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ORDERING CONDITIONAL DISTRIBUTIONS OF GENERALIZED ORDER STATISTICS

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The concept of generalized order statistics was introduced as a unified approach to a variety of models of ordered random variables. The purpose of this article is to establish the usual stochastic and the likelihood ratio orderings of conditional distributions of generalized order statistics from one sample or two samples, strengthening and generalizing the main results in Khaledi and Shaked [15], and Li and Zhao [17]. Some applications of the main results are also given.

1. INTRODUCTION

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the ordinary order statistics of independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_n$. Asadi and Bairamov [1] explored the properties of $\mathbb{E}[X_{n:n} - y|\, X_{r:n} > y]$ for $r = 1, \ldots, n-1$ and each y, and obtained

$$\mathbb{E}[X_{n-1:n-1} - y|X_{1:n-1} > y] \le \mathbb{E}[X_{n:n} - y|X_{1:n} > y] \tag{1.1}$$

and

$$\mathbb{E}[X_{n:n} - y | X_{r:n} > y] \leq \mathbb{E}[X_{n:n} - y | X_{r-1:n} > y] \tag{1.2}$$

for $2 \leq r < n$ and $y \in \Re$. Li and Zhao [17] proved that

$$[X_{s:n} - y | X_{r:n} > y] \leq_{\mathrm{lr}} [X_{s:n} - y | X_{r-1:n} > y] \tag{1.3}$$

for $1 < r < s \leq n$ and $y \in \Re$, generalizing (1.2), where $\leq_{\mathrm{lr}}$ denotes the likelihood ratio order (the formal definitions of some stochastic orders that are mentioned in this section can be found in Section 2).

Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ denote the ordinary order statistics of other i.i.d. random variables $Y_1, Y_2, \ldots, Y_n$. Recently, Khaledi and Shaked [15] and Li and Zhao [17] proved that if $X_1 \leq_{\mathrm{hr}} Y_1$, then

$$[X_{s:n} - y | X_{r:n} > y] \leq_{\mathrm{st}} [Y_{s:n} - y | Y_{r:n} > y] \quad \text{for } y \in \Re \tag{1.4}$$

whenever $1 \leq r \leq s \leq n$. Khaledi and Shojaei [16] established the analogous result of (1.4) for record values. More precisely, let $\{X_{L(n)}, n \geq 1\}$ denote the record values of a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables with a continuous distribution function F, where $L(1) = 1$ and

$$L(n) = \inf\{i : X_i > X_{L(n-1)}, i > L(n-1)\} \quad \text{for } n \geq 2$$

are record times. Similarly, let $\{Y_{M(n)}, n \geq 1\}$ denote the record values based on another sequence of i.i.d. random variables $\{Y_n, n \geq 1\}$ with a continuous distribution function G. Khaledi and Shojaei [16] proved that if $X_1 \leq_{\mathrm{hr}} Y_1$, then

$$[X_{L(s)} - y | X_{L(r)} > y] \leq_{\mathrm{st}} [Y_{M(s)} - y | Y_{M(r)} > y] \quad \text{for } y \in \Re$$

whenever $s \geq r \geq 1$.

Note that the ordinary order statistics and record values are two special models of generalized order statistics (GOSs). The purpose of this article is to establish some results on stochastic comparisons of conditional distributions of GOSs with respect to the usual stochastic and the likelihood ratio orders, strengthening and generalizing (1.4) and (1.5). The main results of this article are given in Section 3. In Section 2 we recall the definitions of GOSs and of some stochastic orders and give some useful lemmas that will be used in Section 3. Some applications of the main results are presented in Section 4.

2. PRELIMINARIES

2.1. Generalized Order Statistics

We first give the definition of GOSs following Kamps [10, 11].

Definition 2.1: *Let* $n \in \mathbb{N}$, $k > 0$, *and* $m_1, \ldots, m_{n-1} \in \Re$ *be parameters such that*

$$\gamma_{r,n} = k + \sum_{j=r}^{n-1} (m_j + 1) > 0, \quad r = 1, \ldots, n-1,$$

and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ *if* $n \geq 2$ ($\tilde{m}$ *arbitrary if* $n = 1$). *If the random variables* $U_{(r,n,\tilde{m},k)}$, $r = 1, \ldots, n$, *possess a joint density of the form*

$$f_{U_{(1,n,\tilde{m},k)},\ldots,U_{(n,n,\tilde{m},k)}}(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_{j,n}\right)\left(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ *of* $\Re^n$, *then they are called uniform GOSs. Now, let F be an arbitrary distribution function. The random variables*

$$X_{(r,n,\tilde{m},k)} = F^{-1}(U_{(r,n,\tilde{m},k)}), \quad r = 1, \ldots, n,$$

are called the GOSs based on F, where

$$F^{-1}(u) = \sup\{x: F(x) \leq u\} \quad \text{for } u \in (0,1).$$

In the particular case $m_1 = \cdots = m_{n-1} = m$, *the above random variables are denoted by* $U_{(r,n,m,k)}$ and $X_{(r,n,m,k)}$, $r = 1, \ldots, n$, *respectively.*

