ &\leq \rho M_T (u-v)^{\alpha+\rho} \left(\int_0^1 t^{\alpha+\rho-1} dt + (1-\rho) \int_1^\infty t^{\alpha+\rho-2} dt \right) \\ &= \rho M_T \left(\frac{1}{\alpha+\rho} + \frac{1-\rho}{1-\alpha-\rho} \right) (u-v)^{\alpha+\rho} \end{aligned}$$

in the case $0 < \alpha + \rho < 1$. Also,

$$\begin{aligned} I_1 &\leq \rho M_T (u-v) \int_0^{v/(u-v)} \left(1 - \left(\frac{t}{t+1} \right)^\alpha \right) dt \\ &\leq \rho M_T (u-v) \int_0^{v/(u-v)} \frac{dt}{t+1} \end{aligned}$$

$$\begin{aligned}
 &= \rho M_T(u - v) \ln \frac{u}{u - v} \\
 &\leq \rho M_T(u - v) \ln \frac{T}{u - v}
 \end{aligned}$$

in the case $\alpha + \rho = 1$.

Further,

$$I_2 \leq \rho M_T \int_0^{u-v} (u - v - y)^\alpha y^{\rho-1} dy = \rho M_T B(\rho, \alpha + 1)(u - v)^{\alpha+\rho}.$$

In the case $\alpha + \rho > 1$ the inequality $(u - v)^{\alpha+\rho} \leq T^{\alpha+\rho-1}(u - v)$ has to be additionally used.

Finally,

$$I_3 \leq M_T v^\alpha (u^\rho - v^\rho) \leq M_T (u^{\alpha+\rho} - v^{\alpha+\rho}).$$

The right-hand side does not exceed $M_T(u - v)^{\alpha+\rho}$ in the case $0 < \alpha + \rho \leq 1$ in view of the subadditivity of $x \mapsto x^{\alpha+\rho}$ on $[0, \infty)$ and $M_T(\alpha + \rho)T^{\alpha+\rho-1}(u - v)$ in the case $\alpha + \rho > 1$.

The proof of Lemma 2.1 is complete. □

3. Applications of Theorem 1.1

In this section we give five examples of particular sequences $(S_k)_{k \in \mathbb{N}_0}$ which satisfy the limit relation (5) with four different Gaussian processes W_α . Throughout the section we always assume, without further notice, that h satisfies the assumptions of Theorem 1.1.

In the case where the sequence $(S_k)_{k \in \mathbb{N}_0}$ is a.s. nondecreasing, the counting process $(N(t))_{t \geq 0}$ is nothing other than a generalized inverse function for (S_k) , that is,

$$N(t) = \inf\{k \in \mathbb{N} : S_k > t\} \quad \text{a.s.,} \quad t \geq 0. \tag{7}$$

In view of this, if a functional limit theorem for $S_{[ut]}$ in the J_1 -topology on D holds, and the limit process is a.s. continuous, then the corresponding functional limit theorem for $N(ut)$ in the J_1 -topology on D is a simple consequence. A detailed discussion of this fact can be found, for instance, in [15]. If the sequence $(S_k)_{k \in \mathbb{N}_0}$ is not monotone (as, for instance, in Subsection 3.2 below), then equality (7) is no longer true, and the proof of a functional limit theorem for $N(ut)$ requires an additional specific argument in every particular case.

3.1. Delayed standard random walk

Let ξ_1, ξ_2, \dots be i.i.d. nonnegative random variables which are independent of a nonnegative random variable S_0 . The random sequence $(S_k)_{k \in \mathbb{N}_0}$ defined by $S_k := S_0 + \xi_1 + \dots + \xi_k$ for $k \in \mathbb{N}_0$ is called a *delayed standard random walk* provided that $P\{S_0 = 0\} < 1$. In the case $S_0 = 0$ a.s., the term *zero-delayed standard random walk* is used. It is well known (see, for instance, [5, Theorem 1b(i)]) that

