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SATURATED FREE ALGEBRAS REVISITED

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Abstract. We give an exposition of results of Baldwin–Shelah [2] on saturated free algebras, at the level of generality of complete first order theories T with a saturated model M which is in the algebraic closure of an indiscernible set. We then make some new observations when M *is* a saturated free algebra, analogous to (more difficult) results for the free group, such as a description of forking.

§1. Introduction. This paper has its origin in joint discussions during the second author's work on his Ph.D. thesis in Leeds. Although the topic of the thesis was the model theory of the free noncommutative group, we were interested in analogies with the much easier situation of saturated free algebras, which had been studied in a paper of Baldwin and Shelah [2]. (But note that free groups, although stable, are never saturated.) In Section 2, we recapitulate, with quick proofs, the main results of [2], in the more general model-theoretic context described in the abstract, which was already alluded to in [2]. These results consist of ω-stability and finite-dimensionality of T, and some refinements involving decompositions of suitable models of T as the algebraic closure of Morley sequences in weight one types. In Section 3, we look in more detail at a basis (or free generating set) I of a saturated free algebra, proving various results which are more specific to the case at hand and not necessarily valid at the level of generality of Section 2. For example we prove that I is a Morley sequence in a stationary type over M, and we describe forking in M in terms of free decompositions. We also ask several questions, some of which may have easy answers. In Section 4 we give a few examples, mainly highlighting the distinction between the context of Section 2 and that of Section 3.

Our model theory notation is standard. For simplicity we will work throughout with countable languages and theories. If L is a language consisting only of function symbols then we will call an L-structure an L-algebra. In that case, by a *variety* V in the language L (in the sense of universal algebra) we mean a class of L-algebras axiomatized by a collection of so-called *identities*, namely universal closures of expressions $t_1(\bar{x}) = t_2(\bar{x})$ where t_1, t_2 are L-terms. Free algebras exist in V: the free algebra F_X on

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generators X is characterized by the property that X generates F_X as an algebra and any map from X to an algebra $A \in V$ extends to a homomorphism from F_X to A, necessarily unique. It is clear that any permutation of X extends to an automorphism of F_X whereby X will be an *indiscernible set* in F_X which of course generates F_X under the terms of L.

Up to and including the 1970's there was considerable interaction between universal algebra and model theory, and it was natural for Baldwin and Shelah to study algebras F which are both free (on some set of generators) and uncountably saturated (as first order structures). In the paper [2] a number of interesting structural results are proved about $Th(F)$, although as John Baldwin mentioned to us, the actual statements of their Theorem 1 and Theorem 2, may need some tweaking to be correct. As Baldwin and Shelah mention in their paper, these structural results should hold with appropriate modifications under the *weaker* assumption that T is what we call below *almost indiscernible*, namely has a model M which is uncountably saturated and in the algebraic closure of some indiscernible set. In any case, working in this slightly more general context of almost indiscernible theories, we give a quick account of the main lemmas of [2] and correct versions of their Theorems 1 and 2.

One would imagine on the other hand that there are model-theoretic or stability theoretic properties of saturated free algebras which are more specific and do not generalize to almost indiscernible theories, and Sections 3 and 4 explore this topic. For example, we conjecture in Section 3 that the theory of a saturated free algebra has finite Morley rank, whereas in Section 4 we give examples of almost indiscernible theories with infinite Morley rank.

Note that in the informal definition above of an almost indiscernible theory, we say *indiscernible set* not *indiscernible sequence*. If we say rather *sequence* then this is a weaker notion which could be explored separately. In fact in [7] Benoist Mariou studies countable first order theories T with a saturated model M which has an expansion M' in a countable language such that M' itself is in the algebraic closure of an indiscernible sequence. Mariou proves that such a theory is *NIP* (does not have the independence property) and moreover among stable theories the property characterizes the ω-stable theories. This whole topic is of course related to the old proof of ω-stability of uncountably categorical theories, using Ehrenfeucht–Mostowski models.

In any case if T is the theory of a free (uncountable) saturated algebra then T is almost indiscernible and really the results in [2] were about such theories.

So in Section 2 we will study almost indiscernible theories and make reasonably free use of stability theoretic notions. Saharon Shelah invented stability theory, and the fundamental notions of the subject trace back to him, although alternative expositions, proofs, and even definitions, have been developed by others. At the time of the writing of [2], Shelah's [13] was the only source in book form for stability theory, although some other important and influential papers were in circulation, such as [5] and [6]. In the meantime several books on stability theory have appeared, and a common vocabulary and conceptual framework has been more or less established among practitioners of the subject. Chapter 1 of [10] is devoted to a summary, with selected proofs, of stability theory, and we will in particular use Section 4 (Miscellaneous facts about stable theories), as a basic reference for the current paper. The reader might also wish to consult [1], [4] as well as the paper [9] which gives an exposition of the computation of the spectrum function for ω-stable nonmultidimensional theories. In any case we complete this introduction with some facts about nonmultidimensional ω-stable theories.