Choosing the parameters appropriately, several other models of ordered random variables are seen to be particular cases. One can refer to Kamps [11] for ordinary order statistics, k-record values, sequential order statistics, and Pfeifer's records, to Cramer and Kamps [5] for progressive type II censored order statistics, and to Belzunce, Mercader, and Ruiz [4] for order statistics under multivariate imperfect repair.

Throughout this article, we consider the special case of GOSs with $m_1 = \cdots = m_{n-1} = m$, in which the marginal distribution and density functions of the rth GOS have closed forms. If F is absolutely continuous with density function f, then the marginal density function of the rth GOS, $X(r, n, m, k)$, based on F is given by

$$f_{X(r,n,m,k)}(x) = \phi_{r,n,m,k}(F(x))f(x), \tag{2.1}$$

where

$$\phi_{r,n,m,k}(u) = \frac{c_{r-1,n}}{(r-1)!}(1-u)^{\gamma_{r,n}-1}[\delta_m(u)]^{r-1}, \qquad u \in (0,1), \tag{2.2}$$

and

$$c_{r-1,n} = \prod_{j=1}^{r} \gamma_{j,n}, \qquad \gamma_{n,n} = k$$

(see Lemma 3.3 in Kamps [11]). Here, the function δ_m: $[0, 1) \to \Re$, $m \in \Re$, is

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defined by

$$\delta_m(x) = \begin{cases} \dfrac{1}{(m+1)}[1-(1-x)^{m+1}], & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases}$$

It is easy to see that

$$\gamma_{r,n} = k + (n-r)(m+1), \qquad r = 1, \ldots, n, \tag{2.3}$$

and that $g_m(x)$ is nonnegative and increasing in $x \in [0, 1)$ for each $m \in \Re$.

Recently, several articles have dealt with stochastic comparisons of (unconditional) GOSs and their spacings, among which are Belzunce, Mercader, and Ruiz [4], Khaledi [14], and Hu and Zhuang [7–9].

2.2. Stochastic Orders

We recall the definitions of three stochastic orders that will be useful in this article. Throughout, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively. $a/0$ is understood to be $+\infty$ whenever $a > 0$. For any distribution function F, $\bar{F} = 1 - F$ denotes its survival function.

DEFINITION 2.2: *Let X and Y be two random variables with respective distribution functions F and G. We say that X is smaller than Y*

- *in the usual stochastic order, denoted by* $X \le_{\text{st}} Y$, if $\bar{F}(t) \le \bar{G}(t)$ *for all t*
- *in the hazard rate order, denoted by* $X \le_{\text{hr}} Y$, *if* $\bar{G}(t)/\bar{F}(t)$ *is increasing in t*
- *in the likelihood ratio order, denoted by* $X \le_{\text{lr}} Y$, if *X and Y have respective density functions (or mass functions) f and g, and if* $g(t)/f(t)$ *is increasing in t*

The relationships among these orders are shown in the following diagram (see Shaked and Shanthikumar [20] and Müller and Stoyan [19]):

$$X \le_{\text{lr}} Y \Longrightarrow X \le_{\text{hr}} Y \Longrightarrow X \le_{\text{st}} Y.$$

We will write $F \le_* G$ if $X \le_* Y$, where $\le_*$ is $\le_{\text{st}}$, $\le_{\text{hr}}$ or $\le_{\text{lr}}$.

2.3. Some Useful Lemmas

To prove the main results in the next section, we will need the following lemmas. The first lemma states that under suitable restrictions on the parameters of GOSs, the conditional distribution of one GOS given another lower indexed one based on a continuous distribution has the same distribution as some GOSs based on the truncated parent distribution. We denote by $[Y|A]$ any random variable whose distribution is the conditional distribution of Y given event A.

LEMMA 2.1: (Keseling [13], Hu and Zhuang [8]): *Let* $X_{(r,n,m,k)}$, $r = 1, \ldots, n$, *be GOSs based on a continuous distribution function F. For each* $u \in \mathrm{Supp}(F)$, *the support of F, denote*

$$F_u(x) = 1 - \frac{\bar{F}(\max\{u+x, u\})}{\bar{F}(u)} \quad \text{for } x \in \Re. \tag{2.4}$$

Then

$$[X_{(r+p-1,n,m,k)} - X_{(r-1,n,m,k)} | X_{(r-1,n,m,k)} = u] \overset{\text{st}}{=} X^u_{(p,n-r+1,m,k)},$$

where $p \geq 1$, $2 \leq r \leq n - p + 1$, $X^u_{(p,n-r+1,m,k)}$ *is a GOS based on* F_u, *and* $\overset{\text{st}}{=}$ *means equal in distribution.*

The second lemma is the special case of Corollary 3.2 of Hu and Zhuang [7] with all m_i equal.

