Proceedings of the Edinburgh Mathematical Society (2017) **60**, 665–687
DOI:10.1017/S0013091516000341

DEFINING COARSENINGS OF VALUATIONS

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(Received 21 January 2015)

Abstract We study the question of which Henselian fields admit definable Henselian valuations (with or without parameters). We show that every field that admits a Henselian valuation with non-divisible value group admits a parameter-definable (non-trivial) Henselian valuation. In equicharacteristic 0, we give a complete characterization of Henselian fields admitting a parameter-definable (non-trivial) Henselian valuation. We also obtain partial characterization results of fields admitting $\emptyset$-definable (non-trivial) Henselian valuations. We then draw some Galois-theoretic conclusions from our results.

Keywords: valuations; Henselian valued fields; definable valuations

2010 *Mathematics subject classification:* Primary 03C40; 12J10
Secondary 03C60; 12L12; 12E30; 13F30

1. Introduction

We study the question of which Henselian fields admit non-trivial Henselian valuations that are *definable*, i.e. those for which the valuation ring is first-order definable in the language of rings. Furthermore, we investigate whether parameters are required for these definitions. Here, we call a field *Henselian* if it admits some non-trivial Henselian valuation. There has been considerable progress in the area of definable Henselian valuations over the last few years. Most recent results are focused on defining a specific *given* Henselian valuation on a Henselian field, sometimes with formulae of low quantifier complexity (see [**1**, **5**, **9**, **10**, **12**, **13**, **22**]). The question considered in this paper, however, is whether a given Henselian field admits at least *some* non-trivial definable Henselian valuation. There are many Henselian fields having both definable and non-definable Henselian valuations (see Example 3.2).

Neither separably closed fields nor real closed fields admit any non-trivial definable valuations. For real closed fields, this follows from quantifier elimination in the language of ordered rings $\mathcal{L}_{\text{ring}} \cup \{<\}$: any definable subset of a real closed field is a finite union of intervals and points, and in particular it is not a valuation ring. The fact that separably closed fields do not admit any definable valuations is explained in the introduction of [**16**, p. 1]. Hence, we focus on Henselian fields that are neither separably closed nor real closed. Any such field K interprets a finite Galois extension F such that for some prime p, the canonical p-Henselian valuation v_F^p is $\emptyset$-definable and non-trivial (see § 2 for the definition of the canonical p-Henselian valuation). This valuation is, in particular, comparable to any Henselian valuation on F. If v_F^p is already Henselian, then its restriction to K gives a non-trivial definable Henselian valuation on K. If v_F^p is non-Henselian, then any Henselian valuation on F is a coarsening of v_F^p. Thus, the task of finding definable Henselian valuations on F (and thus on K) comes down to defining (Henselian) coarsenings of v_F^p.

We use two different methods to define coarsenings of a given (definable) valuation on a field F. In § 3 we introduce *p-antiregular* ordered abelian groups. The case distinction between p-antiregular and non-p-antiregular value groups is a key step in several of our proofs. We also show how to interpret, for any prime p, the maximal p-divisible quotient of an ordered abelian group by defining (without any parameters) the corresponding convex subgroup. The construction should be well known to anyone with a good knowledge of definable convex subgroups of ordered abelian groups. However, our approach is rather short and self-contained and should be easily accessible to anyone with an interest in valuation theory. The main result of the section is Proposition 3.7, which gives conditions on the value group of a Henselian valuation under which some non-trivial coarsening is $\emptyset$-definable. In this section we also discuss the construction of a field that will be helpful in examples and counterexamples at several points later on (see Example 3.8).

The other method we use is introduced in § 4. Here, we discuss a certain class of parameter-definable convex subgroups of ordered abelian groups. Again, our treatment is rather short and self-contained. This gives us the means to find a definable Henselian valuation on K whenever some Henselian valuation on K has a non-divisible value group (Proposition 4.2).

We then proceed to apply these two basic constructions to give criteria for the existence of $\emptyset$-definable and definable Henselian valuations. These criteria are phrased in terms of the value group $v_K K$ and the residue field Kv_K of the canonical Henselian valuation v_K on K (see § 2 for the definition of v_K).

In § 5 we discuss the existence of a non-trivial $\emptyset$-definable Henselian valuation on a field K. Here, our main result is the following.

Theorem A. *Let K be a Henselian field that is not separably closed. Assume that K satisfies at least one of the following conditions:*

(1) *Kv_K is separably closed;*

(2) *Kv_K is not t-Henselian;*

(3) *there is some prime p such that $v_K K$ is non-p-divisible and not p-antiregular.*

Then K admits a $\emptyset$-definable Henselian valuation.

See § 3 for the definition of t-Henselianity. Note that a real closed field K satisfies none of the conditions in the theorem. In this case, Kv_K is an Archimedean ordered real closed field and is hence t-Henselian without being Henselian, and the value group of any Henselian valuation on K is divisible. The theorem implies that every (non-separably or non-real closed) Henselian field of finite transcendence degree over its prime field admits a non-trivial $\emptyset$-definable Henselian valuation (Corollary 5.2). As another consequence, we get a classification of all fields with small absolute Galois group admitting $\emptyset$-definable Henselian valuations, provided that the canonical Henselian valuation has residue characteristic 0 (Corollary 5.3). However, the conditions described in Theorem A are not sufficient for a full characterization of fields admitting $\emptyset$-definable non-trivial Henselian valuations (see Example 5.4 and Proposition 5.5).

In § 6 we discuss the existence of a non-trivial definable Henselian valuation on a field K. Here, we prove the following.

Theorem B. *Let K be a Henselian field that is not separably closed. Assume that K satisfies at least one of the following conditions:*

(1) *Kv_K is separably closed;*

(2) *Kv_K is not t-Henselian;*

(3) *$v_K K$ is not divisible.*

Then K admits a definable non-trivial Henselian valuation (using at most one parameter).

Furthermore, in equicharacteristic 0, this theorem gives rise to a characterization of Henselian fields admitting non-trivial definable Henselian valuations (see Corollary 6.1). We also give an example of a Henselian field without a definable non-trivial Henselian valuation and an example of a Henselian field that admits a definable non-trivial Henselian valuation but no $\emptyset$-definable one.

We study the existence of ($\emptyset$-)definable (p-)Henselian valuations tamely branching at p in the last section of the paper (which also contains the definition of tamely branching valuations). By the results in [**18**], these are exactly the Henselian valuations encoded in the absolute Galois group G_K of a field K. Our main result in this context is as follows.

Theorem C. *Let K be a field and let p be a prime.*

(1) *If K admits a Henselian valuation v tamely branching at p, then K admits a definable such valuation (using at most one parameter).*

(2) *Assume that $\zeta_p \in K$ and, if $p = 2$ and $\mathrm{char}(K) = 0$, assume also that $\sqrt{-1} \in K$. If K admits a p-Henselian valuation tamely branching at p, then K admits a $\emptyset$-definable such valuation.*

This theorem is an immediate consequence of Propositions 7.2 and 7.8. As an application, we also obtain some Galois-theoretic consequences (see Corollaries 7.3 and 7.7).

2. Canonical (p-)Henselian valuations

Throughout the paper we use the following notation. For a valued field (K, v), we write Kv for its residue field and vK for its value group. Furthermore, we denote the valuation ring of v by $\mathcal{O}_v$ and its maximal ideal by $\mathfrak{m}_v$. If $p \neq \mathrm{char}(K)$ is a prime, we write $\zeta_p \in K$ to denote that K contains a primitive pth root of unity. For basic facts about (p-)Henselian valued fields, we refer the reader to [**8**].

2.1. The canonical Henselian valuation

Let K be a *Henselian* field, i.e. assume that K admits some non-trivial Henselian valuation. In general, K may admit many non-trivial Henselian valuations, but unless K is separably closed, they all induce the same topology on K. When we ask which Henselian fields admit a definable non-trivial Henselian valuation, we do not specify which one should be definable. In all our constructions, we define coarsenings of the canonical Henselian valuation. Recall that on a Henselian valued field, any two Henselian valuations with non-separably closed residue field are comparable.

The *canonical Henselian valuation* v_K on K is defined as follows. If K admits a Henselian valuation with separably closed residue field, then v_K is the (unique) coarsest such valuation. In this case, any Henselian valuation with non-separably closed residue field is a proper coarsening of v_K and any Henselian valuation with separably closed residue field is a refinement of v_K. If K admits no Henselian valuations with separably closed residue field, then v_K is the (unique) finest Henselian valuation on K and any two Henselian valuations on K are comparable.

2.2. The canonical p-Henselian valuation

Let K be a field and let p be a prime. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree. A valuation v on K is called *p-Henselian* if v extends uniquely to $K(p)$; furthermore, we say that K is p-Henselian if it admits a non-trivial p-Henselian valuation. Note that every Henselian valuation is p-Henselian for all primes p but, in general, the converse is not true.

