LARGE-DEVIATION RESULTS FOR TRIANGULAR ARRAYS OF SEMIEXPONENTIAL RANDOM VARIABLES

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Abstract

Asymptotics deviation probabilities of the sum $S_n = X_1 + \cdots + X_n$ of independent and identically distributed real-valued random variables have been extensively investigated, in particular when X_1 is not exponentially integrable. For instance, Nagaev (1969a, 1969b) formulated exact asymptotics results for $\mathbb{P}(S_n > x_n)$ with $x_n \to \infty$ when X_1 has a semiexponential distribution. In the same setting, Brosset *et al.* (2020) derived deviation results at logarithmic scale with shorter proofs relying on classical tools of large-deviation theory and making the rate function at the transition explicit. In this paper we exhibit the same asymptotic behavior for triangular arrays of semiexponentially distributed random variables.

Keywords: Large deviations; triangular array; semiexponential distribution; Weibull-like distribution; Gärtner–Ellis theorem; Contraction principle; truncated random variable

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1. Introduction

Moderate and large deviations of the sum of independent and identically distributed (i.i.d.) real-valued random variables have been investigated since the beginning of the 20th century. Kinchin [13] in 1929 was the first to give a result on large deviations of the sum of i.i.d. Bernoulli-distributed random variables. In 1933 Smirnov [23] improved this result, and in 1938 Cramér [5] gave a generalization to sums of i.i.d. random variables satisfying the eponymous Cramér condition which requires the Laplace transform of the common distribution of the random variables to be finite in a neighborhood of zero. Cramér's result was extended by Feller [8] to sequences of not necessarily identically distributed random variables under restrictive conditions (Feller considered only random variables taking values in bounded intervals), thus Cramér's result does not follow from Feller's result. A strengthening of Cramér's theorem was given by Petrov [20] together with a generalization to the case of non-identically distributed random variables. Improvements of Petrov's result can be found in [21]. Deviations for sums

of heavy-tailed i.i.d. random variables were studied by several authors: an early result appears in [15], and more recent references are [2, 3, 7, 16].

In [17, 18], A. V. Nagaev studied the case where the commom distribution of the i.i.d. random variables is absolutely continuous with respect to the Lebesgue measure with density $p(t) \sim \exp\left\{-|t|^{1-\varepsilon}\right\}$ as $|t| \to \infty$, with $\varepsilon \in (0, 1)$. He distinguished five exact-asymptotics results corresponding to five types of deviation speeds. These results were generalized in [19] to the case where the tail writes as $\exp\left\{-t^{1-\varepsilon}L(t)\right\}$, where $\varepsilon \in (0, 1)$ and L is a suitably slowly varying function at infinity. Such results can also be found in [2, 3]. In [4], the authors considered the following setting. Let $\varepsilon \in (0, 1)$ and let X be a Weibull-like (or semiexponential, or stretched exponential) random variable, i.e. there exists $q > 0$ such that $\log \mathbb{P}(X \geqslant x) \sim -qx^{1-\varepsilon}$ as $x \to \infty$. Assume also that there exists $\gamma > 0$ such that $\mathbb{E}[|X|^{2+\gamma}] < \infty$. For all $n \in \mathbb{N}^*$, let $X_1, X_2, \ldots, X_n$ be i.i.d. copies of X and set $S_n = X_1 + \cdots + X_n$. The asymptotic behavior of the large-deviation probability $\mathbb{P}(S_n \geqslant x_n)$ is given for any positive sequence $x_n \gg n^{1/2}$. According to the asymptotics of x_n, three types of behavior emerge for $\log \mathbb{P}(S_n \geqslant x_n)$:

- Maximal jump range [4, Theorem 1]: When $x_n \gg n^{1/(1+\varepsilon)}$, $\log \mathbb{P}(S_n \geqslant x_n) \sim \log \mathbb{P}(\max(X_1, \ldots, X_n) \geqslant x_n)$.

- Gaussian range: [4, Theorem 2]: When $n^{1/2} \ll x_n \ll n^{1/(1+\varepsilon)}$, $\log \mathbb{P}(S_n \geqslant x_n) \sim \log(1 - \phi(n^{-1/2}x_n))$, ϕ being the cumulative distribution function of the standard Gaussian law.

- Transition: [4, Theorem 3]: The case $x_n = \Theta\left(n^{1/(1+\varepsilon)}\right)$ appears to be an interpolation between the Gaussian range and the maximal jump range.

The main contribution of the present paper is a generalization to triangular arrays of the results in [4, 17, 18]. Such a setting appears naturally in some combinatorial problems, such as those presented in [12], including hashing with linear probing. Since the 1980s, laws of large numbers have been established for triangular arrays [9–11]. Lindeberg's condition is standard for the central limit theorem to hold for triangular arrays [1, Theorem 27.2]. Dealing with triangular arrays of light-tailed random variables, the Gärtner–Ellis theorem provides moderate- and large-deviation results. Deviations for sums of heavy-tailed i.i.d. random variables have been studied by several authors [2–4, 15, 17–19], and a good survey can be found in [16]. Here, we focus on the particular case of semiexponential tails (treated in [2–4, 17, 18] for sums of i.i.d. random variables), generalizing the results to triangular arrays. See [14] for an application to hashing with linear probing.

The paper is organized as follows. In Section 2 we state the main results, the proofs of which can be found in Section 3. The assumptions are discussed in Section 4. Section 5 is devoted to the study of the model of a truncated random variable which is a natural model of a triangular array. This kind of model appears in many proofs of large deviations. Indeed, when dealing with a random variable, the Laplace transform of which is not finite, a classical approach consists in truncating the random variable and letting the truncation go to infinity. In this model we exhibit various rate functions, especially non-convex ones.

2. Main results

In the following, for any sequences $(x_n)_{n \geqslant 1}$ and $(y_n)_{n \geqslant 1}$, it will be more convenient to write $x_n \preccurlyeq y_n$ for $x_n = O(y_n)$ as $n \to \infty$. For all $n \geqslant 1$, let Y_n be a centered real-valued random variable, and let N_n be a natural number. We assume that $N_n \to \infty$ as $n \to \infty$. For all $n \geqslant 1$,

let $(Y_{n,i})_{1 \leqslant i \leqslant N_n}$ be a family of i.i.d. random variables distributed as Y_n. Define, for all $k \in [\![1, N_n]\!]$, $T_{n,k} := \sum_{i=1}^{k} Y_{n,i}$. To lighten the notation, let $T_n := T_{n,N_n}$.

Theorem 2.1. (Maximal jump range.) *Let $\varepsilon \in (0, 1)$, $q > 0$, and $\alpha > 1/(1 + \varepsilon)$. Assume that:*

(H1) *For all sequences $(y_n)_{n \geqslant 1}$ such that $N_n^{\alpha\varepsilon} \preccurlyeq y_n \preccurlyeq N_n^{\alpha}$, $\log \mathbb{P}(Y_n \geqslant y_n) \underset{n \to \infty}{\sim} -qy_n^{1-\varepsilon}$.*

(H2) $\mathbb{E}[Y_n^2] = o\left(N_n^{\alpha(1+\varepsilon)-1}\right)$ *as $n \to \infty$.*

Then, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant N_n^{\alpha} y) = -qy^{1-\varepsilon}.$$

In this paper we have chosen to make explicit the deviations in terms of powers of N_n, as is now standard in large-deviation theory. Nevertheless, as in [4, 17, 18], the proof of Theorem 2.1 immediately adapts to show that $\log \mathbb{P}(T_n \geqslant x_n) \underset{n \to \infty}{\sim} -qx_n^{1-\varepsilon}$ as soon as $x_n \gg N_n^{1/(1+\varepsilon)}$, assuming that $\mathbb{E}[Y_n^2] = o(x_n^{1+\varepsilon}/N_n)$ as $n \to \infty$ and that there exists $\delta > 0$ such that, for all sequences $(y_n)_{n \geqslant 1}$ such that $x_n^{\varepsilon} \leqslant y_n \leqslant x_n(1 + \delta)$, $\log \mathbb{P}(Y_n \geqslant y_n) \underset{n \to \infty}{\sim} -qy_n^{1-\varepsilon}$. Theorems 2.2 and 2.3 adapt analogously.

Theorem 2.2. (Gaussian range.) *Let $\varepsilon \in (0, 1)$, $q > 0$, $\sigma > 0$, and $\frac{1}{2} < \alpha < 1/(1 + \varepsilon)$. Suppose that (H1) holds together with:*

(H2′) $\mathbb{E}[Y_n^2] \underset{n \to \infty}{\longrightarrow} \sigma^2$.

(H2⁺) *There exists $\gamma \in (0, 1]$ such that $\mathbb{E}[\, |Y_n|^{2+\gamma}\,] = o\left(N_n^{\gamma(1-\alpha)}\right)$ as $n \to \infty$.*

Then, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^{\alpha} y) = -\frac{y^2}{2\sigma^2}.$$

Theorem 2.3. (Transition.) *Let $\varepsilon \in (0, 1)$, $q > 0$, $\sigma > 0$, and $\alpha = 1/(1 + \varepsilon)$. Suppose that (H1), (H2′), and (H2⁺) hold. Then, for all $y \geqslant 0$,*

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^{\alpha} y) = -\inf_{0 \leqslant t \leqslant 1} \left\{ q(1-t)^{1-\varepsilon} y^{1-\varepsilon} + \frac{t^2 y^2}{2\sigma^2} \right\} =: -I(y).$$

Notice that the speed of deviation is continuous in α since, at the transition ($\alpha = 1/(1 + \varepsilon)$), $2\alpha - 1 = \alpha(1 - \varepsilon)$. Moreover, let us make explicit the rate function I. Let $f(t) := q(1-t)^{1-\varepsilon} y^{1-\varepsilon} + t^2 y^2/(2\sigma^2)$. An easy computation shows that, if $y \leqslant y_0 := ((1-\varepsilon^2)(1 + 1/\varepsilon)^{\varepsilon} q\sigma^2)^{1/(1+\varepsilon)}$, f is increasing and its minimum $y^2/(2\sigma^2)$ is attained at $t = 1$. If $y > y_0$, f has two local minima, at $t(y)$ and at 1; the former corresponds to the smaller of the two roots in [0, 1] of the equation $f'(t) = 0$, which is equivalent to

$$t(1-t)^{\varepsilon} = \frac{(1-\varepsilon)q\sigma^2}{y^{1+\varepsilon}}.$$

If $y_0 < y \leqslant y_1 := (1 + \varepsilon) \left(q\sigma^2/(2\varepsilon)^\varepsilon \right)^{1/(1+\varepsilon)}$, then $f(t(y)) \geqslant f(1)$; if $y > y_1$, $f(t(y)) < f(1)$. As a consequence, for all $y \geqslant 0$,

$$I(y) = \begin{cases} \frac{y^2}{2\sigma^2} & \text{if } y \leqslant y_1, \\ q(1 - t(y))^{1-\varepsilon} y^{1-\varepsilon} + \frac{t(y)^2 y^2}{2\sigma^2} & \text{if } y > y_1. \end{cases}$$

Remark 2.1. Consider the following generalization of (H1). Let L be a slowly varying function and assume that:

(H1$'$) For all sequences $(y_n)_{n \geqslant 1}$ such that $N_n^{\alpha\varepsilon}/L(N_n^\alpha) \preccurlyeq y_n \preccurlyeq N_n^\alpha$, $\log \mathbb{P}(Y_n \geqslant y_n) \sim -L(y_n)y_n^{1-\varepsilon}$.

