INDECOMPOSABILITY GRAPHS OF RINGS

KARIN CVETKO-VAH and DAVID DOLŽAN

Abstract

We define a subgraph of the zero divisor graph of a ring, associated to the ring idempotents. We study its properties and prove that for large classes of rings the connectedness of the graph is equivalent to the indecomposability of the ring and in those cases we also calculate the graph's diameter.

1. Introduction

In the last two decades various papers appeared studying groups and rings via certain graph structures. The zero divisor graphs of commutative rings were introduced by Beck in [2], where the zero divisor graph of a commutative ring R is defined as a simple graph with vertices being the set of all non-zero zero divisors $Z(R)$, and $x, y \in Z(R)$ being adjacent if and only if $xy = 0$. This concept was later generalized to non-commutative rings by Redmond in [6], where non-zero zero divisors x and y are adjacent if and only if $xy = 0$ or $yx = 0$. Zero divisor graphs of non-commutative rings were further studied by Akbari and Mohammadian in [1], where special attention was devoted to semisimple Artinian rings and hence to rings of matrices. It was proved in [1] that if the zero divisor graph of a ring R is isomorphic to the zero divisor graph of some matrix ring $M_n(F)$, where F is a finite field and $n > 0$, then $R \cong M_n(F)$.

On the other hand, there was some research in [4] and [5] on the graphs of complete systems of orthogonal idempotents in finite (possibly non-commutative) rings, proving that the graphs of the indecomposable finite rings are connected and finding some inequalities for the number of elements in indecomposable finite rings according to the number of elements in their Jacobson radical. Here we say that a ring is *indecomposable* if it cannot be written as a non-trivial direct sum of its subrings.

The graphs studied in this paper are subgraphs of the zero divisor graph, defined on the set of idempotents of an arbitrary associative ring with unity, satisfying the

descending chain condition on idempotents. We define the edges of a graph in such a way that they correspond exactly to all the pairs of orthogonal idempotents e and f such that either $eRf \neq 0$ or $fRe \neq 0$. Thus, such a graph is in some sense directly measuring the indecomposability of a ring. We find the conditions for connectedness of such graphs in large classes of rings, and in those cases we also calculate their diameters.

In the next section we study the properties of these graphs, defined only on the complete sets of orthogonal idempotents of a ring. In the following section we expand the results to the graphs, defined on the set of all idempotents in a ring and in the final section we apply these results to some classes of rings.

2. Graphs associated to complete systems of orthogonal idempotents

Let R be an associative ring with unity 1. An element $x \in R$ is called an *idempotent* if $x^2 = x$. The set of all idempotents in R will be denoted by $E(R)$. A *natural partial order* is defined on $E(R)$ by $f \leq e$ if and only if $fe = ef = f$. An idempotent $e \neq 0$ is called a *primitive idempotent* if $f \leq e$ implies $f = 0$ or $f = e$, for any idempotent $f \in E(R)$. Throughout the paper P shall denote the set of all primitive idempotents in R.

A set of idempotents $S = \{e_1, \ldots, e_n\} \subseteq E(R)$ is called a *complete system of orthogonal idempotents* in R if $e_1 + \cdots + e_n = 1$ and $e_i e_j = 0$ for all $i \neq j$; S is a *complete system of primitive orthogonal idempotents* in R if it is a complete system of orthogonal idempotents and all idempotents in S are primitive.

We say that R satisfies the *descending chain condition on idempotents (DCCI)* if any decreasing chain of idempotents in respect to $\leq$ in R must be finite. Note that DCCI implies the ascending chain condition on idempotents because $1 \in R$.

If $S = \{e_1, \ldots, e_n\}$ is a complete system of orthogonal idempotents, assign to S a graph $G(S)$ with vertices being the elements of S and distinct vertices e_i, e_j being adjacent (denoted by $e_i \sim e_j$) if and only if $e_i R e_j \neq 0$ or $e_j R e_i \neq 0$.

The following lemma was proved in [5] for finite rings, but it also holds in a more general setting and we include the proof for the sake of completeness.

