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A NOTE ON CENTRAL IDEMPOTENTS IN GROUP RINGS II

by SÔNIA P. COELHO and C. POLCINO MILIES

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Let G be a group and K a field. We shall denote by $U(KG)$ the group of units of the group ring of G over K. Also, if X is a group, $T(X)$ will denote the torsion subset of X, i.e., the set of all elements of finite order in X.

Group theoretical properties of $U(KG)$ have been studied intensively in recent years and it has been found that some conditions about $U(KG)$ imply that $T=T(G)$ must be a subgroup of G and that every idempotent of KT must be central in KG.

In a previous note [**1**], the first author has studied this fact, and determined what it actually implies about the structure of the original group G and the given field K, when $\mathrm{char}(K)=p>0$.

Here, we carry on this study to the case where $\mathrm{char}(K)=0$ and prove the following:

Theorem. *Let K be a field of characteristic* 0 *and T the set of torsion elements of a group G. Then, every idempotent of KG whose support lies in T is central in KG if and only if:*

(i) *For every $t\in T$ and every $x\in G$ there exists a positive integer j such that $xtx^{-1}=t^j$. Furthermore, for every noncentral element $t\in T$, K contains no root of unity of order equal to $o(t)$.*

(ii) *Either T is abelian or $T=A\times E\times K_8$ where A is an abelian group such that every element in A is of odd order, E is an elementary abelian 2-group, K_8 is the quaternion group of order 8 and for every root of unity ξ in an algebraic closure Ω of K such that $o(\xi)=o(a)$, for some $a\in A$, the field $K(\xi)$ contains no nontrivial solution of the equation $X^2+Y^2+Z^2=0$.*

We shall first establish some lemmas.

Lemma 1. *If every idempotent $e\in KG$ whose support lies in T is central in KG then we have that:*

(i) *For every $t\in T$ and every $x\in G$ there exists a positive integer j such that $xtx^{-1}=t^j$.*

(ii) *T is a subgroup of G.*

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Proof. Set $t \in T$. Then $e = o(t)^{-1}\ (l + t + \cdots + t^{o(t)-1})$ is an idempotent of KG with $\mathrm{supp}(e) \subset T$, so it is central.

Given any element $x \in G$ we must have that $xex^{-1} = e$ and, by considering the support of the elements in both sides of this equation, we see that $xtx^{-1} = t^j$, for some positive integer j.

Now, if $t_1, t_2 \in T$, it is easy to see that $(t_1t_2)^2 = t_2^j t_1^2 t_2$ and, inductively, that $(t_1t_2)^r = t_2^{jr} . t_1^r t_2$. So if $m = o(t_1)$ we have that $(t_1t_2)^m \in \langle t_2 \rangle$ is an element of finite order and hence $t_1t_2 \in T$. □

Lemma 2. *Let F be a field of characteristic O. Then FK_8 contains no noncentral idempotent if and only if the equation $X^2 + Y^2 + Z^2 = 0$ has no nontrivial solution in F.*

Proof. It is well-known that $FK_8 \cong F \oplus F \oplus F \oplus F \oplus S$, where S is either a division ring or $S \cong M_2(F)$, the ring of 2×2 matrices with entries in F.

Hence, FK_8 contains a noncentral idempotent if and only if $S \cong M_2(F)$, i.e., if and only if S contains a nilpotent element.

Our statement now follows directly from [**4**, Prop. VI, 1.13]. □

We conclude our preliminaries with a simple observation. Let $F \subset L$ be fields of characteristic O and H a finite abelian group. Let $\{e_i\}_{1 \leqq i \leqq m}$ and $\{f_j\}_{1 \leqq j \leqq n}$ be the families of idempotents corresponding to the decompositions:

$$FH = A_1 \oplus \cdots \oplus A_m$$

$$LH = B_1 \oplus \cdots \oplus B_n$$

of FH and LH as sums of simple components, respectively.

When we extend coefficients from FH to LH, each simple component A_i of FH splits into a sum of some simple components $\{B_{j_i}\}$ of LH. Hence, we have the following:

Remark. Given $e_r \neq e_s$, we can write $e_r = \Sigma f_{j_r}$ and $e_s = \Sigma f_{j_s}$ where both expressions have no commom summand.

We are now ready to prove our main statement.

Proof of the theorem

Necessity. The first part of condition (i) has been established in Lemma 1. So, let t be a noncentral element in T and let $x \in G$ be such that $xtx^{-1} = t^j$ with $j \not\equiv 1 \pmod{o(t)}$.

