A Non-abelian, Non-Sidon, Completely Bounded Λ(p) Set

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Abstract. The purpose of this note is to construct an example of a discrete non-abelian group G and a subset E of G, not contained in any abelian subgroup, that is a completely bounded Λ(p) set for all $p < \infty$, but is neither a Leinert set nor a weak Sidon set.

1 Introduction

The study of lacunary sets, such as Sidon sets and Λ(p) sets, constitutes an interesting theme in the theory of Fourier series on the circle group $\mathbb{T}$. It has many applications in harmonic analysis and in the theory of Banach spaces, and various combinatorial and arithmetic properties of these sets have been studied extensively. These concepts have also been investigated in the context of more general compact abelian groups (with their discrete dual groups) and compact non-abelian groups; see [5], [9], [15] and the references cited therein. The study of these sets in the setting of discrete non-abelian groups was pioneered by Bożejko [1], Figà-Talamanca [4], and Picardello [10].

In abelian groups, there are various equivalent ways to define Sidon sets, and these sets are plentiful. Indeed, every infinite subset of a discrete abelian group contains an infinite Sidon set. The natural analogues of these definitions in discrete non-abelian groups are known as strong Sidon, Sidon, and weak Sidon sets. It was shown in [10] that every weak Sidon set is Λ(p) for all $p < \infty$. In [8], Leinert introduced the concept of a Λ(∞) set, a notion only of interest in the non-abelian setting because in abelian groups such sets are necessarily finite. In striking contrast to the abelian situation, Leinert showed that the free group with two generators contains an infinite subset that is both weak Sidon and Λ(∞), but does not contain any infinite Sidon subsets.

In [6], Harcharras studied the concept of completely bounded Λ(p) sets, a property more restrictive than Λ(p), but still possessed by Sidon sets. The converse is not true as every infinite discrete abelian group admits a completely bounded Λ(p) set that is not Sidon; see [7].

In this paper, we construct a non-amenable group G and a set E not contained in any abelian subgroup of G, which is completely bounded Λ(p) for every $p < \infty$, but is neither Λ(∞) nor weak Sidon. It remains open if every infinite discrete group contains such a set E.

2 Definitions

Throughout this paper, G will be an infinite discrete group. To define Sidon and $\Lambda(p)$ sets in this setting one requires the concepts of the Fourier algebra, $A(G)$, the von Neumann algebra, $VN(G)$, and the Fourier–Stieljies algebra, $B(G)$, as developed by P. Eymard in [3] for locally compact groups. We also need the concept of a non-commutative L^p-space introduced by I. E. Segal. We refer the reader to [12] for details on these latter spaces.

Definition 2.1   (i) The set $E \subseteq G$ is said to be a *strong (weak) Sidon set* if for all $f \in c_0(E)$ (resp. $l_\infty(E)$) there exists $g \in A(G)$ (resp. $B(G)$) such that $f(x) = g(x)$ for all $x \in E$.

(ii)The set $E \subseteq G$ is said to be a *Sidon set* if there is a constant C such that for all functions f, compactly supported in E, we have $\|f\|_1 \le C\|f\|_{VN(G)}$. The least such constant C is known as the *Sidon constant* of E.

These definitions are well known to be equivalent in the commutative setting. For any discrete group, it is the case that strong Sidon sets are Sidon, and Sidon sets are weak Sidon. Finite sets are always strong Sidon sets. In [10], it was shown that $E \subseteq G$ is Sidon if and only if for every $f \in l_\infty(E)$, there is some $g \in B_\rho(G)$ that extends f, where $B_\rho(G)$ is the dual of the reduced C^* algebra $C_\rho^*(G)$. Since in an amenable group $B_\rho(G) = B(G)$, weak Sidon sets are Sidon in this setting. Very recently, Wang [16] showed that every Sidon set in any discrete group is a strong Sidon set. It remains open if every infinite amenable group contains an infinite Sidon subset.

Picardello [10] defined the notion of $\Lambda(p)$ sets in this setting, and Harcharras [6] introduced completely bounded $\Lambda(p)$ sets. For these, we require further notation. Let λ denote the left regular representation of G into $\mathcal{B}(l_2(G))$ and denote by $L^p(\tau_0)$ the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G)$ with respect to the usual trace τ_0. Let $L^p(\tau)$ denote the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G) \otimes \mathcal{B}(l_2)$ with respect to the trace $\tau = \tau_0 \otimes tr$, where tr denotes the usual trace in $\mathcal{B}(l_2)$. Observe that $L^p(\tau)$ has a canonical operator space structure obtained from complex interpolation in the operator space category. We refer the reader to [11] for more details.