There is a canonical p-Henselian valuation, analogous to the canonical Henselian valuation. Here, one replaces the notion of 'separably closed' by 'admitting no Galois extensions of degree p'. Again, on a p-Henselian field, any two p-Henselian valuations whose residue fields admit Galois extensions of degree p are comparable. The canonical Henselian valuation v_K^p on K is defined as follows: if K admits a p-Henselian valuation with residue field *p-closed*, i.e. not admitting Galois extensions of degree p, then v_K^p is the (unique) coarsest such valuation. In this case, any p-Henselian valuation with residue field not p-closed is a proper coarsening of v_K^p, and any p-Henselian valuation whose residue field is p-closed is a refinement of v_K^p. If there are no p-Henselian valuations with p-closed residue field on K, then v_K^p is the (unique) finest p-Henselian valuation on K. Whenever K admits a non-trivial p-Henselian valuation, v_K^p is non-trivial and comparable to all p-Henselian valuations on K.

Unlike the canonical Henselian valuation, in most cases the canonical p-Henselian valuation is definable in $\mathcal{L}_{\mathrm{ring}}$.

Theorem 2.1 (Main Theorem in [14]). *Let p be a prime. Consider the (elementary) class of fields*

$$\mathcal{K}_p := \{K \neq K(p) \mid \zeta_p \in K \text{ if } \mathrm{char}(K) \neq p, \text{ and } \sqrt{-1} \in K \text{ if } p = 2 \text{ and } \mathrm{char}(K) = 0\}.$$

Then, the canonical p-Henselian valuation is uniformly $\emptyset$-definable in $\mathcal{K}_p$, i.e. there is a parameter-free $\mathcal{L}_{\mathrm{ring}}$-formula $\phi_p(x)$ such that in any $K \in \mathcal{K}_p$ we have

$$\phi_p(K) = \mathcal{O}_{v_K^p}.$$

3. Antiregular value groups

In this section we use specific properties of the value group of the canonical p-Henselian valuation to define (Henselian) coarsenings without parameters. We first recall some work by Hong on defining valuations with regular value groups that we make use of in some of our proofs. We then define a property of ordered abelian groups that we call antiregular and show a definability result for non-antiregular value groups. Throughout the section, all quotients of ordered abelian groups considered are assumed to be quotients by convex subgroups.

Definition. Let Γ be an ordered abelian group and let p be a prime. Then, Γ is *p-regular* if all proper quotients of Γ are p-divisible. Furthermore, Γ is *regular* if it is p-regular for all primes p.

Note that p-regularity is an elementary property of Γ:

$$\Gamma \text{ is } p\text{-regular} \iff \forall \gamma_0, \ldots, \gamma_p \; (\gamma_0 < \cdots < \gamma_p \to \exists \delta \; (\gamma_0 \leqslant p\delta \leqslant \gamma_p)).$$

Furthermore, an ordered abelian group is regular if and only if it is elementarily equivalent to an Archimedean ordered group. See [**25**] for more details on (p-)regular ordered abelian groups. Hong proved the following definability results about (p-)Henselian valuations with (p-)regular value groups.

Theorem 3.1 (Hong [12, Theorems 3 and 4]). *Let (K, v) be a valued field.*

(1) *Assume that (K, v) is p-Henselian and that we have $\zeta_p \in K$ if $\mathrm{char}(K) \neq p$. If vK is p-regular and not p-divisible, then v is definable.*

(2) *If (K, v) is Henselian and vK is regular but not divisible, then v is $\emptyset$-definable.*

We can use this theorem to give an example of a field admitting both definable and non-definable non-trivial Henselian valuations.

Example 3.2. Consider the field $K = \mathbb{R}((\mathbb{Q}))((\mathbb{Z}))$ (for details on power series fields see [**7**, § 4.2]). This field admits exactly two non-trivial Henselian valuations: the power series valuation v_1 with residue field $\mathbb{R}((\mathbb{Q}))$ and value group $\mathbb{Z}$ is Henselian and has no non-trivial coarsenings as its value group has (Archimedean) rank 1. Furthermore, as $\mathbb{R}$ is non-Henselian, the power series valuation u with value group $\mathbb{Q}$ and residue field

https://doi.org/10.1017/S0013091516000341 Published online by Cambridge University Press

$\mathbb{R}$ is the only non-trivial Henselian valuation on the field $\mathbb{R}((\mathbb{Q}))$. Thus, v_1 has exactly one Henselian refinement v_2: namely, the refinement of v_1 by u, with value group $\mathbb{Z} \oplus \mathbb{Q}$ (ordered lexicographically) and residue field $\mathbb{R}$.

As $v_1 K$ is regular and non-divisible, v_1 is $\emptyset$-definable by Theorem 3.1. We claim that v_2 is not $\emptyset$-definable: note that we have $\mathbb{R} \equiv \mathbb{R}((\mathbb{Q}))$ in $\mathcal{L}_{\text{ring}}$ since $\mathbb{R}((\mathbb{Q}))$ is also real closed (see [**8**, Lemma 4.3.6 and Theorem 4.3.7]). Furthermore, there is an elementary equivalence of lexicographically ordered sums $\mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{Q} \equiv \mathbb{Z} \oplus \mathbb{Q}$ in $\mathcal{L}_{\text{oag}} = \{+, <, 0\}$ since finite lexicographic sums preserve elementary equivalence (see the proof of Theorem 3.3 in [**11**]) and the $\mathcal{L}_{\text{oag}}$-theory of divisible ordered abelian groups is complete [**20**, Corollary 3.1.17]. The Ax–Kochen/Ersov theorem [**23**, Theorem 4.6.4] implies that

$$(K, v_2) \equiv (\mathbb{R}((\mathbb{Q}))\underbrace{((\mathbb{Q}))((\mathbb{Z}))}_{w_1}, w_1) \equiv (\mathbb{R}\underbrace{((\mathbb{Q}))((\mathbb{Q}))((\mathbb{Z}))}_{w_2}, w_2)$$

holds. Thus, v_2 cannot be $\emptyset$-definable: any parameter-free first-order definition of v_2 would have to define both w_1 and w_2 on the field $\mathbb{R}((\mathbb{Q}))((\mathbb{Q}))((\mathbb{Z}))$.

Moreover, v_2 is not even definable with parameters: by [**6**, Theorem 4.4 and Remark 3 on p. 1147], on any field K the only possible definable Henselian valuation with real closed residue field is the coarsest such valuation. As v_1 is a proper coarsening of v_2 with real closed residue field, v_2 is not definable.

We now define an antipodal property to p-regularity.

Definition. Let Γ be an ordered abelian group and let p be a prime. Then, Γ is *p-antiregular* if no non-trivial quotient of Γ is p-divisible and Γ has no rank-1 quotient. Furthermore, Γ is *antiregular* if it is p-antiregular for all primes p.

Here, an ordered abelian group has rank 1 if its Archimedean rank is 1. Again, p-antiregularity is an elementary property of Γ:

$$\Gamma \text{ is } p\text{-antiregular} \iff \forall\gamma\ \exists\delta\ \forall\varepsilon\ (|\varepsilon| \leqslant p|\gamma| \to \delta + \varepsilon \notin p\Gamma)$$

with the standard notation $|\gamma| := \max\{\gamma, -\gamma\}$.

Example 3.3 (antiregular ordered abelian groups). For $i \in \mathbb{Z}$, let Z_i be a copy of $\mathbb{Z}$ as an ordered abelian group. Consider the lexicographically ordered sums

$$\Gamma := \bigoplus_{i\in\mathbb{Z}} Z_i \quad \text{and} \quad \Delta := \bigoplus_{i\in\mathbb{Z},\, i\leqslant 0} Z_i.$$

Then both Γ and Δ are antiregular, as all of their non-trivial quotients are either isomorphic to Γ or Δ, so, in particular, neither Γ nor Δ has a rank 1 quotient or a p-divisible quotient for any prime p. The element $(\ldots, 0, 0, 0, 1) \in \Delta$ is a minimal positive element, and Γ has no minimal positive element. Thus, we have

$$\Gamma \not\equiv \Delta$$

as ordered abelian groups. Note that any ordered abelian group that has an antiregular quotient is again antiregular, so there are many examples of elementary classes of antiregular ordered abelian groups.

Question 3.4. Is there a (first-order) classification for antiregular ordered abelian groups that is similar to the one that exists for regular ones?

We now collect some useful facts about antiregular ordered abelian groups.