LEMMA 1. *If R is indecomposable, and $S = \{e_1, \ldots, e_n\}$ is a complete system of orthogonal idempotents in R then $G(S)$ is connected.*

PROOF. We prove this by induction on n. If $n = 1$, then $G(S) = K_1$, the full graph on one vertex. For $n > 1$ assume that $G(\{e_1, \ldots, e_{n-1}\})$ is connected and consider $G(S)$. Indecomposability of R implies that at least one of $f_n R(1 - f_n)$ and $(1 - f_n)Rf_n$ is non-trivial. Assume $f_n R(1 - f_n) \neq 0$. Hence

$$0 \neq f_n R(f_1 + \cdots + f_{n-1}) = f_n Rf_1 + \cdots + f_n Rf_{n-1}$$

which implies $f_n Rf_i \neq 0$ for some $i < n$ and $f_n \sim f_i$.   □

LEMMA 2. *Let $S = \{e_1, \ldots, e_n\}$ be a complete system of primitive orthogonal idempotents in a ring R. Then $G(S)$ is connected if and only if R is indecomposable.*

PROOF. Let $R = R_1 \oplus R_2$. Then $1 = f_1 + f_2$, where $f_1 \in R_1$, $f_2 \in R_2$ are idempotents. For $e_i \in S$ we obtain $e_i = e_i f_1 e_i + e_i f_2 e_i$. Primitivity of e_i implies that exactly one of $e_i f_1 e_i$ and $e_i f_2 e_i$ equals e_i, while the other equals 0. Then $S = S_1 \cup S_2$, where $S_1 = \{e_i \in S \mid e_i f_1 e_i = e_i\}$ and $S_2 = \{e_i \in S \mid e_i f_2 e_i = e_i\}$, induces a separation of $G(S)$.

On the other hand, a separation of $G(S)$ with the set of vertices $S = S_1 \cup S_2$, where there are no edges connecting S_1 to S_2, induces a ring decomposition

$$R = \sum_{e_i \in S_1} e_i R e_i \oplus \sum_{e_i \in S_2} e_i R e_i. \qquad \square$$

PROPOSITION 3. *R decomposes as an orthogonal direct sum of (indecomposable) subrings $R_1 \oplus \cdots \oplus R_r$ if and only if there exists a complete system of (primitive) orthogonal idempotents S in R such that the graph $G(S)$ has r components.*

PROOF. Let $R = R_1 \oplus \cdots \oplus R_r$ be a direct sum. Then there exist $x_i \in R_i$ such that $1 = x_1 + \cdots + x_r$. Multiplying by x_i yields $x_i^2 = x_i$ and x_i are primitive idempotents if the rings R_i are indecomposable. Hence $S = \{x_1, \ldots, x_r\}$ is a complete system of orthogonal idempotents in R and $G(S)$ is an empty graph on r vertices. To prove the converse, assume that $G(S)$ has r components $S_1, \ldots, S_r$ for some complete system of (primitive) orthogonal idempotents in R. Decompose R as

$$R = \sum_{e_k \in S_1} e_k R e_k \oplus \cdots \oplus \sum_{e_k \in S_r} e_k R e_k.$$

Each of the graphs $G(S_i)$ is connected and $\sum_{e_k \in S_i} e_i$ is the unity in $R_i = \sum_{e_k \in S_i} e_k R e_k$. Furthermore, if S_i is a complete orthogonal system of primitive idempotents in R_i then R_i is indecomposable. For $e_i \in S_i$, $e_j \in S_j$, $i \neq j$, we obtain $e_i R e_j = e_j R e_i = 0$ because there are no edges in $G(S)$ between the components S_i and S_j, and hence

$$\left(\sum_{e_k \in S_i} e_k R e_k \right) R \left(\sum_{e_k \in S_j} e_k R e_k \right) = \left(\sum_{e_k \in S_j} e_k R e_k \right) R \left(\sum_{e_k \in S_i} e_k R e_k \right) = 0. \qquad \square$$

LEMMA 4. *Let R be an Artinian ring and $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ two complete orthogonal systems of primitive idempotents. Then $n = m$ and there exists a unit $a \in R$ and a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $e_i = a f_{\pi(i)} a^{-1}$.*

PROOF. This follows from [3, Lemma 3.3] and the fact that R is Artinian, so the decomposition of R into a direct sum of indecomposable modules $R e_i$ is uniquely determined up to an isomorphism and a permutation of the direct summands. $\square$

If G is a graph, then any full subgraph of G is called a *clique*, while any empty subgraph of G is called an *anticlique*.