We fix a complete ω-stable theory T and work in a big saturated model ($\bar{M}$ say).
A *complete type* usually refers to a complete type over some small subset of $\bar{M}$. Sometimes we refer to global types which are complete types over $\bar{M}$. Regular types are assumed, among other things, to be stationary.

Definition 1.1. Recall that T is said to be nonmultidimensional (finite-dimensional) if there are only boundedly many (finitely many) regular types, up to nonorthogonality.

In fact for a general stable theory nonmultidimensionality is defined as any two nonalgebraic stationary types being nonorthogonal, which is equivalent to the definition above for superstable T.

In [2] there are statements to the effect that arbitrary models of an ω-stable finite-dimensional theory are prime over a finite union of indiscernible sequences related to regular types. We want to make this a little more precise (and correct). Let M_0 denote a copy of the prime model as an elementary substructure of $\bar{M}$.

Fact 1.2. *Suppose T is nonmultidimensional. Then up to nonorthogonality every regular type can be chosen as a type over M_0 (which moreover is strongly regular).*

We now assume in addition that T is nonmultidimensional.
Let now $p_i(x)$ for $i \in I$ be a list of regular types over M_0, up to nonorthogonality, and let a_i be a finite tuple from M_0 such that p_i is definable over a_i. Let $(p_i)_{a_i}(x)$ be the restriction of $p_i(x)$ to a_i. Note that $|I| \leq \omega$ (as for example $S(M_0)$ is countable).

Fact 1.3. *Let M be any elementary substructure of $\bar{M}$. Assume that M contains a_i for each i. Let J_i be a maximal independent set of realizations of $(p_i)_{a_i}$ in M, for each i. Then M is prime (and minimal) over $\bigcup_{i\in I} a_i \cup \bigcup_{i \in I} J_i$.*

The cardinality of J_i in M depends only on M and $(p_i)_{a_i}$. We denote this cardinality by $dim((p_i)_{a_i}, M)$.

Remark 1.4. The condition that M contains all the a_i's is minor. In general M contains an isomorphic (elementary) copy M_0' of the prime model M_0, and so will contain a_i' for $i \in I$ such that $tp((a_i' : i \in I)) = tp((a_i : i \in I))$. Then work instead with the "copies" of the $(p_i)_{a_i}$'s over the a_i'. When it comes to counting models, it becomes important to note that if a_i and a_i' have the same strong type over $\emptyset$, then $(p_i)_{a_i}$ and its copy over a_i' have the same "dimension" in a model containing both a_i and a_i'.

The notion of an "a-model" (see Definition 4.2.2 of Chapter 1 of [10]) is important and for ω-stable T coincides with an ω-saturated model.

Lemma 1.5. *Let M be as in Fact* 1.3. *Then M is ω-saturated if and only if $|J_i|$ is infinite for each $i \in I$.*

Proof. Note that there is a unique countable ω-saturated model of T which we will call M_ω and which we can assume to contain M_0. Now suppose M is such that for each i, $dim((p_i)_{a_i}, M)$ is infinite. Let b be a finite tuple from M and $r(y, b)$ a complete type over b. Then there are countably infinite $J_i' \subset J_i$ for $i \in I$ such that b is contained in an elementary submodel M' of M prime over the a_i's together with the J_i'. So M' is isomorphic to M_ω hence ω-saturated too. So $r(x, b)$ is realized in M' so in M. Hence M is ω-saturated. ⊣

Corollary 1.6. *Any elementary extension of an ω-saturated model of T is also ω-saturated.*

Now we can in fact choose a_i to be the canonical base of p_i (as an element of M_0^{eq}), and it is well-known that then a_i is in the definable closure of J whenever J is an infinite Morley sequence in $(p_i)_{a_i}$. We conclude:

Proposition 1.7. *Any ω-saturated model M of T is prime over a union of indiscernible sets.*

Proof. We may assume M contains M_0 so a_i for each $i \in I$. Let J_i be a maximal Morley sequence in M in $(p_i)_{a_i}$. Then J_i is infinite, whereby $a_i \in dcl(J_i)$. So by Fact 1.3, M is already prime over the union of the J_i. ⊣

Remark 1.8. Of course when T is finite-dimensional I is finite. At the current level of generality, Theorem 1 of [2] seems only valid for ω-saturated models of T.