We can write $K\langle t \rangle$ as a sum

$$K\langle t \rangle \overset{\phi}{\cong} K_1 \oplus \cdots \oplus K_s$$

with $K_i = K(\xi_i)$ where ξ_i is a root of unity, $1 \leqq i \leqq s$, and at least one of them, ξ_1 say, is such that $o(\xi_1) = o(t)$.

Here, we have that $\phi(t) = (\xi_1, \ldots, \xi_s)$.

Since every idempotent of KT is central in KG, conjugation by x defines an automorphism $\theta: K\langle t\rangle \to K\langle t\rangle$, which induces automorphisms θ_i on each simple component K_i, $1 \leqq i \leqq s$.

It is easy to see that each K_i contains a copy $\bar{K}_i$ of K, which is fixed under θ_i. On the other hand, since $xtx^{-1} = t^j$, we see that, in particular, $\theta_1(\xi_1) = \xi_1^j \neq \xi_1$ and hence $\xi_1 \notin K$. This shows that K contains no root of unity of order $o(t)$, as desired.

To establish (ii) we note that Lemma 1 implies that all subgroups in T are normal. Consequently, we know from [**2**, Theorem 12.5.4] that T is either abelian or a group of the form $T = K_8 \times A \times E$ as in the statement of our theorem.

So, assume that T is nonabelian and let a be an element in A. Once more we write $K\langle a\rangle = K_1 \oplus \cdots \oplus K_t$ with $K_1 = K(\xi_1)$, where ξ_1 is a primitive root of unity of order $o(\xi_1) = o(a)$.

Then, the group ring $K(\langle a\rangle \times K_8)$ contains a copy of $K(\xi_1)K_8$ and, by hypothesis, every idempotent in $K(\xi_1)K_8$ is central. Hence, Lemma 2 shows that the equation $X^2 + Y^2 + Z^2 = 0$ has no solution in $K(\xi_1)$.

Sufficiency. Assume first that (i) holds that T is an abelian subgroup of G.

Let $e \in KT$ be an idempotent. By considering the subgroup generated by supp(e) we may assume, without loss of generality, that T is finite. Furthermore, since every idempotent of KT is a sum of primitive idempotents we may restrict ourselves to consider the case where e is primitive.

Now, let ζ be a primitive root of unity of order equal to the exponent of T. Then $e \in KT \subset K(\zeta)T$ is a sum of primitive idempotents of $K(\zeta)T$; let f be one of these.

Every K-automorphism of $K(\zeta)$ can be extended in a natural way to $K(\zeta)T$. We consider all such automorphisms which give distinct images when computed on f and denote them by $\phi_1 = I, \phi_2, \ldots, \phi_r$.

We set $e^* = \phi_1(f) + \cdots + \phi_r(f)$ and we wish to prove that $e^* = e$.

Notice that each $\phi_i(f)$, $1 \leqq i \leqq r$, is a primitive idempotent of KT. Hence they are pairwise orthogonal and thus $e^* = \phi_1(f) + \cdots + \phi_r(f)$ is also an idempotent.

Since $\phi_1(f)$ is a summand in the decompositions of both e and e^*, due to the remark above it will suffice to show that e^* is a primitive idempotent of KT.

We shall now prove that the coefficients of e^* belong to K. Let ϕ be any K-automorphism of $K(\zeta)$ and denote also by ϕ its extension to $K(\zeta)T$.

Then $\phi(e^*) = \sum_{i=1}^{r} \phi \circ \phi_i(f)$ and since $\{\phi \circ \phi_i(f)\}_{1 \leqq i \leqq r} = \{\phi_i(f)\}_{1 \leqq i \leqq r}$ it follows that $\phi(e^*) = e^*$. Hence, $e^* \in KT$.

It remains to show that e^* is primitive in KT. So, assume that $e^* = e_1 + e_2$ where $e_1, e_2 \in KT$ are orthogonal idempotents.

When written as sums of primitive orthogonal idempotents in $K(\zeta)T$, e_1 and e_2 can have no common summand, so, reordering if necessary, we have

$$e_1 = \phi_1(f) + \cdots + \phi_{l-1}(f) \quad \text{and} \quad e_2 = \phi_l(f) + \cdots + \phi_r(f).$$

Also, since $e_1 \in KT$, it is fixed by all such K-automorphisms, thus, in particular we have:

$$e_1 = \phi_l(e_1) = \phi_l \circ \phi_1(f) + \cdots + \phi_l \circ \phi_{l-1}(f).$$

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But $\phi_l \circ \phi_1(f) = \phi_l\,(f)$ so e_1 and e_2 would have a common summand, a contradiction.