THEOREM 5. *Let S be a complete system of orthogonal idempotents in an indecomposable ring R and $G(S)$ the associated graph. Then $G(S)$ has a maximal clique of power k if and only if there exists a set $\{f_1, \ldots, f_k\} \subseteq S$ such that:*

(1)   *$(g + h)R(g + h)$ is indecomposable for all $g, h \in \{f_1, \ldots, f_k\}$; and*
(2)   *for all $g \in S \setminus \{f_1, \ldots, f_k\}$ an $i \in \{1, \ldots, k\}$ exists such that $(f_i + g)R(f_i + g) \cong f_i R f_i \oplus g R g$.*

PROOF. Assume that $\{f_1, \ldots, f_k\}$ form a maximal clique in $G(S)$. Choose $g, h \in \{f_1, \ldots, f_k\}$. Then $\{g, h\}$ is a complete system of orthogonal idempotents in $(g + h)R(g + h)$ and the associated graph $G(\{g, h\})$ is connected. Hence $(g + h)R(g + h)$ is indecomposable. This proves (1). To prove (2) choose $g \in S \setminus \{f_1, \ldots, f_k\}$. A vertex f_i in the maximal clique exists such that $g \nsim f_i$. Hence $f_i R g = 0 = g R f_i$ and the assertion follows. □

THEOREM 6. *Let S be a complete system of orthogonal idempotents in an indecomposable ring R and $G(S)$ the associated graph. Then $G(S)$ has an anticlique of power k if and only if $\{f_1, \ldots, f_k\} \in S$ exist such that*

$$(f_1 + \cdots + f_k)R(f_1 + \cdots + f_k) = \bigoplus_{i=1}^{k} f_i R f_i.$$

PROOF. If $\{f_1, \ldots, f_k\}$ is the anticlique in $G(S)$, then $f_i R f_j = f_j R f_i = 0$ for all $i, j = 1, \ldots, k$. □

Hence $G(S)$ in a sense 'measures' the decomposability of R.

3. Graphs associated to $E(R)$

Let $A \subseteq E(R) \setminus \{0, 1\}$ be any set of non-trivial idempotents in R. Define the graph $G(A)$ to be a simple graph with the set of vertices being A, and where $e, f \in A$ are connected by an edge, denoted $e \sim f$, if and only if:

(1) $ef = fe = 0$; and
(2) $eRf \neq 0$ or $fRe \neq 0$.

Furthermore, denote $G(R) = G(E(R) \setminus \{0, 1\})$.

LEMMA 7. *If R is indecomposable and $e \in E(R) \setminus \{0, 1\}$ then e is adjacent to $1 - e$ in $G(R)$.*

PROOF. The contrary would imply the decomposition

$$R = eRe \oplus (1 - e)R(1 - e). \qquad □$$

LEMMA 8. *Let R be a ring that satisfies the DCCI, and $e, f \in E(R)$ with the property $e \sim f$. Then a primitive idempotent g exists such that $g \leq f$ and $g \sim e$.*

PROOF. Choose a primitive idempotent g_1 with $g_1 \leq f$ and observe that $f - g_1 \in E(R)$. Then $eg_1 = e(fg_1) = 0 = (g_1 f)e = g_1 e$. Hence g_1 is orthogonal to e, and similarly $f - g_1$ is orthogonal to e. Without any loss of generality, we can assume that $eRf \neq 0$ and obtain

$$0 \neq eRf = eRg_1 + eR(f - g_1).$$

Hence at least one of eRg_1, $eR(f - g_1)$ is non-trivial. If $eRg_1 \neq 0$, take $g = g_1$. Otherwise, proceed in the same way with $f - g_1$ in the place of f to obtain $g \in P$, $g \sim e$ in finitely many steps. $\square$

LEMMA 9. *For any non-trivial idempotent $e \in E(R)$ a primitive idempotent $e' \in P$ exists with $e \sim e'$.*

PROOF. Lemma 7 implies $e \sim 1 - e$, while Lemma 8 provides a primitive idempotent $e' \leq 1 - e$ with $e \sim e'$. $\square$

PROPOSITION 10. *The graph $G(R)$ is connected if and only if the graph $G(P)$ is connected in which case* diam $G(R) \leq$ diam $G(P) + 2$.

PROOF. Lemma 8 implies that any path in $G(R)$ connecting primitive idempotents can be realized as a path in $G(P)$. Hence if $G(R)$ is connected then so is $G(P)$. Assume now that $G(P)$ is connected and consider non-trivial idempotents $e, f \in G(R)$. Lemma 9 yields the existence of primitive idempotents e', f' such that $e' \sim e$ and $f' \sim f$. Since $G(P)$ is connected, there exists a path connecting e' and f' in $G(P)$. Hence

$$e \sim e' \sim \cdots \sim f' \sim f$$

is a path connecting e and f in $G(R)$, and the final assertion follows as well. $\square$

The following example shows that diam $G(P) + 2$ is the sharpest general bound for diam $G(R)$.

