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Classification of knotted tori

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For a smooth manifold N denote by $E^m(N)$ the set of smooth isotopy classes of smooth embeddings $N \to \mathbb{R}^m$. A description of the set $E^m(S^p \times S^q)$ was known only for $p = q = 0$ or for $p = 0$, $m \neq q + 2$ or for $2m \geqslant 2(p + q) + \max\{p, q\} + 4$. (The description was given in terms of homotopy groups of spheres and of Stiefel manifolds.) For $m \geqslant 2p + q + 3$ we introduce an abelian group structure on $E^m(S^p \times S^q)$ and describe this group 'up to an extension problem'. This result has corollaries which, under stronger dimension restrictions, more explicitly describe $E^m(S^p \times S^q)$. The proof is based on relations between sets $E^m(N)$ for different N and m, in particular, on a recent exact sequence of M. Skopenkov.

Keywords: embedding; isotopy; links; knotted tori; Stiefel manifolds

2010 *Mathematics subject classification:* 57R40; 57R52

1. Introduction and main results

1.1. Some general motivations

This paper is on the classical Knotting Problem: *for an n-manifold N and a number m, classify isotopy classes of embeddings $N \to \mathbb{R}^m$*. For recent surveys see [**16**, **27**]; whenever possible I refer to these surveys not to original papers.

I consider *smooth* manifolds, embeddings and isotopies. By a classification I mean *a readily calculable* classification.[1] Main results are stated in § 1.2 independently of § 1.1.

Many interesting examples of embeddings are embeddings $S^p \times S^q \to \mathbb{R}^m$, i.e. *knotted tori*. See references in [**17**]. Since the general Knotting Problem is very hard [**16**], it is very interesting to solve it for the important particular case of knotted tori. Classification of knotted tori is a natural next step after the Haefliger link theory [**12**] and the classification of embeddings of highly-connected manifolds [**27**, § 2], [**15**]. Such a step gives some insight or even precise information concerning embeddings of *arbitrary* manifolds [**26**, **31**, **33**], and reveals new interesting relations to algebraic topology.

[1]For a discussion of the adjectives 'smooth', 'readily calculable', and of embeddings into $\mathbb{R}^m$ vs into S^m see [**7**, remark 2.20], [**16**, remarks 1.1 and 1.2].

The Knotting Problem is more accessible for $2m \geqslant 3n+4$, when there are some classical complete readily calculable classifications of embeddings [**27**, § 2, § 3], [**16**]. Cf. (S) of § 1.3.

The Knotting Problem is much harder for $2m < 3n+4$: if N is a closed manifold that is not a disjoint union of homology spheres, then until recently no complete readily calculable isotopy classification was known. This is in spite of the existence of many interesting approaches including methods of Haefliger–Wu, Browder–Wall and Goodwillie–Weiss [**27**, § 5], [**3**, **9**, **36**].

Classification results for $2m < 3n+4$ concern links [**1**, **2**, **12**], embeddings of d-connected n-manifolds for $2m \geqslant 3n+3-d$ [**23**, **24**], embeddings of 3- and 4-dimensional manifolds [**6**–**8**, **28**, **30**], and rational classification of embeddings $S^p \times S^q \to \mathbb{R}^m$ under stronger dimension restriction than $m \geqslant 2p+q+3$ [**4**, **5**] (see footnote 2). The methods of those papers essentially use the restrictions present there.

The new ideas allowing to go beyond the above results follow [**32**] and unpublished work [**25**]. One idea is to find *relations* between different sets of (isotopy classes of) embeddings, invariants of embeddings and geometric constructions of embeddings. Group structures on sets of embeddings are constructed.[2] Then such relations are formulated in terms of exact sequences. The most non-trivial exact sequence is relation of knotted tori to links and *knotted strips* $D^p \times S^q \to S^m$, i.e. the $\nu\sigma(i\zeta\lambda')$-sequence from the proof of theorem 1.2 in § 2.2. This is the main theoretical result [**32**, theorem 1.6] of [**32**], which non-trivally extends [**25**, Restriction lemma 5.2] and [**34**, lemma 2.15.a] (see footnote 2).

This theoretical result yielded rational classification (corollary 1.4.a [**32**, corollary 1.7]). Still, it was expected that embeddings $S^p \times S^q \to \mathbb{R}^m$ are hard to classify for $m \geqslant 2p+q+3 > q+3$. Such a classification is the main result of this paper. The main idea of this paper is, in some sense, *a reduction* of classification of knotted tori to classification of links and knotted strips (rather than *a relation* as in [**32**]). This is obtained by discovering new relations between different sets of embeddings, and, more importantly, *connections* between such relations, formulated in terms of diagrams involving the exact sequences, see § 2.2. These ideas are hopefully interesting in themselves.

1.2. Statements of main results

For a manifold N let $E^m(N)$ be the set of isotopy classes of embeddings $N \to S^m$. Abelian group structures on $E^m(D^p \times S^q)$ for $m \geqslant q+3$ and on $E^m(S^p \times S^q)$ for $m \geqslant 2p+q+3$ are defined analogously to the well-known case $p=0$. The sum operation on $E^m(D^p \times S^q)$ is 'connected sum of q-spheres together with normal p-framings' or 'D^p-parametric connected sum'. The sum operation on $E^m(S^p \times S^q)$ is 'S^p-parametric connected sum', cf. [**19**, **26**, **31**], [**33**, theorem 8]. See accurate definitions in § 2.1; cf. [**34**, remarks 2.3 and 2.4].

Our main results describe the group $E^m(S^p \times S^q)$ up to an extension problem.

[2] Group structures are constructed in [**25**] and, with more details, here and in [**34**, § 3.2]. This is already used in [**4**, **5**, **32**].

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Definitions of $[\cdot]$, *the 'embedded connected sum' or 'local knotting' action*

$$\# : E^m(N) \times E^m(S^n) \to E^m(N),$$

and of $E^m_\#(N)$. By $[\cdot]$ we denote the isotopy class of an embedding or the homotopy class of a map.

Assume that $m \geqslant n+2$ and N is a closed connected oriented n-manifold. Represent elements of $E^m(N)$ and of $E^m(S^n)$ by embeddings $f : N \to S^m$ and $g : S^n \to S^m$ whose images are contained in disjoint balls. Join the images of f, g by an arc whose interior misses the images. Let $[f]\#[g]$ be the isotopy class of the *embedded connected sum* of f and g along this arc, cf. [**11**, theorem 1.7], [**12**, theorem 2.4], [**1**, § 1].

For $N = S^q \sqcup S^n$ this construction is made for an arc joining $f(S^n)$ to $g(S^n)$.

For $m \geqslant n+2$ the operation $\#$ is well-defined.[3] Clearly, $\#$ is an action.

Let $E^m_\#(N)$ be the quotient set of $E^m(N)$ by this action and $q_\# : E^m(N) \to E^m_\#(N)$ the quotient map. A group structure on $E^m_\#(S^p \times S^q)$ is well-defined by $q_\# f + q_\# f' := q_\#(f + f')$, $f, f' \in E^m(S^p \times S^q)$, because $(f\#g) + f' = f + (f'\#g) = (f + f')\#g$ by definition of '+' in § 2.1.

The following result reduces description of $E^m(S^p \times S^q)$ to description of $E^m(S^{p+q})$ and of $E^m_\#(S^p \times S^q)$, cf. [**22**], [**6**, end of § 1].

Lemma 1.1 (Smoothing; proved in § 2.4). *For* $m \geqslant 2p+q+3$ *we have* $E^m(S^p \times S^q) \cong E^m_\#(S^p \times S^q) \oplus E^m(S^{p+q})$.

The isomorphism of lemma 1.1 is $q_\# \oplus \overline{\sigma}$, where $\overline{\sigma}$ is 'surgery of $S^p \times *$' defined in § 2.4. It has the property $(q_\# \oplus \overline{\sigma})(f\#g) = q_\#(f) \oplus (\overline{\sigma}(f) + g)$ for each $f \in E^m(S^p \times S^q)$, $g \in E^m(S^{p+q})$.

Denote by $V_{s,t}$ the Stiefel manifold of t-frames in $\mathbb{R}^s$. Identify $V_{s,1}$ with S^{s-1}.

Known results easily imply (see corollary 1.5.a) that

$$E^m_\#(S^p \times S^q) \cong \pi_q(V_{m-q,p+1}) \quad \text{for } 2m \geqslant 2p+3q+4.$$

Our main result generalizes this for $m \geqslant 2p+q+3$.

