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SPARRE ANDERSEN IDENTITY AND THE LAST PASSAGE TIME

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Abstract

It is shown that the celebrated result of Sparre Andersen for random walks and Lévy processes has intriguing consequences when the last time of the process in $(-\infty, 0]$, say σ, is added to the picture. In the case of no positive jumps this leads to six random times, all of which have the same distribution—the uniform distribution on $[0, \sigma]$. Surprisingly, this result does not appear in the literature, even though it is based on some classical observations concerning exchangeable increments.

Keywords: Last passage; time reversal; exchangeable increments; uniform law

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Secondary 60G50

1. The main observation

The main observation of this short paper is best illustrated by a Lévy process $X_t, t \geq 0$ without positive jumps. A particular example of such X is given by a compound Poisson process with positive linear drift and negative jumps, which occupies a central place in applied probability: in risk theory it is known as a Cramér–Lundberg model [5], and in queueing theory $-X$ drives the workload process in the classical M/G/1 queue [4].

Let $\sigma = \sup\{t \geq 0 \colon X_t \leq 0\}$ be the last time of X in $(-\infty, 0]$, which is finite almost surely (a.s.) when $\mathbb{E}X_1 > 0$. Define the following random times, see Figure 1:

$$N^- = \int_0^\sigma \mathbf{1}_{\{X_s \leq 0\}} \mathrm{d}s, \qquad N^+ = \int_0^\sigma \mathbf{1}_{\{X_s \geq 0\}} \mathrm{d}s, \qquad \overrightarrow{F} = \sup\{t \in [0, \sigma) \colon X_t = \underline{X}_t\},$$
$$\overrightarrow{G} = \sup\{t \in [0, \sigma) \colon X_t = \overline{X}_t\}, \qquad \overleftarrow{F} = \sigma - \overrightarrow{F}, \qquad \overleftarrow{G} = \sigma - \overrightarrow{G},$$

where $\overline{X}_t = \sup\{X_s \colon s \in [0, t]\}$ and $\underline{X}_t = \inf\{X_s \colon s \in [0, t]\}$ are the running supremum and infimum processes, respectively. When $\sigma = 0$ we assume that all these times are 0. In words, N^- is the time spent in the nonpositive half-line, $\overrightarrow{F}$ is the time of the infimum, and $\overleftarrow{F}$ is the time from the infimum to σ.

Proposition 1. *Let X be a Lévy process without positive jumps, such that $\mathbb{E}X_1 > 0$. Then $\overrightarrow{F}$, $\overleftarrow{F}$, $\overrightarrow{G}$, $\overleftarrow{G}$, N^-, and N^+ have the same distribution.*

Note that we can replace σ by ∞ in the definitions of N^- and $\overrightarrow{F}$. The equivalence of laws of these two random variables is known as the Sparre Andersen identity, see, e.g. [6, Lemma VI.15]. This identity for random walks was first established by E. Sparre Andersen in [2] using a combinatorial approach; a simpler proof can be found in [9, Theorem XII.8.2].

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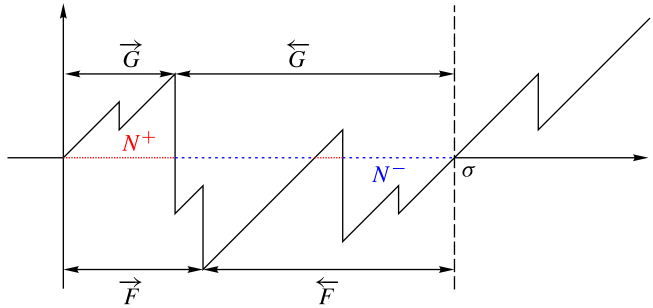

FIGURE 1: A sample path of X and the corresponding random times.

The transform of the single distribution in Proposition 1 is well known. Define the first passage time $\tau_x = \inf\{t \geq 0\colon X_t > x\}$ and let $\psi(s) = \log(\mathbb{E}e^{sX_1})$, $\Phi(s) = -\log(\mathbb{E}e^{-s\tau_1})$ for $s \geq 0$, which are known to satisfy $\psi(\Phi(s)) = s$. Then it follows from [6, Theorem VII.4(ii)] that

$$\mathbb{E}e^{-s\overrightarrow{F}} = \psi'(0)\frac{\Phi(s)}{s}, \qquad s > 0. \tag{1}$$

Alternatively, $\overleftarrow{F}$ is the last passage time of the post-infimum process (known as X conditioned to stay positive) over $I = -\underline{X}_\infty$. It is well known that the post-infimum process is independent of the infimum and by William's representation [6, Theorem VII.18] its last passage time over x has the law of τ_x. Hence, we can add the following identity to Proposition 1:

$$\overleftarrow{F} \stackrel{\text{D}}{=} \hat{\tau}_I, \tag{2}$$

where $\hat{\tau}$ is a copy of τ independent of X. In particular, this readily implies that the transform of $\overleftarrow{F}$ coincides with (1) by way of the generalized Pollaczek–Khinchine formula $\mathbb{E}e^{-sI} = \psi'(0)s/\psi(s)$.