EXAMPLE 11. Consider the ring $R \subseteq M_n(F)$ of all 3×3 matrices of the form

$$\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix},$$

where $*$ denotes an arbitrary element of F. Let e_i be a diagonal element with 1 on the ith diagonal entry and 0s elsewhere. Observe that $S = \{e_1, e_2, e_3\}$ is a complete orthogonal system of idempotents in R, and $e_1 \sim e_3 \sim e_2$, $e_1 \nsim e_2$ in $G(R)$. This implies that graph $G(S)$ is connected and R is indecomposable. Primitive idempotents are of the form

$$x(a) = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y(b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad z(u, v) = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 1 \end{bmatrix}.$$

The edges in $G(P)$ are precisely all $x(a) \sim z(-a, v)$ and all $y(b) \sim z(u, -b)$. Hence $z(u, v) \sim x(u) \sim z(-u, v') \sim y(-v') \sim z(u', v')$ is the shortest path connecting $z(u, v)$ to $z(u', v')$ in $G(P)$ and it is an easy exercise to show that diam $G(P) = 4$. What about $G(R)$? Idempotents of rank two are of the form

$$f(a, b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, \quad g(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad h(u, v) = \begin{bmatrix} 0 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $f(a, b)$ is adjacent only to $z(-a, -b)$ and $z(u, v)$ is not adjacent to any idempotent of the form $g(c)$ or $h(u, v)$. Hence

$$f(a, b) \sim z(-a, -b) \sim x(a) \sim z(-a, -b') \sim y(b') \sim z(-a', -b') \sim f(a', b'),$$

is the shortest path connecting $f(a, b)$ and $f(a', b')$ in $G(R)$, and diam $G(R) = 6$.

4. The application to Artinian rings

Let J denote the Jacobson radical of R.

PROPOSITION 12. *If J commutes with $E(R)$ then $G(R) = G(R/J)$. (Moreover, $S \subseteq R$ is a complete system of orthogonal idempotents in R if and only if $\bar{S} \subseteq R/J$ is a complete system of orthogonal idempotents in R/J.)*

PROOF. We can lift an arbitrary idempotent $\bar{e}$ in R/J to an idempotent $e \in R$. Suppose that $e + j$ is also an idempotent for some $j \in J$. This implies $(e + j)^2 = e + 2ej + j^2 = e + j$, so we get $(1 - 2e - j)j = 0$. But $(1 - 2e)^2 = 1$, therefore $1 - 2e - j$ is an invertible element, and we can conclude that $j = 0$. Thus the lifting of idempotents is unique. Now, if $\overline{e_i e_j} = \bar{0} \in R/J$, we know that $e_i e_j \in J$. However, since J commutes with $E(R)$, $e_i - e_i e_j$ is an idempotent, and therefore $e_i e_j = 0$ by the above argument. We have proved that $\overline{e_i} \sim \overline{e_j}$ implies $e_i \sim e_j$.

On the other hand, if $e_i \sim e_j$ and we have $e_i R e_j \subseteq J$, then $e_i + e_i r e_j$ is an idempotent for every $r \in R$, so $e_i R e_j = 0$ by the above, which is a contradiction. Therefore $e_i \sim e_j$ also implies $\overline{e_i} \sim \overline{e_j}$. □

If R is Artinian, then, by the Wedderburn theorem, the factor ring R/J is a direct sum of fields and full matrix rings over fields.

COROLLARY 13. *Suppose that J commutes with $E(R)$. If R is indecomposable, then R/J is indecomposable.*

PROOF. The result is a direct consequence of the previous proposition and Lemma 2. □

LEMMA 14. *Let $R = M_n(F)$ and for each i let e_i denote the diagonal matrix with the ith diagonal element being 1 and 0 elsewhere. If $e \in R$ is an arbitrary primitive*

idempotent then $e \sim e_i$ if and only if the ith row and the ith column of the matrix e are zero.