Explanation. Theorem 1 of [2] says that any model M of the theory T of of a saturated free algebra is prime over a finite union of indiscernible sets. Once we know T to be ω-stable and finite-dimensional, this follows from Proposition 1.7 when M is ω-saturated. But for arbitrary M one has to also include the a_i as in Fact 1.3.

Finally note by Corollary 1.6 that:

Remark 1.9. Let M be an ω-saturated model of T, and B any set. Then the prime model over $M \cup B$ coincides with the a-prime model (prime model in the category of ω-saturated models) over $M \cup B$.

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§2. **Almost indiscernible theories.**

We work with a countable language L and complete L-theory T.

Definition 2.1. The theory T is said to be almost indiscernible if there is a saturated model of T of cardinality $\aleph_1$ which is in the algebraic closure of an indiscernible *set* of finite tuples I (so I is forced to have cardinality $\aleph_1$ too).

Assumption 2.2. *The theory T is almost indiscernible.*

So we let M denote a saturated model of T of cardinality $\aleph_1$ which is in the algebraic closure of an indiscernible set (which we write as a sequence) $I = (e_\alpha : \alpha < \aleph_1)$ of cardinality $\aleph_1$.

Let $\bar{\kappa}$ be a cardinal much bigger than $\aleph_1$. Let $\bar{M}$ be a $\bar{\kappa}$-saturated elementary extension of M. Let $\bar{I} = (e_\alpha : \alpha < \bar{\kappa})$ be an indiscernible set in $\bar{M}$ extending I. For each infinite $\lambda \leq \bar{\kappa}$, let $I_\lambda = (e_\alpha : \alpha < \lambda)$ and let $M_\lambda = acl(I_\lambda)$ inside $\bar{M}$. So $M_{\aleph_1} = M$ is an elementary substructure of $\bar{M}$ by definition of $\bar{M}$ but on the face of it the other M_λ's are just subsets of $\bar{M}$. Note that M_λ has cardinality λ. We then easily obtain:

Lemma 2.3.

(i) *The M_λ's form an elementary chain.*
(ii) *The structure M_ω is ω-saturated.*

Proof. (i) is left to the reader.
(ii) Let $\Sigma(x)$ be a partial type over a finite subset A of M_ω, consistent with $\bar{M}$. Then A is contained in the algebraic closure of $e_1, \dots, e_n$ say, and $\Sigma(x)$ is realized in $M_{\aleph_1}$ by some d in the algebraic closure of $e_1, \dots, e_n$ together with some other $e_{\alpha_1}, \dots, e_{\alpha_k}$ with $\alpha_i < \aleph_1$. Then as $tp(e_1, \dots, e_n, e_{\alpha_1}, \dots, e_{\alpha_k}) = tp(e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k})$ we can find such a realization in M_ω. ⊣

Remark 2.4. In fact one can also show directly at this stage that each M_λ is λ-saturated, although it will also follow easily from ω-stability, proved next.

Proposition 2.5. *The theory T is ω-stable.*

Proof. By Lemma 2.3(ii), it suffices to show that there only countably many complete 1-types over M_ω. Now any such type is of the form $tp(d/M_\omega)$ for some $d \in M_{\omega_1}$ and $d \in acl(M_\omega \cup I')$ where $I' = I_{\omega_1} \setminus I_\omega$. So $tp(d/M_\omega \cup I')$ is isolated by some formula $\phi(x, \bar{e})$ where $\phi(x, \bar{y})$ has parameters from M_ω and $\bar{e}$ is a finite tuple from I'. As the type over M_ω of such a finite tuple $\bar{e}$ is determined by the cardinality of $\bar{e}$ we see that $tp(d/M_\omega)$ is determined by the formula $\phi(x, \bar{y})$ (which includes the length of $\bar{y}$). As there are countably many possibilities there are countably many such types. ⊣

Concerning the saturation of the M_λ's: let $q(x)$ be a complete type over a subset A of M_λ of cardinality $< \lambda$. We may assume that A contains M_ω.

Let now p denote the so-called average type of I over $\bar{M}$. Namely $p(x) \in S(\bar{M})$ and for $\phi(x)$ over $\bar{M}$, $\phi(x) \in p$ if $\phi(e_i)$ holds for all but finitely many $i < \kappa$. By ω-stability p is definable over $\{e_i : i < n\}$ for some finite n, in particular it is definable over M_ω and moreover $p|M_\omega = tp(e_\omega/M_\omega)$ and $\bar{I}$ is a Morley sequence in $p|M_\omega$.