For $m \geqslant n+3$ denote by

- $\lambda = \lambda^m_{q,n} : E^m(S^q \sqcup S^n) \to \pi_q(S^{m-n-1})$ the linking coefficient that is the homotopy class of the first component in the complement to the second component, see accurate definition in [**18**], [**27**, § 3].
- $E^m_U(S^q \sqcup S^n) \subset E^m(S^q \sqcup S^n)$ the subset formed by the isotopy classes of embeddings whose restriction to *each* component is unknotted.
- $K^m_{q,n} := \ker \lambda \cap E^m_U(S^q \sqcup S^n)$; see geometric description in [**29**, § 3, Definition of $\widehat{DM}^m_{p,q}$].

[3] This is proved analogously to the case $X = D^0_+$ of the Standardization lemma 2.1.b below, because the construction of $\#$ has an analogue for isotopy, cf. [**34**, § 3.2].

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Theorem 1.2 (proved in § 2.2). *For $m \geqslant 2p+q+3$ the group $E^m(D^{p+1}\times S^q)$ has a subgroup $X = X^m_{p,q}$ such that $E^m_\#(S^p\times S^q)$ has a subgroup isomorphic to $X\oplus K^m_{q,p+q}$ whose quotient is isomorphic to $E^m(D^{p+1}\times S^q)/X$.*

Moreover, there are maps forming the following commutative diagram, in which the horizontal sequence is exact:

$$\begin{tikzcd}
 & X \arrow[r, hook] & E^m(D^{p+1}\times S^q) \arrow[d, "q_\# r"'] \arrow[dr, "q_X"] & & \\
0 \arrow[ur] \arrow[r] & X\oplus K^m_{q,p+q} \arrow[r, "\overline{\mu}\oplus\sigma_\#"'] & E^m_\#(S^p\times S^q) \arrow[r, "\overline{\nu}"'] & E^m(D^{p+1}\times S^q)/X \arrow[r] & 0
\end{tikzcd}$$

The subgroup $X^m_{p,q}$ is finite unless $q=4k-1$ and $m=6k+p$ for some $k\geqslant p/2+1$, and $X^{6k+p}_{p,4k-1}$ is the sum of $\mathbb{Z}$ and a finite group for such k.

The subgroup $X^m_{p,q}$ is the kernel of the restriction map to $E^m(D^p\times S^q)$. The maps $\sigma_\#$, $\overline{\mu}$ and $\overline{\nu}$ are defined in § 2.2, r is the restriction map to $E^m(S^p\times S^q)$, and q_X is the quotient map. For relation between $\pi_q(V_{m-q,p+1})$ and $E^m(D^{p+1}\times S^q)$ see theorem 1.7.

Conjecture 1.3. *For $m\geqslant 2p+q+3$*

$$E^m(S^p\times S^q)\cong E^m(D^{p+1}\times S^q)\oplus K^m_{q,p+q}\oplus E^m(S^{p+q}).$$

This is equivalent to $E^m_\#(S^p\times S^q)\cong E^m(D^{p+1}\times S^q)\oplus K^m_{q,p+q}$ by the Smoothing lemma 1.1. For more discussion see [**34**, remark 1.9].

Known cases of theorem 1.2, the Smoothing lemma 1.1 (and of conjecture 1.3) are listed in [**34**, remark 1.8.a]. In particular, these are new results only for

$$1\leqslant p<q \quad \text{and} \quad 2m\leqslant 3q+2p+3.$$

Analogous remark holds for the following corollaries of theorem 1.2 which, under stronger dimension restrictions, describe $E^m(S^p\times S^q)$ more explicitly.

Denote by TG the torsion subgroup of an abelian group G.

Corollary 1.4. *Assume that $m\geqslant 2p+q+3$.*

(a) *[**32**, corollary 1.7]*

$$E^m(S^p\times S^q)\otimes\mathbb{Q}\cong[\pi_q(V_{m-q,p+1})\oplus E^m(S^q)\oplus K^m_{q,p+q}\oplus E^m(S^{p+q})]\otimes\mathbb{Q}.$$

(b)

$$|E^m(S^p\times S^q)| = |E^m(D^{p+1}\times S^q)|\cdot|K^m_{q,p+q}|\cdot|E^m(S^{p+q})|$$

(more precisely, whenever one part is finite, the other is finite and they are equal).

(c)

$$|TE^m(S^p\times S^q)| = |TE^m(D^{p+1}\times S^q)|\cdot|TK^m_{q,p+q}|\cdot|TE^m(S^{p+q})|,$$

unless $m=6k+p$ and $q=4k-1$ for some k.

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(d) *For the diagram of theorem 1.2 any $\mathbb{Z}^s$-direct summand of $K^m_{q,p+q}$ is mapped under $\sigma_\#$ to a $\mathbb{Z}^s$-direct summand in $E^m_\#(S^p \times S^q)$.*

(e) *For the diagram of theorem 1.2 any $\mathbb{Z}^s$-direct summand of $E^m(D^{p+1} \times S^q)/X$ is the image of $\mathbb{Z}^s$-direct summands $\Delta \subset E^m(D^{p+1} \times S^q)$ and $\Sigma \subset E^m_\#(S^p \times S^q)$ such that $r\Delta = \Sigma$.*

Parts (a,b,c) are simplified versions of conjecture 1.3. Part (a) follows by theorem 1.2 and the isomorphism (DF) of § 1.3. Parts (b) and (e) follow by theorem 1.2 in a standard way. Part (c) follows from parts (d,e) and theorem 1.2. Part (d) is proved in § 1.3.

Definition of $\mathbb{Z}_{(s)}$ and the maps pr_k,

$$\tau = \tau^m_{p,q} : \pi_q(V_{m-q,p+1}) \to E^m(D^{p+1} \times S^q).$$

Denote by $\mathbb{Z}_{(s)}$ the group $\mathbb{Z}$ for s even and $\mathbb{Z}_2$ for s odd.

Denote by pr_k the projection of a Cartesian product onto the k-th factor.

Represent an element of $\pi_q(V_{m-q,p+1})$ by a smooth map $x : S^q \to V_{m-q,p+1}$. By the exponential law this map can be considered as a map $x : \mathbb{R}^{p+1} \times S^q \to \mathbb{R}^{m-q}$. The latter map can be normalized to give a map $\widehat{x} : D^{p+1} \times S^q \to D^{m-q}$. Let $\tau[x]$ be the isotopy class of the composition $D^{p+1} \times S^q \overset{\widehat{x}\times\mathrm{pr}_2}{\to} D^{m-q} \times S^q \overset{\mathrm{i}}{\to} S^m$, where i is the standard embedding (see accurate definition in § 2.1) [**17**], [**27**, § 6]. Clearly, τ is well-defined and is a homomorphism.

In this paper the sign $\circ$ of the composition is often omitted.

Corollary 1.5. *Assume that $m \geqslant 2p + q + 3$.*

(a) If $2m \geqslant 2p + 3q + 4$, then $q_\# r\tau : \pi_q(V_{m-q,p+1}) \to E^m_\#(S^p \times S^q)$ is an isomorphism.

(b) *If $2m \geqslant p + 3q + 4$, then $E^m_\#(S^p \times S^q)$ and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients.*

(b') *If $1 \leqslant p < k$, then $E^{6k-p}_\#(S^p \times S^{4k-p-1}) \cong \mathbb{Z} \oplus G_{k,p}$ for a certain group $G_{k,p}$ such that $G_{k,p}$ and $\pi_{4k-p-1}(V_{2k+1,p+1})$ have isomorphic subgroups with isomorphic quotients.*

(c) *If $2m \geqslant 3q + 4$, then $E^m_\#(S^p \times S^q)$ has a subgroup isomorphic to $\pi_{p+2q+2-m}(V_{M+m-q-1,M})$, whose quotient and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients.*

(d) *If $2m = 3q + 3$, then $E^m_\#(S^p \times S^q)$ has a subgroup isomorphic to $\pi_{p+2q+2-m}(V_{M+m-q-1,M})$, whose quotient has a subgroup isomorphic to $\mathbb{Z}_{(m-q-1)}$, whose quotient and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients.*

Corollaries 1.5 are proved at the end of § 1.3, cf. [**34**, § 2.4].