PROOF. Since e is a primitive idempotent, there exists such a unit $a \in R$ that $e = ae_j a^{-1}$ for some j. Therefore the rank of e is one, so $e = xy^{\mathrm{T}}$ for some vectors x and y. Also, let $e_i = f_i f_i^{\mathrm{T}}$, where f_i denotes the vector with 1 on the ith component and 0 elsewhere. If $e \sim e_i$ then $ee_i = (y^{\mathrm{T}} f_i)(x f_i^{\mathrm{T}}) = 0$, so the ith component of y is 0. Similarly $e_i e = (f_i^{\mathrm{T}} x)(f_i y^{\mathrm{T}}) = 0$, so the ith component of x is also 0; therefore both the ith row and the ith column of the matrix e are zero. The other implication is trivial. $\square$

COROLLARY 15. *Let $R = M_n(F)$ and let e_i be the diagonal matrix with the ith diagonal element being 1 and 0 elsewhere and a an arbitrary unit. Then $ae_i a^{-1} \sim e_i$ if and only if $a_{ii} = a_{ii}^{-1} = 0$.*

PROOF. Note that $ae_i a^{-1} = (ae_i)(e_i)a^{-1}$. By the previous lemma, the ith row and the ith column have to be zero and this is true if and only if $a_{ii} = a_{ii}^{-1} = 0$. $\square$

COROLLARY 16. *Let $R = M_2(F)$. Then $G(R)$ is not connected and the number of components for connectivity is equal to the index of the normalizer of any primitive idempotent in R.*

PROOF. If e is an arbitrary idempotent, it follows from the previous lemma that $e \sim e_1$ implies $e = e_2$. Let a and b be arbitrary units. So, if $ae_1 a^{-1} \sim be_2 b^{-1}$ then $b^{-1} ae_1 (b^{-1} a)^{-1} = e_1$, and thus $ae_1 a^{-1} = be_1 b^{-1}$. On the other hand, if $ae_1 a^{-1} \sim be_1 b^{-1}$ then $b^{-1} ae_1 (b^{-1} a)^{-1} = e_2$, and thus $ae_1 a^{-1} = be_2 b^{-1} = 1 - be_1 b^{-1}$, which yields $be_1 b^{-1} = ae_2 a^{-1}$. We have proved that, for each a, the idempotent $ae_1 a^{-1}$ is connected only to the idempotent $ae_2 a^{-1}$, so there are as many components in $G(R)$ as the number of elements conjugated to e_1. $\square$

THEOREM 17. *Let S be a complete orthogonal system of primitive idempotents in an indecomposable ring R. If J commutes with $E(R)$ then:*

(1)   *if $|S| > 2$ then $G(R)$ is connected and* $\operatorname{diam} G(R) = 5$;
(2)   *if $|S| = 2$ then $G(R)$ is not connected and the number of components for connectivity is equal to the index of the normalizer of any primitive idempotent in R.*

PROOF. By the Corollary 13, and the fact that $G(R) = G(R/J)$, we can assume that R is a full matrix ring $M_n(F)$ over a field and we may assume that $S = \{e_1, \ldots, e_n\}$, where e_i is the diagonal matrix with the ith diagonal element being 1 and 0 elsewhere. By Proposition 10 it suffices to show that $G(P)$ is connected. Choose a primitive idempotent $e \in P$. There exists an invertible matrix a such that $e = ae_1 a^{-1}$. Since $a^{-1} e_1 aRb^{-1} e_1 b = a^{-1} e_1 Re_1 b^{-1} \neq 0$ and $e_j Ra^{-1} e_i a = e_j Re_i a \neq 0$, for every i, j, primitive idempotents are adjacent in $G(P)$ if and only if they are orthogonal. Hence $G(S) \cong K_n$ is a full graph.

If $|S| = 2$ then the statement follows directly from Corollary 16. Therefore, we can now assume that $|S| > 2$.

Orthogonality of idempotents $a^{-1}e_i a$, e_j is equivalent to $e_i a e_j = 0 = e_j a^{-1}e_i$ which is further equivalent to $e_i a e_j = 0 = \det a[i|j]$, where $a[i|j]$ denotes the submatrix in a with ith row and jth column being erased.

Consider any primitive idempotents $a^{-1}e_1 a, b^{-1}e_2 b \in P$. (Note that any idempotent can be written in either form.) Denote $c = ab^{-1}$. Gauss elimination yields the existence of matrices p and q such that $(cpq)_{11} = 0$ and $(cpq)_{22} = \cdots = (cpq)_{2n} = 0$. The matrix p changes entries $(2, 2), \ldots, (2, n)$ in c to 0 and is of the form

$$p = \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

while q changes the $(1, 1)$-entry in cp to 0 and is of the form

$$q = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Hence $e_1(ab^{-1}pq)e_1 = 0 = e_1(ab^{-1}pq)^{-1}e_1$, $e_1 q^{-1}e_3 = e_3 q e_1$ and $e_3 p^{-1}e_2 = 0 = e_2 p e_3$. This implies that

$$ba^{-1}e_1 ab^{-1} \sim pqe_1(pq)^{-1} \sim pe_3 p^{-}1 \sim e_2,$$

or equivalently

$$a^{-1}e_1 a \sim b^{-1}pqe_1(pq)^{-1}b \sim b^{-1}pe_3 p^{-}1b \sim b^{-1}e_2 b.$$