Definition 2.2   (i) Let $2 < p < \infty$. The set $E \subseteq G$ is said to be a $\Lambda(p)$ *set* if there exists a constant $C_1 > 0$ such that for all finitely supported functions f, we have

$$(2.1) \qquad \Big\| \sum_{t \in E} f(t) \lambda(t) \Big\|_{L^p(\tau_0)} \le C_1 \Big(\sum_{t \in E} |f(t)|^2 \Big)^{\frac{1}{2}}.$$

(ii) The set $E \subseteq G$ is said to be a *completely bounded* $\Lambda(p)$ *set*, denoted $\Lambda^{cb}(p)$, if there exists a constant $C_2 > 0$ such that

$$(2.2) \qquad \Big\| \sum_{t \in E} \lambda(t) \otimes x_t \Big\|_{L^p(\tau)} \le C_2 \max \Big(\Big\| \Big(\sum_{t \in E} x_t^* x_t \Big)^{1/2} \Big\|_{S_p}, \Big\| \Big(\sum_{t \in E} x_t x_t^* \Big)^{1/2} \Big\|_{S_p} \Big),$$

where x_t are finitely supported families of operators in S_p, the p-Schatten class on l_2.

The least such constants C_1 (resp. C_2) are known as the $\Lambda(p)$ (resp. $\Lambda^{cb}(p)$) *constants* of E.

It is known that every infinite set contains an infinite $\Lambda(p)$ set [1] and that every weak Sidon set is a $\Lambda(p)$ set for each $p < \infty$[10]. Completely bounded $\Lambda(p)$ sets are clearly $\Lambda(p)$, but the converse is not true, as seen in [6].

Extending these notions to $p = \infty$ gives the Leinert and *L*-sets.

Definition 2.3   (i) The set $E \subseteq G$ is called a *Leinert or* $\Lambda(\infty)$ *set* if there exists a constant $C > 0$ such that for every function $f \in l_2(E)$, we have $\|f\|_{VN(G)} \le C\|f\|_2$.

(ii) The sets of interpolation for the completely bounded multipliers of $A(G)$ are called *L-sets* (*i.e.,* $l_\infty(E) \subseteq M_{cb}(A(G))$).

It is well known that the Leinert sets are the sets of interpolation for multipliers of $A(G)$, so any *L*-set is Leinert; see [14]. The set E is said to satisfy the *Leinert condition* if every tuple $(a_1, \ldots, a_{2s}) \in E^{2s}$, with $a_i \ne a_{i+1}$, satisfies the independence-like relation

$$(2.3) \qquad\qquad a_1 a_2^{-1} a_3 \ldots a_{2s-1} a_{2s}^{-1} \ne e.$$

Here e is the identity of G. It can be shown [14] that any set that satisfies the Leinert condition is an *L*-set.

It was seen in [7] that in abelian groups, there are sets that are completely bounded $\Lambda(p)$ for all $p < \infty$, but not Sidon. Thus, the inclusion, weak Sidon is $\Lambda^{cb}(p)$, is strict for groups with infinite abelian subgroups. The purpose of this paper is to show the existence of sets not contained in *any* abelian subgroup that also have this strict inclusion. In fact, we prove, more generally, the following result.

Theorem 2.4   *There is a discrete group G that admits both infinite L-sets and weak Sidon sets, and an infinite subset E of G that is $\Lambda^{cb}(p)$ for all $p < \infty$, but not a Leinert set, an L-set or a weak Sidon set. Moreover, any subset of E consisting of commuting elements is finite.*

3   Results and Proofs

3.1   Preliminary Results

To show that the set we will construct is not a Leinert or weak Sidon set, it is helpful to first establish some arithmetic properties of $\Lambda(p)$ and Leinert sets. We recall that a set $E \subseteq G$ is said to be *quasi-independent* if all the sums

$$\left\{ \sum_{x \in A} x : A \subset E, |A| < \infty \right\}$$

are distinct. Quasi-independent sets in abelian groups are the prototypical Sidon sets.

The first part of the following lemma is well known for abelian groups and even in our non-commutative setting is essentially a commutative argument, since the von Neumann algebra of a subgroup is a subalgebra of the von Neumann algebra of the

group. With this observation, part (i) is essentially [15, 3.5], but we have included the proof here for completeness.

Lemma 3.1 *Let G be a discrete group.*
(i) Suppose $q > 2$ and $E \subseteq G$ is a $\Lambda(q)$ set with $\Lambda(q)$ constant A. If $a \in G$ has order $p_n \geq 2n$, then

$$\left| E \cap \{a, a^2, \ldots, a^n\} \right| \leq 10 A^2 n^{2/q}.$$

(ii) Suppose $E \subseteq G$ is a Leinert set with Leinert constant B and let $F \subseteq E$ be a finite commuting, quasi-independent subset. Then $|F| \leq 6^3 B^2$.