Similarly to the classical identity, Proposition 1 can be reformulated for a Lévy process on a finite interval $[0, T]$, see Proposition 2 below. Yet another possibility is to consider a general Lévy process and to condition on the event $\{X_\sigma = 0\}$, assuming it has positive probability. In Corollary 1 we present this type of result for random walks. Note that if local extrema are not necessarily distinct then $\overleftarrow{F}$ and $\overleftarrow{G}$ must be defined in a slightly different way; see Section 4.

Proposition 2. *Let X be a Lévy process without positive jumps, such that $\mathbb{P}(X_T > 0) > 0$, and let $\sigma = \sup\{t \in [0, T]\colon X_t \leq 0\}$. On the event $\{X_T > 0\}$ the random times $\overrightarrow{F}$, $\overleftarrow{F}$, $\overrightarrow{G}$, $\overleftarrow{G}$, N^-, and N^+ have the same distribution.*

In general, when jumps of both signs are allowed, the above equivalence of laws does not hold. Instead, we can partition these times into two classes of three elements in each according to their laws. We state this general result for random walks and provide its short proof in Section 3. Its standard extension to Lévy processes is discussed briefly in Section 4.

2. Intuitive explanation and further consequences

There is a simple explanation of the above results: the Sparre Andersen identity holds for the random time interval $[0, \sigma]$ (applied to $-X$), and the process seen from σ (backwards in time and downwards in space) has the same law as the original process up to σ. The fundamental

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reason behind these observations is that the increments of the approximating random walk are exchangeable random variables conditioned on $\{\sigma = n\}$, see Section 3 and Section 4 for details.

By considering the approximating random walk we observe some interesting further consequences. Firstly, we note that the above six random times have the same distribution conditional on σ, and so the pairs

$$(\overrightarrow{G}, \overleftarrow{G}),\ (\overrightarrow{F}, \overleftarrow{F}),\ (N^+, N^-)$$

have the same distribution with exchangeable components. The corresponding transform (under the assumptions of Proposition 1) can be obtained in a similar way as above:

$$\mathbb{E}\mathrm{e}^{-s\overrightarrow{F}-t\overleftarrow{F}} = \mathbb{E}\mathrm{e}^{-s\overrightarrow{F}-t\hat{\tau}_I} = \mathbb{E}\mathrm{e}^{-s\overrightarrow{F}+\Phi(t)\underline{X}_\infty} = \psi'(0)\frac{\Phi(s)-\Phi(t)}{s-t}$$

using (2) and the explicit form of the Wiener–Hopf factor corresponding to the infimum; see, e.g. [6, Theorem VII.4(ii)]. Taking $t \uparrow s$, we obtain $\mathbb{E}\mathrm{e}^{-s\sigma} = \psi'(0)\Phi'(s)$ confirming the result of [8].

Finally, another result by Sparre Andersen [3], see also [9, Theorem XII.8.3], states that the time of the maximum of a random walk 'conditioned' to hit 0 at its terminal time, cf. Brownian bridge, has a uniform distribution. This is a simple consequence of cyclical rearrangements of increments. In our setting the cyclical rearrangement argument shows that our six random times have a uniform distribution on $[0, \sigma]$, i.e.

$$\mathbb{P}(\overrightarrow{F} \in \mathrm{d}x \mid \sigma = t) = \frac{1}{t}\mathbf{1}_{\{x\in[0,t]\}}\mathrm{d}x, \qquad t > 0.$$

This result complements well-known uniform laws for Lévy bridges [7], [10] stemming from the same result of Sparre Andersen; see also [1] and [11] for an extension of the cyclical rearrangement idea. In general, however, we have to assume that X is a Lévy process with distinct extrema conditioned on $\{X_\sigma = 0\}$.