Then, the proof of Theorem 2.1 (resp. Theorem 2.2) immediately adapts to show that if Assumptions (H1$'$) and (H2) (resp. (H1$'$), (H2$'$), and (H2$^+$)) hold, then, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{L(N_n^\alpha)N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -y^{1-\varepsilon}.$$

(resp. the conclusion of Theorem 2.2 holds). However, Theorem 2.3 requires additional assumptions on L to handle such a generalization of (H1).

3. Proofs

3.1. Preliminary known results

First, we present a classical result, known as the principle of the largest term. The proof is standard (see, e.g., [6, Lemma 1.2.15]).

Lemma 3.1. (Principle of the largest term) *Let* $(v_n)_{n \geqslant 1}$ *be a positive sequence diverging to* ∞, *let* r *be a positive integer, and, for* $i \in [\![1, r]\!]$, *let* $(a_{n,i})_{n \geqslant 1}$ *be a sequence of non-negative numbers. Then,*

$$\limsup_{n \to \infty} \frac{1}{v_n} \log \left(\sum_{i=1}^r a_{n,i} \right) = \max_{i=1,\dots,r} \left(\limsup_{n \to \infty} \frac{1}{v_n} \log a_{n,i} \right).$$

The next theorem is a unilateral version of the Gärtner–Ellis theorem, which was proved in [22].

Theorem 3.1. (Unilateral Gärtner–Ellis theorem) *Let* $(Z_n)_{n \geqslant 1}$ *be a sequence of real-valued random variables, and let* $(v_n)_{n \geqslant 1}$ *be a positive sequence diverging to* ∞. *Suppose that there exists a differentiable function* Λ *defined on* $\mathbb{R}_+$ *such that* Λ' *is a (increasing) bijective function from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$ *and, for all* $\lambda \geqslant 0$,

$$\frac{1}{v_n} \log \mathbb{E}\left[e^{v_n \lambda Z_n} \right] \xrightarrow[n \to \infty]{} \Lambda(\lambda).$$

Then, for all $z \geqslant 0$,

$$- \inf_{t > z} \Lambda^*(t) \leqslant \liminf_{n \to \infty} \frac{1}{v_n} \log \mathbb{P}(Z_n > z) \leqslant \limsup_{n \to \infty} \frac{1}{v_n} \log \mathbb{P}(Z_n \geqslant z) \leqslant - \inf_{t \geqslant z} \Lambda^*(t),$$

where, for all $t \geqslant 0$, $\Lambda^*(t) := \sup\{\lambda t - \Lambda(\lambda); \lambda \geqslant 0\}$.

The proofs of Theorems 2.1, 2.2, and 2.3 are adaptations of those in [4] to the case of triangular arrays. We decided to detail these proofs here to introduce Sections 4 and 5, where the proofs are mainly sketched.

3.2. Proof of Theorem 2 (Maximal jump range)

Assume that Theorem 2.1 has been proved for $y > 0$. Then, the case $y = 0$ follows by monotony. Indeed, for all $y > 0$,

$$0 \geqslant \liminf_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant 0) \geqslant \lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -q y^{1-\varepsilon} \xrightarrow[y \to 0]{} 0.$$

From now on, we assume that $y > 0$. First, we define

$$\begin{aligned}
\mathbb{P}(T_n \geqslant N_n^\alpha y) &= \mathbb{P}(T_n \geqslant N_n^\alpha y; \text{ for all } i \in [\![1, N_n]\!], \ Y_{n,i} < N_n^\alpha y) \\
&\quad + \mathbb{P}(T_n \geqslant N_n^\alpha y; \text{ there exists } i \in [\![1, N_n]\!] \text{ such that } Y_{n,i} \geqslant N_n^\alpha y) \\
&=: P_n + R_n..
\end{aligned}$$

Theorem 2.1 is a direct consequence of Lemmas 3.1, 3.2, and 3.3.

Lemma 3.2. *Under (H1) and (H2), for $\alpha > \frac{1}{2}$ and $y > 0$,*

$$\lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log R_n = -q y^{1-\varepsilon}.$$

Proof of Lemma 3.2. Using (H1),

$$\limsup_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log R_n \leqslant \lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \left(N_n \mathbb{P}(Y_n \geqslant N_n^\alpha y) \right) = -q y^{1-\varepsilon}.$$

Let us prove the converse inequality. Let $\delta > 0$. We have $R_n \geqslant \mathbb{P}\left(T_n \geqslant N_n^\alpha y, Y_{n,1} \geqslant N_n^\alpha y \right) \geqslant \mathbb{P}\left(T_{n,N_{n-1}} \geqslant -N_n^\alpha \delta \right) \mathbb{P}(Y_n \geqslant N_n^\alpha(y + \delta))$. By Chebyshev's inequality, observe that

$$\mathbb{P}(T_{n,N_{n-1}} \geqslant -N_n{}^\alpha \delta) \geqslant 1 - \frac{\mathrm{Var}(Y_n)}{N_n^{2\alpha - 1} \delta^2} \to 1,$$

using (H2). Finally, by (H1), we get

$$\liminf_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log R_n \geqslant \lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(Y_n \geqslant N_n^\alpha(y + \delta)) = -q(y + \delta)^{1-\varepsilon}.$$

We conclude by letting $\delta \to 0$. $\qquad \square$

Lemma 3.3. *Under (H1) and (H2), for $\alpha > 1/(1 + \varepsilon)$ and $y > 0$,*

$$\limsup_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log P_n \leqslant -q y^{1-\varepsilon}.$$

Proof of Lemma 3.3. For all $q' \in (0, q)$, we have

$$\begin{aligned}
P_n &= \mathbb{E}\left[\mathbf{1}_{T_n \geqslant N_n^\alpha y} \mathbf{1}_{\forall i \in [\![1, N_n]\!], Y_{n,i} < N_n^\alpha y} \right] \\
&\leqslant \exp\left\{ -q'(N_n^\alpha y)^{1-\varepsilon} \right\} \mathbb{E}\left[\exp\left\{ \frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n \right\} \mathbf{1}_{Y_n < N_n^\alpha y} \right]^{N_n},
\end{aligned}$$

using the inequality $\mathbf{1}_{x \geqslant 0} \leqslant e^x$ for the first indicator function above. If we prove that $\mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{Y_n < N_n^\alpha y}\right] \leqslant 1 + o\left(N_n^{\alpha(1-\varepsilon)-1}\right)$, then $\log P_n \leqslant -q'(N_n^\alpha y)^{1-\varepsilon} + o\left(N_n^{\alpha(1-\varepsilon)}\right)$ and the conclusion follows by letting $q' \to q$. Write

$$\mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{Y_n < N_n^\alpha y}\right] = \mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right]$$
$$+ \mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{N_n^{\alpha\varepsilon} \leqslant Y_n < N_n^\alpha y}\right].$$

First, using the fact that, for all $x < q'y^{-\varepsilon}$, $e^x \leqslant 1 + x + \frac{1}{2}e^{q'y^{-\varepsilon}}x^2$, and (H2), we get

$$\mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right] \leqslant \mathbb{E}\left[\left(1 + \frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n + \frac{(q')^2 e^{q'y^{-\varepsilon}}}{2(N_n^\alpha y)^{2\varepsilon}} Y_n^2\right) \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right]$$
$$\leqslant 1 + \frac{(q')^2 e^{q'y^{-\varepsilon}}}{2} \cdot \frac{\mathbb{E}[Y_n^2]}{(N_n^\alpha y)^{2\varepsilon}}$$
$$= 1 + o(N_n^{\alpha(1-\varepsilon)-1}),$$

the second inequality stemming from the following lemma.

Lemma 3.4. *Under (H1), for all $y > 0$, for all $q' < q$ there exists $n_0(q') \geqslant 1$ such that, for all $n \geqslant n_0(q')$ and for all $u \in \left[N_n^{\alpha\varepsilon}, N_n^\alpha y\right]$, $\log \mathbb{P}(Y_n \geqslant u) \leqslant -q'u^{1-\varepsilon}$.*

Proof of Lemma 3.4. By contraposition, if the conclusion of the lemma is false, we can construct a sequence $(u_n)_{n \geqslant 1}$ such that, for all $n \geqslant 1$, $u_n \in \left[N_n^{\alpha\varepsilon}, N_n^\alpha y\right]$ and $\log \mathbb{P}(Y_n \geqslant u_n) > -q'u_n^{1-\varepsilon}$, whence (H1) is not satisfied. □

Secondly, integrating by parts (Lebesgue–Stieltjes version), we get

$$\mathbb{E}\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} Y_n\right\} \mathbf{1}_{N_n^{\alpha\varepsilon} \leqslant Y_n < N_n^\alpha y}\right] = \int_{N_n^{\alpha\varepsilon}}^{N_n^\alpha y} \exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} u\right\} \mathbb{P}(Y_n \in du)$$
$$= -\left[\exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} u\right\} \mathbb{P}(Y_n \geqslant u)\right]_{N_n^{\alpha\varepsilon}}^{N_n^\alpha y} + \frac{q'}{(N_n^\alpha y)^\varepsilon} \int_{N_n^{\alpha\varepsilon}}^{N_n^\alpha y} \exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} u\right\} \mathbb{P}(Y_n \geqslant u)\, du$$
$$\leqslant e^{q'y^{-\varepsilon}} \mathbb{P}(Y_n \geqslant N_n^{\alpha\varepsilon}) + \frac{q'}{(N_n^\alpha y)^\varepsilon} \int_{N_n^{\alpha\varepsilon}}^{N_n^\alpha y} \exp\left\{\frac{q'}{(N_n^\alpha y)^\varepsilon} u - q''u^{1-\varepsilon}\right\} du$$
$$\leqslant (1 + q'(N_n^\alpha y)^{1-\varepsilon}) e^{q'y^{-\varepsilon} - q''N_n^{\alpha\varepsilon(1-\varepsilon)}}$$
$$= o\left(N_n^{\alpha(1-\varepsilon)-1}\right)$$

for n large enough, using (H1) and Lemma 3.4 with $q'' \in (q', q)$, and taking the supremum of $u \mapsto q'(N_n^\alpha y)^{-\varepsilon} u - q''u^{1-\varepsilon}$ over $\left[N_n^{\alpha\varepsilon}, N_n^\alpha y\right]$. The proof of Lemma 3.3 is now complete. □

3.3. Proof of Theorems 2.2 (Gaussian range) and 2.3 (Transition)

The conclusions of Theorems 2.2 and 2.3 follow from Lemmas 3.5, 3.6, and 3.9 below, and the principle of the largest term (Lemma 3.1): Lemma 3.5 (resp. Lemma 3.6) provides the

leading term in Theorem 2.2 (resp. in Theorem 2.3), and Lemma 3.9 bounds the remaining terms. The structure of the proof follows [4].