(a) if $\sigma^2 := \text{Var } \xi_1 \in (0, \infty)$, then

$$\frac{N(t \cdot) - \mu^{-1}t(\cdot)}{(\sigma^2 \mu^{-3}t)^{1/2}} \xrightarrow{J_1} B(\cdot), \quad t \rightarrow \infty, \tag{8}$$

where $\mu := E(\xi_1) < \infty$, and $(B(u))_{u \geq 0}$ is a standard Brownian motion (so that relation (5) holds with $b(t) = \mu^{-1}t$ and $a(t) = (\sigma^2 \mu^{-3}t)^{1/2}$);

(b) if

$$\sigma^2 = \infty \quad \text{and} \quad \int_{[0,x]} y^2 P\{\xi_1 \in dy\} \sim L(x), \quad x \rightarrow \infty \tag{9}$$

for some L slowly varying at ∞ , then

$$\frac{N(t \cdot) - \mu^{-1}t(\cdot)}{\mu^{-3/2}c(t)} \xrightarrow{J_1} B(\cdot), \quad t \rightarrow \infty, \tag{10}$$

where c is a positive measurable function satisfying $\lim_{t \rightarrow \infty} c(t)^{-2}tL(c(t)) = 1$ (so that relation (5) holds with $b(t) = \mu^{-1}t$ and $a(t) = \mu^{-3/2}c(t)$; since c is asymptotically inverse for $t \mapsto t^2/L(t)$, an application of [6, Proposition 1.5.15] enables us to conclude that c , and hence also a , are regularly varying at ∞ of index $1/2$).

Thus, according to Theorem 1.1, we have

$$\frac{X(t \cdot) - \mu^{-1} \int_0^t h(y)dy}{(\sigma^2 \mu^{-3}t)^{1/2}h(t)} \xrightarrow{J_1} \beta \int_0^{(\cdot)} (\cdot - y)^{\beta-1} B(y)dy = R_\beta(\cdot), \quad t \rightarrow \infty,$$

provided that $\sigma^2 \in (0, \infty)$ (in the case $\beta = 0$, the limit process is $R_0 = B$), and

$$\frac{X(t \cdot) - \mu^{-1} \int_0^t h(y)dy}{\mu^{-3/2}c(t)h(t)} \xrightarrow{J_1} R_\beta(\cdot), \quad t \rightarrow \infty$$

provided that conditions (9) hold. In particular, irrespective of whether the variance is finite or not, the limit process is a fractionally integrated Brownian motion with parameter β . As far as zero-delayed standard random walks are concerned, the aforementioned results can be found in Theorem 1.1 (A1, A2) of [16].

3.2. Perturbed random walks

Let $(\xi_1, \eta_1), (\xi_2, \eta_2) \dots$ be i.i.d. random vectors with nonnegative coordinates. Put

$$S_1 := \eta_1, \quad S_n := \xi_1 + \dots + \xi_{n-1} + \eta_n, \quad n \geq 2.$$

The so defined sequence $(S_n)_{n \in \mathbb{N}}$ is called a *perturbed random walk*. Various properties of perturbed random walks are discussed in [17].

Assume that $\sigma^2 = \text{Var } \xi_1 \in (0, \infty)$ and $E(\eta^a) < \infty$ for some $a > 0$. Put $F(x) := P\{\eta_1 \leq x\}$ for $x \in \mathbb{R}$. According to [2, Theorem 3.2],

$$\frac{N(t \cdot) - \mu^{-1} \int_0^t F(y)dy}{\sqrt{\sigma^2 \mu^{-3}t}} \xrightarrow{J_1} B(\cdot), \quad t \rightarrow \infty,$$

where $\mu = E(\xi_1) < \infty$. Therefore, by Theorem 1.1,

$$\frac{X(t \cdot) - \mu^{-1} \int_0^t h(y)F(y)dy}{(\sigma^2 \mu^{-3}t)^{1/2}h(t)} \xrightarrow{J_1} R_\beta(\cdot), \quad t \rightarrow \infty.$$

