Piecing it together

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1. *Introduction*

'All the King's horses and all the King's men couldn't put Humpty together again'. Such is how it goes in the nursery rhyme books about Humpty Dumpty and his great fall from the top of a wall. Oddly enough, that is how it seems also to be going now in respect of efforts to piece together the dynamics of Felix Baumgartner's great fall (flight) from the capsule of a very high altitude balloon at nearly 40 km MSL (above Mean Sea Level). During his flight he reached a maximum velocity fairly early on, after which he slowed down until deployment of a drogue that arrested his free fall. Graham Hoare raised issues in the Letters column of *Mathematics Today* [1] about the mathematical modelling of the fall and this stimulated discussions that resulted in various efforts seeking to reconcile the dynamics of the fall to facts that were known about it. Such facts have been refined over time and the purpose of this note is to present a very simple mathematical model incorporating the latest data that an A level student familiar with Newton's laws of motion can well understand. Briefly, the problem is treated as one of gravitational motion through a series of adjacent, horizontally stratified layers wherein resistance to motion at any point within a layer is presumed to vary with the square of the velocity at that point. To this end, a multiplicative motion resistance factor is introduced as a staircase type function, one which is presumed to be constant within each layer but varying from one layer to the next. Following a description based on Newton's equations of motion, outputs from one layer provide inputs to the next, and so on. In this respect the approach to the problem is much the same as one commonly adopted in other sciences, for example in Electromagnetic Theory, where plane wave propagation through a layered media, such as a Radome used to protect an antenna, is treated in a similar fashion, again to very good effect.

2. *The equations of motion*

These are well known and the relevant ones for a typical layer appear below. In such a layer, the nth one say, it will be assumed that the distance fallen to any point is denoted by x, $(x_{n-1} \leqslant x \leqslant x_n)$. The forces acting on the body are downwards (gravitational) and $mk_n v^2$ upwards (motion resistive), where k_n is the constant motion resistance factor for the layer, $v(x)$ denotes the velocity, m denotes the point mass of the falling body and g (= 9.78 ms^{-2}) is the acceleration due to gravity. In the usual notation, these equations are

$$\frac{dv}{dt} = k_n\left(u_n^2 - v^2\right) \tag{1}$$

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and

$$v\frac{dv}{dx} = k_n\left(u_n^2 - v^2\right) \tag{2}$$

where $u_n = \sqrt{g/k_n}$.

Solutions to these equations involve standard integration techniques. It is assumed that, for example, the falling body reaches the layer entry point x_{n-1} at a velocity v_{n-1} after a time t_{n-1}, and exits the layer with a velocity v_n at the point x_n after a time t_n. Accordingly, from (1) it follows that the time spent in the nth layer, $T_n \left(= t_n - t_{n-1}\right)$ say, is given by

$$T_n = \frac{u_n}{2g}\ln\left(\frac{(u_n + v_n)(u_n - v_{n-1})}{(u_n - v_n)(u_n + v_{n-1})}\right). \tag{3}$$

Similarly, denoting the thickness of the nth layer by D_n, it may be deduced from (2) that

$$v_n^2 = u_n^2\left(1 - \exp\left(-2k_nD_n\right)\right) + v_{n-1}^2\exp\left(-2k_nD_n\right). \tag{4}$$

Provided starting information about the fall is available and provided also that the motion resistance term k_n (and hence the term u_n) is known it is possible to track the velocity and associated times at the entry and departure points through each layer. For example, if the start velocity v_0 and the motion resistance term k_1 in the first layer are known together with the thickness D_1, then (4) will determine the exit velocity v_1, and (3) will determine subsequently the associated time T_1 taken to traverse the layer. Equation (4) can then be used again with an input velocity v_1 (the previous exit velocity) to determine an output velocity v_2 for the second layer, after having taken into account the other input parameters (thickness and motion resistance) associated with it. Then, as before, the time parameter T_2 for the second layer can be determined from (3). Thus it is possible to put together the bits and pieces of the motion. However, for this process to be successful it is necessary to determine or at least have available across the spectrum a description of the discrete motion resistance factor k_n. This issue is addressed below.