Let us fix $y > 0$. The result for $y = 0$ follows by monotony (see the beginning of the proof of Theorem 2.1).

3.3.1. *Principal estimates* For all $m \in [\![0, N_n]\!]$, we define

$$\Pi_{n,m} = \mathbb{P}\big(T_n \geqslant N_n^\alpha y;$$
$$\text{for all } i \in [\![1, m]\!], \, Y_{n,i} \geqslant N_n^{\alpha\varepsilon}; \text{ for all } i \in [\![m+1, N_n]\!], \, Y_{n,i} < N_n^{\alpha\varepsilon}\big), \quad (3.1)$$

so that

$$\mathbb{P}(T_n \geqslant N_n^\alpha y) = \sum_{m=0}^{N_n} \binom{N_n}{m} \Pi_{n,m}. \quad (3.2)$$

Lemma 3.5. *Under (H1), (H2′), and (H2$^+$), for $\frac{1}{2} < \alpha \leqslant 1/(1+\varepsilon)$ and $y > 0$,*

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \Pi_{n,0} = -\frac{y^2}{2\sigma^2}.$$

Proof of Lemma 3.5. For all $n \geqslant 1$, we introduce the variable $Y_n^<$ distributed as $\mathcal{L}(Y_n \mid Y_n < N_n^{\alpha\varepsilon})$. Let $T_n^< = \sum_{i=1}^{N_n} Y_{n,i}^<$, where the $Y_{n,i}^<$ are independent random variables distributed as $Y_n^<$. Then $\Pi_{n,0} = \mathbb{P}(T_n^< \geqslant N_n^\alpha y)\mathbb{P}(Y_n < N_n^{\alpha\varepsilon})^{N_n}$. On the one hand, $\mathbb{P}(Y_n < N_n^{\alpha\varepsilon})^{N_n} \to 1$ by (H1). On the other hand, in order to apply the unilateral version of the Gärtner–Ellis theorem (Theorem 3.1), we compute, for $u \geqslant 0$,

$$\Lambda_n(u) = \frac{1}{N_n^{2\alpha-1}} \log \mathbb{E}\left[\exp\left\{\frac{u}{N_n^\alpha} T_n^<\right\}\right]$$
$$= N_n^{2(1-\alpha)} \log \mathbb{E}\left[\exp\left\{\frac{u}{N_n^{1-\alpha}} Y_n\right\} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right] - N_n^{2(1-\alpha)} \log \mathbb{P}(Y_n < N_n^{\alpha\varepsilon}). \quad (3.3)$$

By (H1), the second term above goes to 0. As for the first term, for $\gamma \in (0, 1)$ given by (H2$^+$), there exists a constant $c > 0$ such that, for all $t \leqslant u$, $\big|e^t - 1 - t - t^2/2\big| \leqslant c\,|t|^{2+\gamma}$, whence

$$\left|\left(\exp\left\{\frac{u}{N_n^{1-\alpha}} Y_n\right\} - 1 - \frac{u}{N_n^{1-\alpha}} Y_n - \frac{u^2}{2N_n^{2(1-\alpha)}} Y_n^2\right) \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right|$$
$$\leqslant \frac{cu^{2+\gamma}}{N_n^{(2+\gamma)(1-\alpha)}} |Y_n|^{2+\gamma}, \quad (3.4)$$

since $\alpha(1+\varepsilon) \leqslant 1$. Now,

$$\left|\mathbb{E}\left[\exp\left\{\frac{u}{N_n^{1-\alpha}} Y_n\right\} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right] - \exp\left\{\frac{u^2\sigma^2}{2N_n^{2(1-\alpha)}}\right\}\right|$$
$$\leqslant \left|\mathbb{E}\left[\left(\exp\left\{\frac{u}{N_n^{1-\alpha}} Y_n\right\} - 1 - \frac{u}{N_n^{1-\alpha}} Y_n - \frac{u^2}{2N_n^{2(1-\alpha)}} Y_n^2\right) \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right]\right|$$
$$+ \left|\mathbb{E}\left[\left(1 + \frac{u}{N_n^{1-\alpha}} Y_n + \frac{u^2}{2N_n^{2(1-\alpha)}} Y_n^2\right) \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\right] - \left(1 + \frac{u^2\mathbb{E}[Y_n^2]}{2N_n^{2(1-\alpha)}}\right)\right|$$

$$+ \left| \left(1 + \frac{u^2 \mathbb{E}[Y_n^2]}{2N_n^{2(1-\alpha)}} \right) - \exp\left\{ \frac{u^2 \sigma^2}{2N_n^{2(1-\alpha)}} \right\} \right|$$

$$= \left| \mathbb{E}\left[\left(1 + \frac{u}{N_n^{1-\alpha}} Y_n + \frac{u^2}{2N_n^{2(1-\alpha)}} Y_n^2 \right) \mathbf{1}_{Y_n \geqslant N_n^{\alpha\varepsilon}} \right] \right| + o(N_n^{-2(1-\alpha)}), \qquad (3.5)$$

using (3.4), (H2$^+$), the fact that $\mathbb{E}[Y_n] = 0$, a Taylor expansion of order 2 of the exponential function, and (H2$'$). Finally, for n large enough, applying Hölder's inequality,

$$\left| \mathbb{E}\left[\left(1 + \frac{u}{N_n^{1-\alpha}} Y_n + \frac{u^2}{2N_n^{2(1-\alpha)}} Y_n^2 \right) \mathbf{1}_{Y_n \geqslant N_n^{\alpha\varepsilon}} \right] \right| \leqslant \mathbb{E}[Y_n^2 \mathbf{1}_{Y_n \geqslant N_n^{\alpha\varepsilon}}]$$

$$\leqslant \mathbb{E}[|Y_n|^{2+\gamma}]^{2/(2+\gamma)} \mathbb{P}(Y_n \geqslant N_n^{\alpha\varepsilon})^{\gamma/(2+\gamma)}$$

$$= o(N_n^{-2(1-\alpha)}) \qquad (3.6)$$

as a consequence of (H1) and (H2$^+$). Combining (3.3), (3.5), and (3.6), we get

$$\Lambda_n(u) \underset{n\to\infty}{\longrightarrow} \frac{u^2 \sigma^2}{2} =: \Lambda(u),$$

and the proof of Lemma 3.5 follows from the fact that $\Lambda^*(y) = y^2/(2\sigma^2)$. $\qquad\square$

Lemma 3.6. *Under (H1), (H2$'$), and (H2$^+$), for $\alpha = 1/(1+\varepsilon)$ and $y > 0$,*

$$\liminf_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \Pi_{n,1} \geqslant -I(y).$$

Proof of Lemma 3.6. Remember that for $\alpha = 1/(1+\varepsilon)$ we have $2\alpha - 1 = \alpha(1-\varepsilon)$. So, for all $t \in (0, 1)$,

$$\frac{1}{N_n^{2\alpha-1}} \log \Pi_{n,1} \geqslant \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_{n,N_{n-1}} \geqslant N_n^\alpha ty; \text{ for all } i \in [\![1, N_n-1]\!], Y_{n,i} < N_n^{\alpha\varepsilon})$$

$$+ \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(N_n^\alpha (1-t)y \leqslant Y_n^<, N_n < N_n^\alpha y)$$

$$\underset{n\to\infty}{\longrightarrow} -\frac{t^2 y^2}{2\sigma^2} - q(1-t)^{1-\varepsilon} y^{1-\varepsilon},$$

by Lemma 3.5 (applied to the array $(Y_{n,i})_{1 \leqslant i \leqslant N_n-1}$) and by (H1). Optimizing in $t \in (0, 1)$ provides the conclusion. $\qquad\square$

3.3.2. *Two uniform bounds*

Lemma 3.7. *Under (H2$'$) and (H2$^+$), for all $\delta \in (0, 1)$ and $y > 0$, there exists $n(\delta, y) \geqslant 1$ such that, for all $n \geqslant n(\delta, y)$, for all $m \in [\![0, N_n]\!]$, for all $u \in [0, N_n^\alpha y]$,*

$$\log \mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} < N_n^{\alpha\varepsilon}) \leqslant -\frac{(1-\delta)u^2}{2N_n\sigma^2}.$$

Proof of Lemma 3.7. Using the fact that $\mathbf{1}_{t \geqslant 0} \leqslant e^t$, for all $\lambda > 0$,

$$\mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], \quad Y_{n,i} < N_n^{\alpha\varepsilon}) \leqslant e^{-\lambda u} \mathbb{E}[e^{\lambda Y_n} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}]^m.$$