Proof   We will write 1_X for the characteristic function of a set X.
(i) Define the function K_n on G by

$$K_n(x) = \sum_{j=-2n}^{2n} \left(1 - \frac{|j|}{2n}\right) 1_{\{a^j\}}(x).$$

Let J_n denote the function on $\mathbb{Z}_{p_n}$ (or $\mathbb{Z}$ if $p_n = \infty$) defined in the analogous fashion. It is well known that the $A(G)$ and $VN(G)$ norms for the function K_n are dominated by the corresponding norms of the function J_n on $\mathbb{Z}_{p_n}$.

As $L^{q'}(\tau_0)$ (for q' the dual index to q) is an interpolation space between $A(G)$ and $VN(G)$, it follows that

$$\|K_n\|_{L^{q'}(\tau_0)} \leq \|K_n\|_{A(G)}^{1/q'} \|K_n\|_{VN(G)}^{1/q}$$
$$= \|J_n\|_{A(\mathbb{Z}_{p_n})}^{1/q'} \|J_n\|_{VN(\mathbb{Z}_{p_n})}^{1/q} \leq (4n+1)^{1/q}.$$

Suppose $E \cap \{a, a^2, \ldots, a^n\}$ consists of the M elements $\{a^{s_j}\}_{j=1}^M$ and put

$$k_n(x) = \sum_{j=1}^M 1_{\{a^{s_j}\}}(x).$$

Since E has $\Lambda(q)$ constant A, the generalized Hölder inequality implies

$$\frac{M}{2} \leq \sum_{j=1}^M K_n(a^{s_j}) = \sum_{x \in G} K_n(x) k_n(x)$$
$$\leq \|K_n\|_{L^{q'}(\tau_0)} \|k_n\|_{L^q(\tau_0)} \leq (4n+1)^{1/q} A \|k_n\|_2$$
$$= (4n+1)^{1/q} A \sqrt{M}.$$

Consequently, $M \leq 2(4n+1)^{2/q} A^2 \leq 10 A^2 n^{2/q}$, as claimed.

(ii) Let H be the abelian group generated by F. Being quasi-independent, F is a Sidon subset of H with Sidon constant at most $6\sqrt{6}$ [5, p. 115]. Consider the function $h = 1_F$ defined on H and $g = 1_F$ defined on G. The Sidon property, together with the fact that $\|h\|_{VN(H)} = \|g\|_{VN(G)}$, ensures that

$$|F| = \|h\|_{\ell^1} \leq 6\sqrt{6} \|h\|_{VN(H)} = 6\sqrt{6} \|g\|_{VN(H)}.$$

Since E has Leinert constant B, we have $\|f\|_{VN(G)} \leq B\|f\|_2$ for any function f defined on G and supported on E. In particular, this is true for the function g, hence

$$|F| \leq 6\sqrt{6}\|g\|_{VN(H)} \leq 6\sqrt{6}B\sqrt{|F|}. \qquad \blacksquare$$

3.2 Proof of Theorem 2.1

Proof   We will let G be the free product of the cyclic groups Z_{p_n}, $n \in N$, where $p_n > 2^{n+1}$ are distinct odd primes. If a_n is a generator of Z_{p_n}, then $\{a_n\}_{n=1}^{\infty}$ is both a weak Sidon and Leinert set, as shown in [10]. The set E will be the union of finite sets $E_n \subseteq Z_{p_n}$, where $|E_n| = n^2$ and $E_n \subset \{a_n, \ldots, a_n^{2^n}\}$. The fact that any commuting subset of E is finite is obvious from the definition of E.

We recall the following notation from [6]. We say that a subset $\Lambda \subseteq G$ has the $Z(p)$ property if $Z_p(\Lambda) < \infty$ where

$$Z_p(\Lambda) = \sup_{x \in G} \left| \left\{ (x_1, \ldots, x_p) \in \Lambda^p : x_i \neq x_j, x_1^{-1}x_2 x_3^{-1} \ldots x_p^{(-1)^p} = x \right\} \right|.$$

In [6], Harcharras proved that if $2 < p < \infty$, then every subset Λ of G with the $Z(p)$ property is a $\Lambda^{cb}(2p)$.

We will construct the sets E_n so that they have the property that for every even $s \geq 2$, there is an integer n_s such that $Z_s(\bigcup_{n \geq n_s} E_n) \leq s!$. Consequently, $\bigcup_{n \geq n_s} E_n$ will be $\Lambda^{cb}(2s)$ for all $s < \infty$. As finite sets are $\Lambda^{cb}(p)$ for all $p < \infty$, and a finite union of $\Lambda^{cb}(p)$ sets is again $\Lambda^{cb}(p)$, it will follow that E is $\Lambda^{cb}(p)$ for all $p < \infty$.