The smallest m for which there are p, q such that $1 \leqslant p < q$ and $2p + q + 3 \leqslant m \leqslant (3q + 2p + 3)/2$ are $m = 10, 11, 12$. Then $p = 1$ and $q = m - 5$. Hence by the Smoothing lemma 1.1, theorem 1.2, corollaries 1.5.b,b',c,d and [**11**, **21**]

- $|E^{10}(S^1 \times S^5)| = |E^{10}_{\#}(S^1 \times S^5)| = 4$. Cf. [32, example 1.4].
- $E^{11}_{\#}(S^1 \times S^6) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $E^{11}(S^1 \times S^6) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus E^{11}(S^7)$, of which $E^{11}(S^7)$ is rank one infinite.
- $E^{12}_{\#}(S^1 \times S^7) \cong \mathbb{Z}^2 \oplus G$, where $|G|$ is a divisor of 8, and $E^{12}(S^1 \times S^7) \cong \mathbb{Z}^2 \oplus G \oplus E^{12}(S^8)$, of which $E^{12}(S^8)$ is finite.

1.3. Calculations and proofs of corollaries

The group $\pi_r(V_{m,n})$ is calculated for many cases, see e.g. [21], [2, lemma 1.12].

(V) $\pi_r(V_{m,n}) = 0$ for $r > m - n$.

(V') $\pi_{m-n}(V_{m,n}) \cong \mathbb{Z}_{(m-n)}$ for $n > 1$.

(VF) $\pi_r(V_{m,n})$ is finite if and only if either $r = m - n$ is even, or $r = m - 1$ is odd, or $4|r + 1 \neq m$ and $\frac{r}{2} + 1 < m < n + \frac{r}{2} + 1$.

The group $E^m(S^n)$ is calculated for some cases when $m \geqslant n + 3$ [11, 20]. In particular,

(S) $E^m(S^n) = 0$ for $2m \geqslant 3n + 4$.

(S') $E^m(S^n) \cong \mathbb{Z}_{(m-n-1)}$ for $2m = 3n + 3$.

(SF) $E^m(S^n)$ is finite if and only if $n \equiv 3 \mod 4$ and $2m < 3n + 4$ [11, corollary 6.7].

Theorem 1.6. *For $m - 3 \geqslant q, n$ we have $E^m_U(S^q \sqcup S^n) \cong \pi_q(S^{m-n-1}) \oplus K_{q,n}$. [12, theorem 2.4 and the text before corollary 10.3]*

The group $K^m_{q,p+q}$ (or, equivalently, $E^m_U(S^q \sqcup S^{p+q})$) is calculated in terms of homotopy groups of spheres and Whitehead products [12, 29], [2, theorem 1.9]. In particular,

(L) $K^m_{q,p+q} = 0$ for $2m \geqslant 3q + p + 4$;

(L') $K^m_{q,p+q} \cong \pi_{p+2q+2-m}(V_{M+m-q-1,M})$ for $m \geqslant (2p + 4q)/3 + 2$ and M large.

This holds by the Haefliger theorems [27, theorems 3.1 and 3.6]. Also (L) follows by (L'). The isomorphism of (L') from the left to the right is defined in [11].

The group $E^m(D^{p+1} \times S^q)$ can be calculated using theorem 1.7 below. For example, by theorem 1.7, (S), (S') and since for $2m \geqslant 3q + 2$ the normal bundle of any embedding $S^q \to \mathbb{R}^m$ is trivial [14], we have the following.

(D) $\tau : \pi_q(V_{m-q,p+1}) \to E^m(D^{p+1} \times S^q)$ is an isomorphism for $2m \geqslant 3q + 4$.

(D') $E^m(D^{p+1} \times S^q)$ has a subgroup $\mathbb{Z}_{(m-q-1)}$ whose quotient is $\pi_q(V_{m-q,p+1})$ for $2m = 3q + 3$.

(DF) $E^m(D^{p+1} \times S^q) \otimes \mathbb{Q} \cong [\pi_q(V_{m-q,p+1}) \oplus E^m(S^q)] \otimes \mathbb{Q}$ [2, lemma 2.15].

A *p-framing* in a vector bundle is a system of p ordered orthogonal normal unit vector fields on the zero section of the bundle.

THEOREM 1.7. *For $m \geqslant q+3$ the following sequence is exact:*

$$\dots \to E^{m+1}(S^{q+1}) \xrightarrow{\xi} \pi_q(V_{m-q,p+1}) \xrightarrow{\tau} E^m(D^{p+1}\times S^q) \xrightarrow{\rho} E^m(S^q) \to \dots$$

Here ρ is the restriction map and $\xi[f]$ is the obstruction to the existence of a normal $(p+1)$-framing of an embedding $f : S^{q+1} \to S^{m+1}$, see accurate definition in [34, after theorem 1.7]. [2, theorem 2.14], [32, theorem 2.5], cf. [11, corollary 5.9]

Proof of corollary 1.4.d. If $m = 6k+p$ and $q = 4k-1$ for some k, then by (L) $K^m_{q,p+q} = 0$, hence the corollary is trivial. So assume that there are no k such that $m = 6k+p$ and $q = 4k-1$.

Then by theorem 1.2 X is finite.

Denote by E and $E_\#$ the quotients of $E^m(D^{p+1}\times S^q)$ and of $E^m_\#(S^p\times S^q)$ by the maximal summands Δ, Σ of corollary 1.4.e. Then E/X is well-defined and is finite. Hence E is finite.

By theorem 1.2 we have the following commutative diagram, in which the horizontal sequence is exact:

$$\begin{tikzcd}
 & X \arrow[r, "\subset"] & E \arrow[d, "q_\# r"'] \arrow[dr, "q_X"] & & \\
0 \arrow[ur] \arrow[r] & X\oplus K^m_{q,p+q} \arrow[r, "\varphi"'] & E_\# \arrow[r, "\overline{\nu}"'] & E/X \arrow[r] & 0
\end{tikzcd} .$$

Here we denote by $q_\# r, q_X, \varphi, \overline{\nu}$ the maps corresponding to $q_\# r, q_X, \overline{\mu}\oplus\sigma_\#, \overline{\nu}$.

Since E is finite, we have $q_\# rE \subset TE_\#$, so $\overline{\nu}|_{TE_\#}$ is surjective.

Denote by F the maximal free direct summand of $K^m_{q,p+q}$. The corollary follows because in the next paragraph we prove that if $x \in F$ and $\varphi x \neq 0$ is divisible by an integer n, then x is divisible by n in F.

Take $y \in E_\#$ such that $\varphi x = ny \neq 0$. Since $\overline{\nu}|_{TE_\#}$ is surjective, there is $z \in TE_\#$ such that $\overline{\nu}z = \overline{\nu}y$. By exactness $y - z = \varphi t$ for some $t \in X\oplus K^m_{q,p+q}$. Then $t = t_F + t_T$ for some $t_F \in F$ and a finite order element t_T. Hence $y - \varphi t_F = z + \varphi t_T \in TE_\#$. So $TE_\# \ni n(y-\varphi t_F) = \varphi(x - nt_F)$. Therefore $n_1\varphi(x-nt_F) = 0$ for some integer $n_1 > 0$. Since φ is injective and $x, t_F \in F$, we have $x = nt_F$. □

Proof of corollaries 1.5.b,c,d. These corollaries follow from theorem 1.2 and (D,L), (D,L'), (D',L'), respectively. Here (L') is applicable because $\max\{2p+q+3, (3q+3)/2\} \geqslant (2p+4q)/3+2$ (indeed, the opposite inequalities imply $4p+3 < q < 4p+3$). □

Proof of corollary 1.5.b'. Denote $m = 6k-p$ and $q = 4k-p-1$. Since $p < k$, we have $m \geqslant 2p+q+3$ and $m \geqslant (2p+4q)/3+2$. Hence by (L') $K^m_{q,p+q} \cong \mathbb{Z}$ is free. Since $2m = p+3q+3 \geqslant 3q+4$, by (D) $E^m(D^{p+1}\times S^q) \cong \pi_q(V_{m-q,p+1})$. So the corollary follows from corollary 1.4.d. □

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Deduction of corollary 1.5.a from known results. Consider the following diagram

$$\begin{tikzcd} \pi_q(V_{m-q,p+1}) \arrow[r, "\tau"] & E^m(D^{p+1}\times S^q) \arrow[r, "r"] & E^m(S^p\times S^q) \arrow[r, "q_\#"] \arrow[ll, bend right, "\widehat{\alpha}"'] & E^m_\#(S^p\times S^q) \end{tikzcd}$$

Here $\widehat{\alpha}$ is a map such that $\widehat{\alpha}r\tau = \mathrm{id}$ and $\widehat{\alpha}(f\#g)=\widehat{\alpha}(f)$ for each $f\in E^m(S^p\times S^q)$ and $g\in E^m(S^{p+q})$; such a map exists by [**24**, Torus lemma 6.1] ($\widehat{\alpha} := \rho^{-1}\sigma^{-1}\,\mathrm{pr}_1\,\gamma\alpha$ in the notation of that lemma). Hence $r\tau$ is injective and $q_\# r\tau$ is injective.