For $\gamma \in (0, 1)$ given by (H2$^+$), there exists $c(y) > 0$ such that, for all $s \leqslant y\sigma^{-2}$, we have $e^s \leqslant 1 + s + \frac{1}{2}s^2 + c(y)|s|^{2+\gamma}$. Hence, for $\lambda := u(N_n\sigma^2)^{-1} \leqslant N_n^{-\alpha\varepsilon}y\sigma^{-2}$,

$$\mathbb{E}\big[e^{\lambda Y_n}\mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\big] \leqslant 1 + \frac{\lambda^2}{2}\mathbb{E}\big[Y_n^2\big] + c(y)\lambda^{2+\gamma}\mathbb{E}\big[\,|Y_n|^{2+\gamma}\,\big] = 1 + \frac{\lambda^2\sigma^2}{2}(1 + \delta_n),$$

where

$$\delta_n := \left(\frac{\mathbb{E}\big[Y_n^2\big]}{\sigma^2} - 1\right) + \frac{2c(y)\lambda^\gamma}{\sigma^2}\mathbb{E}\big[\,|Y_n|^{2+\gamma}\,\big].$$

By (H2$'$), the first term in δ_n goes to 0 as $n \to \infty$. Moreover, $\lambda \leqslant N_n^{\alpha-1}y\sigma^{-2}$ so, using (H2$^+$), we obtain that, for $n \geqslant n(\delta, y)$ large enough, $|\delta_n| \leqslant \delta$. Finally, since $m \leqslant N_n$, we have, for $n \geqslant n(\delta, y)$,

$$\log \mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} < N_n^{\alpha\varepsilon})$$
$$\leqslant -\lambda u + \frac{N_n\lambda^2\sigma^2}{2}(1 + \delta) = -\frac{(1 - \delta)u^2}{2N_n\sigma^2}. \qquad \square$$

Lemma 3.8. *Under (H1), for all $\delta \in (0, 1)$ and $y > 0$, there exists $n(\delta, y) \geqslant 1$ such that, for all $n \geqslant n(\delta, y)$, for all $m \in [\![1, N_n]\!]$, for all $u \in \big[0, N_n^\alpha y\big]$, $\log \mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon}) \leqslant -(1 - \delta)q\big(u^{1-\varepsilon} + (m - 1)(1 - 2^{-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)}\big)$.*

Proof of Lemma 3.8. Let q' be such that $(1 - \delta)q < q' < q$. First, we establish that there exists $n_0(\delta)$ such that, for all $n \geqslant n_0(\delta)$,

$$\mathbb{P}\big(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon}\big) \tag{3.7}$$
$$\leqslant \begin{cases} \exp\big[-q'mN_n^{\alpha\varepsilon(1-\varepsilon)}\big] & \text{if } u < mN_n^{\alpha\varepsilon}, \\ (u^m + m)\exp\big[-q'\big((u - (m - 1)N_n^{\alpha\varepsilon})^{1-\varepsilon} + (m - 1)N_n^{\alpha\varepsilon(1-\varepsilon)}\big)\big] & \text{if } mN_n^{\alpha\varepsilon} \leqslant u \leqslant N_n^\alpha y. \end{cases}$$

The result is trivial for $u < mN_n^{\alpha\varepsilon}$ or $m = 1$, using Lemma 3.4. Now, we suppose $u \geqslant mN_n^{\alpha\varepsilon}$ and $m \geqslant 2$. We have

$$\mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon})$$
$$\leqslant \mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], N_n^{\alpha\varepsilon} \leqslant Y_{n,i} < u)$$
$$+ \mathbb{P}(\text{there exists } i_0 \in [\![1, m]\!] \text{ such that } Y_{n,i_0} \geqslant u - (m - 1)N_n^{\alpha\varepsilon};$$
$$\text{for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon}).$$

First,

$$\mathbb{P}(\text{there exists } i_0 \in [\![1, m]\!] \text{ such that } Y_{n,i_0} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon})$$
$$\leqslant m\mathbb{P}(Y_n \geqslant u - (m - 1)N_n^{\alpha\varepsilon})\mathbb{P}(Y_n \geqslant N_n^{\alpha\varepsilon})^{m-1}$$
$$\leqslant m\exp\big[-q'\big((u - (m - 1)N_n^{\alpha\varepsilon})^{1-\varepsilon} + (m - 1)N_n^{\alpha\varepsilon(1-\varepsilon)}\big)\big] \tag{3.8}$$

as soon as $n \geqslant n_1(\delta) \geqslant n_0(\delta)$, where $n_1(\delta)$ is the integer $n_0(q')$ given by Lemma 3.4 (remember that $N_n^{\alpha\varepsilon} \leqslant mN_n^{\alpha\varepsilon} \leqslant u \leqslant N_n^\alpha y$). Secondly, denoting by a_i integers and q'' a number such that $q' < q'' < q$,

$$\mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], N_n^{\alpha\varepsilon} \leqslant Y_{n,i} < u - (m-1)N_n^{\alpha\varepsilon})$$

$$= \int_{\forall i, N_n^{\alpha\varepsilon} \leqslant u_i < u} \mathbf{1}_{u_1 + \cdots + u_m \geqslant u} \prod_{i=1}^{m} \mathbb{P}(Y_n \in \mathrm{d}u_i)$$

$$\leqslant \sum_{\forall i, \lceil N_n^{\alpha\varepsilon} \rceil \leqslant a_i \leqslant \lceil u \rceil} \mathbf{1}_{a_1 + \cdots + a_m \geqslant u} \prod_{i=1}^{m} \mathbb{P}(a_i - 1 < Y_n \leqslant a_i)$$

$$\leqslant \sum_{\forall i, \lceil N_n^{\alpha\varepsilon} \rceil \leqslant a_i \leqslant \lceil u \rceil} \mathbf{1}_{a_1 + \cdots + a_m \geqslant u} \prod_{i=1}^{m} \mathrm{e}^{-q''(a_i - 1)^{1-\varepsilon}}$$

$$\leqslant \int_{\forall i, N_n^{\alpha\varepsilon} \leqslant u_i < u + 2} \mathbf{1}_{u_1 + \cdots + u_m \geqslant u} \prod_{i=1}^{m} \mathrm{e}^{-q''(u_i - 2)^{1-\varepsilon}} \mathrm{d}u_i$$

$$\leqslant \int_{\substack{\forall i, N_n^{\alpha\varepsilon} \leqslant u_i < u + 2 \\ u_1 + \cdots + u_m \geqslant u}} \mathrm{e}^{-q'\left(u_1^{1-\varepsilon} + \cdots + u_m^{1-\varepsilon}\right)} \mathrm{d}u_1 \cdots \mathrm{d}u_m,$$

as soon as n is large enough ($n \geqslant n_2(\delta) \geqslant n_1(\delta)$) so that, for all $v \geqslant N_n^{\alpha\varepsilon}$, $q''(v-2)^{1-\varepsilon} \geqslant q'v^{1-\varepsilon}$. Now, the function $s_m \colon (u_1, \ldots, u_m) \mapsto u_1^{1-\varepsilon} + \cdots + u_m^{1-\varepsilon}$ is concave, so s_m reaches its minimum on the domain of integration at the points where all the u_i equal $N_n^{\alpha\varepsilon}$, except one equal to $u - (m-1)N_n^{\alpha\varepsilon}$. Therefore,

$$\mathbb{P}(T_{n,m} \geqslant u, \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon})$$

$$\leqslant u^m \exp\left[-q'\left((u - (m-1)N_n^{\alpha\varepsilon})^{1-\varepsilon} + (m-1)N_n^{\alpha\varepsilon(1-\varepsilon)}\right)\right]. \quad (3.9)$$

Equations (3.8) and (3.9) yield (3.7).

The conclusion of Lemma 3.8 for $u < mN_n^{\alpha\varepsilon}$ stems from (3.7) and the following easy inequality: for all $m \geqslant 1$, $m^{1-\varepsilon} + (m-1)(1 - 2^{-\varepsilon}) \leqslant m$. As for $u \geqslant mN_n^{\alpha\varepsilon}$, we notice that the function $g(u) := -u^{1-\varepsilon} + (u - (m-1)N_n^{\alpha\varepsilon})^{1-\varepsilon} + (m-1)N_n^{\alpha\varepsilon(1-\varepsilon)}$ is increasing on $\left[mN_n^{\alpha\varepsilon}, \infty\right)$ and $g(mN_n^{\alpha\varepsilon}) = N_n^{\alpha\varepsilon(1-\varepsilon)}m(1 - m^{-\varepsilon}) \geqslant N_n^{\alpha\varepsilon(1-\varepsilon)}(m-1)(1 - 2^{-\varepsilon})$, so, using (3.7) and the fact that, for $m \geqslant 1$, $u^m + m \leqslant (u+1)^m$,

$$\mathbb{P}(T_{n,m} \geqslant u, \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon})$$

$$\leqslant (u+1)^m \exp\left[-q'\left(u^{1-\varepsilon} + (m-1)(1 - 2^{-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)}\right)\right]$$

$$\leqslant \exp\left[-(1-\delta)q\left(u^{1-\varepsilon} + (m-1)(1 - 2^{-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)}\right)\right]$$

as soon as

$$\log(u+1) \leqslant \log(N_n^{\alpha}y + 1)$$

$$\leqslant \frac{1}{2}(q' - (1-\delta)q)(1 - 2^{-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)}$$

$$\leqslant \frac{m-1}{m}(q' - (1-\delta)q)(1 - 2^{-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)},$$

i.e. for $n \geqslant n(\delta, y) \geqslant n_2(\delta)$. $\qquad\qquad \square$

3.3.3. *Upper bound for the sum of the* $\Pi_{n,m}$ Using the uniform bounds of Lemmas 3.7 and 3.8, we are able to bound above the remaining term $\sum_{m=1}^{n} \binom{n}{m} \Pi_{n,m}$.