3. *The motion resistance factor*

Generally, there is a consensus of opinion that resistance to motion in the ethers surrounding Felix Baumgartner's great fall increases more or less exponentially with distance fallen. Thus, it seems reasonable to assume an exponential format for the motion resistance factor. This notion was exploited in [2], where it was shown that with this type of model it is possible to achieve a prescribed maximum velocity at a given height such as was the case in Baumgartner's descent. Specifically, a continuous motion resistance factor $k(x)$ was assumed in the form

$$k(x) = \exp\left(x\alpha - \beta\right) \tag{5}$$

where x denoted the distance fallen, and α and β were parameters chosen to ensure that a specified maximum velocity could be met at a given distance

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of fall. This particular model can thus be sampled to determine a representative or averaged value for the motion resistance factor k_n in the nth layer, as follows:

$$k_n = \frac{k(x_{n-1}) + k(x_n)}{2}. \qquad (6)$$

The k values on the right-hand side of (6) are determined using (5). Having shown how a discrete spectrum of values might be obtained for the motion resistance factor, it remains to address a case study of the fall to appreciate just how well the bits and pieces of it might be put together.

4. *Case study, discussion and conclusion*

The latest known facts about Baumgartner's great fall [3] indicate that his drop off point was at an altitude of 38.9694 km (MSL), and that he reached a maximum velocity of 377.1 ms^{-1} after having fallen a distance of 11.1364 km in about 50 seconds. Drogue deployment occurred at 2.5668 km (MSL) after a total free fall time of about 260 seconds. This implied a free fall distance of 36.4026 km. The analysis in [2] showed that the maximum velocity information (speed and associated distance fallen) could be met here with values for the parameters α and β given by

$$\alpha = 1.703224 \times 10^{-4} \qquad \text{and} \qquad \beta = 11.4814594.$$

Calculations based on the above input values and procedures are accommodated easily on a spreadsheet. The free fall atmosphere was considered to comprise 400 equal-depth layers, so that $x_0 = 0$ and $x_{400} = 36402.6$ m with a layer depth of $D_n = 91.0065$ m. A selection of data rows from the ensuing spreadsheet is shown in the table below. It is largely self explanatory and, not surprisingly, the data confirms that the magnitude of the maximum velocity occurred pretty much at the alleged

									Time T	Total
n	X_n (m)	Alt. (m)	K	K_n	K_nD_n	U_n	V_{n-1}	V_n	T_n (s)	T (s)
0	0	38969.40	1.032E-05					0.00	0.00	0.00
1	91.0065	38878.39	1.048E-05	1.040E-05	9.465E-04	969.72	0.00	42.17	4.31	4.31
2	182.013	38787.39	1.064E-05	1.056E-05	9.613E-04	962.24	42.17	59.61	1.79	6.10
3	273.02	38696.38	1.081E-05	1.073E-05	9.763E-04	954.81	59.61	72.97	1.37	7.48
…										
121	11011.8	27957.61	6.733E-05	6.681E-05	6.080E-03	382.60	377.00	377.07	0.24	49.62
122	11102.8	27866.61	6.838E-05	6.786E-05	6.175E-03	379.64	377.07	377.20	0.24	49.86
123	11193.8	27775.60	6.945E-05	6.892E-05	6.272E-03	376.71	377.20	377.09	0.24	50.10
124	11284.8	27684.39	7.053E-05	6.999E-05	6.370E-03	373.80	377.09	377.05	0.24	50.34
…										
397	36129.6	2839.82	4.885E-03	4.817E-03	4.384E-01	45.67	45.67	45.31	2.00	264.90
398	36220.6	2748.81	4.931E-03	4.893E-03	4.453E-01	44.71	45.31	44.96	2.02	266.92
399	36311.6	2657.81	5.008E-03	4.969E-03	4.522E-01	44.36	44.96	44.61	2.03	268.95
400	36402.6	2566.80	5.086E-03	5.074E-03	5.057E-01	44.02	44.61	44.26	2.05	271.00