Lemma 3.5. *Let $\Gamma \neq \{0\}$ be an ordered abelian group.*

(1) *If Γ is p-antiregular, then we have $[\Gamma : p\Gamma] = \infty$.*

(2) *If $\Gamma \leqslant \Gamma'$ and the index $[\Gamma' : \Gamma]$ is finite, then Γ is p-antiregular if and only if Γ' is p-antiregular.*

Proof. (1) Assume that $[\Gamma : p\Gamma] = n$, and let Δ be a minimal convex subgroup of Γ with $[\Delta : p\Delta] = n$. Choose a system of representatives $\{x_1, \ldots, x_n\}$ for $\Delta/p\Delta$ (and hence also for $\Gamma/p\Gamma$). The quotient Γ/Δ is p-divisible. If the quotient is trivial, then $\Gamma = \Delta$ and Δ has an Archimedean rank-1 quotient as follows: let x_i be maximal among $x_1, \ldots, x_n$ and let Δ_0 be the maximal convex subgroup of Δ not containing x_i; then, by the minimality of Δ, Δ/Δ_0 has Archimedean rank 1. If the quotient is non-trivial, Γ has a non-trivial p-divisible quotient. In either case, Γ is not p-antiregular.

(2) Consider ordered abelian groups $\Gamma \leqslant \Gamma'$ with $[\Gamma' : \Gamma]$ finite. Then, there is a one-to-one correspondence between convex subgroups Δ' of Γ' and convex subgroups Δ of Γ with $\Delta' \cap \Gamma = \Delta$ and $[\Delta' : \Delta]$ finite. In particular, Γ'/Δ' is p-divisible if and only if Γ/Δ is p-divisible and Γ'/Δ' has rank 1 if and only if Γ/Δ has rank 1. Thus, Γ' is p-antiregular if and only if Γ is. □

The next lemma gives the means to define a coarsening of a $\emptyset$-definable valuation with non-antiregular value group without parameters.

Lemma 3.6. *Let Γ be an ordered abelian group and let p be a prime. Define*

$$D := \{\Delta \leqslant \Gamma \mid \Delta \text{ convex and } \Gamma/\Delta \text{ is } p\text{-divisible}\}.$$

Then, we have the following.

(1) *$\Delta_0 := \bigcap_{\Delta \in D} \Delta$ is a convex subgroup of Γ such that Γ/Δ_0 is p-regular.*

(2) *If Γ is not p-divisible and every p-regular quotient of Γ is p-divisible, then Δ_0 is $\emptyset$-definable: for any $\gamma \in \Gamma$, we have*

$$\gamma \in \Delta_0 \iff \exists \varepsilon\, \forall \alpha\, (|\alpha| < |\gamma| \to \varepsilon - \alpha \notin p\Gamma).$$

Proof. (1) Since all convex subgroups of Γ are linearly ordered by inclusion, it is clear that Δ_0 is a convex subgroup of Γ. Every non-trivial convex subgroup of Γ/Δ_0 is of the shape Δ/Δ_0 for some $\Delta \in D$. Hence,

$$(\Gamma/\Delta_0)/(\Delta/\Delta_0) \cong \Gamma/\Delta$$

is p-divisible and so Γ/Δ_0 is p-regular.

(2) As all p-regular quotients of Γ are p-divisible by assumption, Γ/Δ_0 is p-divisible. Assume that $\gamma \in \Delta_0$. Let $\langle\gamma\rangle$ be the convex hull of the subgroup generated by γ in Δ_0.

We claim that $\Delta_0/\langle\gamma\rangle$ is not p-divisible. Assume for a contradiction that $\Delta_0/\langle\gamma\rangle$ was p-divisible. Then, as Γ/Δ_0 is p-divisible, we also get that $\Gamma/\langle\gamma\rangle$ is p-divisible. This implies that $\langle\gamma\rangle \in D$ and hence $\Delta_0 = \langle\gamma\rangle$. In particular, by the definition of Δ_0, there is some $n \geqslant 1$ with $\gamma \notin p^n\Delta_0$. Consider the maximal convex subgroup B_γ of Δ_0 such that $\gamma \notin B_\gamma$, i.e.

$$B_\gamma := \{\delta \in \Delta_0 \mid \forall n \in \mathbb{Z}\colon |n\delta| < \gamma\}.$$

Now, Δ_0/B_γ is a non-p-divisible rank-1 quotient of Δ_0 and thus of Γ. Hence, Γ has a non-p-divisible p-regular quotient, contradicting our assumption on Γ that no such quotient exists. This proves the claim.

By the claim, we can choose some $\varepsilon \in \Delta_0 \setminus \langle\gamma\rangle$ such that

$$\varepsilon + \langle\gamma\rangle \notin p(\Delta_0/\langle\gamma\rangle)$$

holds. Hence, for any $\alpha \in \langle\gamma\rangle$, we have

$$\varepsilon - \alpha \notin p\Delta_0 = p\Gamma \cap \Delta_0.$$

Thus, we have for all $\gamma \in \Delta_0$

$$\Gamma \models \exists\varepsilon\ \forall\alpha\ (|\alpha| < |\gamma| \rightarrow \varepsilon - \alpha \notin p\Gamma).$$

Conversely, if $\gamma \notin \Delta_0$ holds, then we have $\gamma \notin \Delta$ for some $\Delta \in D$. As Γ/Δ is p-divisible, for every $\varepsilon \in \Gamma$ there is some $\alpha \in \Delta$ such that $\varepsilon - \alpha \in p\Gamma$ holds. Thus, we have for all $\gamma \in \Gamma \setminus \Delta_0$

$$\Gamma \models \forall\varepsilon\ \exists\alpha\ (|\alpha| < |\gamma| \wedge \varepsilon - \alpha \in p\Gamma).$$

□

Remark. Let Γ be an ordered abelian group and let p be a prime as in the assumptions of Lemma 3.6 (2), i.e. assume that $\Gamma \neq p\Gamma$ holds and that every p-regular quotient of Γ is p-divisible. Define Δ_0 as before. An alternative way to show that Δ_0 is $\emptyset$-definable is to check that one has

$$\Delta_0 = \bigcup_{\alpha \in \mathcal{S}_p} \Gamma_\alpha$$

for $\mathcal{S}_p$ and Γ_α as defined in [**4**, Definition 1.1].

We can now prove our first result on defining Henselian valuations without parameters.

Proposition 3.7. *Let (K, v) be a Henselian field and let p be a prime. If the value group vK is not p-divisible and not p-antiregular, then some non-trivial (Henselian) coarsening of v is $\emptyset$-definable on K.*

Proof. Assume that vK is not p-divisible and not p-antiregular. If $\mathrm{char}(K) \neq p$, we may assume that K contains a primitive pth root of unity ζ_p: as v is Henselian, it extends uniquely to a Henselian valuation w on $F := K(\zeta_p)$. By Lemma 3.5, the value group wF of the prolongation is again non-p-divisible and not p-antiregular. Note that $K(\zeta_p)$ is $\emptyset$-*definably interpretable in* K in the following sense. We identify the additive groups $(K(\zeta_p), +)$ and $(K^d, +)$, where $d = [K(\zeta_p) : K]$. The appropriate multiplicative structure on K^d can be defined using only the (parameter-free) minimal polynomial f of a chosen primitive pth root of unity ζ_p over K: note that for any other choice of ζ_p (and, hence, possibly of f) this defines the same multiplication on K^d. To see that there can, in fact, be different such polynomials, consider the decomposition of $x^4 + x^3 + x^2 + x + 1$ over $\mathbb{F}_{19}$ as $(x^2 - 4x + 1)(x^2 + 5x + 1)$. As $K(\zeta_p)$ is $\emptyset$-definably interpretable in K, any parameter-free definition of a non-trivial coarsening of w gives rise to a parameter-free definition of a non-trivial coarsening of v. Assuming that $\zeta_p \in K$ if $\mathrm{char}(K) \neq p$, the non-p-divisibility of vK now implies $K \neq K(p)$.

If vK admits a non-p-divisible rank-1 quotient, then the corresponding coarsening is $\emptyset$-definable by Theorem 3.1.

Otherwise, vK admits some non-trivial p-divisible quotient by assumption. If vK admits a non-p-divisible p-regular quotient, then the corresponding coarsening is definable by Theorem 3.1, say via the formula $\phi(x, t)$ for some parameter $t \in K$. Note that vK has at most one non-p-divisible p-regular quotient and that no proper refinement of v has p-regular value group. In particular, there is only one p-Henselian valuation with non-p-divisible p-regular value group on K. By [**17**, Theorem 1.5], p-Henselianity is an elementary property of a valued field in the language $\mathcal{L}_{\mathrm{val}} := \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$. Thus, the set

$$X = \{t \in K \mid \mathcal{O}_{w_t} := \phi(K, t) \text{ is a } p\text{-Henselian valuation ring with } w_tK \neq pw_tK \text{ and } w_tK\ p\text{-regular}\}$$

is $\emptyset$-definable. Hence, the parameter-free formula

$$\psi(x) \equiv \exists t \in X\ (x \in \phi(K, t))$$

defines the unique p-Henselian valuation on K with non-p-divisible p-regular value group that is a non-trivial coarsening of v.