Take any $f\in E^m(S^p\times S^q)$. Let $f' := r\tau\widehat{\alpha}(f)$. Then $\widehat{\alpha}(f')=\widehat{\alpha}(f)$. Then by [**24**, corollary 1.6.i] and since the smoothing obstruction assuming values in $E^m(S^{p+q})$ is changed by $[g]\in E^m(S^{p+q})$ if f is changed to $f\#g$, we obtain $q_\# f = q_\# f' = q_\# r\tau\widehat{\alpha}(f)$. Since $q_\#$ is surjective, we see that $q_\# r\tau$ is surjective. $\square$

2. Proofs

2.1. Standardization and group structure

Definition of the inclusion $\mathbb{R}^q\subset\mathbb{R}^m$ and of $\mathbb{R}^m_\pm, D^m_\pm, 0_k, 1_k, l, T^{p,q}, T^{p,q}_\pm$. For each $q\leqslant m$ identify the space $\mathbb{R}^q$ with the subspace of $\mathbb{R}^m$ given by the equations $x_{q+1}=x_{q+2}=\cdots=x_m=0$ [**11**] (note that the notation in [**12**, **32**] is slightly different). Analogously identify D^q, S^q with the subspaces of D^m, S^m.

Define $\mathbb{R}^m_+, \mathbb{R}^m_-\subset\mathbb{R}^m$ and $D^m_+, D^m_-\subset S^m$ by equations $x_1\geqslant 0$ and $x_1\leqslant 0$, respectively. Then $S^m = D^m_+\cup D^m_-$. Note that $0\times S^{m-1} = \partial D^m_+ = \partial D^m_- = D^m_+\cap D^m_- \neq S^{m-1}$. Denote by 0_k the vector of k zero coordinates,

$$1_k := (1,0_k)\in S^k,\quad l := m-p-q-1,\quad T^{p,q} := S^p\times S^q \quad\text{and}\quad T^{p,q}_\pm := D^p_\pm\times S^q.$$

Assume that $m>p+q$. Informally, *the standard embedding* is the smoothing of the composition

$$D^{p+1}\times D^{q+1}\cong D^{q+1}\times D^{p+1}\cong D^{q+1}\times 0_l\times\frac{1}{2}D^{p+1}\overset{\subset}{\to} D^{q+1}\times D^l\times D^{p+1}\cong D^{m+1}.$$

Formally, define *the standard embedding*

$$\mathrm{i}=\mathrm{i}_{m,p,q}: D^{p+1}\times D^{q+1}\to D^{m+1}\quad\text{by}\quad \mathrm{i}(x,y) := (y\sqrt{2-|x|^2}, 0_l, x)/\sqrt{2}.$$

See [**34**, footnote 9]. Note that $\mathrm{i}(D^{p+1}\times S^q)\subset S^m$, $\mathrm{i}(D^{p+1}\times D^q_\pm)\subset D^m_\pm$ and $\mathrm{i}_{m,p,q}$ is the restriction of $\mathrm{i}_{m+1,p+1,q}$ but not of $\mathrm{i}_{m+1,p,q+1}$. Denote by the same notation ‘i’ restrictions of i (it would be clear from the context, to which sets).

Take a subset $X\subset S^p$. A map $f: X\times S^q\to S^m$ is called *standardized* if

$$f(X\times \mathrm{Int}\, D^q_+)\subset \mathrm{Int}\, D^m_+\quad\text{and}\quad f|_{X\times D^q_-} = \mathrm{i}_{m,p,q}.$$

Cf. [**26**, remark after definition of the standard embedding in § 2].

A homotopy $F : X \times S^q \times I \to S^m \times I$ is called *standardized* if

$$F(X \times \mathrm{Int}\, D^q_+ \times I) \subset \mathrm{Int}\, D^m_+ \times I \quad \text{and} \quad F|_{X\times D^q_- \times I} = \mathrm{i} \times \mathrm{id}\, I.$$

LEMMA 2.1 (Standardization lemma; proved in § 2.3). *Let X denote either D^p_+ or S^p. For $X = S^p$ assume that $m \geqslant 2p + q + 3$.*

(a) *Each embedding $X \times S^q \to S^m$ is isotopic to a standardized embedding.*

(b) *If standardized embeddings $X \times S^q \to S^m$ are isotopic, then there is a standardized isotopy between them.*

Definition of the reflections R, R_j. Let $R : \mathbb{R}^m \to \mathbb{R}^m$ be the reflection of $\mathbb{R}^m$ with respect to the hyperplane given by equations $x_1 = x_2 = 0$, i.e., $R(x_1, x_2, x_3, \dots, x_m) := (-x_1, -x_2, x_3, \dots, x_m)$. Let R_j be the reflection of $\mathbb{R}^m$ with respect to the hyperplane $x_j = 0$, i.e., $R_j(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m) := (x_1, x_2, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_m)$.

LEMMA 2.2 (Group Structure lemma). *Let X denote either D^p_+ or S^p. For $X = D^p_+$ assume that $m \geqslant q + 3$, for $X = S^p$ assume that $m \geqslant 2p + q + 3$. Then a commutative group structure on $E^m(X \times S^q)$ is well-defined by the following construction.*

Let $0 := [\mathrm{i}]$. Let $-[f] := [\overline{f}]$, where $\overline{f}(x, y) := R_2 f(x, R_2 y)$. For standardized embeddings $f, g : X \times S^q \to S^m$ let $[f] + [g]$ be the isotopy class of the embedding s_{fg} defined by

$$s_{fg}(x, y) := \begin{cases} f(x, y) & y \in D^q_+ \\ R(g(x, Ry)) & y \in D^q_- \end{cases}.$$

The two formulas agree on $X \times (D^q_+ \cap D^q_-)$ because $\mathrm{i}(x, y) = R\,\mathrm{i}(x, Ry)$.

The proof modulo lemma 2.1 is given in [**25**, § 3] and, with more details, in [**34**, § 3.2].

Define the 'embedded connected sum' or 'local knotting' map

$$_{\mathrm{i}}\# : E^m(S^{p+q}) \to E^m(T^{p,q}) \quad \text{by } {}_{\mathrm{i}}\#(g) := 0\#g = [\mathrm{i}]\#g.$$

Identify $1 \times S^q$ and $-1 \times S^q$ with the *first* and the *second* component of $S^q \sqcup S^q$, respectively. Clearly, for $m \geqslant 2p + q + 3$ the map $_{\mathrm{i}}\#$ is a homomorphism.

2.2. Proof of theorem 1.2 using lemmas 2.1, 2.2

Before reading this subsection a reader might want to grasp the idea by reading the proof of a simpler result in [**34**, § 2.4] (although the proof here is formally independent of [**34**, § 2.4]).