3. Random walk

Consider a random walk $S_i = \sum_{j=1}^{i} \zeta_j$ for $i = 0, \dots, n$, where $\zeta_1, \dots, \zeta_n$ be independent and identically distributed random variables. Let us condition this random walk on the (positive probability) event $\{S_n \in B\}$ for some Borel set B; later we will take $B = \mathbb{R}$ and $B = [0, \infty)$. Let $\sigma = \max\{i \le n \colon S_i \le 0\}$ be the last time of S_i in the nonpositive half-line. Also let $\underline{S} = \min\{S_i : i \le \sigma\}$, $\overline{S} = \max\{S_i : i \le \sigma\}$, and define the following eight quantities:

$$N^- = \sum_{i=1}^{\sigma} \mathbf{1}_{\{S_i \le 0\}}, \qquad N^+ = \sum_{i=1}^{\sigma} \mathbf{1}_{\{S_i \ge 0\}},$$

$$\widetilde{N}^- = \sum_{i=0}^{\sigma-1} \mathbf{1}_{\{S_i \le S_\sigma\}}, \qquad \widetilde{N}^+ = \sum_{i=0}^{\sigma-1} \mathbf{1}_{\{S_i \ge S_\sigma\}},$$

$$\overrightarrow{F} = \max\{i \le \sigma : S_i = \underline{S}\}, \qquad \overrightarrow{G} = \max\{i \le \sigma : S_i = \overline{S}\},$$

$$\overleftarrow{F} = \sigma - \min\{i \le \sigma : S_i = \underline{S}\}, \qquad \overleftarrow{G} = \sigma - \min\{i \le \sigma : S_i = \overline{S}\};$$

see Figure 2. Moreover, we define a process $\hat{S}_i$, $i = 0, \dots, \sigma$ by $\hat{S}_i = S_\sigma - S_{\sigma-i}$, which is just $-S$ time reversed at σ.

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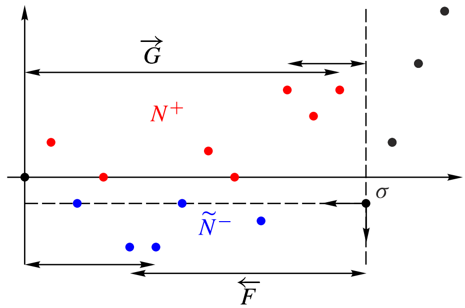

FIGURE 2: A realization of a random walk S and the corresponding random times.

Proposition 3. *For a random walk S_i, $i = 0, \dots, n$ conditioned on $\{S_n \in B\}$, it holds that*

- *$\hat{S}$ has the law of S considered up to σ;*
- *N^-, $\widetilde{N}^+$, $\overrightarrow{F}$, and $\overleftarrow{G}$ have the same distribution;*
- *N^+, $\widetilde{N}^-$, $\overleftarrow{F}$, and $\overrightarrow{G}$ have the same distribution.*

Proof. For fixed $k = 0, \dots, n$ consider an event $\{\sigma = k\} = \{S_k \le 0,\ S_i > 0 \text{ for all } k < i \le n\}$ (assuming it has a positive probability). Note that on the event $\{\sigma = k,\ S_n \in B\}$ the sequences $\zeta_1, \dots, \zeta_k$ and $\zeta_k, \dots, \zeta_1$ have the same law, and so S_i, $i = 0, \dots, k$ and $\hat{S}_i$, $i = 0, \dots, k$ have the same laws. Now the first statement follows by conditioning on σ.

From the law equivalence of S and $\hat{S}$, we obtain

$$N^- \stackrel{\mathrm{D}}{=} \widetilde{N}^+, \qquad N^+ \stackrel{\mathrm{D}}{=} \widetilde{N}^-, \qquad \overrightarrow{F} \stackrel{\mathrm{D}}{=} \overleftarrow{G}, \qquad \overleftarrow{F} \stackrel{\mathrm{D}}{=} \overrightarrow{G},$$

which is easily understood from Figure 2. More precisely, observe that $S_i \ge S_\sigma$ is the same as $0 \ge S_\sigma - S_i = \hat{S}_{\sigma - i}$ and so $\widetilde{N}^+ = \sum_{i=0}^{\sigma-1} \mathbf{1}_{\{\hat{S}_{\sigma-i} \le 0\}} = \sum_{i=1}^{\sigma} \mathbf{1}_{\{\hat{S}_i \le 0\}}$. This proves the first equality, and the second follows similarly. Also

$$\begin{aligned}\overleftarrow{G} &= \max\{\sigma - i \in [0, \sigma] \colon S_i = \overline{S}\} \\ &= \max\{j \in [0, \sigma] \colon S_\sigma - S_{\sigma - j} = S_\sigma - \overline{S}\} \\ &= \max\{j \in [0, \sigma] \colon \hat{S}_j = \min\{\hat{S}_i : i \le \sigma\}\}.\end{aligned}$$

This proves the third statement and the fourth follows similarly.

Next, note that $\zeta_1, \dots, \zeta_n$ conditioned on the event $\{S_n \in B\}$ are exchangeable random variables. Thus, it follows from the Sparre Andresen identity, see [9, Theorem XII.8.2], that N^- and $\overrightarrow{F}$ have the same distribution (note that in their definitions σ can be replaced by n).