LEMMA 2.2: *Let* $\{X_{(r,n,m,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,m,k)}, r = 1, \ldots, n\}$ *be the GOSs based on distribution functions F and G, respectively, with* $m \geq -1$. *If* $F \leq_{\text{hr}} G$, *then*

$$X_{(r,n,m,k)} \leq_{\text{hr}} Y_{(r',n',m,k)} \quad \text{whenever } r' - r \geq \max\{0, n' - n\}.$$

The next lemma follows from Corollary 3.2 of Hu and Zhuang [7] and from a proof similar to that of Lemma 2.3 in Hu and Zhuang [9].

LEMMA 2.3: *Let* $\{X_{(r,n,m,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,m,k)}, r = 1, \ldots, n\}$ *be the GOSs based on distribution functions F and G with* $k \geq 1$ *and* $m \geq -1$, *and denote by* $\lambda(x)$ *and* $\eta(x)$ *the hazard rate functions of F and G, respectively. If*

$$F \leq_{\text{lr}} G \quad \text{for } m \geq 0, \tag{2.5}$$

and if

$$F \leq_{\text{hr}} G \quad \text{and} \quad \frac{\eta(x)}{\lambda(x)} \text{ is increasing in } x \text{ for } m \in [-1, 0), \tag{2.6}$$

then

(i) $\delta_m(G(x))/\delta_m(F(x))$ *is increasing in* $x \in \Re$
(ii) $X(r, n, m, k) \leq_{\text{lr}} Y(r', n', m, k)$ *whenever* $r' - r \geq \max\{0, n' - n\}$.

It is well known that the usual stochastic order is closed under increasing transformations. From Theorem 3.1 of Hu and Zhuang [7] and Theorem 3.1 of Belzunce, Mercader, and Ruiz [4], we obtain the following lemma.

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LEMMA 2.4: *Let* $\{X_{(r,n,m,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,m,k)}, r = 1, \ldots, n\}$ *be the GOSs based on distribution functions F and G, respectively, with* $m \geq -1$. *If* $F \leq_{\text{st}} G$, *then*

$$X_{(r,n,m,k)} \leq_{\text{st}} Y_{(r',n',m,k)} \quad \text{whenever } r' - r \geq \max\{0, n' - n\}.$$

The last lemma, which is due to Misra and van der Meulen [18], is useful in establishing the monotonicity of a fraction with its numerator and denominator being integrals or summations.

LEMMA 2.5: *Let* Θ *be a subset of the real line* $\Re$ *and let W be a variable having a distribution function belonging to the family* $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$, *which satisfies that*

$$H(\cdot|\theta_1) \leq_{\text{st}} H(\cdot|\theta_2) \quad \text{whenever } \theta_1, \theta_2 \in \Theta \text{ and } \theta_1 < \theta_2.$$

Let $\Psi(w, \theta)$ *be a real-valued function defined on* $\Re \times \Theta$. *If* $\Psi(w, \theta)$ *is increasing in* (w, θ), *then* $\mathbb{E}_\theta[\Psi(W, \theta)]$ *is increasing in* θ.

3. MAIN RESULTS

Throughout this section, let $\{X_{(r,n,m,k)}, r = 1, \ldots, n\}$ and $\{Y_{(r,n',m,k)}, r = 1, \ldots, n'\}$ be the GOSs based on distribution functions F and G, respectively. We will investigate conditions on F and G and on parameters $(r, s, n, r', s', n', m, k)$ such that

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\text{st}} [\leq_{\text{lr}}] \; [Y_{(s',n',m,k)} - y | Y_{(r',n',m,k)} > y] \quad \text{for } y \in \Re.$$

3.1. Usual Stochastic Ordering

Our first result generalizes (1.4) and (1.5) to GOSs in the more general form.

THEOREM 3.1: *If* $m \geq -1$ *and* $k > 0$ *and if* $F \leq_{\text{hr}} G$, *then*

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\text{st}} [Y_{(s',n',m,k)} - y | Y_{(r',n',m,k)} > y] \quad \text{for } y \in \Re \qquad \textbf{(3.1)}$$

whenever $s > r$ *and* $s' - s \geq r' - r \geq \max\{0, n' - n\}$.