LEMMA 2.3. *For $m \geqslant p + q + 3$ the following is exact sequence of groups:*

$$\cdots \to E^{m+1}(T^{p,q+1}_+) \xrightarrow{\lambda'} \pi_q(S^l) \xrightarrow{\mu'} E^m(T^{p+1,q}_+) \xrightarrow{\nu'} E^m(T^{p,q}_+) \to \dots$$

Here ν' is the restriction-induced map; $\lambda'[f]$ is the obstruction to the existence of a vector field on $f(1_p \times S^{q+1})$ normal to $f(T^{p,q+1}_+)$ (see accurate definition in

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*[**34**, after lemma 2.6]), and μ' is the composition of τ and the map $\mu'' : \pi_q(S^l) = \pi_q(V_{l+1,1}) \to \pi_q(V_{m-q,p+1})$ induced by 'adding p vectors' inclusion.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
E^{m+1}(T_+^{p,q+1}) \arrow[r, squiggly, "\overline{\rho}"] \arrow[d, dotted, "\lambda'"] & E^{m+1}(S^{q+1}) \arrow[d, "\xi"] \arrow[dr, squiggly, "\overline{\xi}"] & & & \\
\pi_q(S^l) \arrow[r, dashed, "\mu''"] \arrow[dr, dotted, "\mu'"] & \pi_q(V_{m-q,p+1}) \arrow[r, dashed, "\nu''"] \arrow[d, "\tau"] & \pi_q(V_{m-q,p}) \arrow[d, squiggly, "\overline{\tau}"] \arrow[dr, dashed, "\lambda''"] & & \\
& E^m(T_+^{p+1,q}) \arrow[r, dotted, "\nu'"] \arrow[dr, "\rho"] & E^m(T_+^{p,q}) \arrow[r, dotted, "\lambda'"] \arrow[d, squiggly, "\overline{\rho}"] & \pi_{q-1}(S^l) \arrow[d, dashed, "\mu''"] \arrow[dr, dotted, "\mu'"] & \\
& & E^m(S^q) \arrow[r, "\xi"] \arrow[dr, squiggly, "\overline{\xi}"] & \pi_{q-1}(V_{m-q,p+1}) \arrow[r, "\tau"] \arrow[d, dashed, "\nu''"] & E^{m-1}(T_+^{p+1,q-1}) \arrow[d, dotted, "\nu'"] \\
& & & \pi_{q-1}(V_{m-q,p}) \arrow[r, squiggly, "\overline{\tau}"] & E^{m-1}(T_+^{p,q-1})
\end{tikzcd}$$

Here

- the $\mu''\nu''\lambda''$ sequence is the exact sequence of the 'forgetting the last vector' bundle $S^l \to V_{m-q,p+1} \to V_{m-q,p}$;
- the exact $\tau\rho\xi$- and $\overline{\tau\rho\xi}$-sequences are defined in theorem 1.7.

Let us prove the commutativity.

Let us prove that $\xi\overline{\rho} = \mu''\lambda'$ for the left upper square. By the Standardization lemma 2.1.a each element of $E^{m+1}(T_+^{p,q+1})$ is representable by a standardized embedding $f : D_+^p \times S^{q+1} \to S^{m+1}$. Since $f|_{D_+^p \times D_-^{q+1}} = \mathrm{i}$, there is a normal $(m-q)$-framing of $f(D_-^{q+1})$ extending $f|_{D_+^p \times D_-^{q+1}}$ and a normal $(p+1)$-framing of $f(D_+^{q+1})$ extending $f|_{D_+^p \times D_+^{q+1}}$. Then $\xi[f|_{0_p \times S^q}] = \mu''\lambda'[f]$ by definitions of λ' and ξ.

Relation $\lambda'' = \overline{\tau}\lambda'$ follows by definitions of λ' (§ 2.2) and of λ'' (recalled in [**34**, § 3.4]). The commutativity of other squares and triangles is obvious.

Clearly, $\lambda'\nu' = 0$. So the exactness of the $\lambda'\mu'\nu'$ sequence follows by the Snake lemma, cf. [**11**, proof of (6.5)]. □

Definition of the Zeeman homomorphism

$$\zeta = \zeta_{m,n,q} : \pi_q(S^{m-n-1}) \to E_U^m(S^q \sqcup S^n) \quad \text{for } q \leqslant n,$$

cf. [**32**, Definition of Ze in p.9]. Denote by $\mathrm{i}_{m,q} : S^q \to S^m$ the standard embedding. For a map $x : S^q \to S^{m-n-1}$ representing an element of $\pi_q(S^{m-n-1})$ let

$$\overline{\zeta}_x : S^q \to S^m \quad \text{be the composition} \quad S^q \overset{x \times \mathrm{i}_{n,q}}{\to} S^{m-n-1} \times S^n \overset{\mathrm{i}}{\to} S^m,$$
$$\text{where} \quad \mathrm{i} := \mathrm{i}_{\,m,m-n-1,n}.$$

We have $\overline{\zeta}_x(S^q) \cap \mathrm{i}_{m,n}(S^n) \subset \mathrm{i}(S^{m-n-1} \times S^n) \cap \mathrm{i}(0_{m-n} \times S^n) = \emptyset$. Let $\zeta[x] := [\overline{\zeta}_x \sqcup \mathrm{i}_{m,n}]$.

Clearly, ζ is well-defined, is a homomorphism, and $\lambda\zeta = \mathrm{id}\,\pi_q(S^{m-n-1})$.

Note that $\zeta_{m,q,q} = r\tau^m_{0,q} = r\mu'$.

Definition of the homomorphism

$$\sigma = \sigma_{m,p,q} : E^m_0(S^q \sqcup S^{p+q}) \to E^m(T^{p,q}) \quad \text{for } m \geqslant p+q+3 \quad \text{and} \quad q > 0,$$

cf. [**32**, § 3, Definition of σ^*]. Denote by $E^m_0(S^q \sqcup S^n) \subset E^m(S^q \sqcup S^n)$ the subset formed by the isotopy classes of embeddings whose restriction to the *first* component is unknotted. Represent an element of $E^m_0(S^q \sqcup S^{p+q})$ by an embedding

$$f : S^q \sqcup S^{p+q} \to S^m \quad \text{such that} \quad f|_{S^q} = \mathrm{i}\,|_{0_{p+1} \times S^q}$$
$$\text{and} \quad f(S^{p+q}) \cap \mathrm{i}(D^{p+1} \times S^q) = \emptyset.$$

Join $f(S^{p+q})$ to $\mathrm{i}(-1_p \times S^q)$ by an arc whose interior misses $f(S^{p+q}) \cup \mathrm{i}(D^{p+1} \times S^q)$. Let $\sigma[f]$ be the isotopy class of the embedded connected sum of $\mathrm{i}\,|_{S^p \times S^q}$ and $f|_{S^{p+q}}$ along the arc. (The images of these embeddings are not necessarily contained in disjoint balls.) For $p = 0$ the orientation on $\mathrm{i}(-1_p \times S^q)$ is 'parallel' to the orientation on $\mathrm{i}(1_p \times S^q)$.

The map σ is well-defined for $m \geqslant p+q+3$ and is a homomorphism for $m \geqslant 2p+q+3$ [**32**, lemmas 3.1–3.3]. For $p = 0$ an interpretation of σ and some results on σ are presented in [**34**, § 2.3]. We have $\sigma(f) +_{\mathrm{i}} \#g = \sigma(f\#g)$ [**32**, remark after lemma 3.3]

Proof of theorem 1.2. Clearly, the first sentence follows from the 'moreover' part. So let us prove the 'moreover' part. Consider the following diagram.

$$\begin{tikzcd}
 & \pi_q(S^l) \arrow[r, "\mu'"] \arrow[d, "\zeta"'] & E^m(T^{p+1,q}_+) \arrow[d, dotted, "q_\# r"'] \arrow[dr, "\nu'"] & & \pi_{q-1}(S^l) \arrow[d, "\zeta"'] \\
E^{m+1}(T^{p,q+1}_+) \arrow[ur, "\lambda'"] \arrow[r, "\zeta\lambda'"'] \arrow[dr, "i\zeta\lambda'"'] & E^m_U(S^q \sqcup S^{p+q}) \arrow[r, "\sigma_\#"'] \arrow[d, "i"] \arrow[u, bend right, dashed, "\lambda"'] & E^m_\#(T^{p,q}) \arrow[r, "\overline{\nu}"'] & E^m(T^{p,q}_+) \arrow[ur, "\lambda'"] \arrow[r, "\zeta\lambda'"'] \arrow[dr, "i\zeta\lambda'"'] & E^{m-1}_U(S^{q-1} \sqcup S^{p+q-1}) \arrow[d, "i"] \arrow[u, bend right, dashed, "\lambda"'] \\
 & E^m_0(S^q \sqcup S^{p+q}) \arrow[r, "\sigma"] & E^m(T^{p,q}) \arrow[u, "q_\#"] \arrow[ur, "\nu"] & & E^{m-1}_0(S^{q-1} \sqcup S^{p+q-1}) \\
 & E^m(S^{p+q}) \arrow[u, "j\#"'] \arrow[ur, "\mathrm{i}\#"'] & & &
\end{tikzcd}$$

Here the $\lambda'\mu'\nu'$-sequence is defined in lemma 2.3, maps ζ and σ are defined above,

- i is the inclusion,
- ν is the restriction-induced map,
- $\sigma_\# := q_\#\sigma$,
- the map $\overline{\nu}$ is well-defined by $\overline{\nu}q_\#(f) := \nu(f)$,

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- ${}_j\#g := j\#g$, where the 'standard embedding' $j : S^q \sqcup S^{p+q} \to S^m$ is any embedding whose components are contained in disjoint balls and are isotopic to the inclusions.