So, we have established that $e = e^*$.

Now, let x be a n arbitrary element in G and write $T = \langle t_1 \rangle x \ldots x \langle t_s \rangle$, a direct product of cyclic groups. We can find a positive integer j such that $xt_kx^{-1} = t_k^j$, for every index k, $1 \leqq k \leqq s$.

According to [3, Theorem 2.12] we can write f in the form:

$$f = \frac{1}{|T|} \sum_{t \in T} \chi(t^{-1})t$$

where χ is an irreducible character with values in $K(\zeta)$.

Then:

$$e = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i(\chi(t^{-1}) \right) t$$

and hence:

$$xex^{-1} = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i(\chi(t^{-1}) \right) t^j.$$

So, it will suffice to prove that $\sum_i \phi_i(\chi(t)) = \sum_i \phi_i(\chi(t^j))$, for every $t \in T$.

Since T is abelian and $K(\zeta)$ is a splitting field for T, it is easy to see that $\chi(t)$ is a primitive root of unity whose order divides $\exp(T)$. So, we may assume that $\chi(t) = \xi$ where ξ is some power of ζ.

By (i), if t is not central (in which case, there would be nothing to prove) then $\zeta \notin K$ and we can define a K-automorphism of $K(\zeta)$ such $\phi(\zeta) = \zeta^j$ and extend it to a K-automorphism of $K(\zeta)T$ in the usual way, which we still denote by ϕ.

Now, since $\{\phi_i\}_{1 \leqq i \leqq r} = \{\phi_i \circ \phi\}_{i \leqq 1 \leqq r}$, we have:

$$\sum_i \phi_i(\chi(t)) = \sum_i \phi_i(\xi) = \sum_i \phi_i \circ \phi(\xi) = \sum_i \phi_i(\xi^j) = \sum_i \phi_i(\chi(t^j))$$

as we intend to prove.

Finally assume that (i) and (ii) hold and that T is not abelian. As before, we may suppose that T is finite.

Once again, let ζ be a primitive root of unity, of order $o(\zeta) = \exp(A)$. Then, we can write $K(A \times E) = \oplus_i K_i$, a direct sum of fields, all of which are contained in $K(\zeta)$.

Hence, $KT = K(A \times E \times K_8) \cong K(A \times E)K_8 \cong \oplus_i K_iK_8$. By our hypothesis, the equation $X^2 + Y^2 + Z^2 = 0$ has no nontrivial solution in any of the fields K_i, so every idempotent of K_iK_8 is central in KT and our given idempotent e is a sum of central idempotents in some of these components.

To show that e is also central in KG consider an arbitrary element $x \in G$. We wish to show that any idempotent in a given component of the form K_iK_8 commutes with x. To see this, denote by μ_i the identity element of K_i. Notice that each μ_i is an idempotent in $K(A \times E)$ and hence is central in KG because of our earlier argument. Then, the idempotents in K_iK_8 are precisely those corresponding to its simple components and can be written explicitly. If we give K_8 by its presentation:

$$K_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^3 \rangle$$

these idempotents are:

$$e_1 = \frac{\mu_i}{8}(1 + a + a^2 + a^3 + b + ab + a^2b + a^3b)$$

$$e_2 = \frac{\mu_i}{8}(1 + a + a^2 + a^3 - b - ab - a^2b - a^3b)$$

$$e_3 = \frac{\mu_i}{8}(1 - a + a^2 - a^3 + b - ab + a^2b - a^3b)$$

$$e_4 = \frac{\mu_i}{8}(1 - a + a^2 - a^3 - b + ab - a^2b + a^3b)$$

$$e_5 = \frac{\mu_i}{8}(1 - a^2).$$

Condition (i) implies that either $xax^{-1} = a$ or $xax^{-1} = a^3$ and that either $xbx^{-1} = b$ or $xbx^{-1} = a^2b$. In all cases, it follows easily that all the idempotents above are central and hence, e itself is also central. □

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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO
CAIXA POSTAL 20.570—AG. IGUATEMI
01000—SÃO PAULO—BRAZIL