Finally, assume that the value group vK of v only has p-divisible p-regular quotients; in particular, vK is not p-regular. As v is Henselian, v is comparable to the canonical p-Henselian valuation v_K^p. If v_K^p is a coarsening of v, we have found a $\emptyset$-definable coarsening of v. Otherwise, the value group of the canonical p-Henselian valuation v_K^pK also admits only p-divisible p-regular quotients. Thus, Lemma 3.6 applies and Δ_0 is $\emptyset$-definable in v_K^pK. Now, the corresponding non-trivial $\emptyset$-definable coarsening w of v_K^p has p-divisible value group and is hence also a coarsening of v.

Note that any coarsening of a Henselian valuation is again Henselian. Thus, we have shown that if v is Henselian and vK is non-p-divisible and not p-antiregular, then some non-trivial, Henselian coarsening of v is $\emptyset$-definable. □

Next, we repeat the construction given in [**24**] of a field that is elementarily equivalent in $\mathcal{L}_{\mathrm{ring}}$ to a Henselian field but which does not admit any non-trivial Henselian valuation. Following [**24**], we give the following definition.

Definition. A field K is called *t-Henselian* if there is some Henselian field L with $L \equiv K$.

Fields that are t-Henselian but non-Henselian play an important role in several of the examples in this paper. Consider a field K that is t-Henselian but not Henselian. Clearly, no field elementarily equivalent to K can admit a *∅-definable* non-trivial Henselian valuation. However, for the field K as discussed in the following example, any Henselian field elementarily equivalent to K admits a *parameter-definable* Henselian valuation (see Example 5.4). This follows from the fact that the canonical 2-Henselian valuation v_K^2 is ∅-definable and has an antiregular value group.

Example 3.8 (a t-Henselian field that is not Henselian). Let $K_0 := \mathbb{Q}^{\mathrm{alg}}$ and let v_0 be the trivial valuation on K_0. For $n \geqslant 1$, one iteratively constructs valued fields (K_n, v_n) with $v_n K_n = \mathbb{Z}$ and $K_n v_n = K_{n-1}$ and such that Hensel's Lemma holds for polynomials of degree at most n as follows.

Choose a minimal algebraic extension K_n of $K_{n-1}(X_{n-1})$ with

$$K_{n-1}(X_{n-1}) \subseteq K_n \subsetneq K_{n-1}((X_{n-1}))$$

such that Hensel's Lemma holds on (K_n, v_n) for polynomials of degree at most n, where v_n is the restriction of the power series valuation on $K_{n-1}((X_{n-1}))$ to K_n. One can of course choose $K_1 = K_0(X_0)$ with v_1 the X_0-adic valuation, as Hensel's Lemma holds trivially for all polynomials of degree 1. Note that we get a place $p_n \colon K_n \to K_{n-1} \cup \{\infty\}$ that is p-Henselian for all primes $p \leqslant n$.

The field K is then taken as the inverse limit of

$$(K_n \cup \{\infty\}, p_n) \text{ with projections } s_n \colon K \cup \{\infty\} \to K_{n-1} \cup \{\infty\}.$$

It follows from the arguments given in [**24**, p. 338] that K admits no non-trivial Henselian valuation.

The canonical 2-Henselian valuation v_K^2 on K now corresponds to the place

$$s_2 \colon K \to K_1 \cup \{\infty\}$$

as p_n is 2-Henselian if and only if $n \geqslant 2$. As usual, the quotients of $v_K^2 K$ correspond to the value groups of coarsenings of v_K^2. Since the coarsenings of v_K^2 correspond to the places s_n for $n \geqslant 2$ and none of them has a p-divisible value group for any prime p or has a value group of rank 1, we conclude that the group $v_K^2 K$ is antiregular.

4. Defining coarsenings of valuations using subgroups

In this section we discuss a class of parameter-definable convex subgroups of an ordered abelian group. The motivation for this comes from [**2**]. We then apply our construction to show that a field admitting a Henselian valuation with non-divisible value group admits a non-trivial parameter-definable Henselian valuation.

Lemma 4.1. *Let Γ be an ordered abelian group and let p be a prime. Take any $\gamma \in \Gamma$ with $\gamma > 0$ and define*

$$\Delta_\gamma := \{\delta \in \Gamma \mid [0, p|\delta|] \subseteq [0, p\gamma] + p\Gamma\},$$

where $|\delta| = \max\{\delta, -\delta\}$. Then Δ_γ is a convex $\{\gamma\}$-definable subgroup of Γ with $\gamma \in \Delta_\gamma$. Furthermore, no non-trivial convex subgroup of Γ/Δ_γ is p-divisible.

Proof. By definition, Δ_γ is a $\{\gamma\}$-definable convex subset of Γ containing γ with $\Delta_\gamma = -\Delta_\gamma$. We now show that Δ_γ is a subgroup of Γ. As Δ_γ is convex, it suffices to show that for all $\delta \in \Delta_\gamma$ we have $\delta + \delta \in \Delta_\gamma$. Since we have $\Delta_\gamma = -\Delta_\gamma$, it suffices to consider the case $\delta > 0$. Take any $\delta \in \Delta_\gamma$ with $\delta > 0$ and $\beta \in \Gamma$ with

$$0 \leqslant |\beta| \leqslant p(\delta + \delta).$$

In case when we have $|\beta| \leqslant p\delta$ we immediately get $|\beta| \in [0, p\gamma] + p\Gamma$. Otherwise, we have $p\delta < |\beta| \leqslant p(\delta + \delta)$, so we get $|\beta| - p\delta \leqslant p\delta$. This implies again $|\beta| \in [0, p\gamma] + p\Gamma$. Overall, we get $[0, p(\delta + \delta)] \subseteq [0, p\gamma] + p\Gamma$, i.e. $\delta + \delta \in \Delta_\gamma$ as required.

Let $\tilde{\Delta} \leqslant \Gamma$ be a convex subgroup with $\Delta_\gamma \subseteq \tilde{\Delta}$. If $\tilde{\Delta}/\Delta_\gamma$ is p-divisible, then for any $\tilde{\delta} \in \tilde{\Delta}$ there is some $\delta \in \Delta_\gamma$ with $\tilde{\delta} - \delta \in p\Gamma$. Fix some $\tilde{\delta} \in \tilde{\Delta}$ and take any $\tilde{\beta} \in [0, p|\tilde{\delta}|] \subseteq \tilde{\Delta}$. Then, there is some $\beta \in \Delta_\gamma$ with

$$\tilde{\beta} \in \beta + p\Gamma \subseteq [0, p\gamma] + p\Gamma.$$

Thus, we get $\tilde{\delta} \in \Delta_\gamma$ and hence $\tilde{\Delta} = \Delta_\gamma$. As any convex subgroup of Γ/Δ_γ corresponds to a subgroup $\tilde{\Delta} \leqslant \Gamma$ as above, we conclude that Γ/Δ_γ has no non-trivial p-divisible convex subgroup. □

If Γ is the value group of a definable valuation v on a field K, the construction in the lemma gives rise to a definable coarsening of v. As discussed in the next remark, this is a special case of a construction introduced by Arason, Elman and Jacob (see [**2**]).

Remark. Let (K, v) be a valued field and let $t \in \mathfrak{m}_v$. Consider the set

$$T_t := \{x \in K^\times \mid \exists z\colon v(t^{-p}) \leqslant v(xz^p) \leqslant v(t^p)\}.$$

It is straightforward to check that if v is $\emptyset$-definable, T_t is a t-definable subgroup of $K^\times$. In [**2**] the authors introduce a method for obtaining definable valuation rings from certain definable subgroups of $K^\times$, and they discuss conditions under which this valuation ring is non-trivial. Using the notation and machinery from [**2**] (in particular, Theorem 2.10 and Lemma 3.1), one can show that there is a valuation ring $\mathcal{O}(T_t, T_t) \subseteq K$ that is trivial if and only if $T_t = K^\times$.

Now, let $\Delta_{v(t)}$ be the convex subgroup of vK as defined in Lemma 4.1. The valuation ring $\mathcal{O}(T_t, T_t)$ is exactly the coarsening of v that is obtained by quotienting vK by the convex subgroup $\Delta_{v(t)}$. This valuation can also be described as the finest coarsening w of v such that we have $t \in \mathcal{O}_w^\times$ and such that no non-trivial convex subgroup of wK is p-divisible.

Proposition 3.7 and Lemma 4.1 are the two main ingredients needed to show that on any field admitting a Henselian valuation with non-divisible value group there is a non-trivial definable Henselian valuation.