The commutativity of the triangles is clear, except for ${}_{\mathrm{i}}\# = \sigma_j\#$, which follows by $\sigma[j] = [\mathrm{i}]$.

The map $\overline{\mu}$ of theorem 1.2 is well-defined by $\overline{\mu}(\mu' x) := \sigma_\# \zeta x$. Let $X := \ker \nu' = \operatorname{im} \mu'$.

Recall the Serre theorem: *the group $\pi_q(S^l)$ is finite unless $q = 4k-1$ and $l = 2k$ for some k, and $\pi_{4k-1}(S^{2k})$ is the sum of $\mathbb{Z}$ and a finite group.* This and (VF) of § 1.3 imply the assertion on the finiteness of $X = \operatorname{im}(\tau\mu'')$. Then using the exact sequence of the 'forgetting the last vector' bundle $S^l \to V_{m-q,p+1} \to V_{m-q,p}$ we obtain the assertion on $X^{6k+p}_{p,4k-1}$.

It suffices to prove that the horizontal sequence of theorem 1.2 is exact.

The exactness of the $\nu\sigma(i\zeta\lambda')$-sequence is [**32**, theorem 1.6].

The map $i \oplus {}_j\#$ is an isomorphism [**12**, theorem 2.4]. Hence by the Smoothing lemma 1.1 and ${}_{\mathrm{i}}\# = \sigma_j\#$, the first two third-line groups both have $E^m(S^{p+q})$-summands mapped one to the other under σ. Taking quotients by these summands one obtains the exactness of the second line.

Since $\lambda\zeta = \mathrm{id}$, we have that ζ is injective and $E^m_U(S^q \sqcup S^{p+q}) = \operatorname{im}\zeta \oplus K^m_{q,p+q}$ (the mutually inverse isomorphisms are given by $x \mapsto (\zeta\lambda x, x - \zeta\lambda x)$ and $(y, z) \mapsto y + z$).

Since the right ζ is injective, we have $\operatorname{im}\nu' = \ker\lambda' = \ker\zeta\lambda' = \operatorname{im}\overline{\nu}$.

The restriction $\sigma_\#|_{K^m_{q,p+q}}$ is injective because

$$\ker\sigma_\# \cap K^m_{q,p+q} = \operatorname{im}(\zeta\lambda') \cap K^m_{q,p+q} \subset \operatorname{im}\zeta \cap K^m_{q,p+q} = 0.$$

If $\overline{\mu}\mu' x = \sigma_\#\zeta x = 0$, then $x \in \operatorname{im}\lambda' = \ker\mu'$, so $\mu' x = 0$. Hence $\overline{\mu}$ is injective.

Also

$$\ker\overline{\nu} = \operatorname{im}\sigma_\# = \sigma_\# \operatorname{im}\zeta \oplus \sigma_\# K^m_{q,p+q} = \operatorname{im}\overline{\mu} \oplus \sigma_\# K^m_{q,p+q}.$$

Thus the horizontal sequence of theorem 1.2 is exact. □

2.3. Proof of the Standardization lemma 2.1

Proof of (a) for $X = D^p_+$. Take an embedding $g : T^{p,q}_+ \to S^m$. Since every two embeddings of a disk into S^m are isotopic, we can make an isotopy of S^m and assume that $g = \mathrm{i}$ on $D^p_+ \times D^q_-$.

The ball D^m_- is contained in a tubular neighbourhood of $\mathrm{i}(D^p_+ \times D^q_-)$ in S^m relative to $\mathrm{i}(D^p_+ \times \partial D^q_-)$. The image $g(D^p_+ \times \operatorname{Int} D^q_+)$ is disjoint from some tighter such tubular neighbourhood. Hence by the Uniqueness of Tubular Neighbourhood Theorem we can make an isotopy of S^m and assume that $g(D^p_+ \times \operatorname{Int} D^q_+) \cap D^m_- = \emptyset$. Then g is standardized. □

Proof of (b) for $X = D^p_+$. Take an isotopy g between standardized embeddings $T^{p,q}_+ \to S^m$. By the 1-parametric version of 'every two embeddings of a disk into S^m are isotopic' we can make a self-isotopy of $\mathrm{id}\, S^m$, i.e. a level-preserving autodiffeomorphism of $S^m \times I$ identical on $S^m \times \{0, 1\}$, and assume that $g = \mathrm{i} \times \mathrm{id}\, I$ on $D^p_+ \times D^q_- \times I$.

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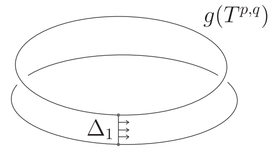

Figure 1. To the proof of the Standardization lemma 2.1.a for $X = S^p$.

The ball $D^m_- \times I$ is contained in a tubular neighbourhood V of $\mathfrak{i}(D^p_+ \times D^q_-) \times I$ in $S^m \times I$ relative to $\mathfrak{i}(D^p_+ \times \partial D^q_-) \times I$. We may assume that $V \cap S^m \times k$ is 'almost D^m_-' for each $k = 0, 1$.

The image $g(D^p_+ \times \mathrm{Int}\, D^q_+ \times I)$ is disjoint from some tighter such tubular neighbourhood, whose intersection with $S^m \times k$ is $V \cap S^m \times k$ for each $k = 0, 1$. Hence by the Uniqueness of Tubular Neighbourhood theorem we can make an isotopy of $S^m \times I$ relative to $S^m \times \{0, 1\}$ and assume that $g(D^p_+ \times \mathrm{Int}\, D^q_+ \times I) \cap D^m_- \times I = \emptyset$. Then g is standardized. $\square$

Extend $\mathfrak{i}$ to $\sqrt{2}D^{p+1} \times D^{q+1}$ by the same formula as in the definition of $\mathfrak{i}$. For $\gamma \leqslant \sqrt{2}$ denote $\Delta_\gamma := \mathfrak{i}(\gamma D^{p+1} \times \{-1_q\}) \subset \mathrm{Int}\, D^m_-$.

Proof of (a) for $X = S^p$. See figure 1. Take an embedding $g : T^{p,q} \to S^m$. Since $m > 2p + q$, every two embeddings $S^p \times D^q \to S^m$ are isotopic (this is a trivial case of theorem 1.7). So we can make an isotopy and assume that $g = \mathfrak{i}$ on $S^p \times D^q_-$.

Since $m > 2p + q + 1$, by general position we may assume that $\mathrm{im}\, g \cap \Delta_1 = \partial \Delta_1$. Then there is γ slightly greater than 1 such that $\mathrm{im}\, g \cap \Delta_\gamma = \partial \Delta_1$. Take the 'standard' q-framing on Δ_γ tangent to $\mathfrak{i}(\gamma D^{p+1} \times S^q)$ whose restriction to $\partial \Delta_1$ is the 'standard' normal q-framing of $\partial \Delta_1$ in $\mathrm{im}\, g$. Then the 'standard' $(m - p - q - 1)$-framing normal to $\mathfrak{i}(\gamma D^{p+1} \times S^q)$ is an $(m - p - q - 1)$-framing on $\partial \Delta_1$ normal to $\mathrm{im}\, g$. Using these framings we construct

- an orientation-preserving embedding $H : D^m_- \to D^m_-$ onto a tight neighbourhood of Δ_1 in D^m_-, and
- an isotopy h_t of $\mathrm{id}\, T^{p,q}$ shrinking $S^p \times D^q_-$ to a tight neighbourhood of $S^p \times \{-1_q\}$ in $S^p \times D^q_-$ such that

$$H(\Delta_{\sqrt{2}}) = \Delta_\gamma, \quad H\,\mathfrak{i}(S^p \times D^q_-) = H(D^m_-) \cap \mathrm{im}\, g$$
$$\text{and} \quad H\,\mathfrak{i} = \mathfrak{i}\, h_1 \quad \text{on} \quad S^p \times D^q_-.$$

Embedding H is isotopic to $\mathrm{id}\, D^m_-$ by [**13**, theorem 3.2]. This isotopy extends to an isotopy H_t of $\mathrm{id}\, S^m$ by the Isotopy Extension theorem [**13**, theorem 1.3]. Then $H_t^{-1} g h_t$ is an isotopy of g. Let us prove that embedding $H_1^{-1} g h_1$ is standardized.