Lemma 2.6. *Any complete type over M_ω is nonorthogonal to p, hence nonweakly orthogonal to p as M_ω is an a-model.*

Proof. Let $q(y) \in S(M_\omega)$. Then as $M = M_{\aleph_1}$ is $\aleph_1$-saturated q is realized by some $d \in acl(M_\omega, \bar{e})$ for some finite $\bar{e}$ from I. As $\bar{e}$ is an independent set of realizations of $p|M_\omega$ it follows that q is nonorthogonal to p. ⊣

Proposition 2.7. *The theory T is finite-dimensional.*

Proof. For any regular type r over M_ω, by the previous lemma there is a realization a_r of r such that e_ω forks with a_r over M_ω. If $r_1, \dots, r_n$ are pairwise orthogonal regular types then the a_{r_i} are independent over M_ω and each forks with e_ω over M_ω. So the weight of p gives a bound on n. Hence there are only finitely many regular types over M_ω up to nonorthogonality. By ω-stability and the fact that M_ω is an a-model, this implies that T is finite-dimensional. ⊣

So we see by Proposition 1.7 and its proof that any ω-saturated model of T is prime over a finite union of indiscernible sets each of which comes from a nonorthogonality class of a (strongly) regular type of T. This is (suitably adapted) Theorem 1 of [2], as remarked earlier. We now make a few refinements.

Proposition 2.8. *The structure $M_{\omega+1} = acl(M_\omega, e_\omega)$ is prime and a-prime (and minimal) over $(M_\omega, c_1, \dots, c_n)$ where $tp(c_i/M_\omega)$ is regular, $\{c_1, \dots, c_n\}$ is M_ω-independent, and each regular $q \in S(M_\omega)$ appears up to nonorthogonality among the $tp(c_i/M_\omega)$.*

Proof. Now $M_{\omega+1}$ is clearly prime over (M_ω, e_ω) and by Remark 1.9 is also a-prime over (M_ω, e_ω). Let $\{c_1, \dots, c_n\}$ be a maximal independent over M_ω subset of $M_{\omega+1}$ such that each $tp(c_i/M_\omega)$ is regular. By the proof of Corollary 4.5.7 of Chapter 1 of [10], $M_{\omega+1}$ is a-prime and so also prime over $(M_\omega, c_1, \dots, c_n)$. It remains to be seen that every regular $q \in S(M_\omega)$ appears among the $tp(c_i/M_\omega)$ up to nonorthogonality. But by Lemma 2.5, the type $p|M_\omega$ dominates q, so q is realized in $M_{\omega+1}$ by some d, and then d forks with some c_i over M. ⊣

We now aim for a stronger result which decomposes $p|M_\omega$ into a product of weight one types in a stronger sense. The following proposition is essentially Lemma 13 of [2], although they have in (iii) only one direction of the interalgebraicity result, namely that e_ω is algebraic over $M_\omega \cup \{d_1, .., d_n\}$. (But the other direction follows automatically as we point out.) Our proof follows the same general line of argument as in [2] with a few simplifications.

Proposition 2.9. *There are tuples $d_1, \dots, d_n$ such that:*

(i) *the type $tp(d_i/M_\omega)$ has weight one and $c_i \in acl(M_\omega, d_i)$, for each i;*
(ii) *the set $\{d_1, \dots, d_n\}$ is M_ω-independent; and*
(iii) *the element e_ω is interalgebraic with $(d_1, \dots, d_n)$ over M_ω.*

Proof.

Claim I. There are $d_1, \dots, d_n$ such that $\{d_1, \dots, d_n\}$ is M_ω-independent, and $tp(d_i, c_i/M_\omega) = tp(e_\omega, c_i/M_\omega)$ for each i. In particular $c_i \in acl(M_\omega, d_i)$ for each i, and $(d_1, \dots, d_n)$ realizes $(p|M_\omega)^{(n)}$.

Proof (of Claim I). Simply choose d_i to realize $tp(e_\omega/M_\omega, c_i)$ such that the d_i's are as independent as possible over M_ω. For example, inductively choose the d_i such that d_{i+1} is independent from $M_\omega, d_1, \ldots, d_i$ over c_{i+1}. Then the independence of the c_i plus forking calculus guarantees the independence of the d_i.

Claim II. There are $d_1, \ldots, d_n$ as in Claim I, such that $e_\omega \in acl(M_\omega, d_1, \ldots, d_n)$.

Proof (of Claim II). Let $M^n = acl(M_\omega, d_1, \ldots, d_n)$. Then by Lemma 2.2 M^n is the prime model over $(M_\omega, d_1, \ldots, d_n)$ so contains a copy of the prime model over $M_\omega, c_1, \ldots, c_n$. Thus, by Proposition 2.7, we find e'_ω in M^n realizing $tp(e_\omega/M_\omega, c_1, \ldots, c_n)$, which suffices.