PROOF: Let X and Y be two random variables with distribution functions F and G, respectively. For each $u \in \Re$, let $X^u_{(s-r,n-r,m,k)}$ denote the $(s-r)$th GOS based on the same distribution function F_u as $[X - u | X > u]$. Similarly, let $Y^u_{(s'-r',n'-r',m,k)}$ denote the $(s' - r')$th GOS based on the same distribution function G_u as $[Y - u | Y > u]$. Fix a $y \in \Re$. Consider now two families of random variables $\{T_1(\theta), \theta \in \Re\}$ and $\{T_2(\theta), \theta \in \Re\}$ such that, for all $\theta \in \Re$,

$$T_1(\theta) = X^\theta_{(s-r,n-r,m,k)} + \theta - y$$

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and

$$T_2(\theta) = Y^{\theta}_{(s'-r',n'-r',m,k)} + \theta - y.$$

Define two random variables Θ_1 and Θ_2 such that

$$\Theta_1 \stackrel{\text{st}}{=} [X_{(r,n,m,k)} | X_{(r,n,m,k)} > y]$$

and

$$\Theta_2 \stackrel{\text{st}}{=} [Y_{(r',n',m,k)} | Y_{(r',n',m,k)} > y],$$

where $\{\Theta_1, \Theta_2\}$ are independent of $\{T_i(\theta), \theta \in \Re, i = 1, 2\}$. Let F_W denote the distribution function of a random variable W. By Lemma 2.1, we get that for $x \in \Re_+$ and $s > r$,

$$\begin{aligned}
\mathbb{P}[X_{(s,n,m,k)} - y > x | X_{(r,n,m,k)} > y] &= \frac{\mathbb{P}[X_{(s,n,m,k)} - y > x, X_{(r,n,m,k)} > y]}{\mathbb{P}[X_{(r,n,m,k)} > y]} \\
&= \frac{1}{\mathbb{P}[X_{(r,n,m,k)} > y]} \int_y^{\infty} \mathbb{P}[X_{(s,n,m,k)} - y > x | X_{(r,n,m,k)} = u] \, dF_{X_{(r,n,m,k)}}(u) \\
&= \frac{1}{\mathbb{P}[X_{(r,n,m,k)} > y]} \int_y^{\infty} \mathbb{P}[X^u_{(s-r,n-r,m,k)} + u - y > x] \, dF_{X_{(r,n,m,k)}}(u) \\
&= \int_{-\infty}^{\infty} \mathbb{P}[T_1(u) > x] \, dF_{\Theta_1}(u) \\
&= \mathbb{P}[T_1(\Theta_1) > x], \qquad \textbf{(3.2)}
\end{aligned}$$

which means that

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \stackrel{\text{st}}{=} T_1(\Theta_1). \qquad \textbf{(3.3)}$$

Similarly,

$$[Y_{(s',n',m,k)} - y | Y_{(r',n',m,k)} > y] \stackrel{\text{st}}{=} T_2(\Theta_2). \qquad \textbf{(3.4)}$$

By Theorem 1.A.11 of Shaked and Shanthikumar [20], we see that

$$[Y | Y > \theta] \leq_{\text{st}} [Y | Y > \theta'] \quad \text{whenever } \theta \leq \theta',$$

which implies, by Lemma 2.4, that

$$Y^{\theta}_{(s'-r',n'-r',m,k)} + \theta \leq_{\text{st}} Y^{\theta'}_{(s'-r',n'-r',m,k)} + \theta' \quad \text{whenever } \theta \leq \theta'.$$

Here, we use the observation that $Y^{\theta}_{(s'-r',\, n'-r',m,k)} + \theta$ is the $(s' - r')$th GOS based the distribution function of $[Y|Y>\theta]$. Thus,

$$T_2(\theta) \leq_{\text{st}} T_2(\theta') \quad \text{whenever } \theta \leq \theta'. \qquad \textbf{(3.5)}$$

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By Lemma 2.2, $F \leq_{\text{hr}} G$ implies that $X_{(r,n,m,k)} \leq_{\text{hr}} Y_{(r',n',m,k)}$ and, hence,

$$\Theta_1 \leq_{\text{st}} \Theta_2 \quad \text{whenever } r' \geq r \text{ and } r' - r \geq n' - n. \tag{3.6}$$