We have $H_1^{-1} g h_1 = H_1^{-1}\, \mathfrak{i}\, h_1 = \mathfrak{i}$ on $S^p \times D^q_-$. Also if $H_1^{-1} g h_1(S^p \times \mathrm{Int}\, D^q_+) \not\subset \mathrm{Int}\, D^m_+$, then there is $x \in S^p \times \mathrm{Int}\, D^q_+$ such that $g h_1(x) \in H(D^m_-)$. Then $g h_1(x) = H\,\mathfrak{i}(y) = \mathfrak{i}\, h_1(y) = g h_1(y)$ for some $y \in S^p \times D^q_-$. This contradicts to the fact that $g h_1$ is an embedding. $\square$

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An embedding $F : N \times I \to S^m \times I$ is a *concordance* if $N \times k = F^{-1}(S^m \times k)$ for each $k = 0, 1$. Embeddings are called *concordant* if there is a concordance between them.

Proof of (b) for $X = S^p$. Take an isotopy g between standardized embeddings. The restriction $g|_{S^p \times D^q_-}$ is an isotopy between standard embeddings. So this restriction gives an embedding $g' : S^p \times D^q_- \times S^1 \to S^m \times S^1$ homotopic to $\mathrm{i}|_{S^p \times 0} \times \mathrm{id}\, S^1$. Since $m + 1 > 2(p + 1)$, by general position $g'|_{S^p \times 0 \times S^1}$ is isotopic to $\mathrm{i}|_{S^p \times 0} \times \mathrm{id}\, S^1$. Since $m > 2p + q + 1$, the Stiefel manifold $V_{m-p,q}$ is $(p + 1)$-connected. Hence every two maps $S^1 \times S^p \to V_{m-p,q}$ are homotopic. Therefore g' is isotopic to $\mathrm{i} \times \mathrm{id}\, S^1$. So we can make a self-isotopy of $\mathrm{id}\, S^m$, i.e. a level-preserving autodiffeomorphism of $S^m \times I$ identical on $S^m \times \{0, 1\}$, and assume that $g = \mathrm{i} \times \mathrm{id}\, I$ on $S^p \times D^q_- \times I$.

Since $m > 2p + q + 2$, by general position we may assume that $\mathrm{im}\, g \cap \Delta_1 \times I = \partial\Delta_1 \times I$. Then there is a disk $\Delta \subset D^m_- \times I$ such that

$$\mathrm{Int}\, \Delta \supset \Delta_1 \times (0, 1), \quad \Delta \cap D^m_- \times \{0, 1\} = \Delta_{\sqrt{2}} \times \{0, 1\}$$
$$\text{and} \quad \mathrm{im}\, g \cap \Delta = \partial\Delta_1 \times I.$$

Take the ‘standard’ q-framing on Δ tangent to $\mathrm{i}(\sqrt{2}D^{p+1} \times S^q) \times I$ whose restriction to $\partial\Delta_1 \times I$ is the ‘standard’ normal q-framing of $\partial\Delta_1 \times I$ in $\mathrm{im}\, g$. Then the ‘standard’ $(m - p - q - 1)$-framing on $\partial\Delta_1 \times I$ normal to $\mathrm{i}(\sqrt{2}D^{p+1} \times S^q) \times I$ is an $(m - p - q - 1)$-framing on $\partial\Delta_1 \times I$ normal to $\mathrm{im}\, g$. Using these framings we construct

- an orientation-preserving embedding $H : D^m_- \times I \to D^m_- \times I$ onto a neighbourhood of $\Delta_1 \times I$ in $D^m_- \times I$, and
- an isotopy h_t of $\mathrm{id}\, T^{p,q} \times I$ shrinking $S^p \times D^q_- \times I$ to a neighbourhood of $S^p \times \{-1_q\} \times I$ in $S^p \times D^q_- \times I$ such that

$$H(\Delta_{\sqrt{2}} \times I) = \Delta, \quad H(\mathrm{i}(S^p \times D^q_-) \times I) = H(D^m_- \times I) \cap \mathrm{im}\, g$$
$$\text{and} \quad H \circ (\mathrm{i} \times \mathrm{id}\, I) = (\mathrm{i} \times \mathrm{id}\, I) \circ h_1 \quad \text{on} \quad S^p \times D^q_- \times I.$$

Analogously to the proof of (a) embedding H is isotopic to $\mathrm{id}(D^m_- \times I)$, such an isotopy extends to an isotopy H_t of $\mathrm{id}(S^m \times I)$, and $H_t^{-1} g h_t$ is an isotopy from g to a standardized isotopy $H_1^{-1} g h_1$. □

2.4. Proof of the Smoothing lemma 1.1

Lemma 2.4 (proved below). *For $m \geqslant 2p + q + 3$ there is a homomorphism*

$$\overline{\sigma} : E^m(T^{p,q}) \to E^m(S^{p+q}) \quad \textit{such that } \overline{\sigma} \circ {}_{\mathrm{i}}\# = \mathrm{id}\, E^m(S^{p+q}).$$

The Smoothing lemma 1.1 follows because lemma 2.4 and $q_\# \circ {}_{\mathrm{i}}\# = 0$ imply that $q_\# \oplus \overline{\sigma}$ is an isomorphism. Lemma 2.4 is known [**5**, proposition 5.6] except for the non-trivial assertion that $\overline{\sigma}$ is a homomorphism.

Proof of lemma 2.4. Definition of $\overline{\sigma}$ and proof that $\sigma \circ {}_{\mathrm{i}}\# = \mathrm{id}\, E^m(S^{p+q})$. The map $\overline{\sigma}$ is ‘embedded surgery of $S^p \times 1_q$’, cf. equivalent definition below. We give an

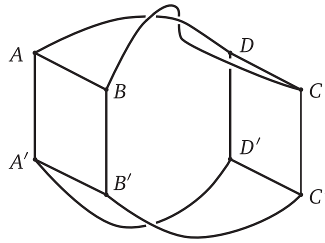

Figure 2. *To the proof that $\overline{\sigma}$ is a homomorphism.* This picture illustrates the proof by the case $p = 0$, $q = 1$ and $m = 3$ (these values are not within the dimension range $m \geqslant 2p + q + 3$). The part above plane $ABCD$ stands for $\widehat{D^m_+}$. The part below plane $A'B'C'D'$ stands for $\widehat{D^m_-}$. The part between the planes stands for $S^{m-1} \times D^1$. The upper curved lines stand for $f_+(S^p \times S^{q-1}) = u(S^p \times S^{q-1})$. The bottom curved lines stand for $f_-(S^p \times S^{q-1}) = u(S^p \times S^{q-1})$. The union of segments $A'A, B'B, C'C$ and $D'D$ stands for $u(S^p \times S^{q-1} \times D^1)$. The union of segments $A'A$ and $B'B$ stands for $\mathrm{i}(S^p \times 1_{q-1}) \times D^1$. The quadrilateral $A'ABB'$ stands for the 'surgery disk' $\mathrm{i}(D^{p+1} \times D^{q-1}_+) \times D^1$. The union of the upper curved lines and the segment AB stands for the $(p + q)$-disk Δ_+. Analogously for Δ_-. The union of Δ_+, Δ_- and the segments $C'C$ and $D'D$ stands for the $(p + q)$-sphere that is the image of a representative of $\overline{\sigma}[u]$. The union of Δ_+ and CD stands for Σ_+. Analogously for Σ_-. The quadrilateral $C'CDD'$ stands for the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$.

alternative detailed construction following [**5**, proposition 5.6]. Take $f \in E^m(T^{p,q})$. By the Standardization lemma 2.1.a there is a standardized representative $f' : T^{p,q} \to S^m$ of f. Identify

$$S^{p+q} \quad \text{and} \quad S^p \times D^q_+ \bigcup_{S^p \times \partial D^q_+} D^{p+1} \times \partial D^q_+$$

by a diffeomorphism. Define an embedding

$$i' : D^{p+1} \times \partial D^q_+ \to D^m_- \quad \text{by } i'(x, (0, y)) := (-\sqrt{1 - |x|^2}, y, 0_l, x)/\sqrt{2}.$$

Then i' is an extension of the restriction $S^p \times \partial D^q_+ \to \partial D^m_+$ of $\mathrm{i}_{m,p,q}$. Infinite derivative for $|x| = 1$ means that i' meets the boundary regularly. Hence i' and $f'|_{S^p \times D^q_+}$ form together a (C^1-smooth) embedding $g : S^{p+q} \to S^m$. Let $\overline{\sigma}(f) := [g]$.