3.3. Random walks with long memory

Let ξ_1, ξ_2, \dots be i.i.d. positive random variables with finite mean. Assume that these are independent of random variables $\theta_1, \theta_2, \dots$ which form a centered stationary Gaussian sequence with $E(\theta_1\theta_{k+1}) \sim k^{2d-1}\ell(k)$ as $k \rightarrow \infty$ for some $d \in (0, 1/2)$. Put $S_0 := 0$ and

$$S_n - S_{n-1} = \xi_n e^{\theta_n}, \quad n \in \mathbb{N}.$$

Recall that a fractional Brownian motion with Hurst index $H \in (0, 1)$ is a centered Gaussian process $B_H := (B_H(u))_{u \geq 0}$ with covariance $E(B_H(u)B_H(v)) = 2^{-1}(u^{2H} + v^{2H} - (u - v)^{2H})$ for $u, v \geq 0$. This process has stationary increments and is self-similar of index H . Therefore, for any $p > 0$,

$$E|B_H(u) - B_H(v)|^p = (u - v)^{Hp} E|B_H(1)|^p, \quad u, v \geq 0.$$

According to the Kolmogorov–Chentsov sufficient conditions, there exists a version of B_H (which we also denote by B_H) which is a.s. Hölder continuous with exponent smaller than $H - 1/p$ for any $p > 0$, and hence also smaller than H .

According to [3, Example 4.25, p. 357],

$$\frac{N(t \cdot) - m_1^{-1}t(\cdot)}{(d(2d + 1))^{-1/2}m_1^{-3/2-d}m_2 t^{d+1/2}(\ell(t))^{1/2}} \xrightarrow{J_1} B_{d+1/2}(\cdot), \quad t \rightarrow \infty,$$

where $m_1 := E(S_1) = E(\xi_1)E(e^{\theta_1})$ and $m_2 := E(\xi_1)E(\theta_1 e^{\theta_1})$. An application of Theorem 1.1 yields, as $t \rightarrow \infty$,

$$\frac{X(t \cdot) - m_1^{-1} \int_0^t h(y)dy}{(d(2d + 1))^{-1/2}m_1^{-3/2-d}m_2 t^{d+1/2}(\ell(t))^{1/2}h(t)} \xrightarrow{J_1} \beta \int_0^{(\cdot)} (\cdot - y)^{\beta-1} B_{d+1/2}(y)dy$$

if $\beta > 0$. If $\beta = 0$, the limit process is $B_{d+1/2}$.

3.4. Counting process in a branching random walk

Assume that the random variables ξ_k defined in Subsection 3.1 are a.s. positive. For some integer $k \geq 2$, we take in the role of $N(t)$ the number of the k th-generation individuals with positions $\leq t$ in a branching random walk in which the first-generation individuals are located at the points $S_1 = \xi_1, S_2 = \xi_1 + \xi_2, \dots$ (a more precise definition can be found in [19, Section 1.2]).

Assume that $\sigma^2 = \text{Var } \xi_1 \in (0, \infty)$. Theorem 1.3 in [19] implies that

$$\frac{N(t \cdot) - (t \cdot)^k / (k! \mu^k)}{((k - 1)!)^{-1} \sqrt{\sigma^2 \mu^{-2k-1} t^{2k-1}}} \xrightarrow{J_1} R_{k-1}(\cdot), \quad t \rightarrow \infty,$$

where $\mu = E(\xi_1) < \infty$. Of course, for $k = 1$ this limit relation is also valid and amounts to (8), as it must be. By Lemma 2.1, the process R_{k-1} is a.s. locally Hölder continuous with any positive exponent smaller than $k - 1/2$. Thus, Theorem 1.1 applies and gives