On the other hand, $F \leq_{\text{hr}} G$ is equivalent to $[X - \theta | X > \theta] \leq_{\text{st}} [Y - \theta | Y > \theta]$ for all $\theta \in \Re$, which, by Lemma 2.4, implies that

$$X^{\theta}_{(s-r,n-r,m,k)} \leq_{\text{st}} Y^{\theta}_{(s'-r',n'-r',m,k)} \quad \text{whenever } s' - r' \geq s - r > 0 \text{ and } s' - s \geq n' - n$$

and, hence

$$T_1(\theta) \leq_{\text{st}} T_2(\theta) \quad \text{whenever } s' - r' \geq s - r > 0 \text{ and } s' - s \geq n' - n \tag{3.7}$$

for all $\theta \in \Re$. By Theorems 1.A.3(d) and 1.A.6 of Shaked and Shanthikumar [20], it follows from (3.5)–(3.7) that

$$T_1(\Theta_1) \leq_{\text{st}} T_2(\Theta_1) \leq_{\text{st}} T_2(\Theta_2) \tag{3.8}$$

whenever $s > r$ and $s' - s \geq r' - r \geq \max\{0, n' - n\}$. The desired result now follows from (3.3), (3.4), and (3.8). ■

3.2. Likelihood Ratio Ordering

In this subsection, we first investigate conditions on the parameters that enable one to compare GOSs based on the same distribution and then based on two different distributions.

THEOREM 3.2: *If $m \geq -1$ and $k \geq 1$ and if F is absolutely continuous, then*

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\text{lr}} [X_{(s',n',m,k)} - y | X_{(r',n',m,k)} > y] \quad \textit{for } y \in \Re \tag{3.9}$$

whenever $s > r$ and $s' - s = r' - r \geq \max\{0, n' - n\}$.

PROOF: We use the same notations as in the proof of Theorem 3.1. For any random variable W, denote by f_W the density function of W. Suppose that $s > r$ and $s' - s = r' - r \geq \max\{0, n' - n\}$. To prove (3.9), it suffices to verify that

$$\Delta_1(\theta) \equiv \frac{f_{[X_{(s',n',m,k)} - y | X_{(r',n',m,k)} > y]}(\theta)}{f_{[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y]}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

From (3.2), we get that

$$\begin{aligned} \Delta_1(\theta) &= \frac{\mathbb{P}[X_{(r,n,m,k)} > y]}{\mathbb{P}[X_{(r',n',m,k)} > y]} \frac{\int_y^\infty f_{X^u_{(s'-r',n'-r',m,k)}+u}(\theta + y) f_{X_{(r',n',m,k)}}(u)\, du}{\int_y^\infty f_{X^u_{(s-r,n-r,m,k)}+u}(\theta + y) f_{X_{(r,n,m,k)}}(u)\, du} \\ &= \frac{\mathbb{P}[X_{(r,n,m,k)} > y]}{\mathbb{P}[X_{(r',n',m,k)} > y]} \mathbb{E}_\theta[\Psi_1(U, \theta)], \end{aligned} \tag{3.10}$$

where

$$\Psi_1(u,\theta) = \frac{f_{X^u_{(s'-r',n'-r',m,k)}+u}(\theta+y) f_{X_{(r',n',m,k)}}(u)}{f_{X^u_{(s-r,n-r,m,k)}+u}(\theta+y) f_{X_{(r,n,m,k)}}(u)} \quad \text{for } u \geq y,$$

and the random variable U has a distribution function belonging to the family $\mathcal{P} = \{H(\cdot|\theta),\ \theta \in \Re_+\}$ with corresponding densities given by

$$h(u|\theta) = d(\theta) f_{X^u_{(s-r,n-r,m,k)}+u}(\theta+y) f_{X_{(r,n,m,k)}}(u) \quad \text{for } u \geq y;$$

here, $d(\theta)$ is the normalizing constant.

For any $u' > u \geq y$, by Theorem 1.C.17 of Shaked and Shanthikumar [20], we get that

$$[X|X>u] \leq_{\text{lr}} [X|X>u'].$$

Then the conditions in Lemma 2.3 are all satisfied since the failure rate of $[X|X>u]$ is given by $\lambda_{[X|X>u]}(x) = \lambda(x) 1_{\{x \geq u\}}$, where 1_A is the indicator function of set A. Observe that $X^u_{(s-r,n-r,m,k)} + u$ is the $(s-r)$th GOS based on distribution function $F_{[X|X>u]}$. Thus, by Lemma 2.3(ii), we have

$$X^u_{(s-r,n-r,m,k)} + u \leq_{\text{lr}} X^{u'}_{(s-r,n-r,m,k)} + u' \quad \text{for } u' > u \geq y,$$

which implies that

$$\frac{h(u'|\theta)}{h(u|\theta)} = \frac{f_{X^{u'}_{(s-r,n-r,m,k)}+u'}(\theta+y)}{f_{X^u_{(s-r,n-r,m,k)}+u}(\theta+y)} \frac{f_{X_{(r,n,m,k)}}(u')}{f_{X_{(r,n,m,k)}}(u)}$$

is increasing in $\theta \in \Re_+$; that is, $H(\cdot|\theta) \leq_{\text{lr}} H(\cdot|\theta')$ whenever $0 \leq \theta < \theta'$ and, hence,

$$H(\cdot|\theta) \leq_{\text{st}} H(\cdot|\theta') \quad \text{whenever } 0 \leq \theta < \theta'. \tag{3.11}$$