Moreover, $\zeta_1, \dots, \zeta_k$ conditioned on the event $\{\sigma = k,\ S_n \in B\}$ are exchangeable random variables. Thus, on this event N^+ and $\overrightarrow{G}$ have the same distribution, which by conditioning also holds on the event $\{S_n \in B\}$. □

Corollary 1. *Assume that $\mathbb{P}(S_\sigma = 0) > 0$, then on the event $\{S_\sigma = 0\}$ it holds that N^-, N^+, $\overrightarrow{F}$, $\overleftarrow{F}$, $\overrightarrow{G}$, and $\overleftarrow{G}$ have the same distribution.*

Proof. The above proof requires only a small modification: we need to condition on $\{\sigma = k\}$ in the proof of $N^- \stackrel{\mathrm{D}}{=} \overrightarrow{F}$. Finally, it follows that $N^- = \widetilde{N}^-$ and $N^+ = \widetilde{N}^+$, showing that there is a single distribution. □

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4. Lévy process

Extension of Proposition 3 to the case of a Lévy process is standard, and hence only a sketch of it is presented in this paper. Consider a general Lévy process X, and without loss of generality assume that $T = 1$. Define $\sigma = \sup\{t \in [0, T]\colon X_t \leq 0\}$ and the corresponding time reversed process by

$$\hat{X}_t = X_{\sigma-} - X_{(\sigma-t)-}, \qquad t \in [0, \sigma),$$

where X_{t-} denotes the left limit of X at t. Define the random times as in Section 1 and in addition put

$$\widetilde{N}^- = \int_0^\sigma \mathbf{1}_{\{X_t \leq X_{\sigma-}\}} \mathrm{d}t, \qquad \widetilde{N}^+ = \int_0^\sigma \mathbf{1}_{\{X_t \geq X_{\sigma-}\}} \mathrm{d}t.$$

Consider a sequence of random walks $S^{(n)}$, defined by $S_i^{(n)} = X_{i/n}$, $i = 0, \dots, n$, and the corresponding sequence of continuous approximations $X^{(n)}$ of X, where points $(i/n, X_{i/n})$, $i = 0, \dots, n$ are connected by line segments (the appropriate topology is M_1; see [13, Chapter 3.3]). This setup and the law equivalence of $\hat{S}$ and S readily show that $\hat{X}$ has the same law as X_t, $t \in [0, \sigma)$. Alternatively, this result can be obtained using Nagasawa's time reversal theory for Markov processes [12].

Finally, we need to show that $n^{-1}N^{(n)-}$ (corresponding to $S^{(n)}$) converges to N^- (corresponding to X) a.s., and the same for the other quantities. The case of a compound Poisson process is rather obvious, but requires us to use another definition of $\overleftarrow{F}$ and $\overleftarrow{G}$:

$$\overleftarrow{F} = \sigma - \min\{t \in [0, \sigma)\colon X_t = \underline{X}_T\}, \qquad \overleftarrow{G} = \sigma - \min\{t \in [0, \sigma)\colon X_t = \overline{X}_{\sigma-}\}.$$

Now suppose that X is not a compound Poisson process. Then $\int_0^\sigma \mathbf{1}_{\{X_t=0\}}\mathrm{d}t = 0$ a.s, see [6, Proposition I.15], and then also $\int_0^\sigma \mathbf{1}_{\{X_t=X_{\sigma-}\}}\mathrm{d}t = 0$ a.s., because of the law equivalence of X and $\hat{X}$. In addition, local extrema of X are all distinct; see [6, Proposition VI.4]. Now the convergence of the scaled times for random walks to their Lévy counterparts is clear; see also the proof of [6, Lemma VI.15] where an extension of the Sparre Andersen identity to the Lévy process case was presented.

In conclusion, N^-, $\widetilde{N}^+$, $\overrightarrow{F}$, and $\overleftarrow{G}$ have the same distribution, and the same is true for N^+, $\widetilde{N}^-$, $\overleftarrow{F}$, and $\overrightarrow{G}$. Moreover, Proposition 1 follows from Proposition 2, and the latter follows immediately from the general result, by noting that $X_{\sigma-} = 0$ and, thus, $N^- = \widetilde{N}^-$, $N^+ = \widetilde{N}^+$.

Acknowledgements

This short paper started from an observation that $N^- \overset{\mathrm{D}}{=} \tau_{\hat{I}}$ for a spectrally-negative Lévy process, which together with (2) asked for further investigations. I would like to thank Hansjörg Albrecher for discussions which led to this observation, and Victor Rivero for drawing my attention to [1] and [11]. Support from the Swiss National Science Foundation (project number 200020_143889) is gratefully acknowledged.

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