$$\begin{aligned} \frac{X(t \cdot) - ((k - 1)! \mu^k)^{-1} \int_0^t h(t \cdot - y) y^{k-1} dy}{((k - 1)!)^{-1} \sqrt{\sigma^2 \mu^{-2k-1} t^{2k-1} h(t)}} &\xrightarrow{J_1} \beta \int_0^{(\cdot)} (\cdot - y)^{\beta-1} R_{k-1}(y) dy \\ &= (k - 1)B(k - 1, \beta + 1)R_{\beta+k-1}(\cdot) \end{aligned}$$

as $t \rightarrow \infty$, where $B(\cdot, \cdot)$ is the beta function. The latter equality can be checked as follows: for $u > 0$,

$$\begin{aligned} \beta \int_0^u (u - y)^{\beta-1} R_{k-1}(y) dy &= \beta \int_0^u (u - y)^{\beta-1} (k - 1) \int_0^y (y - x)^{k-2} B(x) dx dy \\ &= \beta(k - 1) \int_0^u B(x) \int_x^u (u - y)^{\beta-1} (y - x)^{k-2} dy dx \\ &= \beta(k - 1) \int_0^u B(x) \int_0^{u-x} (u - x - y)^{\beta-1} y^{k-2} dy dx \\ &= \beta(k - 1) B(k - 1, \beta) \int_0^u (u - x)^{\beta+k-2} B(x) dx \\ &= \frac{\beta(k - 1) B(k - 1, \beta)}{\beta + k - 1} R_{\beta+k-1}(u) \\ &= (k - 1) B(k - 1, \beta + 1) R_{\beta+k-1}(u). \end{aligned}$$

3.5. Inhomogeneous Poisson process

Let $(N(t))_{t \geq 0}$ be an inhomogeneous Poisson process with $E[N(t)] = m(t)$ for a nondecreasing function $m(t)$ satisfying

$$m(t) \sim ct^w, \quad t \rightarrow \infty, \tag{11}$$

where $c, w > 0$. Without loss of generality we can identify $(N(t))_{t \geq 0}$ with the process $(N^*(m(t)))_{t \geq 0}$, where $(N^*(t))_{t \geq 0}$ is a homogeneous Poisson process with $E[N^*(t)] = t, t \geq 0$. According to (8),

$$\frac{N^*(t \cdot) - (t \cdot)}{t^{1/2}} \xrightarrow{J_1} B(\cdot), \quad t \rightarrow \infty. \tag{12}$$

Dini’s theorem in combination with (11) ensures that

$$\lim_{t \rightarrow \infty} \sup_{u \in [0, T]} \left| \frac{m(tu)}{ct^w} - u^w \right| = 0 \tag{13}$$

for all $T > 0$. It is known (see, for instance, [14, Lemma 2.3, p. 159]) that the composition mapping $(x, \varphi) \mapsto (x \circ \varphi)$ is continuous on continuous functions $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ and continuous nondecreasing functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Using this fact in conjunction with (12), (13), and the continuous mapping theorem, we infer that

$$\frac{N(t \cdot) - m(t \cdot)}{(ct^w)^{1/2}} \xrightarrow{J_1} B((\cdot)^w), \quad t \rightarrow \infty.$$

Thus, the limit process is a time-changed Brownian motion. An application of Theorem 1.1 yields

$$\frac{X(t \cdot) - \mu^{-1} \int_0^{t \cdot} h(y) dm(y)}{(ct^w)^{1/2} h(t)} \xrightarrow{J_1} \beta \int_0^{(\cdot)} (\cdot - y)^{\beta-1} B(y^w) dy, \quad t \rightarrow \infty$$

if $\beta > 0$. If $\beta = 0$, the limit process is $B((\cdot)^w)$.

4 Proof of Theorem 1.1

To prove weak convergence of finite-dimensional distributions we need an auxiliary lemma.