Again, from Lemma 2.3(ii), it follows that

$$X^u_{(s-r,n-r,m,k)} + u \leq_{\text{lr}} X^u_{(s'-r',n'-r',m,k)} + u \quad \text{for } u \geq y,$$

which implies that $\Psi_1(u,\theta)$ is increasing in $\theta \in \Re_+$ for each $u \geq y$. On the other hand, from (2.1)–(2.3), we get that

$$\begin{aligned}\Psi_1(u,\theta) &= \frac{c_{s'-r'-1,n'-r'}\, c_{r'-1,n'}}{c_{s-r-1,n-r}\, c_{r-1,n}} \frac{(s-r-1)!(r-1)!}{(s'-r'-1)!(r'-1)!} \\ &\quad \times \left[\frac{\bar{F}(\max\{y+\theta,u\})}{\bar{F}(u)}\right]^{\gamma_{s'-r',n'-r'}-\gamma_{s-r,n-r}} [\bar{F}(u)]^{\gamma_{r',n'}-\gamma_{r,n}} [\delta_m(F(u))]^{r'-r} \\ &= \frac{c_{s'-r'-1,n'-r'}\, c_{r'-1,n'}}{c_{s-r-1,n-r}\, c_{r-1,n}} \frac{(s-r-1)!(r-1)!}{(s'-r'-1)!(r'-1)!} \\ &\quad \times [\bar{F}(\max\{y+\theta,u\})]^{(n'-n-s'+s)(m+1)} [\delta_m(F(u))]^{r'-r},\end{aligned}$$

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which is increasing in $u \in [y, +\infty)$ for each $\theta \in \Re_+$ since $s' - s = r' - r \geq \max\{0, n' - n\}$ and $\delta_m(F(u))$ is increasing in u.

Therefore, applying Lemma 2.5 in (3.10) yields that $\Delta_1(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof. ■

It is still unknown whether, under the same conditions as in Theorem 3.2,

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\mathrm{lr}} [X_{(s+1,n,m,k)} - y | X_{(r,n,m,k)} > y]$$

and

$$[X_{(s,n,m,k)} - y | X_{(r+1,n,m,k)} > y] \leq_{\mathrm{lr}} [X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y]$$

hold for $1 \leq r < s < n$ and any y.

If, instead, F is not assumed to be absolutely continuous in Theorem 3.2, we can prove the following theorem by using the fact that the hazard rate order is closed under increasing transforms (see Theorem 1.B.2 of Shaked and Shanthikumar [20]).

THEOREM 3.3: *If $m \geq -1$ and $k \geq 1$, then*

$$[X_{(s,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\mathrm{hr}} [X_{(s',n',m,k)} - y | X_{(r',n',m,k)} > y] \quad \textit{for } y \in \Re$$

whenever $s > r$ and $s' - s = r' - r \geq \max\{0, n' - n\}$.

Now, we turn to consider stochastic comparisons of GOSs based on two different distributions.

THEOREM 3.4: *If $m \geq -1$ and $k \geq 1$ and if conditions* (2.5) *and* (2.6) *are satisfied, then*

$$[X_{(r+1,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\mathrm{lr}} [Y_{(r+1,n,m,k)} - y | Y_{(r,n,m,k)} > y] \tag{3.12}$$

for $y \in \Re$ and $r = 1, \ldots, n-1$.

PROOF: We use the same notations as in the proof of Theorem 3.1. Let f and g denote the density functions of F and G, respectively, and denote by f_W or g_W the density function of any random variable W. Fix an r, $1 \leq r < n$. To prove (3.12), it suffices to verify that

$$\Delta_2(\theta) \equiv \frac{g_{[Y_{(r+1,n,m,k)} - y | Y_{(r,n,m,k)} > y]}(\theta)}{f_{[X_{(r+1,n,m,k)} - y | X_{(r,n,m,k)} > y]}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

From (3.2), we get that

$$\Delta_2(\theta) = \frac{\mathbb{P}[X_{(r,n,m,k)} > y]}{\mathbb{P}[Y_{(r,n,m,k)} > y]} \mathbb{E}_\theta[\Psi_2(U, \theta)], \tag{3.13}$$

where

$$\Psi_2(u,\theta) = \frac{g_{Y^u_{(1,n-r,m,k)}+u}(\theta+y)g_{Y_{(r,n,m,k)}}(u)}{f_{X^u_{(1,n-r,m,k)}+u}(\theta+y)f_{X_{(r,n,m,k)}}(u)} \quad \text{for } u \geq y$$

and U has a distribution function belonging to the family $\mathcal{P} = \{H(\cdot|\theta),\ \theta \in \Re_+\}$, defined in the proof of Theorem 3.2, with $s = r+1$. This means that (3.11) holds.