Lemma 3.9. *Assume (H1), (H2$'$), and (H2$^+$). If* $\frac{1}{2} < \alpha < 1/(1+\varepsilon)$, *then, for all* $y > 0$,

$$\limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{n} \binom{n}{m} \Pi_{n,m} \leqslant -\frac{y}{2\sigma^2}.$$

If $\alpha = 1/(1+\varepsilon)$, *then, for all* $y > 0$,

$$\limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{n} \binom{n}{m} \Pi_{n,m} \leqslant -I(y).$$

Proof of Lemma 3.9. Fix some integer $r \geqslant 1$. Noticing that

$$\left\{ (a,b) \in (\mathbb{R}_+)^2 \;\middle|\; a+b \geqslant 1 \right\} \subset \bigcup_{k=1}^{r} \left\{ (a,b) \in (\mathbb{R}_+)^2 \;\middle|\; a \geqslant \frac{k-1}{r}, \; b \geqslant 1-\frac{k}{r} \right\},$$

we have, for all $m \in [\![1, N_n]\!]$,

$$
\begin{aligned}
\Pi_{n,m} = \mathbb{P}\Big(& T_n \geqslant N_n^{\alpha} y; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon}; \\
& \text{for all } i \in [\![m+1, N_n]\!], Y_{n,i} < N_n^{\alpha\varepsilon} \Big) \\
\leqslant & \sum_{k=1}^{r} \mathbb{P}\Big(T_{n,N_n-m} \geqslant \frac{k-1}{r} N_n^{\alpha} y; \text{ for all } i \in [\![1, n-m]\!], Y_{n,i} < N_n^{\alpha\varepsilon} \Big) \\
& \times \mathbb{P}\Big(T_{n,m} \geqslant \Big(1-\frac{k}{r}\Big) N_n^{\alpha} y; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon} \Big) \\
\leqslant & \sum_{k=1}^{r} \exp\Big[-(1-\delta)\Big(\frac{(k-1)^2(N_n^{\alpha} y)^2}{2N_n\sigma^2 r^2} \\
& \qquad + q\Big(\Big(1-\frac{k}{r}\Big) N_n^{\alpha} y\Big)^{1-\varepsilon} + q(m-1)(1-2^{-\varepsilon}) N_n^{\alpha\varepsilon(1-\varepsilon)} \Big) \Big],
\end{aligned}
$$

for n large enough, applying Lemmas 3.7 and 3.8. Hence,

$$
\begin{aligned}
\log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \leqslant{} & \log \sum_{k=1}^{r} \exp\Big[-(1-\delta)\Big(N_n^{2\alpha-1} \frac{((k-1)/r)^2 y^2}{2\sigma^2} \\
& \qquad\qquad + q N_n^{\alpha(1-\varepsilon)} \Big(\Big(1-\frac{k}{r}\Big) y\Big)^{1-\varepsilon} \Big) \Big] \\
& + \log \sum_{m=1}^{N_n} \binom{N_n}{m} e^{-(1-\delta)q(m-1)(1-2^{-\varepsilon}) N_n^{\alpha\varepsilon(1-\varepsilon)}},
\end{aligned}
$$

where the latter sum is bounded.

For $\alpha \in \left(\frac{1}{2}, 1/(1+\varepsilon)\right)$, we have $2\alpha - 1 < \alpha(1-\varepsilon)$. Therefore, applying the principle of the largest term (Lemma 3.1), we get

$$\limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \leqslant -(1-\delta)\Big(\frac{r-1}{r}\Big)^2 \frac{1}{2\sigma^2},$$

so, letting $r \to \infty$ and $\delta \to 0$,

$$\limsup_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \leqslant -\frac{1}{2\sigma^2}.$$

For $\alpha = 1/(1+\varepsilon)$, remember that $2\alpha - 1 = \alpha(1-\varepsilon)$. Therefore, applying the principle of the largest term (Lemma 3.1), we get

$$\limsup_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \leqslant -(1-\delta) \min_{k=1}^{r} \left(\left(\frac{k-1}{r} \right)^2 \frac{y^2}{2\sigma^2} + q \left(1 - \frac{k}{r} \right)^{1-\varepsilon} y^{1-\varepsilon} \right),$$

so, letting $r \to \infty$ and $\delta \to 0$,

$$\limsup_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \leqslant - \min_{t \in [0,1]} \left(\frac{t^2 y^2}{2\sigma^2} + q(1-t)^{1-\varepsilon} y^{1-\varepsilon} \right) = -I(y). \quad \square$$

Remark 3.1. Notice that, using the contraction principle, we can show that, for all fixed m, if $\alpha \in \left(\frac{1}{2}, 1/(1+\varepsilon) \right)$, then

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \Pi_{n,m} = -\frac{y^2}{2\sigma^2}, \; ;$$

and if $\alpha = 1/(1+\varepsilon)$, then

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \Pi_{n,m} = -I(y).$$

4. About the assumptions

Looking into the proof of Theorem 2.1, one can see that Assumption (H1) can be weakened and we may only assume the two conditions that follow.

Theorem 4.1. *The conclusion of Theorem 2.1 holds under (H2) and:*

(H1a) *For all $y_n = \Theta(N_n^\alpha)$, $\log \mathbb{P}(Y_n \geqslant y_n) \underset{n \to \infty}{\sim} -q y_n^{1-\varepsilon}$.*

(H1b) *For all $N_n^{\alpha\varepsilon} \preccurlyeq y_n \preccurlyeq N_n^\alpha$, $\limsup_{n \to \infty} y_n^{-(1-\varepsilon)} \log \mathbb{P}(Y_n \geqslant y_n) \leqslant -q$.*

Lemma 4.1. *(H1a) is equivalent to:*

(H1a′) *For all $y > 0$, $\log \mathbb{P}(Y_n \geqslant N_n^\alpha y) \underset{n \to \infty}{\sim} -q(N_n^\alpha y)^{1-\varepsilon}$.*

Proof of Lemma 4.1. If $N_n^\alpha c_1 \leqslant y_n \leqslant N_n^\alpha c_2$, then $-q c_2 \leqslant N_n^{-\alpha(1-\varepsilon)} \log \mathbb{P}(Y_n \geqslant y_n) \leqslant -q c_1$. First, extract a convergent subsequence; then, again extract a subsequence such that $N_n^{-\alpha} y_n$ is convergent and use (H1a) to show that $N_n^{-\alpha(1-\varepsilon)} \log \mathbb{P}(Y_n \geqslant y_n)$ is convergent. $\quad \square$

The following lemma is straightforward.

Lemma 4.2. *(H1b) is equivalent to the conclusion of Lemma 3.4:*

(H1b′) *For all $y > 0$, for all $q' < q$, there exists n_0 such that for all $n \geqslant n_0$ and for all $u \in \left[N_n^{\alpha\varepsilon}, N_n^\alpha y \right]$, $\log \mathbb{P}(Y_n \geqslant u) \leqslant -q' u^{1-\varepsilon}$.*

Theorem 4.2. *The conclusion of Theorem 2.1 holds under (H1a), (H1b), and*

$$\mathbb{E}\big[\,|Y_n|^{2+\gamma}\,\big]/\mathbb{E}\big[\,|Y_n|^2\,\big]^{1+\gamma/2} = o\big(N_n^{\gamma/2}\big)$$

as $n \to \infty$.

Proof of Theorem 4.2. The only modification in the proof is the minoration of R_n: $R_n \geqslant \mathbb{P}\big(T_{n,N_{n-1}} \geqslant 0\big)\mathbb{P}(Y_n \geqslant N_n^\alpha y)$. Now Lyapunov's theorem [1, Theorem 27.3] applies, so $\mathbb{P}\big(T_{n,N_{n-1}} \geqslant 0\big) \to \tfrac{1}{2}$ as $n \to \infty$. □

As for Theorem 2.2, Assumption (H1) can be weakened and we may only assume (H1b), or even the following weaker assumption.

Theorem 4.3. *The conclusion of Theorem 2.2 holds under (H2′), (H2⁺), and:*

(H1c) *For all $y > 0$ there exists $q > 0$ and n_0 such that, for all $n \geqslant n_0$ and for all $u \in \big[N_n^{\alpha\varepsilon}, N_n^\alpha y\big]$, $\log \mathbb{P}(Y_n \geqslant u) \leqslant -qu^{1-\varepsilon}$.*

Finally, in Theorem 2.3, Assumption (H1) can be weakened and we may only assume (H1a) and (H1b).

5. Application: Truncated random variable

Let us consider a centered real-valued random variable Y, admitting a finite moment of order $2 + \gamma$ for some $\gamma > 0$. Set $\sigma^2 := \mathbb{E}\big[Y^2\big]$. Now, let $\beta > 0$ and $c > 0$. For all $n \geqslant 1$, let us introduce the truncated random variable Y_n defined by $\mathcal{L}(Y_n) = \mathcal{L}\big(Y \mid Y < N_n^\beta c\big)$. Such truncated random variables naturally appear in proofs of large-deviation results. For all $n \geqslant 1$, let N_n be a natural number. We assume that $N_n \to \infty$ as $n \to \infty$. For all $n \geqslant 1$, let $\big(Y_{n,i}\big)_{1 \leqslant i \leqslant N_n}$ be a family of i.i.d. random variables distributed as Y_n. Define, for all $k \in [\![1, N_n]\!]$, $T_{n,k} := \sum_{i=1}^k Y_{n,i}$, and let $T_n := T_{n,N_n}$.

If Y has a light-tailed distribution, i.e. $\Lambda_Y(\lambda) := \log \mathbb{E}\big[e^{\lambda Y}\big] < \infty$ for some $\lambda > 0$, then (the unilateral version of) the Gärtner–Ellis theorem (Theorem 3.1) applies: if $\alpha \in (1/2, 1)$, then

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -\frac{y^2}{2\sigma^2},$$

and if $\alpha = 1$, then

$$\lim_{n \to \infty} \frac{1}{N_n} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -\Lambda_Y^*(y) = -\sup_{\lambda \geqslant 0} \{\lambda y - \Lambda_Y(\lambda)\}.$$

Note that we recover the same asymptotics as for the non-truncated random variable Y. In other words, the truncation does not affect the deviation behavior.