Proposition 4.2. *Assume that some Henselian valuation v on K has a non-divisible value group. Then, some non-trivial (Henselian) coarsening of v is definable on K (using at most one parameter).*

Proof. Let (K, v) be Henselian such that vK is not p-divisible for some prime p. If vK is not p-antiregular, then it admits a $\emptyset$-definable non-trivial coarsening by Proposition 3.7. Thus, we may assume that vK is p-antiregular, which means that it has no non-trivial p-divisible quotient and no rank-1 quotient.

Consider $F := K(\zeta_p)$ in the case when $\mathrm{char}(K) \neq p$: by Lemma 3.5, the unique prolongation w of v to $K(\zeta_p)$ will again have non-p-divisible and p-antiregular value group. If we define a coarsening of w with parameters from K on F, its restriction to K is also definable (with parameters from K). If $\mathrm{char}(K) = p$, we set $F := K$.

We now have $F \neq F(p)$ and, as $[F : K]$ is prime to p, for any $t \in \mathfrak{m}_v$ with $p \nmid v(t)$ we get $p \nmid w(t)$. By construction, v_F^p is $\emptyset$-definable on F. Note that, by Henselianity, w is comparable to v_F^p. If v_F^p is a coarsening of w, we have found a non-trivial $\emptyset$-definable coarsening of w (and thus of v). Hence, we may assume that v_F^p refines w.

Choose any $t \in \mathfrak{m}_v \subseteq \mathfrak{m}_w$ with $p \nmid v(t)$. Then, we also have $p \nmid w(t) =: \gamma$. Define $\Gamma := wF$ and consider the convex subgroup

$$\Delta_\gamma = \{\delta \in \Gamma \mid [0, p|\delta|] \subseteq [0, p\gamma] + p\Gamma\}$$

of Γ as in Lemma 4.1. We claim that $\Delta_\gamma \neq \Gamma$ holds. Assume for a contradiction that we have $\Delta_\gamma = \Gamma$. Let $\langle\gamma\rangle$ denote the convex subgroup of Γ generated by γ. Then, we have for all $\delta \in \Delta_\gamma = \Gamma$ that

$$|\delta| \in [0, p\gamma] + p\Gamma \subseteq \langle\gamma\rangle + p\Gamma$$

holds. Thus, $\Gamma/\langle\gamma\rangle$ is p-divisible, and thus, as Γ is p-antiregular, trivial. Now, the maximal convex subgroup of Γ not containing γ, i.e.

$$B_\gamma := \{\delta \in \Gamma \mid \forall n \in \mathbb{Z}\colon |n\delta| < \gamma\},$$

is a proper convex subgroup of Γ such that Γ/B_γ has rank 1. This contradicts the p-antiregularity of Γ. Thus, we conclude that $\Gamma \neq \Delta_\gamma$.

Hence, the coarsening of w that corresponds to quotienting wF by Δ_γ is a non-trivial $\{t\}$-definable coarsening of w. Its restriction to K is a non-trivial $\{t\}$-definable coarsening of v. □

5. Definitions without parameters

We are now in a position to prove our main theorem on the existence of a parameter-free definable Henselian valuation on a Henselian field, as stated in the introduction.

Theorem A. *Let K be a Henselian field that is not separably closed. Assume that K satisfies at least one of the following conditions:*

(1) *Kv_K is separably closed;*

(2) *Kv_K is not t-Henselian;*

(3) *there is some prime p such that $v_K K$ is non-p-divisible and not p-antiregular.*

Then K admits a $\emptyset$-definable Henselian valuation.

Proof. Let $K \neq K^{\mathrm{sep}}$ be a Henselian field.

(1) If $Kv_K = Kv_K^{\mathrm{sep}}$, then K admits a $\emptyset$-definable Henselian valuation by [**13**, Theorem 3.10].

(2) If Kv_K is not t-Henselian, then K admits a $\emptyset$-definable Henselian valuation by [**10**, Proposition 5.5].

(3) If $v_K K$ is non-p-divisible and is not p-antiregular for some p, then some non-trivial (Henselian) coarsening of v_K is $\emptyset$-definable by Proposition 3.7.

Hence, if K satisfies one of the assumptions of the theorem, then K admits a non-trivial $\emptyset$-definable Henselian valuation. □

Note that Henselian real closed fields do not satisfy any of the conditions in the theorem: if K is Henselian and real closed, then Kv_K is real closed (and thus t-Henselian and not separably closed) and $v_K K$ is divisible.

We first draw some conclusions from Theorem A. Recall that we use G_K to denote the absolute Galois group of a field K. We say that G_K is *small* if K has only finitely many Galois extensions of degree n for any natural number n.

Corollary 5.1. *Let K be a non-separably closed Henselian field with G_K small or with finite transcendence degree. Then K admits a $\emptyset$-definable Henselian valuation unless $v_K K$ is divisible and Kv_K is t-Henselian but not separably closed.*

Proof. Let K be Henselian and assume that $K \neq K^{\mathrm{sep}}$. If K admits no $\emptyset$-definable Henselian valuation, then, by Theorem A, Kv_K is t-Henselian but not separably closed. If $v_K K$ is not divisible, Theorem A implies that $v_K K$ is p-antiregular and not p-divisible for at least one prime p.

Let K be a field of finite transcendence degree or such that G_K is small. The index $[v_K K : p v_K K]$ is then finite for any prime p. Hence, Lemma 3.5 implies that $v_K K$ is not p-antiregular. □

Corollary 5.2. *Let K be a Henselian field, neither separably closed nor real closed, and assume that the transcendence degree of K is finite. Then K admits a $\emptyset$-definable Henselian valuation.*

https://doi.org/10.1017/S0013091516000341 Published online by Cambridge University Press

Proof. Assume that $\mathrm{trdeg}(K)$ is finite and that $K \neq K^{\mathrm{sep}}$. By Corollary 5.1, K admits a $\emptyset$-definable Henselian valuation unless Kv_K is t-Henselian but not Henselian. However, [**8**, Theorem 3.4.2] implies that $\mathrm{trdeg}(Kv_K)$ is also finite. By [**19**, Lemma 3.5], every t-Henselian field of finite transcendence degree is Henselian. Thus, Kv_K cannot be t-Henselian but not Henselian. □

Corollary 5.3. *Let K be a Henselian field with G_K small and $\mathrm{char}(Kv_K) = 0$. Then K admits no $\emptyset$-definable Henselian valuation if and only if $K \equiv Kv_K$.*

Proof. Let K be a Henselian field with G_K small, $\mathrm{char}(Kv_K) = 0$, which does not admit a $\emptyset$-definable Henselian valuation. Corollary 5.1 implies that $v_K K$ is divisible and Kv_K is t-Henselian. By [**24**, Lemma 3.3], there is some Henselian $L \succ Kv_K$. Note that G_{Kv_K}, and hence G_L, is also small. Using Corollary 5.1 once more, we get that Lv_L is t-Henselian and $v_L L$ is divisible. Since the restriction of v_L to Kv_K is trivial, we have $\mathrm{char}(Lv_L) = 0$. Using the Ax–Kochen/Ersov theorem [**23**, Theorem 4.6.4] several times, we conclude that

$$Kv_K \equiv L \equiv Lv_L((\mathbb{Q})) \equiv Lv_L((\mathbb{Q}))((\mathbb{Q})) \equiv L((\mathbb{Q})) \equiv Kv_K((\mathbb{Q})) \equiv K.$$

On the other hand, if $K \equiv Kv_K$, we have that K is either separably closed (and hence admits no non-trivial $\emptyset$-definable Henselian valuation) or, by the definition of v_K, that Kv_K is t-Henselian but not Henselian. In the latter case, K cannot admit a $\emptyset$-definable non-trivial Henselian valuation, lest Kv_K be Henselian. □

Note that by [**10**, Construction 6.5 and Proposition 6.7], there are fields with small absolute Galois group that are t-Henselian but not Henselian. Hence, there are Henselian fields with small absolute Galois group that admit no non-trivial $\emptyset$-definable Henselian valuation. Furthermore, Example 6.2 shows that there are Henselian fields with small absolute Galois group not admitting any non-trivial definable Henselian valuation.

We now give an example illustrating that, in general, Theorem A does not give rise to a full classification of which Henselian fields admit $\emptyset$-definable Henselian valuations.

Example 5.4 (a field admitting a $\emptyset$-definable Henselian valuation satisfying conditions (1)–(3) in Theorem A). Let K be the field as constructed in Example 3.8, so K is elementarily equivalent to a Henselian field but not Henselian and v_K^2 is a $\emptyset$-definable and p-antiregular value group for all p.