Finally we massage the situation in a routine manner to get the full statement of Proposition 2.8. For each $i = 1, \ldots, n$ let f_i be a tuple such that c_i is independent from c_i over M_ω and the Morley rank of $tp(d_i/M_\omega, f_i)$ is minimized. Then we know that c_i dominates d_i over M_ω, f_i, whereby $tp(d_i/M_\omega, f_i)$ has weight one. Now choosing the f_i's as free as possible, we can ensure that $(f_1, \ldots, f_n)$ is independent from $(c_1, \ldots, c_n)$ over M_ω from which we conclude that $(c_1, \ldots, c_n)$ dominates $(d_1, \ldots, d_n)$ over $(M_\omega, f_1, \ldots, f_n)$. Let $\bar{f} = (f_1, \ldots, f_n)$. Note that as $(c_1, \ldots, c_n)$ dominates e_ω over M_ω, we have that:
(*) e_ω is independent from $\bar{f}$ over M_ω.
Now choose finite $A \subset M_\omega$ such that $tp(e_\omega, c_1, \ldots, c_n, d_1, \ldots, d_n)/M_\omega, \bar{f})$ does not fork over $A, \bar{f}$, and bearing in mind (*), we may assume that $tp(e_\omega/M_\omega \bar{f})$ is also definable over A. Note that we have that $(c_1, \ldots, c_n)$ dominates $(d_1, \ldots, d_n)$ over $(A, \bar{f})$, and $e_\omega \in acl(d_1, \ldots, d_n, A, \bar{f})$. As M_ω is ω-saturated, we may choose $\bar{f}'$ in M_ω such that $tp(\bar{f}/A) = tp(\bar{f}'/A)$. So:
(a) $tp(a_\omega, c_1, \ldots, c_n, \bar{f}/A) = tp(a_\omega, c_1, \ldots, c_n, \bar{f}'/A)$.
So we can choose $(d'_1, \ldots, d'_n)$ such that
(b) $tp(e_\omega, c_1, \ldots, c_n, d'_1, \ldots, d'_n, \bar{f}'/A) = tp(e_\omega, c_1, \ldots, c_n, d_1, \ldots, d_n, \bar{f}/A)$.
In particular:
(c) $e_\omega \in acl(d'_1, \ldots, d'_n, A, \bar{f}')$, and
(d) $(c_1, \ldots, c_n)$ dominates $(d'_1, \ldots, d'_n)$ over $(A, \bar{f}')$ and $tp(d_i/A, \bar{f}')$ has weight 1.
But $(c_1, \ldots, c_n)$ is independent from M_ω over $(A, \bar{f}')$, so by (d) we see that each d_i is independent from M_ω over $(A, \bar{f}')$ whereby
(e) each $tp(d'_i/M_\omega)$ has weight 1, c_i dominates d_i over M, $(c_1, \ldots, c_n)$ dominates $(d'_1, \ldots, d'_n)$ over M_ω, and $\{d'_1, \ldots, d'_n\}$ is M_ω-independent.
So renaming d'_i as d_i, we have (i) and (ii) of Proposition 2.8, as well as e_ω being algebraic over $(M_\omega, d_1, \ldots, d_n)$. To see that the d_i are in $acl(M_\omega, e_\omega)$, we do the following. As $(c_1, \ldots, c_n)$ dominates $(d_1, \ldots, d_n)$ over M_ω we can find a copy M' of the a-prime (so prime) model over $(M_\omega, c_1, \ldots, c_n)$ which contains $(d_1, \ldots, d_n)$. By (c), $e_\omega \in M'$. By Proposition 2.7, M' is also prime over (M_ω, e_ω) so by uniqueness equals $M_{\omega+1} = acl(M_\omega, e_\omega)$ so each $d_i \in acl(M_\omega, e_\omega)$. ⊣

We obtain the following "structure theorem", which is our version of Theorem 2 from [2].

Proposition 2.10. *Let M be a model of T containing M_ω. Then there are $J_1, \dots, J_k$ each being a Morley sequence in some weight* 1 *type over M_ω such that M is the* algebraic closure *of M_ω union the J_i.*

Proof. For simplicity we assume that in Proposition 2.7 the (strongly) regular types $q_i = tp(c_i/M_\omega)$ are pairwise orthogonal. In Proposition 2.8, we may assume that c_i is a subtuple of d_i for $i = 1, \dots, n$. As c_i dominates d_i over M_ω it follows that $tp(d_i/M_\omega, c_i)$ is actually isolated, by the formula $\phi_i(y_i, c_i)$ say ($\phi(y, z)$ over M_ω). Let $r_i = tp(d_i/M_\omega)$. Now let M' be any model containing M_ω. For $i = 1, \dots, n$, let K_i be a Morley sequence of q_i in M'. Note that q_i might not be realized in M, in which case K_i is empty. So as in Proposition 1.7, M' is prime over $M_\omega \cup \bigcup_i K_i$. Now for each $c_{i,j} \in K_i$, let $d_{i,j} \in M'$ be such that $\models \phi_i(d_{i,j}, c_{i,j})$. So $d_{i,j}$ realizes r_i and $\{d_{i,j} : i, j\}$ is M_ω-independent. In any case let $J_i = (d_{i,j})_j$, a Morley sequence in the weight 1-type r_i, which is contained in M'.