Any two primitive idempotents in R can therefore be connected by a path of length at most 3, which implies diam $G(P) \leq 3$ and diam $G(R) \leq 5$ by Proposition 10. It is an easy exercise to find an example proving that in fact diam $G(R) = 5$. □

COROLLARY 18. *Assume that R is not local and that $eJf = 0$ for every pair of idempotents e and f such that $ef = 0$. Then $G(R)$ is connected if and only if R/J is a matrix ring of dimension at least 3. In that case, diam $G(R) = 5$.*

PROOF. Let $\overline{(\cdot)}$ denote the canonical map from R onto R/J. Since the ideal J is nilpotent, an element $a \in R$ is invertible if and only if $\overline{a} \in R/J$ is invertible. Therefore the set of all invertible elements in R is equal to $\overline{(\cdot)}^{-1}(R/J)$. Now, let $\overline{S}$ denote a complete orthogonal system of primitive idempotents in R/J and

let S denote the 'lifted' complete orthogonal system of primitive idempotents in R. Also let $G_{R/J}$ denote the subgraph of $G(R)$ induced by all the lifted idempotents from R/J. By the above remark, if e' is an arbitrary idempotent in R, we know that $e' = (1 + j)^{-1} e(1 + j)$ for some $j \in J$ and some $e \in G_{R/J}$.

If $e \nsim f$ for an idempotent $f \in G_{R/J}$, then either $ef \neq 0$ or $eRf = 0$. If $ef \neq 0$, then $e'f \neq 0$, since $\overline{e'} = \overline{e}$ and $eJf = 0$. On the other hand, $eRf = 0$ implies that also $e'Rf = 0$, so $e' \nsim f$. So we see that e' can only be connected to those idempotents in $G_{R/J}$ that e is connected to.

If $e \sim f$ for an idempotent $f \in G_{R/J}$, then $eRf \neq 0$ and thus also $e'Rf \neq 0$, again since $\overline{e'} = \overline{e}$. However, $ef = 0$ implies $e'f = 0$, since $eJf = 0$, and thus $e' \sim f$.

Therefore $e \sim f$ for two idempotents in $G(R)$ if and only if $\overline{e} \sim \overline{f}$ in $G(R/J)$. By Theorem 17, $G(R/J)$ is connected if and only if it is a matrix ring of dimension at least three.

We also note that, for connected graphs, the diameter of $G(R)$ equals the diameter of $G(R/J)$, so the equality also follows directly from Theorem 17. □

EXAMPLE 19. If we omit the condition $eJf = 0$ in the previous corollary the assertion no longer holds. Namely, consider the ring of upper triangular matrices of dimension at least three. Its graph is connected, but since the Jacobson radical is equal to the set of all strictly upper triangular matrices, the factor ring R/J is a direct sum of fields. Of course, $e_i J e_j \neq 0$ for every $i < j$ (where e_i, e_j denote the canonical idempotent matrices).

References

[1] S. Akbari and A. Mohammadian, 'Zero-divisor graphs of non-commutative rings', *J. Algebra* **296** (2006), 462–479.

[2] I. Beck, 'Coloring of a commutative ring', *J. Algebra* **116** (1988), 208–226.

[3] G. M. Benkart and J. M. Osborn, 'Derivations and automorphisms of nonassociative matrix algebras', *Trans. Amer. Math. Soc.* **263**(2) (1981), 411–430.

[4] Y. Hirano and T. Sumiyama, 'On orders of directly indecomposable finite rings', *Bull. Austral. Math. Soc.* **46**(3) (1992), 353–359.

[5] D. Mainwaring and K. R. Pearson, 'Decomposability of finite rings', *J. Austral. Math. Soc. Ser. A* **28** (1979), 136–138.

[6] S. P. Redmond, 'The zero-divisor graph of a non-commutative ring', *Internat. J. Commutative Rings* **1**(4) (2002), 203–211.

Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 21
1000 Ljubljana
Slovenia

Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 21
1000 Ljubljana
Slovenia
e-mail: david.dolzan@fmf.uni-lj.si