The map $\overline{\sigma}$ is well-defined for $m \geqslant 2p + q + 3$ by the Standardization lemma 2.1.b because the above construction of $\overline{\sigma}$ has an analogue for isotopy, cf. [**34**, beginning of § 3.2].

Clearly, $\overline{\sigma} \circ \mathrm{i}\#(g) = \overline{\sigma}(0\#g) = \overline{\sigma}(0) + g = 0 + g = g$. $\square$

Proof of lemma 2.4. Beginning of the proof that $\overline{\sigma}$ is a homomorphism. See figure 2. For each n identify

$$S^n \quad \text{and} \quad \widehat{D^n_+} \bigcup_{\widehat{\partial D^n_+} = S^{n-1} \times 1} S^{n-1} \times D^1 \bigcup_{S^{n-1} \times \{-1\} = \widehat{\partial D^n_-}} \widehat{D^n_-},$$

where $\widehat{A}$ is a copy of A.

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Then $S^{n-1} = S^{n-1} \times 0 \subset S^n$. Let $\mathrm{i} = \mathrm{i}_{m-1,p,q-1}$. Under the identifications $\widehat{\partial D^n_\pm} = S^{n-1} \times \{\pm 1\}$, $n \in \{m, q\}$, the embedding $\mathrm{i}_{m,p,q}$ goes to $\mathrm{i}|_{S^p \times S^{q-1}}$. Hence analogously to (or by) the Standardization lemma 2.1.a each element in $E^m(T^{p,q})$ has a representative f such that

- $f(S^p \times \widehat{D^q_+}) \subset \widehat{D^m_+}$;
- $f = \mathrm{i}_{m,p,q}$ on $S^p \times \widehat{D^q_-}$ (the image of this embedding lies in $\widehat{D^m_-}$);
- $f = \mathrm{i}|_{S^p \times S^{q-1}} \times \mathrm{id}\, D^1$ on $S^p \times S^{q-1} \times D^1$ (the image of this embedding lies in $S^{m-1} \times D^1$).

  Take embeddings $f_\pm : T^{p,q} \to S^m$ satisfying the above properties. Then $[f_+] + [f_-]$ has a representative $u : T^{p,q} \to S^m$ such that

- $u = f_+$ on $S^p \times \widehat{D^q_+}$;
- $u = (\mathrm{id}\, S^p \times R) \circ f_- \circ (\mathrm{id}\, S^p \times R)$ on $S^p \times \widehat{D^q_-}$;
- $u = \mathrm{i}|_{S^p \times S^{q-1}} \times \mathrm{id}\, D^1$ on $S^p \times S^{q-1} \times D^1$.

For completion of the proof that $\overline{\sigma}$ is a homomorphism we need an equivalent definition of $\overline{\sigma}$.

First we assume that $p = 0$, i.e. define the embedded connected sum of embeddings $f_{-1}, f_1 : S^q \to S^m$ whose images are disjoint. Take an embedding $l : D^1 \times D^q_- \to S^m$ such that

$$l = f_k \quad \text{on} \quad k \times D^q_- \quad \text{and} \quad l(D^1 \times D^q_-) \cap f_k(S^q) = l(k \times D^q_-) \quad \text{for } k = \pm 1.$$

Define $h : S^q \to S^m$ by

$$h(x) := \begin{cases} f_0(x) & x \in \widehat{D^q_+} \\ l(x) & x \in D^1 \times \partial D^q_+ \\ f_1(x) & x \in \widehat{D^q_-} \end{cases}.$$

Then a representative of $[f_0] + [f_1]$ is obtained from h by smoothing of the 'dihedral corner' along $h(S^0 \times \partial D^q_+)$. This smoothing is local replacement of embedded $(I \times 0 \cup 0 \times I) \times D^{q-1}$ by embedded $C \times D^{q-1}$, where $C \subset I^2$ is a smooth curve joining $(0, 1)$ to $(1, 0)$ and such that $C \cup [1, 2] \times 0 \cup 0 \times [1, 2]$ is smooth. This smoothing is 'canonical', i.e. does not depend on the choice of C. Cf. [**10**, Proof of 3.3] and, for non-embedded version, [**35**].

Let us generalize this definition to arbitrary p. Given embedding $f : T^{p,q} \to S^m$, take an embedding $l : D^{p+1} \times D^q_- \to S^m$ such that

$$l = f \quad \text{on } S^p \times D^q_- \quad \text{and} \quad l(D^{p+1} \times D^q_-) \cap f(T^{p,q}) = l(S^p \times D^q_-).$$

Define $h : S^{p+q} \to S^m$ by

$$h(x) := \begin{cases} f(x) & x \in S^p \times D^q_+ \\ l(x) & x \in D^{p+1} \times \partial D^q_+ \end{cases}.$$

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Then a representative of $\overline{\sigma}(f)$ is obtained from h by 'canonical' smoothing of the 'dihedral corner' along $h(S^p \times \partial D^q_+)$ analogous to the above case $p = 0$.

This definition is equivalent to that from the beginning of proof of lemma 2.4 because there are a closed ε-neighbourhood U of the image of l (for some small $\varepsilon > 0$) and a self-diffeomorphism $G : S^m \to S^m$ such that $G(D^m_-, \mathrm{i}(T^{p,q}_-), i'(D^{p+1} \times \partial D^q_+)) = (U, U \cap f(T^{p,q}), U \cap h(S^{p+q}))$.

The result of the above surgery does not depend on the choices involved because $\overline{\sigma}(f)$ is well-defined.

Completion of the proof that $\overline{\sigma}$ is a homomorphism.[4] Recall that a representative of $\overline{\sigma}[u]$ is obtained from u by 'embedded surgery of $\mathrm{i}(S^p \times 1_{q-1}) \times 0$'. Recall that the isotopy class of an embedding $g : S^{p+q} \to S^m$ is defined by the image of g and an orientation on the image.

Denote

$$\begin{aligned}\Delta_\pm :=\ & u(S^p \times D^q_\pm) \ \cup\ \mathrm{i}(D^{p+1} \times D^{q-1}_+) \times \{\pm 1\} \\ \cong\ & S^p \times D^q \cup D^{p+1} \times D^{q-1}_+ \underset{PL}{\cong} D^{p+q}.\end{aligned}$$

Then the oriented image of the representative of $\overline{\sigma}[u]$ is obtained by 'canonical' smoothing of corners from

$$(u(T^{p,q}) - \mathrm{i}(S^p \times D^{q-1}_+) \times D^1) \cup (\mathrm{i} \times \mathrm{id}\, D^1) \left(D^{p+1} \times \partial(D^{q-1}_+ \times D^1)\right) =$$

$$\begin{aligned}&= \Delta_- \cup \mathrm{i}\,\partial(D^{p+1} \times D^{q-1}_-) \times D^1 \cup \Delta_+ \underset{PL}{\cong} \\ &D^{p+q} \times 0 \cup S^{p+q-1} \times I \cup D^{p+q} \times 1 \underset{PL}{\cong} S^{p+q}.\end{aligned}$$

This oriented $(p+q)$-sphere is a connected sum of oriented $(p+q)$-spheres

$$\Sigma_\pm := \Delta_\pm \cup \mathrm{i}(D^{p+1} \times D^{q-1}_-) \times \{\pm 1\} \underset{PL}{\cong} 0 \times D^{p+q} \cup D^{p+q}_+ \underset{PL}{\cong} S^{p+q}$$

along the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$. The image of a representative of $\overline{\sigma}[f_\pm]$ is obtained from $\Sigma_\pm$ by 'canonical' smoothing of the 'dihedral corner'. The corners of the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$ can be 'canonically' smoothed to obtain an embedding $D^{p+q} \times D^1 \to S^m$. Thus $\overline{\sigma}[u] = \overline{\sigma}[f_+] + \overline{\sigma}[f_-]$. □

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[4]This argument appeared after a discussion with A. Zhubr, cf. [**37**] and [**34**, footnote 20].

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