We now proceed to construct the sets E_n by an iterative argument. Temporarily fix n and take $g_1 = a_n$. Inductively assume that for $N < n^2$, $\{g_i\}_{i=1}^N \subseteq \{a_n, \ldots, a_n^{2^n}\}$ have been chosen with the property that if

$$(\mathcal{P}_N) \qquad \prod_{j=1}^N g_j^{\varepsilon_j} = 1 \text{ for } \varepsilon_j = 0, \pm 1, \pm 2, \sum_j |\varepsilon_j| \leq 2s, \text{ then all } \varepsilon_j = 0.$$

Now choose

$$g_{N+1} \neq \prod_{j=1}^N g_j^{\varepsilon_j} \text{ for any } \varepsilon_j = 0, \pm 1, \pm 2 \text{ and } \sum_j |\varepsilon_j| \leq 2s$$

and

$$g_{N+1}^2 \neq \prod_{j=1}^N g_j^{\varepsilon_j} \text{ for any } \varepsilon_j = 0, \pm 1, \pm 2 \text{ and } \sum_j |\varepsilon_j| \leq 2s.$$

There are at most $\binom{N}{2s}5^{2s} \leq C_s N^{2s}$ terms that g_{N+1} must avoid and similarly for g_{N+1}^2. Provided $2C_s N^{2s} \leq 2^n$, we can make such a choice of $g_{N+1} \in \{a_n, \ldots, a_n^{2^n}\}$. Of course, it is immediate that property ($\mathcal{P}_{N+1}$) then holds. This can be done for every $N < n^2$ as long as n is suitably large, say for $n \geq n_s$. The set E_n will be taken to be $\{g_j\}_{j=1}^{n^2}$.

Now we need to check the claim that $Z_s\left(\bigcup_{n \geq n_s} E_n\right) \leq s!$. Towards this, suppose

$$(3.1) \qquad\qquad x_1 x_2^{-1} \ldots x_s^{-1} = y_1 y_2^{-1} \ldots y_s^{-1}$$

where x_i are all distinct, y_j are all distinct, and all $x_i, y_j \in \bigcup_{n \geq n_s} E_n$. The free product property guarantees that if this is true, then it must necessarily be the case that if we consider only the elements x_{i_k} and y_{j_l} that belong to a given E_n, we must have $\prod_k x_{i_k}^{\delta_k} = \prod_l y_{j_l}^{\varepsilon_l}$ for the appropriate choices of $\delta_k, \varepsilon_l \in \{\pm 1\}$. As there are at most s choices for each of x_{i_k} and y_{i_l}, our property ($\mathcal{P}_N$) ensures that this can happen only if $\{x_{i_k} : \delta_k = 1\} = \{y_{j_l} : \varepsilon_l = 1\}$ and similarly for the terms with -1 exponents. Hence, we can only satisfy (3.1) if upon reordering, $\{x_1, x_3, \ldots, x_{s-1}\} = \{y_1, y_3, \ldots, y_{s-1}\}$, and similarly for the terms with even labels. (We remark that for non-abelian groups, this is only a necessary but not, in general, sufficient condition for (3.1).) This suffices to establish that

$$Z_s\left(\bigcup_{n \geq n_s} E_n \right) \leq \left((s/2)! \right)^2 \leq s!,$$

and hence, as explained above, E is a $\Lambda^{cb}(p)$ set for all $p < \infty$.

Next, we will verify that E is not a weak Sidon set. We proceed by contradiction. According to [10], if it was, then E would be a $\Lambda(p)$ set for each $p > 2$, with $\Lambda(p)$ constant bounded by $C\sqrt{p}$ for a constant C independent of p. Appealing to Lemma 3.1(i), we have

$$n^2 = |E_n| = \left| E \cap \{a_n, \ldots, a_n^{2^n}\} \right| \leq 10 C^2 p 2^{2n/p}.$$

Taking $p = 2n$ for sufficiently large n gives a contradiction.

Finally, to see that E is not a Leinert set, we first observe that an easy combinatorial argument shows that any set of N distinct elements contains a quasi-independent subset of cardinality at least $\log N / \log 3$. Thus, we can obtain quasi-independent subsets $F_n \subseteq E_n$ with $|F_n| \to \infty$. But according to Lemma 3.1(ii), this would be impossible if E was a Leinert set. As E is not Leinert, it is also not an L set.

This concludes the proof.   ∎

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