Now we consider the case where $\log \mathbb{P}(Y \geqslant y) \sim -qy^{1-\varepsilon}$ for some $q > 0$ and $\varepsilon \in (0, 1)$. In this case, the Gärtner–Ellis theorem does not apply directly; indeed, the Gärtner–Ellis theorem always provides a convex rate function (Λ^* in Theorem 3.1) but, as can be seen in Figures 1 to 3, some of the rate functions obtained are not convex. Observe that, as soon as $y_n \to \infty$,

$$\limsup_{n \to \infty} \frac{1}{y_n^{1-\varepsilon}} \log \mathbb{P}(Y_n \geqslant y_n) = \limsup_{n \to \infty} \frac{1}{y_n^{1-\varepsilon}} \big(\log \mathbb{P}\big(y_n \leqslant Y < N_n^\beta c\big) - \log \mathbb{P}\big(Y < N_n^\beta c\big)\big)$$
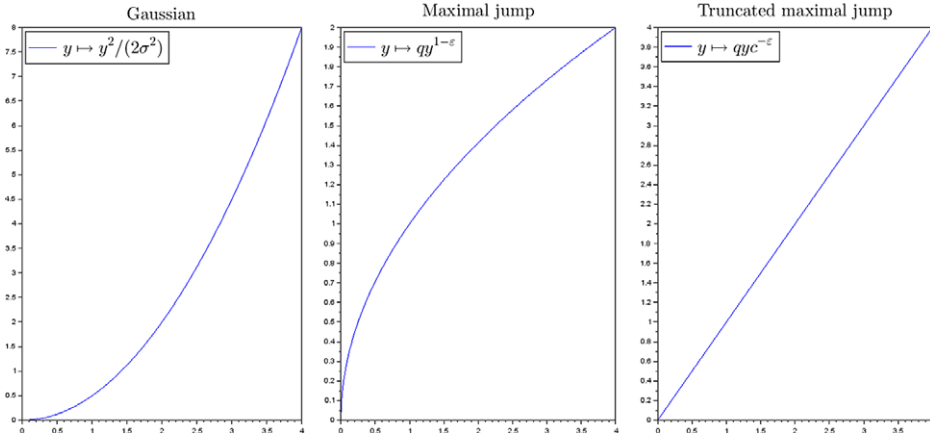
$$\leqslant -q,$$

FIGURE 1. Representation of the rate functions. Here, $q = 1$, $\sigma^2 = 2$, $\varepsilon = 1/2$, and $c = 1$. Left: Gaussian range. The typical event corresponds to the case where all the random variables are small but their sum has a Gaussian contribution. Center: Maximal jump range. The typical event corresponds to the case where one random variable contributes to the total sum ($N_n^\alpha y$), regardless of the others. We recover the random variable tail. Right: Truncated maximal jump range. The typical event corresponds to the case where $N_n^{\alpha - \beta} y/c$ variables take the saturation value $N_n^\beta c$, regardless of the others.

so (H1b) is satisfied. If, moreover, $y_n \leqslant N_n^\beta c'$ with $c' < c$, then $\log \mathbb{P}(Y_n \geqslant y_n) \sim -q y_n^{1-\varepsilon}$, so (H1) is satisfied for $\alpha < \beta$. In addition, $\mathbb{E}[Y_n]$, $\mathbb{E}\big[(Y_n - \mathbb{E}[Y_n])^2\big] - \mathbb{E}\big[Y^2\big]$, and $\mathbb{E}\big[\,|Y_n - \mathbb{E}[Y_n]|^{2+\gamma}\,\big] - \mathbb{E}\big[Y^{2+\gamma}\big]$ are exponentially decreasing to zero. Therefore, (H2′) and (H2$^+$) are satisfied, and our Theorems 2.1, 2.2, 2.3, and 4.3 directly apply (to $Y_n - \mathbb{E}[Y_n]$) for $\alpha < \max(\beta, 1/(1+\varepsilon))$.

For $\alpha \geqslant \max(\beta, 1/(1+\varepsilon))$, the proofs easily adapt to cover all cases. To expose the results, we separate the three cases $\beta > 1/(1+\varepsilon)$, $\beta < 1/(1+\varepsilon)$, and $\beta = 1/(1+\varepsilon)$. We provide the graphs of the exhibited rate functions (Figures 1 and 3) and a synthetic diagram (Figure 4).

5.1. Case $\beta > 1/(1+\varepsilon)$

5.1.1. *Gaussian range* When $\alpha < 1/(1+\varepsilon)$ Theorem 2.2 applies and, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha - 1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -\frac{y^2}{2\sigma^2}.$$

5.1.2. *Transition 1* When $\alpha = 1/(1+\varepsilon)$ Theorem 2.3 applies and, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha - 1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -I_1(y) := -I(y) = -\inf_{0 \leqslant t \leqslant 1} \left\{ q(1-t)^{1-\varepsilon} y^{1-\varepsilon} + \frac{t^2 y^2}{2\sigma^2} \right\}.$$

5.1.3. *Maximal jump range* When $1/(1+\varepsilon) < \alpha < \beta$ Theorem 2.1 applies and, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -q y^{1-\varepsilon}.$$

5.1.4. *Transition 2* When $\alpha = \beta$, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -I_2(c, y) := -q \left(\lfloor y/c \rfloor c^{1-\varepsilon} + (y - \lfloor y/c \rfloor c)^{1-\varepsilon} \right).$$
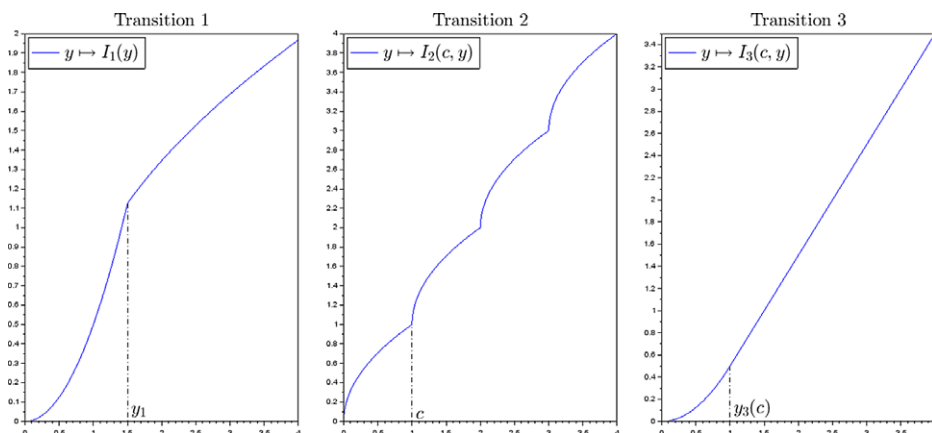
FIGURE 2. Representation of the rate functions. Here, $q = 1$, $\sigma^2 = 1$, $\varepsilon = 1/2$, and $c = 1$. Left: Transition 1. The typical event corresponds to the case where one random variable is large ($N_n^\alpha(1 - t(y))y$) and the sum of the others has a Gaussian contribution (two competing terms). Center: Transition 2. The typical event corresponds to the case where $\lfloor y/c \rfloor$ random variables take the saturation value $N_n^\beta c$ and one completes to get the total sum. Right: Transition 3. The typical event corresponds to the case where some random variables (a number of order $N_n^{1-\beta(1+\varepsilon)}$) take the saturation value $N_n^\beta c$, and the sum of the others has a Gaussian contribution (two competing terms).

Observe that I_2 is continuous on $(0, \infty) \times [0, \infty)$. For all $t > 0$,

$$
\begin{aligned}
\Pi_{n,0} &= \mathbb{P}\big(T_n \geqslant N_n^\alpha y; \text{ for all } i \in [\![1, N_n]\!], Y_{n,i} < N_n^{\alpha\varepsilon}\big) \\
&\leqslant e^{-tyN_n^{\alpha(1-\varepsilon)}} \mathbb{E}\big[e^{tN_n^{-\alpha\varepsilon}Y_n} \mathbf{1}_{Y_n < N_n^{\alpha\varepsilon}}\big]^{N_n} \\
&= e^{-tyN_n^{\alpha(1-\varepsilon)}(1+o(1))},
\end{aligned}
$$

(see the proof of Theorem 2.1). Therefore, letting $t \to \infty$, Lemma 3.5 updates into

$$
\frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \Pi_{n,0} \xrightarrow[n \to \infty]{} -\infty.
$$

For all fixed $m \geqslant \lfloor y/c \rfloor + 1$, let $c' < c$ be such that $\lfloor y/c' \rfloor + 1 \leqslant m$. Then, $\Pi_{n,m} \geqslant \mathbb{P}\big(\text{for all } i \in [\![1, \lfloor y/c' \rfloor]\!], Y_{n,i} \geqslant N_n^\alpha c'; \big[Y_{n,\lfloor y/c' \rfloor} + 1\big] \geqslant N_n^\alpha(y - \lfloor y/c' \rfloor c')\big)$, so

$$
\liminf_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \Pi_{n,m} \geqslant -I_2(c', y).
$$

Letting $c' \to c$, Lemma 3.6 updates into: for all $m \geqslant \lfloor y/c \rfloor + 1$,

$$
\liminf_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \Pi_{n,m} \geqslant -I_2(c, y),
$$

which provides a lower bound for the sum of the $\Pi_{n,m}$. To upper bound the sum of the $\Pi_{n,m}$, Lemma 3.7 still applies. Lemma 3.8 (more precisely (3.7)) adapts as follows: for all $\delta \in (0, 1)$,
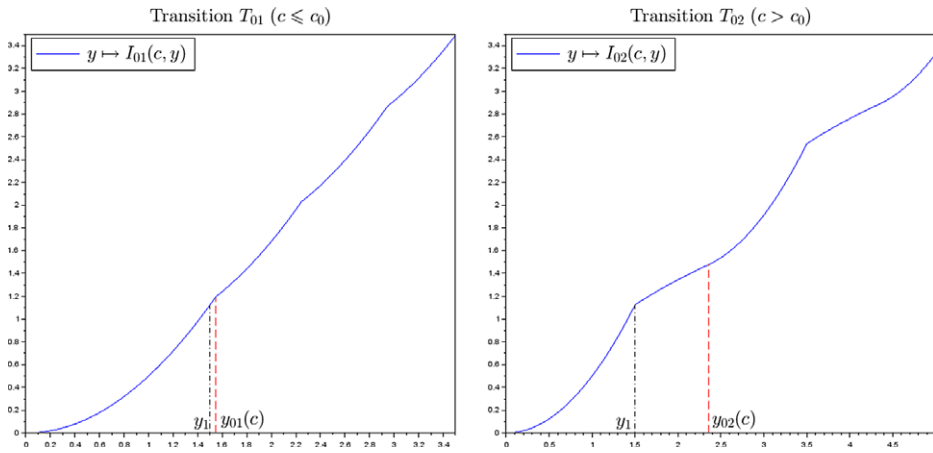
FIGURE 3. Representation of the rate functions. Here, $q = 1$, $\sigma^2 = 1$, and $\varepsilon = 1/2$ (so $c_0 = 1$). Left: Transition 1, for $c \leqslant c_0$ (here, $c = 0.7$). The typical event corresponds to the case where $k_3(c, y)$ variables take the saturation value $N^\beta c$, and the sum of the others has a Gaussian contribution. Right: Transition 1, for $c > c_0$ (here, $c = 2$). The typical event corresponds to the case where $k_2(c, y)$ variables take the saturation value $N^\beta c$, one is also large $\left(N_n^\beta (1 - t(y - k_2(c, y)c))(y - k_2(c, y)c) \right)$, and the sum of the others has a Gaussian contribution.
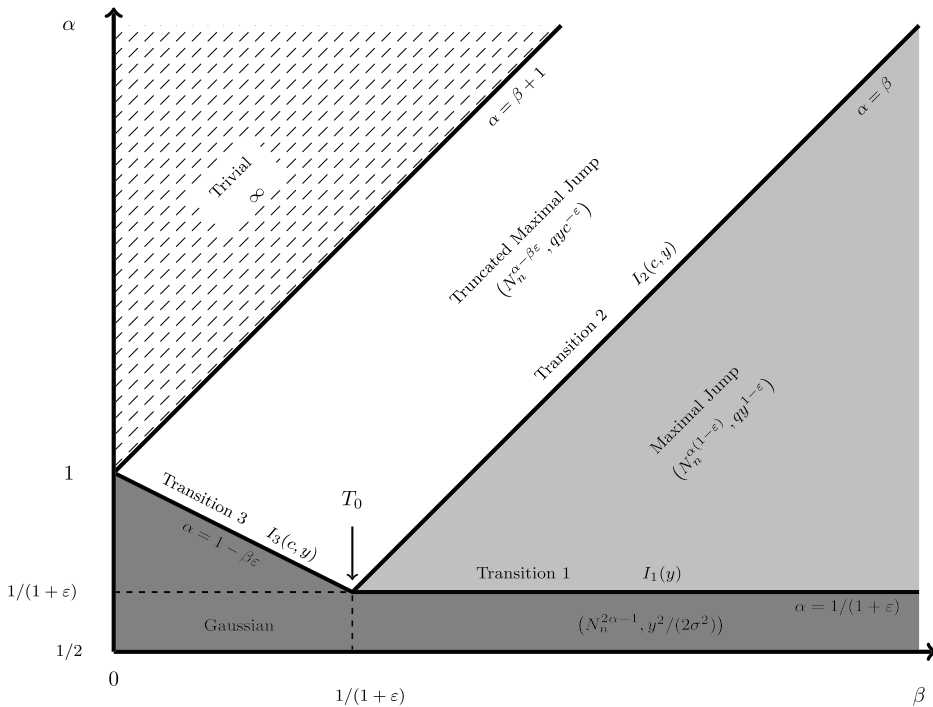