Lemma 1. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow (0, \infty)$ be a function which is nondecreasing in the second coordinate and satisfies $\lim_{t \rightarrow \infty} f(t, x) = x^\beta$ for all $x > 0$ and some $\beta \geq 0$. For $t, x \geq 0$, set*

$$\begin{aligned} Z(t, x) &:= \int_{[0, x]} W_\alpha(x - y) d_y f(t, y), \\ Z(x) &:= \int_{[0, x]} W_\alpha(x - y) d_y f^\beta \text{ for } \beta > 0, \\ Z(x) &:= W_\alpha(x) \text{ for } \beta = 0. \end{aligned}$$

Then, for any $u, v > 0$,

$$\lim_{t \rightarrow \infty} E[Z(t, u)Z(t, v)] = E[Z(u)Z(v)].$$

Proof. Fix any $u, v > 0$. For each $t > 0$, denote by $Q_t^{(u)}$ and $Q_t^{(v)}$ independent random variables with the distribution functions

$$P\{Q_t^{(u)} \leq y\} = \begin{cases} 0, & \text{if } y < 0, \\ \frac{f(t, y)}{f(t, u)}, & \text{if } y \in [0, u], \\ 1, & \text{if } y > u; \end{cases} \quad P\{Q_t^{(v)} \leq y\} = \begin{cases} 0, & \text{if } y < 0, \\ \frac{f(t, y)}{f(t, v)}, & \text{if } y \in [0, v], \\ 1, & \text{if } y > v. \end{cases}$$

Also, denote by $Q^{(u)}$ and $Q^{(v)}$ independent random variables with the distribution functions

$$P\{Q^{(u)} \leq y\} = \begin{cases} 0, & \text{if } y < 0, \\ (\frac{y}{u})^\beta, & \text{if } y \in [0, u], \\ 1, & \text{if } y > u; \end{cases} \quad P\{Q^{(v)} \leq y\} = \begin{cases} 0, & \text{if } y < 0, \\ (\frac{y}{v})^\beta, & \text{if } y \in [0, v], \\ 1, & \text{if } y > v. \end{cases}$$

By assumption,

$$(Q_t^{(u)}, Q_t^{(v)}) \xrightarrow{d} (Q^{(u)}, Q^{(v)}), \quad t \rightarrow \infty.$$

Define the function $r(x, y) := E[W_\alpha(x)W_\alpha(y)]$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Using the a.s. continuity of W_α , Lebesgue’s dominated convergence theorem, and the fact that, according to [1, Theorem 3.2, p. 63], $E(\sup_{z \in [0, T]} W_\alpha(z))^2 < \infty$, we conclude that r is continuous, hence also bounded on $[0, T] \times [0, T]$ for all $T > 0$. This entails

$$r(u - Q_t^{(u)}, v - Q_t^{(v)}) \xrightarrow{d} r(u - Q^{(u)}, v - Q^{(v)}), \quad t \rightarrow \infty,$$

and thus

$$\lim_{t \rightarrow \infty} E[r(u - Q_t^{(u)}, v - Q_t^{(v)})] = E[r(u - Q^{(u)}, v - Q^{(v)})]$$

by Lebesgue’s dominated convergence theorem. Further,

$$\begin{aligned} &E[Z(t, u)Z(t, v)] \\ &= f(t, u)f(t, v) \int_{[0, u]} \int_{[0, v]} E W_\alpha(u - y)W_\alpha(v - z) d_y \left(\frac{f(t, y)}{f(t, u)}\right) d_z \left(\frac{f(t, z)}{f(t, v)}\right) \\ &= f(t, u)f(t, v)E[r(u - Q_t^{(u)}, v - Q_t^{(v)})] \xrightarrow{t \rightarrow \infty} (uv)^\beta E[r(u - Q^{(u)}, v - Q^{(v)})]. \end{aligned}$$