Consider the canonical Henselian valuation v_L on $L = K((\mathbb{Q}))$. Note that v_L is the power series valuation on L, thus $v_L L = \mathbb{Q}$ is divisible and $Lv_L = K$ is t-Henselian but not separably closed. In particular, it does not follow from Theorem A that L admits a $\emptyset$-definable Henselian valuation.

We claim that v_L is $\emptyset$-definable. Fix any prime p. As K is 2-Henselian, v_L^2 refines v_L. Thus, $v_L^2 L$ has a p-divisible quotient (namely $\mathbb{Q}$) and is therefore not p-antiregular. Furthermore, v_L^2 is the composition of v_L and v_K^2, so, as $v_K^2 K$ is p-antiregular, $\mathbb{Q}$ is the only p-regular quotient of $v_L^2 L$. Hence, $v_L^2 L$ has no non-p-divisible p-regular quotient and so some non-trivial convex subgroup with p-divisible quotient is $\emptyset$-definable in $v_L^2 L$ by

Lemma 3.6. However, Lv_L is the only such quotient and v_L^2 is $\emptyset$-definable by Theorem 2.1. Thus, v_L is $\emptyset$-definable.

The arguments given in the example above can in general be used to prove the following addition to Theorem A.

Proposition 5.5. *Let K be a Henselian field with* $\mathrm{char}(Kv_K) = 0$. *If*

(1) *there is some Henselian $L \succ Kv_K$ with $v_L L$ non-divisible and*

(2) *$v_K K$ is divisible,*

then K admits a $\emptyset$-definable non-trivial Henselian valuation.

Proof. Let K be a Henselian field such that $Kv_K \prec L$ for some Henselian L with $v_L L$ non-divisible. Fix a prime p with $v_L L \neq p v_L L$. Then, in particular, L is not separably closed and hence neither are Kv_K nor K. Since Kv_K is t-Henselian but not Henselian, L admits no $\emptyset$-definable non-trivial Henselian valuation. Thus, by Theorem A, $v_L L$ is p-antiregular.

Consider the field $M := L((v_K K))$ with the power series valuation w. By the Ax–Kochen/Ersov theorem [**23**, Theorem 4.6.4], we have

$$(K, v_K) \equiv (M, w).$$

Note that $v_M M \equiv v_K K \oplus v_L L$ holds (with the sum ordered lexicographically). Therefore, $v_M M$ is not p-divisible and not p-antiregular. Hence, M admits a $\emptyset$-definable non-trivial Henselian valuation by Theorem A. Thus, K also admits a $\emptyset$-definable non-trivial Henselian valuation. □

It would be very interesting to have a complete classification for the existence of non-trivial $\emptyset$-definable Henselian valuations. A necessary condition is that any elementarily equivalent field also admits a non-trivial Henselian valuation. We now ask whether this condition is also sufficient.

Question 5.6. Let K be a Henselian field such that any $L \equiv K$ is Henselian. Does K admit a non-trivial $\emptyset$-definable Henselian valuation?

It follows immediately from Corollaries 5.2 and 5.3 that if K is a Henselian field of finite transcendence degree or if it has a small absolute Galois group such that additionally $(\mathrm{char}(K), \mathrm{char}(Kv_K)) = (0, 0)$ holds, the answer to this question is positive.

6. Definitions with parameters

Theorem B. *Let K be a Henselian field that is not separably closed. Assume that K satisfies at least one of the following conditions:*

(1) *Kv_K is separably closed;*

(2) *Kv_K is not t-Henselian;*

(3) *$v_K K$ is not divisible.*

Then K admits a definable non-trivial Henselian valuation (using at most one parameter).

https://doi.org/10.1017/S0013091516000341 Published online by Cambridge University Press

Proof. Let $K \neq K^{\text{sep}}$ be a Henselian field. If Kv_K is not separably closed, or not t-Henselian, or there is a prime p with $v_K K$ non-p-divisible and not p-antiregular, we get a $\emptyset$-definable non-trivial Henselian valuation on K by Theorem A.

If $v_K K$ is not p-divisible for some prime p, some non-trivial (Henselian) coarsening of v_K is definable using at most one parameter by Proposition 4.2.

Thus, if K satisfies any of the assumptions (1)–(3), then K admits a definable non-trivial Henselian valuation (using at most one parameter). $\square$

In equicharacteristic 0, Theorem 1 gives rise to a classification of fields admitting definable Henselian valuations.

Corollary 6.1. *Let K be a non-separably closed Henselian field with* $\text{char}(Kv_K) = 0$. *Then K admits a definable non-trivial Henselian valuation if and only if at least one of the following conditions hold:*

(1) *Kv_K is separably closed;*

(2) *Kv_K is not t-Henselian, or for some Henselian $L \succ Kv_K$, the value group $v_L L$ is not divisible;*

(3) *$v_K K$ is not divisible.*

Proof. Let $K \neq K^{\text{sep}}$ be a Henselian field with $\text{char}(Kv_K) = 0$. If K satisfies at least one of the three conditions in the corollary, then K admits a definable non-trivial Henselian valuation by Theorem 1 and Proposition 5.5. For the other direction, assume that K satisfies

(1) $Kv_K \neq Kv_K^{\text{sep}}$ *and*

(2) $v_L L$ is divisible for all Henselian $Kv_K \prec L$ *and*

(3) $v_K K$ is divisible.

We need to show that K admits no definable non-trivial Henselian valuation. The key argument of the proof is relative quantifier elimination in the Denef–Pas language; however, first we need to do some work to set the situation up.

Since we have $Kv_K \neq Kv_K^{\text{sep}}$, any Henselian valuation is a coarsening of v_K. Take $L \succ Kv_K$ with $v_L L$ divisible. Note that as the extension $Kv_K \subset L$ is regular, the restriction of v_L to Kv_K is Henselian and hence trivial. Thus, we also get $\text{char}(Lv_L) = 0$.

Consider an $\aleph_0$-saturated elementary extension (M, v) of (K, v_K) in $\mathcal{L}_{\text{val}} = \mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a unary predicate that is interpreted as the valuation ring. Then, vM is a divisible ordered abelian group and $F := Mv$ is an $\aleph_0$-saturated elementary extension of Kv_K in $\mathcal{L}_{\text{ring}}$ and is thus Henselian [**24**, Lemma 3.3]. In particular, v_F is non-trivial and hence v_M is a proper refinement of v: namely, the composition of v and v_F. By our assumption, we get that $v_F F$ is divisible (and thus $v_M M$ is too). Note that the restriction of v_M to Kv_K is trivial, thus we get $\text{char}(Mv_M) = 0$.

We want to consider (M, v_M) as a structure in the Denef–Pas language $\mathcal{L}_{\text{DP}}$, which is an extension of $\mathcal{L}_{\text{ring}}$ (see [**21**] for details). A valued field (N, w) can be made into an

$\mathcal{L}_{\mathrm{DP}}$-structure if and only if there is an *angular component map* ac on N, i.e. a multiplicative map $\mathrm{ac}\colon N \to Nw$ with $\mathrm{ac}(0) = 0$ and which coincides on $\mathcal{O}_w^\times$ with the residue map. If there is no such map for (M, v_M), any sufficiently saturated $\mathcal{L}_{\mathrm{val}}$-elementary extension (N, w) of (M, v_M) has a cross-section since (M, v_M) is Henselian of equicharacteristic 0. Hence, (N, w) has an angular component map and can be considered as an $\mathcal{L}_{\mathrm{DP}}$-structure. In particular, we have $\mathrm{char}(Nw) = 0$ and wN divisible.

Assume for a contradiction that K admits a definable non-trivial Henselian valuation, i.e. that some non-trivial coarsening of v_K is definable. Then, via the elementary embedding

$$(K, v_K) \prec (M, v),$$

some non-trivial coarsening of v is $\mathcal{L}_{\mathrm{ring}}$-definable on M (using the same $\mathcal{L}_{\mathrm{ring}}$-formula and the same parameters from $K \subseteq M$). Thus, some proper coarsening of v_M is $\mathcal{L}_{\mathrm{ring}}$-definable in the Henselian valued field (M, v_M). Furthermore, via the elementary embedding

$$(M, v_M) \prec (N, w),$$

some proper coarsening of w is $\mathcal{L}_{\mathrm{ring}}$-definable in N.

In particular, this induces a definition of a proper, non-trivial convex subgroup of the divisible ordered abelian group wN.