Claim. $M' = acl(M_\omega \cup \bigcup_i J_i))$.

Proof (of claim). In fact it is enough to prove that $acl(M_\omega \cup \bigcup_i J_i)$ is a model (elementary substructure of $\bar{M}$), because it will then be prime over $(M_\omega \cup \bigcup_i K_i)$ so isomorphic to M' (in fact equal to M' as M' is not only prime but also minimal over $M_\omega \cup \bigcup_i K_i$). Note that in general the K_i's may have different cardinalities for different $i = 1, \dots, n$. Let J_i' for $i = 1, \dots, n$ be a Morley sequence in r_i extending J_i such that all the J_i' have the same cardinality κ say. For each $\alpha < \kappa$, let a_α be a realization of $p|M_\omega$ interalgebraic with $(d_{i,_\alpha}) : i = 1, \dots, n)$ (where $J_i = (d_{i,\alpha} : \alpha < \kappa)$). Then $(a_\alpha : \alpha < \kappa)$ is a Morley sequence in $p|M_\omega$ so by Lemma 2.2 its algebraic closure over M_ω is a model. But this coincides with $acl(M_\omega, \bigcup_i J_i')$ which is therefore a model. Now as $\bigcup_i J_i'$ is independent over M_ω, for each tuple b from $\bigcup_i J_i' \setminus \bigcup_i J_i$, $tp(b/M_\omega \cup \bigcup_i J_i)$ is finitely satisfiable in M_ω. So using Tarski–Vaught it follows that $acl(M_\omega \cup \bigcup_i J_i)$ is an elementary substructure of $\bar{M}$, as required. ⊣

Remark 2.11. In Section 4 we give a few examples of almost indiscernible theories of infinite rank. But one can check by inspection that any almost indiscernible theory of abelian groups (in the group language) has finite Morley rank.

§3. Free Algebras. The reader is referred to [3] for background on universal algebra (of which not much is needed). As mentioned before we work with algebras in a countable language (or signature) L. Fix a variety V. Then for any set X, F_X denotes the free algebra in V on generators X, and we call X a *basis* of F_X. In general it is possible that F_X and F_Y are isomorphic even though X and Y have different cardinalities, so there is no well-defined notion of dimension for a free algebra. But this can only happen if both X, Y are finite. On the other hand it is clear that any bijection between X and Y extends to an isomorphism between F_X and F_Y and conversely any isomorphism between F_X and F_Y takes X to another basis of F_Y.

In general if A is an algebra and X a subset of A then $\langle X \rangle$ denotes the subalgebra of A generated by X.

Remark 3.1. Suppose that the algebra A is free on $X_1 \cup X_2$, and A_1 is the subalgebra of A (freely) generated by X_1. Let Y_1 be another basis of A_1. Then A is freely generated by $Y_1 \cup X_2$.

Proof. Let B be an algebra in V and $f : Y_1 \cup X_2 \to B$. So $f|Y_1$ extends uniquely to a homomorphism $f_1 : A_1 \to B$. Let g be the restriction of f_1 to X_1. Then as A_1 is free on X_1, f_1 is also the unique extension of g to a homomorphism from A_1 to B. Now $g \cup f|X_2$ is a map from $X_1 \cup X_2$ to B hence extends to a homomorphism h from A to B. Now the restriction of h to A_1 must coincide with f_1 hence the restriction of h to Y_1 coincides with $f|Y_1$. So h extends f. ⊣

Assumption 3.2. *The algebra M is a free algebra for V on a set $I = (e_\alpha : \alpha < \aleph_1)$ and is moreover $\aleph_1$-saturated.*

So I is an uncountable indiscernible set in M, $dcl(I) = M$ and M is saturated, whereby all of Section 2 applies to $T = Th(M)$. But we will prove some results which are specific to the “free saturated algebra” setting.

It is also not hard to see that if I' is either a shrinking or stretching of I to another infinite indiscernible set in the sense of T, then the algebraic closure of I' (in the ambient model of T) is precisely $\langle I' \rangle$ and is moreover free on I' in the variety V.

Definition 3.3. We call a subset A of M *basic* if A is a subset of a basis of M. And we call an element $a \in M$ basic if $\{a\}$ is basic.