It remains to note that while in the case $\beta > 0$ we have

$$\begin{aligned} &(uv)^\beta E[r(u - Q^{(u)}, v - Q^{(v)})] \\ &= \int_{[0, u]} \int_{[0, v]} r(u - y, v - z) d_y(u^\beta P\{Q^{(u)} \leq y\}) d_z(v^\beta P\{Q^{(v)} \leq z\}) \\ &= \int_{[0, u]} \int_{[0, v]} r(u - y, v - z) d_y^\beta d_z^\beta = E \int_{[0, u]} W_\alpha(u - y) d_y^\beta \int_{[0, v]} W_\alpha(v - z) d_z^\beta \\ &= E[Z(u)Z(v)], \end{aligned}$$

in the case $\beta = 0$ we have

$$(uv)^\beta E[r(u - Q^{(u)}, v - Q^{(v)})] = r(u, v) = E[W_\alpha(u)W_\alpha(v)] = E[Z(u)Z(v)].$$

The proof of Lemma 1 is complete. □

Proof of Theorem 1.1. Since h is eventually nondecreasing, there exists $t_0 > 0$ such that $h(t)$ is nondecreasing for $t > t_0$. Being a regularly varying function of nonnegative index, h is eventually positive. Hence, increasing t_0 if needed, we can ensure that $h(t) > 0$ for $t > t_0$. We first show that the behavior of the function of bounded variation h on $[0, t_0]$ does not affect the weak convergence of the general shot noise process. Once this is done, we can assume, without loss of generality, that $h(0) = 0$ and that h is nondecreasing on \mathbb{R}^+ .

Integrating by parts yields

$$\begin{aligned} X(tu) - \int_{[0, tu]} (h(tu - y)db(y)) &= \int_{[0, u]} (h(tu - y)d_y(N(ty) - b(ty))) \\ &= (h(tu) - h((tu) -))(N(0) - b(0)) \\ &\quad + \int_{(0, u]} (N(ty) - b(ty))d_y(-h(t(u - y))). \end{aligned} \tag{14}$$

For all $T > 0$,

$$\frac{\sup_{u \in [0, T]} |h(tu) - h((tu) -)||N(0) - b(0)|}{a(t)h(t)} \leq \frac{h(tT)}{h(t)} \frac{|N(0) - b(0)|}{a(t)} \xrightarrow{P} 0, \quad t \rightarrow \infty,$$

because h is regularly varying at ∞ . Denote by $V_0^{t_0}(h)$ the total variation of h on $[0, t_0]$. By assumption, $V_0^{t_0}(h) < \infty$. For all $T > 0$,

$$\begin{aligned} &\frac{\sup_{u \in [0, T]} \int_{[u-t_0/t, u]} (N(ty) - b(ty))d_y(-h(t(u - y)))}{a(t)h(t)} \\ &\leq \frac{\sup_{u \in [0, T]} |N(tu) - b(tu)|}{a(t)} \frac{V_0^{t_0}(h)}{h(t)} \xrightarrow{P} 0, \quad t \rightarrow \infty. \end{aligned}$$

The convergence to 0 is justified by the facts that, according to (5), the first factor converges in distribution to $\sup_{u \in [0, T]} |W_\alpha(u)|$, whereas the second trivially converges to 0. Recall that, when $\beta = 0$, $\lim_{t \rightarrow \infty} h(t) = \infty$ holds by assumption, whereas when $\beta > 0$, it holds automatically. Thus, as was claimed, while investigating the asymptotic behavior of the second summand on the right-hand side of (14) we can and do assume that $h(0) = 0$ and that h is nondecreasing on \mathbb{R}^+ .

Skorokhod's representation theorem ensures that there exist versions $(\hat{N}(t))_{t \geq 0}$ and $(\hat{W}_\alpha(t))_{t \geq 0}$ of the processes $(N(t))_{t \geq 0}$ and $(W_\alpha(t))_{t \geq 0}$ such that, for all $T > 0$,

$$\lim_{t \rightarrow \infty} \sup_{u \in [0, T]} |\hat{W}_\alpha^{(t)}(u) - \hat{W}_\alpha(u)| = 0 \quad \text{a.s.}, \quad (15)$$

where $\hat{W}_\alpha^{(t)}(u) := \frac{\hat{N}(tu) - b(tu)}{a(t)}$ for $t > 0$ and $u \geq 0$. For each $t > 0$, set

$$\begin{aligned} h_t(x) &:= h(tx)/h(t), \quad x \geq 0, \\ X_t(u) &:= (a(t))^{-1} \int_{(0, u]} (N(ty) - b(ty)) d_y(-h_t(u-y)), \quad u \geq 0, \\ \hat{X}_t^*(u) &:= \int_{(0, u]} \hat{W}_\alpha^{(t)}(y) d_y(-h_t(u-y)), \quad u \geq 0. \end{aligned}$$