Suppose that conditions (2.5) and (2.6) are satisfied. It is well known that if $F \leq_{\text{hr}} G$ and $\eta(x)/\lambda(x)$ is increasing in x, then $F \leq_{\text{lr}} G$ (Belzunce, Lillo, Ruiz, and Shaked [3, Lemma 3]) which implies that $[X|X>u] \leq_{\text{lr}} [Y|Y>u]$ for each u. From Lemma 2.3(ii), it follows that

$$X^u_{(1,n-r,m,k)} + u \leq_{\text{lr}} Y^u_{(1,n-r,m,k)} + u \quad \text{for each } u.$$

Hence, $\Psi_2(u,\theta)$ is increasing in θ for each $u \geq y$.

It remains to prove that $\Psi_2(u,\theta)$ is increasing in $u \in [y,+\infty)$ for each $\theta \in \Re_+$. First, observe that the support of U is in $[y, y+\theta)$ since the density function $h(u|\theta) = 0$ for $u > y+\theta$. Thus, we have to verify that $\Psi_2(u,\theta)$ is increasing in $u \in [y, y+\theta)$ for each $\theta \in \Re_+$. From (2.1)–(2.3), we get that

$$\begin{aligned}\Psi_2(u,\theta) &= \left[\frac{\bar{G}(y+\theta)/\bar{G}(u)}{\bar{F}(y+\theta)/\bar{F}(u)}\right]^{\gamma_{1,n-r}-1} \frac{g(y+\theta)/\bar{G}(u)}{f(y+\theta)/\bar{F}(u)} \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\gamma_{r,n}-1} \left[\frac{\delta_m(G(u))}{\delta_m(F(u))}\right]^{r-1} \frac{g(u)}{f(u)} \\ &= \left[\frac{\bar{G}(y+\theta)}{\bar{F}(y+\theta)}\right]^{\gamma_{1,n-r}-1} \frac{g(y+\theta)}{f(y+\theta)} \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m} \left[\frac{\delta_m(G(u))}{\delta_m(F(u))}\right]^{r-1} \frac{g(u)}{f(u)} \qquad \textbf{(3.14)} \\ &= \left[\frac{\bar{G}(y+\theta)}{\bar{F}(y+\theta)}\right]^{\gamma_{1,n-r}-1} \frac{g(y+\theta)}{f(y+\theta)} \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m+1} \left[\frac{\delta_m(G(u))}{\delta_m(F(u))}\right]^{r-1} \frac{\eta(u)}{\lambda(u)} \qquad \textbf{(3.15)}\end{aligned}$$

for $u \in [y, y+\theta)$. Note that the likelihood ratio order is stronger than the hazard rate order. For $m \geq 0$, it follows from (2.5), (3.14), and Lemma 2.3(i) that $\Psi_2(u,\theta)$ is increasing in $u \in [y, y+\theta)$ for each $\theta \in \Re_+$. For $m \in [-1, 0)$, it follows from (2.6), (3.15) and Lemma 2.3(i) that $\Psi_2(u,\theta)$ is increasing in $u \in [y, y+\theta)$ for each $\theta \in \Re_+$.

Therefore, applying Lemma 2.5 in (3.13) yields that $\Delta_2(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof. ∎

An immediate consequence of Theorems 3.2 and 3.4 is the following:

Theorem 3.5: *If $m \geq -1$ and $k \geq 1$ and if conditions (2.5) and (2.6) are satisfied, then*

$$[X_{(r+1,n,m,k)} - y | X_{(r,n,m,k)} > y] \leq_{\text{lr}} [Y_{(r'+1,n',m,k)} - y | Y_{(r',n',m,k)} > y]$$

for $y \in \Re$ whenever $r' - r \geq \max\{0, n'-n\}$.

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4. APPLICATIONS

In this section, some applications of the main results in Section 3 are presented.

4.1. Ordinary Order Statistics

For a collection $\{X_1, X_2, \ldots, X_n\}$ of i.i.d. random variables with a common distribution function F, the ordinary order statistics $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ correspond to the GOSs based on F with parameters $k = 1$ and $m_1 = \cdots = m_{n-1} = 0$. From Theorems 3.1–3.5, we have the following two corollaries.