So a basic element is what in the context of a free group is called a primitive element.

Lemma 3.4. *There is a complete type $p_0(x)$ over $\emptyset$ such that for any $a \in M$, a is basic if and only if a realizes p_0.*

Proof. Note that all elements of I have the same type over $\emptyset$ which we take to be $p_0(x)$. Suppose first that a is basic. So a extends to a basis X for M. The basis X has cardinality $\aleph_1$ too and any bijection between X and I induces an automorphism of M, so a realizes p_0. Conversely if a realizes p_0 in M and $e \in I$ then there is an automorphism of M taking e to a (as M is saturated so homogeneous) and the image of I will be a basis of M containing a. ⊣

For the rest of this section, $p_0(x)$ denotes the type given by Lemma 3.3, namely the type of some/any element of I.

Remark 3.5. As remarked above, if I_0 is a countable subset of I and M_0 is the subalgebra of M generated by I_0, then M_0 is free on basis I_0, and is moreover an ω-saturated elementary substructure of M. In particular p_0 is the type of any element of I_0 in M_0, and Lemma 3.3 also applies to M_0 with the same proof.

Question 3.6. *Is $p_0(x)$ of maximal Morley rank among complete 1-types of T?*

Lemma 3.7. *Suppose $a \in M$ is basic and a is a term in $e_{\alpha_1}, \dots, e_{\alpha_n}$, then for any countable subset C of $I \setminus \{e_{\alpha_1}, \dots, e_{\alpha_n}\}$, $C \cup \{a\}$ is a basic set.*

Proof. Extend $e_{\alpha_1}, \ldots, e_{\alpha_n}$ to a countable subset I_0 of I, avoiding C. Let M_0 be generated by I_0. Then $a \in M_0$ and by Remark 3.4 is basic in M_0. By Remark 3.1 $\{a\} \cup C$ is basic in M, and also basic in the (free) algebra it generates. ⊣

Lemma 3.8. *The type p_0 is stationary (as therefore is $p_0^{(n)}$ for any n).*

Proof. We have to show that p_0 determines a unique strong type over $\emptyset$. So suppose a, b are both realizations of p_0. So a is part of a basis I of M and b part of a basis J of M. By Lemma 3.6 there is $b' \in J$ such that $\{a, b'\}$ is a basic set, namely extends to another basis J' of M. But then, as J' is indiscernible in M, a and b' have the same strong type. As for the same reason b and b' have the same strong type it follows that a and b do too. ⊣

Proposition 3.9. *The sequence I is a Morley sequence in p_0, namely not only indiscernible but also independent over $\emptyset$.*

Proof. Let $I_0 = \{e_\alpha : \alpha < \omega\}$. Let a realizes p_0 such that a is independent from I_0 over $\emptyset$. By Lemma 3.6 we can find an infinite subset I_0' of I_0 such that $I_0' \cup \{a\}$ is a basic. But then this is an indiscernible set with the same "Ehrenfeucht–Mostoswki" type as I. Hence for example e_ω is independent from I_0 over $\emptyset$ which is enough. ⊣

Corollary 3.10. *In T, $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$.*

Proof. Suppose $a \in acl^{eq}(\emptyset)$. Then $a \in dcl^{eq}(\bar{e})$ for some finite tuple from I. But by Lemma 3.7 and Proposition 3.8, $tp(\bar{e}/\emptyset)$ is stationary, whereby $tp(a/\emptyset)$ is stationary whereby $a \in dcl^{eq}(\emptyset)$. ⊣

We can prove in a similar manner.

Remark 3.11. Let E be any subset of I (or in fact any basic set). Then $acl^{eq}(E) = dcl^{eq}(E)$.

Proposition 3.12. *If $\bar{a}, \bar{b}$ are tuples from M. Then $\bar{a}$ is independent from $\bar{b}$ over $\emptyset$ if and only if there is a basis $B_1 \cup B_2$ of M such that $\bar{a}$ is contained in $\langle B_1 \rangle$ and $\bar{b}$ is contained in $\langle B_2 \rangle$.*

Proof. Right implies left is clear as by a basis (or basic subset) of M is independent over $\emptyset$. For the converse. Suppose $\bar{a}$ and $\bar{b}$ are independent over $\emptyset$. Without loss, $\bar{a}$, $\bar{b}$ are both terms in $e_1, \ldots, e_n$ and write $\bar{a} = \bar{t}(e_1, \ldots, e_n)$. Let $\bar{a}'\bar{t}(e_{n+1}, \ldots, e_{2n})$. Then $\bar{a}'$ is independent from $\bar{b}$ and $tp(\bar{a}) = tp(\bar{a}')$. By stationarity $tp(\bar{a}, \bar{b}) = tp(\bar{a}', \bar{b})$, so by automorphism we can find the suitable basis. ⊣

Remark 3.13. Proposition 3.11 extends naturally to describing independence *over* any basic set B.