The distributions of the processes $(X_t(u))_{u \geq 0}$ and $(\hat{X}_t^*(u))_{u \geq 0}$ are the same. Hence, it remains to check that

$$\lim_{t \rightarrow \infty} \int_{(0, u]} (\hat{W}_\alpha^{(t)}(y) - \hat{W}_\alpha(y)) d_y(-h_t(u-y)) = 0 \quad \text{a.s.} \quad (16)$$

in the J_1 -topology on D , and

$$\int_{(0, \cdot]} W_\alpha(y) d(-h_t(\cdot - y)) \xrightarrow{J_1} Y_{\alpha, \beta}(\cdot), \quad t \rightarrow \infty. \quad (17)$$

In view of (15) and the monotonicity of h_t , we have, for all $T > 0$ as $t \rightarrow \infty$,

$$\begin{aligned} \sup_{u \in [0, T]} \left| \int_{(0, u]} (\hat{W}_\alpha^{(t)}(u) - \hat{W}_\alpha(u)) d_y(-h_t(u-y)) \right| \\ \leq \sup_{u \in [0, T]} \left| \hat{W}_\alpha^{(t)}(u) - \hat{W}_\alpha(u) \right| h_t(T) \longrightarrow 0 \quad \text{a.s.}, \end{aligned}$$

which proves (16).

Since W_α is a Gaussian process, the convergence of the finite-dimensional distributions in (17) which is equivalent to the convergence of covariances follows from Lemma 1. While applying the lemma we use the equalities

$$Y_{\alpha, \beta}(u) = \int_{(0, u]} W_\alpha(y) d_y(-(u-y)^\beta)$$

when $\beta > 0$ and $Y_{\alpha, \beta}(u) = W_\alpha(u)$ when $\beta = 0$. Our next step is to prove the tightness on $D[0, T]$, for all $T > 0$, of

$$\hat{X}_t(u) := \int_{(0, u]} W_\alpha(y) d_y(-h_t(u-y)) = \int_{[0, u]} W_\alpha(u-y) d_y h_t(y), \quad u \geq 0.$$

By [4, Theorem 15.5], it is enough to show that, for any $r_1, r_2 > 0$, there exist $t_0, \delta > 0$ such that, for all $t \geq t_0$,

$$\mathbb{P} \left\{ \sup_{0 \leq u, v \leq T, |u-v| \leq \delta} |\hat{X}_t(u) - \hat{X}_t(v)| > r_1 \right\} \leq r_2. \quad (18)$$

Put $l := \max(u, v)$. Recalling that $W_\alpha(s) = 0$ for $s < 0$ (see the beginning of Section 2), we have, for $0 \leq u, v \leq T$ and $|u - v| \leq \delta$,

$$\begin{aligned} |\hat{X}_t(u) - \hat{X}_t(v)| &= \left| \int_{[0, t)} (W_\alpha(u - y) - W_\alpha(v - y)) d_y h_t(y) \right| \\ &\leq M_T |u - v|^\alpha h_t(T) \leq M_T |u - v|^\alpha \lambda \end{aligned}$$

for large enough t and a positive constant λ . The existence of λ is justified by the relation $\lim_{t \rightarrow \infty} h_t(T) = T^\beta < \infty$. Decreasing δ if needed, we ensure that the inequality (18) holds for any positive r_1 and r_2 . The proof of Theorem 1.1 is complete. \square

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