FIGURE 4. Speed and rate function diagram.

there exists $n(\delta) \geqslant 1$ such that, for all $n \geqslant n(\delta)$, for all $m \geqslant 1$, for all $u \in \big[0, N_n^\alpha y\big]$,

$$
\begin{aligned}
&\log \mathbb{P}(T_{n,m} \geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\alpha\varepsilon}) \\
&\leqslant
\begin{cases}
-q' m N_n^{\alpha\varepsilon(1-\varepsilon)} & \text{if } u < m N_n^{\alpha\varepsilon}, \\
-q'\big(l(N_n^\alpha c + 2)^{1-\varepsilon} + (u - l(N_n^\alpha c + 2))^{1-\varepsilon} \\
\quad + (m - l - (m - l)^{1-\varepsilon}) N_n^{\alpha\varepsilon(1-\varepsilon)}\big) & \text{if } m N_n^{\alpha\varepsilon} \leqslant u < m N_n^\alpha c, \\
-\infty & \text{if } m N_n^\alpha c \leqslant u
\end{cases} \\
&=: -M(n, m, u),
\end{aligned}
$$

with

$$
l = \left\lfloor \frac{u - m N_n^{\alpha\varepsilon}}{N_n^\alpha c + 2 - N_n^{\alpha\varepsilon}} \right\rfloor.
$$

Here, following the proof of Lemma 3.8, the concave function $s_m \colon (u_1, \ldots, u_m) \mapsto u_1^{1-\varepsilon} + \cdots + u_m^{1-\varepsilon}$ attains its minimum on the domain of integration at the points with all coordinates equal to $N_n^{\alpha\varepsilon}$, except for l coordinates equal to $N_n^\alpha c + 2$ and one coordinate equal to $u - l(N_n^\alpha c + 2) - (m - l - 1) N_n^{\alpha\varepsilon}$. Finally, Lemma 3.9 adapts as follows: for $\alpha(1-\varepsilon)^2 < \gamma < \alpha(1-\varepsilon)$,

$$
\begin{aligned}
&\limsup_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \\
&\leqslant \limsup_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{k=1}^{r} \sum_{m=1}^{N_n} \exp\left[-(1-\delta) N_n^{2\alpha-1} \frac{(k-1)^2 y^2}{2\sigma^2 r^2} - M(n, m, (1 - k/r) N_n^\alpha y) \right] \\
&= \limsup_{n \to \infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m=1}^{N_n} \exp\left[-M(n, m, (1 - 1/r) N_n^\alpha y) \right],
\end{aligned}
$$

since $2\alpha - 1 > \alpha(1-\varepsilon)$ and, for n large enough,

$$
\begin{aligned}
\frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m > N_n^\gamma} \exp\big[-M(n, m, (1 - 1/r) N_n^\alpha y) \big] \\
\leqslant \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m > N_n^\gamma} \exp\big[-q'(m - l - (m - l)^{1-\varepsilon}) N_n^{\alpha\varepsilon(1-\varepsilon)} \big] \\
\leqslant \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m > N_n^\gamma} \exp\left[-\frac{q}{2} m N_n^{\alpha\varepsilon(1-\varepsilon)} \right] \\
\leqslant -\frac{q}{4} N_n^{\gamma - \alpha(1-\varepsilon)^2} \\
\xrightarrow[n \to \infty]{} -\infty.
\end{aligned}
$$

Now, for $m \leqslant N_n^\gamma$,

$$
\begin{aligned}
M(n, m, &(1 - 1/r)N_n^\alpha y) \\
&\geqslant N_n^{\alpha(1-\varepsilon)}(1 - \delta)q \left(\lfloor (1 - 1/r)y/c \rfloor \, c^{1-\varepsilon} + ((1 - 1/r)y - \lfloor (1 - 1/r)y/c \rfloor \, c)^{1-\varepsilon} \right) \\
&\quad + N_n^{\alpha\varepsilon(1-\varepsilon)}(1 - \delta)q(m - \lfloor y/c \rfloor - m^{1-\varepsilon}) \\
&\geqslant N_n^{\alpha(1-\varepsilon)}(1 - \delta)I_2(c, (1 - 1/r)y) + N_n^{\alpha\varepsilon(1-\varepsilon)}(1 - \delta)q(m - \lfloor y/c \rfloor - m^{1-\varepsilon}).
\end{aligned}
$$

Eventually,

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} &\log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \\
&\leqslant -(1 - \delta)I_2(c, (1 - 1/r)y) \\
&\quad + \limsup_{n\to\infty} \frac{1}{N_n^{\alpha(1-\varepsilon)}} \log \sum_{m \leqslant N_n^\gamma} \exp\left[-(1 - \delta)q(m - \lfloor y/c \rfloor - m^{1-\varepsilon})N_n^{\alpha\varepsilon(1-\varepsilon)} \right] \\
&= -(1 - \delta)I_2(c, (1 - 1/r)y),
\end{aligned}
$$

since the latter series is convergent. The result follows letting $r \to \infty$ and $\delta \to 0$.

5.1.5. *Truncated maximal jump range* When $\beta < \alpha < \beta + 1$ and $y \geqslant 0$, or $\alpha = \beta + 1$ and $y < c$, the proof of Theorem 2.1 adapts and provides

$$
\lim_{n\to\infty} \frac{1}{N_n^{\alpha - \beta\varepsilon}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -qyc^{-\varepsilon}.
$$

The upper bound stems from

$$
\mathbb{P}(T_n \geqslant N_n^\alpha y) \leqslant \mathrm{e}^{-qyc^{-\varepsilon}N_n^{\alpha-\beta\varepsilon}} \mathbb{E}\left[\mathrm{e}^{yc^{-\varepsilon}N_n^{-\beta\varepsilon}Y_n}\left(\mathbf{1}_{Y_n < N_n^{\beta\varepsilon}} + \mathbf{1}_{N_n^{\beta\varepsilon} \leqslant Y_n < N_n^\beta c} \right) \right]^{N_n}
$$

and the same lines as in the proof of Theorem 2.1. As for the lower bound, we write, for $c' < c$,

$$
\begin{aligned}
\log \mathbb{P}(T_n \geqslant N_n^\alpha y) &\geqslant \log \mathbb{P}(\text{for all } i \in [\![1, \lceil N_n^{\alpha-\beta}y/c' \rceil]\!], \, Y_{n,i} \geqslant N_n^\beta c') \\
&\sim -N_n^{\alpha-\beta}y(c')^{-1}q(N_n^\beta c')^{1-\varepsilon} \\
&= -N_n^{\alpha-\beta\varepsilon}qy(c')^{-\varepsilon},
\end{aligned}
$$

and we recover the upper bound when $c' \to c$.

5.1.6. *Trivial case* When $\alpha = \beta + 1$ and $y \geqslant c$, or $\alpha > \beta + 1$, we obviously have $\mathbb{P}(T_n \geqslant N_n^\alpha y) = 0$.

5.2. Case $\beta = 1/(1 + \varepsilon)$

Here, Theorem 2.2 applies for $\alpha < 1/(1 + \varepsilon)$. The notable fact is that the Gaussian range is extended: it spreads up to $\alpha < 1 - \beta\varepsilon$.