The interested reader can consult [8] for the analogous result for noncommutative free groups. As a matter of fact our proof is a straight adaptation of the proof there, whose main ingredients had been the homogeneity of noncommutative free groups and the stationarity of every type over the empty set and of every type over any primitive element.

Question 3.14. *Let T be the theory of saturated free algebra*:

(i) *must T have finite Morley rank*?
(ii) *must T be* 1-*based*.

Given (i), one could prove (ii) by showing that inside suitable strongly minimal sets, algebraic closure equals definable closure, so we have "unimodularity" so one-basedness. Probably (i) is easy for saturated free algebras in a variety of R-modules.

One could also specialize to the context where V is a variety of groups (in the language of groups including an inverse function).

Question 3.15. *Suppose V is a variety of groups and G is free in V as well as being saturated. Is G commutative and of finite Morley rank* (*in which one can explicitly list the possibilities*).

Remark 3.16. In the context where T is the theory of a saturated free group G in a variety of groups, we have that G is connected and p_0 is the generic type over $\emptyset$. In order to see this one can use Poizat's argument and show that a definable set is generic if and only if it contains all but finitely many elements of a fixed basis (see [11], [12]).

§4. Examples.

We typically work with one-sorted structures, where the relevant indiscernible set witnessing almost indiscernibility is a set of n-tuples for some n (rather than working in a many sorted theory T^{eq}).

Example 4.1. Any almost strongly minimal theory is almost indiscernible (after adding additional parameters to witness the almost strong minimality).

The next two examples give almost indiscernible theories of infinite rank (which we conjectured could not happen for the theory of a saturated free algebra).

Example 4.2. Consider the theory T of infinitely many disjoint infinite unary predicates $P_1, P_2, \ldots$ equipped with, for each n, a bijection f_n between P_1^n and P_n. The theory is complete. P_n has Morley rank n, whereby the Morley rank of the universe $x = x$ is ω. P_1 is an indiscernible set. Let $q(x)$ be the "type at infinity": $\{\neg P_n(x) : n = 1, 2, \ldots\}$. Then q is complete with U-rank 1 (and Morley rank ω), and its set of realizations in any model is also an indiscernible set. Let M be a κ-saturated model of T of cardinality κ. Let $(a_i : i < \kappa)$ be an enumeration of P_1 in M, and let $(b_i : i < \kappa)$ be an enumeration of the set of realizations of q in M. Then $((a_i b_i) : i < \kappa)$ is an indiscernible set in M whose definable closure is precisely M. This is the simplest example of an almost indiscernible theory with Morley rank of $x = x$ being infinite.

Example 4.3. This is a kind of group version of the above. Let T be the theory of $\mathbb{Q}$-vector spaces equipped with a new predicate P for an infinite $\mathbb{Q}$-linearly independent set. Then T is complete, and ω-stable. The predicate P is strongly minimal (and its solution set in any model is an indiscernible set). $nP = P + \cdots + P$ (n-times) has Morley rank n. The formula

$x = x$ has Morley rank ω again, but also U-rank ω: Let q be the type saying $\{x \notin nP : n = 1, 2, \dots\}$. Then q has U-rank and Morley rank ω. Let M be a κ-saturated model of T. Let $P(M) = (a_i : i < \kappa)$ and let $(b_i : i < \kappa)$ be a maximal independent set of realizations of q in M. Then M is in the definable closure of $((a_i b_i) : i : \kappa)$.

The same thing can be done with the theory of algebraically closed fields in place of $\mathbb{Q}$-vector spaces.

Example 4.4. This is actually a negative example. Let T be theory of $(\mathbb{Z}^{\omega}_{p^\infty}, +)$, where $\mathbb{Z}_{p^\infty}$ is the group of roots of unity of order a power of p. So T is a theory of abelian groups of Morley rank ω. A κ-saturated model M of T is of the form $(\mathbb{Z}_{p^\infty})^\kappa \oplus \mathbb{Q}^\kappa$. M is *not* in the algebraic closure of an indiscernible set of finite tuples of cardinality κ. But it *is* visibly in the definable closure of an indiscernible set of ω-tuples of cardinality κ.

It is easy to produce an almost indiscernible theory, witnessed by an infinite indiscernible set I such that $tp(a/\emptyset)$, for $a \in I$ is not stationary. For example the theory of an equivalence relation with two classes, both infinite. But we would like an example of an almost indiscernible $\aleph_1$-categorical theory, such that there is no Morley sequence (so infinite, indiscernible, and independent) witnessing almost indiscernibility.

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