Note that we have $\mathrm{char}(Nw) = 0$. By the relative quantifier elimination result in $\mathcal{L}_{\mathrm{DP}}$ the following holds in a Henselian valued field (N, w) of equicharacteristic 0 (see [**21**, Theorem 4.1]): any $\mathcal{L}_{\mathrm{DP}}$-definable subset of wN (using parameters from N) is already definable in the ordered abelian group wN (using parameters from wN). However, in a divisible ordered abelian group (like wN), there can be no proper, non-trivial convex definable subgroups. Hence, no non-trivial proper coarsening of w is definable on N and thus there can be no non-trivial definable Henselian valuation on K. □

Example 6.2 (a Henselian field that does not admit any non-trivial definable Henselian valuation). Refining the construction by Prestel and Ziegler as repeated in Example 3.8, one can construct a t-Henselian non-Henselian field k of characteristic 0 with $k \neq k^{\mathrm{sep}}$ and G_k small (see [**10**, Construction 6.5 and Proposition 6.7]). By [**10**, Proposition 5.8], $v_L L$ is divisible for any Henselian $L \succ k$. Consider the field $K := k((\mathbb{Q}))$. By Corollary 6.1, K does not admit a non-trivial definable Henselian valuation.

Example 6.3 (a field L admitting a non-trivial definable Henselian valuation such that there is some non-Henselian $K \equiv L$). Consider the field K as constructed in Example 3.8, so K is t-Henselian but not Henselian, v_K^2 is $\emptyset$-definable and has an antiregular value group.

By [**24**, Lemma 3.3], there is some elementary extension $L \succ K$ such that L is Henselian. We claim that the canonical Henselian valuation v_L on L has a non-divisible value group. By Theorem 2.1, v_L^2 and v_K^2 are defined by the same parameter-free formula. As antiregularity is an elementary property of an ordered abelian group, $v_L^2 L$ is also antiregular. Since $v_L L$ is a quotient of $v_L^2 L$, it cannot be p-divisible for any prime p.

Thus, by Theorem 1, L admits a non-trivial definable Henselian valuation. Since we have $L \equiv K$ and K is t-Henselian but not Henselian, L does not admit any non-trivial $\emptyset$-definable Henselian valuation.

7. Tamely branching (p-)Henselian valuations

In this section we study (p-)Henselian valuations tamely branching at p. In the first part we show that every field that admits a p-Henselian valuation tamely branching at p admits a $\emptyset$-definable such valuation and we draw some Galois-theoretic conclusions. In the second part we show that every field that admits a Henselian valuation tamely branching at p admits a definable such valuation; however, in general, parameters are required for the definition. We conclude that admitting a tamely branching Henselian valuation is not an elementary property in $\mathcal{L}_{\mathrm{ring}}$, which again has some Galois-theoretic consequences.

First, we recall the definition of tamely branching valuations.

Definition. Let (K, v) be a valued field and let p be a prime. We call v *tamely branching* at p if

(1) $\mathrm{char}(Kv) \neq p$ and

(2) vK is not p-divisible; and

(3) if $[vK : pvK] = p$, then Kv has a finite separable field extension of degree divisible by p^2.

7.1. Defining tamely branching p-Henselian valuations

We first consider the problem of defining p-Henselian valuations tamely branching at p. The existence of these valuations is encoded in the maximal pro-p quotient of the absolute Galois group of a field, as described by the following theorem.

Theorem 7.1 (Engler, Koenigsmann and Nogueira [18, Theorem 2.15]). *Let p be a prime, let K be a field containing a primitive pth root of unity (in particular, $\mathrm{char}(K) \neq p$), and assume that $G_K(p) \ncong \mathbb{Z}_p$ and, if $p = 2$, also that $G_K(p) \ncong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}/2\mathbb{Z}$. Then K admits a p-Henselian valuation tamely branching at p if and only if $G_K(p)$ has a non-trivial normal abelian subgroup.*

We now turn to the definability of such valuations.

Proposition 7.2. *Let K be a field and let p be a prime such that $\mathrm{char}(K) \neq p$ holds. Assume that we have $\zeta_p \in K$ and, furthermore, $\sqrt{-1} \in K$ if $p = 2$ and $\mathrm{char}(K) = 0$. If K admits a p-Henselian valuation tamely branching at p, then K admits a $\emptyset$-definable such valuation.*

Proof. Let v be a p-Henselian valuation tamely branching at p. We split the proof into cases.

(1) *If $Kv = Kv(p)$*, then we have $\mathcal{O}_v \subseteq \mathcal{O}_{v_K^p}$, so the canonical p-Henselian valuation v_K^p is also tamely branching at p: the fact that $\mathrm{char}(Kv_K^p) \neq p$ is immediate. Furthermore, we have $v_K^p K = vK/\Delta$, where Δ is the value group of the valuation $\bar{v}$ induced by v on Kv_K^p. Since we have $Kv = Kv(p)$, we also get $Kv_K^p = Kv_K^p(p)$ by the definition of the canonical

https://doi.org/10.1017/S0013091516000341 Published online by Cambridge University Press

Henselian valuation. Thus, Δ is p-divisible and, as vK is not p-divisible, v_K^p is not p-divisible. Moreover, we get $[vK : pvK] = [v_K^p K : pv_K^p K]$. Thus, if $[v_K^p K : pv_K^p K] = p$, Kv admits a finite separable extension of degree divisible by p^2: generated by an irreducible polynomial $f(X) \in Kv[X]$, say. Any lift of this polynomial to $\mathcal{O}_{\bar{v}}[X] \subseteq Kv_K^p[X]$ is still irreducible and separable and thus also generates a finite separable extension of degree divisible by p^2. Hence, in this case v_K^p is also tamely branching at p as claimed and, since v_K^p is $\emptyset$-definable, we have found a $\emptyset$-definable p-Henselian valuation on K.

(2) $Kv \neq Kv(p)$ *and* $\mathrm{char}(Kv_K^p) \neq p$. We then have $\mathcal{O}_{v_K^p} \subseteq \mathcal{O}_v$, and thus $v_K^p K$ is not p-divisible. If $Kv_K^p \neq Kv_K^p(p)$ holds, then v_K^p is again tamely branching at p. Now assume that we have $Kv_K^p = Kv_K^p(p)$ and $[v_K^p K : pv_K^p K] = p$. Then either vK is p-divisible or the value group of the valuation $\bar{v}_K^p$ induced by v_K^p on Kv is p-divisible. The first case cannot happen since v is tamely branching at p by assumption. Hence, assume that v_K^p induces a valuation with p-divisible value group on Kv. As $\bar{v}_K^p$ is p-Henselian of residue characteristic different from p and its residue field admits no Galois extension of degree p, this implies $Kv = Kv(p)$. Thus, we get $v = v_K^p$ and so in either case v_K^p is a $\emptyset$-definable p-Henselian valuation tamely branching at p.

(3) $Kv \neq Kv(p)$ *and* $\mathrm{char}(Kv_K^p) = p$. Define $v_K^p K =: \Gamma$ and $v(p) =: \gamma$. Consider the convex subgroup

$$\Delta_\gamma := \{\delta \in \Gamma \mid [0, p|\delta|] \subseteq [0, p\gamma] + p\Gamma\}$$

of Γ as in Lemma 4.1. We claim that $D_\gamma \neq \Gamma$ holds. Let $\langle\gamma\rangle$ be the convex subgroup of Γ generated by γ. Then, for any $\delta \in \Delta_\gamma$ there is some $\beta \in \Gamma$ with

$$\delta - p\beta \in \langle\gamma\rangle.$$

Note that $v(p) = 0$ holds, so vK is a quotient of $\Gamma/\langle\gamma\rangle$. As vK is not p-divisible, $\Gamma/\langle\gamma\rangle$ is not p-divisible. Hence, we get

$$\Delta_\gamma \subseteq \langle\gamma\rangle + p\Gamma \subsetneq \Gamma.$$

This proves the claim.

By the claim, there is a non-trivial $\emptyset$-definable coarsening u of v_K^p on K with value group $uK = \Gamma/\Delta_\gamma$. Lemma 4.1 implies that $uK \neq puK$ and $\mathrm{char}(Ku) \neq p$. In particular, u is a proper coarsening of v_K^p. Therefore, u is p-Henselian and $Ku \neq Ku(p)$ holds. Hence, u is a $\emptyset$-definable p-Henselian valuation tamely branching at p. □

We now give a Galois-theoretic consequence of the above. Together with Theorem 7.1, Proposition 7.2 yields the following.

Corollary 7.3. *Let p be a prime and let K be a field with* $\mathrm{char}(K) \neq p$ *and* $\zeta_p \in K$. *Take some* $L \equiv K$. *Then, if* $G_K(p)$ *has a non-trivial normal abelian subgroup, so does* $G_L(p)$.

Proof. Assume that $L \equiv K$. By [**3**, Lemma 17], this implies that $G_K \equiv G_L$ in the language of inverse systems introduced in [**3**, § 2]. Moreover, as the maximal pro-p quotient of a profinite group is interpretable in this language, we even get $G_K(p) \equiv G_L(p)$. If $G_K(p) \cong \mathbb{Z}_p$ or $p = 2$ and either $G_K(p) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}/2\mathbb{Z}$ holds, then, as all these groups are small, we conclude that $G_K(p) \cong G_L(p)$ [**3**, Proposition 27]. Hence, $G_L(p)$ also has a non-trivial abelian normal subgroup.