TABLE: Data selection from the spreadsheet

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altitude after about 50 seconds. The results show also that drogue deployment occurred at about 271 seconds, which is 11 seconds later than when it was observed. This represents a time error of about 4.2%, which is not unreasonable given the assumptions implicit in the argument. For example, Baumgartner's falling body has been treated simply as a point mass under the influences only of gravity and an air resistance that has been relatively easy to model. The following figure showing how the velocity varied as a function of time during the fall was produced from data in the complete table:

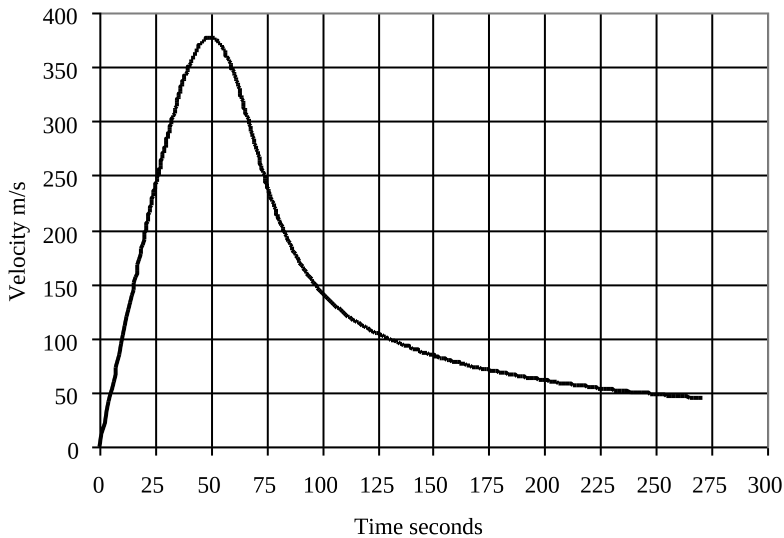

FIGURE 1: Descent velocity as a function of time

The result in the figure is similar to that which may be found on various web sites concerning the fall and it is of course also possible to produce a plot of the altitude variation with time from the spreadsheet data. This is left as an exercise for the reader. The same exercise was repeated using data gleaned from other sources [4], where Baumgartner's drop off point was assumed to be at an altitude of 39.045 km (MSL) and his maximum velocity was 372.6 ms^{-1} after having fallen a distance of 11.165 km in about 50 seconds; drogue deployment occurred at 2.516 km (MSL) after a total free fall time of, again, 260 seconds. This implied a free fall distance of 36.529 km. The analysis in [2] showed that the maximum velocity information now given could be met with the following values for the parameters α and β:

$$\alpha = 1.556362 \times 10^{-4}, \qquad \beta = 11.298350.$$

This description differs marginally from the one cited earlier, but the ramifications in respect of time are pertinent. For example, whilst the reader might care to employ this latest data in a spreadsheet to see that the maximum velocity conditions have been met, it will be seen also that the

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free fall time is now predicted to be about 248.9 seconds. Again there is a difference compared to the reality of 260 seconds, indicating that the predicted arrival time for drogue deployment is about 11 seconds early, whereas the previous results showed a late arrival by a similar amount. Thus it seems that this particular model is sensitive to variations in input data, at least as far as time calculations are concerned. Nonetheless, it is a simple model that goes some considerable way to accommodating the facts and it is particularly amenable to treatments using a spreadsheet, for example in a classroom setting, provided the required input parameters (α and β) obtained elsewhere are taken on trust. Not least, this latest great fall provides for the opportunity to analyse and appreciate a recent and historic event using some very simple sums in the field of elementary dynamics.

References

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