5.2.1. *Gaussian range* When $\alpha < 1 - \beta\varepsilon$ the proof of Theorem 2.2 adapts and, for all $y \geqslant 0$,

$$
\lim_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -\frac{y^2}{2\sigma^2}.
$$

As we said, the result for $\alpha < 1/(1+\varepsilon)$ is a consequence of Theorem 2.2. Now, suppose $\beta < 1/(1+\varepsilon) \leqslant \alpha < 1 - \beta\varepsilon$. Here, we adapt the decomposition (3.1) and (3.2) as

$$\mathbb{P}(T_n \geqslant N_n^\alpha y) = \sum_{m=0}^{N_n} \binom{N_n}{m} \Pi_{n,m},$$

where, for all $m \in [\![0, N_n]\!]$,

$$\Pi_{n,m} = \mathbb{P}\big(T_n \geqslant N_n^\alpha y; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\beta\varepsilon}; \text{ for all } i \in [\![m+1, N_n]\!], Y_{n,i} < N_n^{\beta\varepsilon}\big).$$

Lemma 3.5 works for $\alpha < 1 - \beta\varepsilon$, adapting the proof with $\mathcal{L}(Y_n^<) = \mathcal{L}(Y_n \mid Y_n < N_n^{\beta\varepsilon})$. To upper bound the sum of the $\Pi_{n,m}$, Lemma 3.7 still applies. Lemma 3.8 (more precisely (3.7)) adapts as follows: for all $\delta \in (0, 1)$, there exists $n(\delta) \geqslant 1$ such that, for all $n \geqslant n(\delta)$, for all $m \geqslant 1$, for all $u \in [0, N_n^\alpha y]$,

$$\begin{aligned}
\log \mathbb{P}(T_{n,m} &\geqslant u; \text{ for all } i \in [\![1, m]\!], Y_{n,i} \geqslant N_n^{\beta\varepsilon}) \\
&\leqslant \begin{cases}
-q' m N_n^{\beta\varepsilon(1-\varepsilon)} & \text{if } u < m N_n^{\beta\varepsilon}, \\
-q'\big(l\big(N_n^\beta c + 2\big)^{1-\varepsilon} + \big(u - l\big(N_n^\beta c + 2\big)\big)^{1-\varepsilon} \\
\quad + (m - l - (m-l)^{1-\varepsilon}) N_n^{\beta\varepsilon(1-\varepsilon)}\big) & \text{if } m N_n^{\beta\varepsilon} \leqslant u < m N_n^\beta c, \\
-\infty & \text{if } m N_n^\beta c \leqslant u
\end{cases} \\
&=: -M(n, m, u),
\end{aligned}$$

with

$$l = \left\lfloor \frac{u - m N_n^{\beta\varepsilon}}{N_n^\beta c + 2 - N_n^{\beta\varepsilon}} \right\rfloor.$$

Here, following the proof of Lemma 3.8, the concave function $s_m \colon (u_1, \ldots, u_m) \mapsto u_1^{1-\varepsilon} + \cdots + u_m^{1-\varepsilon}$ reaches its minimum on the domain of integration at the points with all coordinates equal to $N_n^{\beta\varepsilon}$, except for l coordinates equal to $N_n^\beta c + 2$ and one coordinate equal to $u - l(N_n^\beta c + 2) - (m - l - 1)N_n^{\beta\varepsilon}$. Finally, Lemma 3.9 adapts as follows: for $2\alpha - 1 - \beta\varepsilon(1 - \varepsilon) < \gamma < \alpha - \beta\varepsilon$ (note that $2\alpha - 1 < \alpha - \beta\varepsilon$),

$$\begin{aligned}
\limsup_{n\to\infty} &\frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{k=1}^r \exp\left[-(1-\delta)\left(N_n^{2\alpha-1} \frac{((k-1)/r)^2 y^2}{2\sigma^2}\right)\right] \\
&\qquad\qquad \times \sum_{m=1}^{N_n} e^{-M(n,m,(1-k/r)N_n^\alpha y)}.
\end{aligned}$$

For $k \leqslant r - 1$ and n large enough,

$$\begin{aligned}
\frac{1}{N_n^{2\alpha-1}} \log &\sum_{m > N_n^\gamma} e^{-M(n,m,(1-k/r)N_n^\alpha y)} \\
&\leqslant \frac{1}{N_n^{2\alpha-1}} \log \sum_{m > N_n^\gamma} \exp\big[-q'(m - l - (m-l)^{1-\varepsilon})N_n^{\beta\varepsilon(1-\varepsilon)}\big]
\end{aligned}$$

$$\leqslant \frac{1}{N_n^{2\alpha-1}} \log \sum_{m > N_n^{\gamma}} \exp\left[-\frac{q}{2} m N_n^{\beta\varepsilon(1-\varepsilon)}\right]$$

$$\leqslant -\frac{q}{4} N_n^{\gamma-(2\alpha-1-\beta\varepsilon(1-\varepsilon))}$$

$$\xrightarrow[n\to\infty]{} -\infty$$

and

$$\frac{1}{N_n^{2\alpha-1}} \log \sum_{m \leqslant N_n^{\gamma}} \mathrm{e}^{-M(n,m,(1-k/r)N_n^{\alpha}y)}$$

$$\leqslant -(1-\delta)q N_n^{\alpha-\beta\varepsilon-(2\alpha-1)} \lfloor y/(rc) \rfloor \, c^{1-\varepsilon}$$

$$+ \frac{1}{N_n^{2\alpha-1}} \log \sum_{m \leqslant N_n^{\gamma}} \exp\left[-(1-\delta)q(m-l-(m-l)^{1-\varepsilon})N_n^{\beta\varepsilon(1-\varepsilon)}\right]$$

$$\xrightarrow[n\to\infty]{} -\infty,$$

since the latter series is convergent. Eventually,

$$\limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \binom{N_n}{m} \Pi_{n,m}$$

$$\leqslant -(1-\delta)\frac{(1-1/r)^2 y^2}{2\sigma^2} + \limsup_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \sum_{m=1}^{N_n} \mathrm{e}^{-q' m N_n^{\beta\varepsilon(1-\varepsilon)}}$$

$$= -(1-\delta)\frac{(1-1/r)^2 y^2}{2\sigma^2},$$

since the latter series is convergent. The result follows letting $r \to \infty$ and $\delta \to 0$.

5.2.2. *Transition 3* When $\alpha = 1 - \beta\varepsilon$ the proof of Theorem 2.3 adapts and, for all $y \geqslant 0$,

$$\lim_{n\to\infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^{\alpha}y) = -I_3(c,y) := -\inf_{0 \leqslant t \leqslant 1}\left\{q(1-t)yc^{-\varepsilon} + \frac{t^2 y^2}{2\sigma^2}\right\}$$

$$= -\begin{cases} \frac{y^2}{2\sigma^2} & \text{if } y \leqslant y_3(c), \\ \frac{qy}{c^{\varepsilon}} - \frac{q^2\sigma^2}{2c^{2\varepsilon}} & \text{if } y > y_3(c), \end{cases}$$

with $y_3(c) := q\sigma^2 c^{-\varepsilon}$.

5.2.3. *Truncated maximal jump range* When $1 - \beta\varepsilon < \alpha < 1 + \beta$ and $y \geqslant 0$, or $\alpha = 1 + \beta$ and $y < c$, as before, the proof of Theorem 2.1 adapts and

$$\lim_{n\to\infty} \frac{1}{N_n^{\alpha-\beta\varepsilon}} \log \mathbb{P}(T_n \geqslant N_n^{\alpha}y) = -qyc^{-\varepsilon}.$$

5.2.4. *Trivial case* When $\alpha = \beta + 1$ and $y \geqslant c$, or $\alpha > \beta + 1$, we obviously have $\mathbb{P}(T_n \geqslant N_n^{\alpha}y) = 0$.

5.3. Case $\beta = 1/(1+\varepsilon)$

5.3.1. *Gaussian range* When $\alpha < 1/(1+\varepsilon) = \beta$ Theorem 2.2 applies and, for all $y \geqslant 0$,

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -\frac{y^2}{2\sigma^2}.$$

5.3.2. *Transition T_0* As in Section 2 after the statement of Theorem 2.3, we define $t(y)$ and y_1 for the function $f(t) = q(1-t)^{1-\varepsilon} y^{1-\varepsilon} + t^2 y^2/(2\sigma^2)$. Define $\tilde{t}(y) := 1$ for $y < y_1$ and $\tilde{t}(y) := t(y)$ for $y \geqslant y_1$, and notice that $\tilde{t}$ is decreasing on $[y_1, \infty)$ (and $\tilde{t}(y) \to 0$ as $y \to \infty$). Set $c_0 := (1 - \tilde{t}(y_1))y_1 = (2\varepsilon q\sigma^2)^{1/(1+\varepsilon)}$.

When $\alpha = 1/(1+\varepsilon) = \beta$ and $c \leqslant c_0$, then

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -qk_{0,1}(c, y)c^{1-\varepsilon} + \frac{(y - k_{0,1}(c, y)c)^2}{2\sigma^2} =: -I_{0,1}(c, y),$$

where

$$k_{0,1}(c, y) := \max \left(\left\lfloor \frac{y - y_{0,1}(c)}{c} \right\rfloor + 1, 0 \right), \qquad \text{and} \quad y_{0,1}(c) := \frac{c}{2} + q\sigma^2 c^{-\varepsilon}$$

($y_{0,1}(c)$ is the unique solution in y of $y^2 - (y - c)^2 = 2\sigma^2 qc^{1-\varepsilon}$).

When $\alpha = 1/(1+\varepsilon) = \beta$ and $c \geqslant c_0$, then

$$\lim_{n \to \infty} \frac{1}{N_n^{2\alpha-1}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -qk_{0,2}(c, y)c^{1-\varepsilon} + I(y - k_{0,2}(c, y)c) =: -I_{0,2}(c, y),$$

where

$$k_{0,2}(c, y) := \max \left(\left\lfloor \frac{y - y_{0,2}(c)}{c} \right\rfloor + 1, 0 \right), \qquad \text{and} \quad y_{0,2}(c) := c + (1-\varepsilon)q\sigma^2 c^{-\varepsilon}$$

($y_{0,2}(c)$ is the unique solution in y of $(1 - \tilde{t}(y))y = c$).

We remark that, for all $c < c_0$, $y_{0,1}(c) > y_1$, so the Gaussian range in the truncated case is extended compared with the range in the non-truncated case (where it stops at y_1). Moreover, $y_{0,1}(c_0) = y_1 = y_{0,2}(c_0)$ and $I_{0,1}(c_0, \cdot) = I_{0,2}(c_0, \cdot)$ (since $I_1(y) = y^2/(2\sigma^2)$ for $y \leqslant y_1$).

5.3.3. *Truncated maximal jump range* When $1/(1+\varepsilon) = \beta < \alpha < \beta + 1$ and $y \geqslant 0$, or $\alpha = 1 + \beta$ and $y < c$, as before, the proof of Theorem 2.1 adapts and

$$\lim_{n \to \infty} \frac{1}{N_n^{\alpha-\beta\varepsilon}} \log \mathbb{P}(T_n \geqslant N_n^\alpha y) = -qyc^{-\varepsilon}.$$

5.3.4. *Trivial case* When $\alpha = \beta + 1$ and $y \geqslant c$, or $\alpha > \beta + 1$, we obviously have $\mathbb{P}(T_n \geqslant N_n^\alpha y) = 0$.

Competing Interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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