Otherwise, K admits a p-Henselian valuation tamely branching at p by Theorem 7.1. Thus, K admits a $\emptyset$-definable such valuation by Proposition 7.2, so L also admits a p-Henselian valuation tamely branching at p. Using Theorem 7.1 once more, we get that $G_L(p)$ has a non-trivial normal abelian subgroup. □

7.2. Defining tamely branching Henselian valuations

The main motivation to study Henselian valuations tamely branching at some prime p is the fact that they are encoded in the absolute Galois group of the field.

Theorem 7.4 (Koenigsmann [18, Theorem 1]; see also [8, Theorem 5.4.3]). *Let p be a prime. A field K admits a Henselian valuation, tamely branching at p if and only if G_K has a non-procyclic p-Sylow subgroup $P \not\cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ with a non-trivial abelian normal closed subgroup N of P.*

The absolute Galois group of a field K is encoded up to elementary equivalence (when considered in a language for profinite groups) in the theory of K. Hence, for a field K with small absolute Galois group, admitting a Henselian valuation tamely branching at some prime p is an elementary property in $\mathcal{L}_{\text{ring}}$: when G_K is small, we have $G_K \cong G_L$ (as profinite groups) for any L with $L \equiv K$ [**15**, Proposition 4.2]. The next proposition gives an alternative way of seeing this.

Proposition 7.5. *Let K be a field and let p be a prime. Assume that G_K is small. If K admits a Henselian valuation v tamely branching at p, then there is some $\emptyset$-definable coarsening of v that tamely branches at p.*

Proof. We may assume that K contains a primitive pth root of unity ζ_p: as K admits a Henselian valuation v tamely branching at p, we have $\mathrm{char}(K) \neq p$ and so $K(\zeta_p)$ is a $\emptyset$-definably interpretable extension of K in the sense of the proof of Proposition 3.7. Let u be a Henselian valuation on K and let u' be its unique extension to $K(\zeta_p)$. Now, as the index $[K(\zeta_p) : K]$ is prime to p, u is tamely branching at p if and only if u' is tamely branching at p. Thus, any parameter-free definition of a coarsening of v' on $K(\zeta_p)$ that tamely branches at p induces a $\emptyset$-definable such coarsening of v on K.

Let v be a Henselian valuation on K that tamely branches at p. Then vK is not p-divisible and, as G_K is small, it is not p-antiregular (see the proof of Corollary 5.1). Thus, some non-trivial coarsening w of v is $\emptyset$-definable by Proposition 3.7. Following the proof of Proposition 3.7, we get that either the value group of w is p-regular and non-p-divisible or w is the finest coarsening of v with p-divisible value group.

We claim that there is a $\emptyset$-definable coarsening w' of v with non-p-divisible value group. Assume first that wK is p-regular and non-p-divisible; then, we can choose $w' = w$.

Assume now that w is the finest coarsening of v with p-divisible value group. Then, v induces a Henselian valuation $\bar{v}$ on Kw such that its value group $\bar{v}(Kw)$ is not p-divisible and has no non-trivial p-divisible quotient. In particular, $\bar{v}(Kw)$ is either p-antiregular or has finite (Archimedean) rank. As G_{Kw} is a quotient of G_K [**8**, Lemma 5.2.6], G_{Kw} is also small and hence Kw admits no Henselian valuation with non-p-divisible p-antiregular value group (see again the proof of Corollary 5.1). Thus, $\bar{v}$ is a Henselian valuation of finite (Archimedean) rank on Kw such that no non-trivial coarsening of it has p-divisible value group. In particular, $\bar{v}$ has a (Henselian) rank-1 coarsening u such that the value group $u(Kw)$ is not p-divisible. Hence, by [**19**, Lemma 3.6] (or Theorem 3.1), u is $\emptyset$-definable on Kw. Thus, the composition $w' = u \circ w$ is a $\emptyset$-definable Henselian valuation on K with non-p-divisible value group. This proves the claim.

We have now found a $\emptyset$-definable coarsening w' of v such that $w'K$ is not p-divisible. We claim that w' is tamely branching at p. Since w' coarsens v, we have $\mathrm{char}(Kw') \neq p$. Assume that $p^2 \nmid G_{Kw'}$. Then, as G_{Kv} is a quotient of $G_{Kw'}$ [**8**, Lemma 5.2.6], we also get $p^2 \nmid G_{Kv}$. As v is tamely branching at p, we get $[vK : pvK] \neq p$. Furthermore, $p^2 \nmid G_{Kw'}$ implies that all valuations on Kw' have p-divisible value group. Thus, we get $[w'K : pw'K] = [vK : pvK] \neq p$. Therefore, w' is tamely branching at p. □

However, admitting a Henselian valuation tamely branching at some prime p is in general not an elementary property.

Example 7.6. Consider the field K as constructed in Example 3.8; note that K is elementarily equivalent to a Henselian field but is not Henselian, v_K^2 is $\emptyset$-definable and its value group $v_K^2 K$ is a p-antiregular value group for all primes p.

By [**24**, Lemma 3.4], there exists some elementary extension $L \succ K$ such that L is Henselian. We now show that the canonical Henselian valuation v_L on L is tamely branching at all primes p.

Note that the restriction of v_L to K is Henselian and thus trivial. In particular, we get $\mathrm{char}(Lv_L) = 0$. Furthermore, v_L is comparable to v_L^p. Since v_L^p and v_K^p are definable by the same formula and p-antiregularity is an elementary property of an ordered abelian group, $v_L^p L$ is p-antiregular. Thus, $v_L L$ is p-antiregular but not p-divisible. By Lemma 3.5, we have $[v_L L : pv_L L] = \infty$.

Overall, we get that v_L is tamely branching at any prime p. In particular, L admits no $\emptyset$-definable non-trivial Henselian valuation. Proposition 7.8 below shows that L nonetheless admits for every prime p a parameter-definable Henselian valuation that is tamely branching at p.

We immediately get the following.

Corollary 7.7. *Admitting a Henselian valuation tamely branching at p is not an elementary property, i.e. there are fields $K \prec L$ such that G_L has a non-procyclic Sylow subgroup $P_L \ncong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ admitting a non-trivial abelian normal closed subgroup, but G_K does not.*

As a consequence, not every field that admits a Henselian valuation tamely branching at p admits a $\emptyset$-definable such valuation. The next proposition shows that there is, nevertheless, always a definable such valuation.

Proposition 7.8. *Let K be a field and let p be a prime. Assume that K admits a Henselian valuation v tamely branching at p. Then K admits a definable such valuation (using at most one parameter).*

Proof. As in the proof of Proposition 7.5 we may assume that $\zeta_p \in K$. We split the proof into two cases.

(1) If $Kv_K = Kv_K(p)$, then we have $v_K \subseteq v_K^p$. Proposition 7.2 shows that there is a $\emptyset$-definable p-Henselian valuation w that is a coarsening of v_K^p and that tamely branches at p. As w is a coarsening of v_K, this gives a $\emptyset$-definable Henselian valuation tamely branching at p.

(2) If $Kv_K \neq Kv_K(p)$, then we have $v_K^p \subseteq v_K \subseteq v$. Define $\Gamma = v_K^p K$. For any $\gamma \in vK$ let $\langle\gamma\rangle$ be the convex subgroup generated by γ in Γ. We consider once more the convex subgroup

$$\Delta_\gamma = \{\delta \in \Gamma \mid [0, p|\delta|] \subseteq [0, p\gamma] + p\Gamma\}$$

of Γ as in Lemma 4.1. Note that, as in the proof of Proposition 4.2, $\Gamma = \Delta_\gamma$ implies that the quotient $\Gamma/\langle\gamma\rangle$ is p-divisible. Thus, if there is some $\gamma \in vK$ such that $\Gamma/\langle\gamma\rangle$ is not p-divisible, then we get a definable coarsening u of v with $uK = \Gamma/\Delta_\gamma$ that tamely branches at p.

On the other hand, if $\Gamma/\langle\gamma\rangle$ is p-divisible for all $\gamma \in vK$, then $vK/\langle\gamma\rangle$ is also p-divisible for all $\gamma \in vK$. This implies that $vK/\tilde{\Delta}$ is p-divisible for all convex subgroups $\tilde{\Delta} \leqslant vK$. Thus, vK is p-regular but not p-divisible and thus is $\emptyset$-definable by Theorem 3.1. $\square$

Acknowledgements. The authors thank Will Anscombe, Martin Bays, Raf Cluckers and Immanuel Halupczok for helpful discussions and comments.

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