Corollary 4.1: *Let* $\{X_n, n \ge 1\}$ *and* $\{Y_n, n \ge 1\}$ *be two sequences of i.i.d. random variables.*

(a) *If* $X_1 \le_{\text{hr}} Y_1$, *then*

$$[X_{s:n} - y | X_{r:n} > y] \le_{\text{st}} [Y_{s':n'} - y | Y_{r':n'} > y] \quad \textit{for } y \in \Re$$

whenever $s > r$ *and* $s' - s \ge r' - r \ge \max\{0, n' - n\}$.

(b) *If* $X_1 \le_{\text{lr}} Y_1$, *then*

$$[X_{r+1:n} - y | X_{r:n} > y] \le_{\text{lr}} [Y_{r'+1:n'} - y | Y_{r':n'} > y] \quad \textit{for } y \in \Re$$

whenever $r' - r \ge \max\{0, n' - n\}$.

Theorem 2.1 of Khaledi and Shaked [15] and Theorem 3.1 of Li and Zhao [17] are the special case of Corollary 4.1(a) with $s = s'$ and $r = r'$.

Corollary 4.2: *Let* $\{X_n, n \ge 1\}$ *be a sequence of i.i.d. random variables. Then the following hold*:

(*a*) $[X_{s:n} - y | X_{r:n} > y] \le_{\text{st}} [X_{s':n'} - y | X_{r':n'} > y]$ for $y \in \Re$ *whenever* $s > r$ *and* $s' - s \ge r' - r \ge \max\{0, n' - n\}$.

(*b*) $[X_{s:n} - y | X_{r:n} > y] \le_{\text{hr}} [X_{s':n'} - y | X_{r':n'} > y]$ for $y \in \Re$ *whenever* $s > r$ *and* $s' - s = r' - r \ge \max\{0, n' - n\}$.

(*c*) $[X_{s:n} - y | X_{r:n} > y] \le_{\text{lr}} [X_{s':n'} - y | X_{r':n'} > y]$ for $y \in \Re$ *whenever* $s > r$ *and* $s' - s = r' - r \ge \max\{0, n' - n\}$ *if* X_1 *has an absolutely continuous distribution. In particular, for* $y \in \Re$ *and* $s \ge r$, *we have*

$$[X_{s:n} - y | X_{r:n} > y] \le_{\text{lr}} [X_{s+1:n} - y | X_{r+1:n} > y],$$
$$[X_{s:n} - y | X_{r:n} > y] \le_{\text{lr}} [X_{s:n-1} - y | X_{r:n-1} > y],$$

and

$$[X_{s:n} - y | X_{r:n} > y] \le_{\text{lr}} [X_{s+1:n+1} - y | X_{r+1:n+1} > y].$$

4.2. Record Values

Record values based on a sequence of i.i.d. random variables are another particular model of GOSs with $k = 1$ and $m_i = -1$ for each i. For this model, we have the following corollary.

Corollary 4.3: *Let $\{X_{L(n)}, n \geq 1\}$ and $\{Y_{M(n)}, n \geq 1\}$ be record values based on two sequences of i.i.d. random variables with respective distribution functions F and G and denote by $\lambda(x)$ and $\eta(x)$ the hazard rate functions of F and G, respectively.*

(*a*) *If $F \leq_{\text{hr}} G$, then*

$$[X_{L(s)} - y | X_{L(r)} > y] \leq_{\text{st}} [Y_{M(s')} - y | Y_{M(r')} > y]$$

for $y \in \Re$ whenever $s > r$ and $s' - s \geq r' - r \geq 0$.

(*b*) *If $F \leq_{\text{hr}} G$ and $\eta(x)/\lambda(x)$ is increasing in x, then*

$$[X_{L(r+1)} - y | X_{L(r)} > y] \leq_{\text{lr}} [Y_{M(r'+1)} - y | Y_{M(r')} > y]$$

for $y \in \Re$ whenever $r' \geq r$.

(*c*) *If F is absolutely continuous, then*

$$[X_{L(r+p)} - y | X_{L(r)} > y] \leq_{\text{lr}} [X_{L(r'+p)} - y | X_{L(r')} > y]$$

for $y \in \Re$ whenever $p \geq 1$ and $r' \geq 1$.

Theorem 2.3 of Khaledi and Shojaei [16] is the special case of Corollary 4.3(a) with $s = s'$ and $r = r'$.

Notice that the epoch times of a nonhomogeneous Poisson process with intensity function $\lambda(t)$ are the record values of a sequence of i.i.d. nonnegative random variables with the hazard rate being $\lambda(t)$, where

$$\int_t^\infty \lambda(u)\, du = \infty \quad \text{for all } t \in \Re_+.$$

Therefore, Corollary 4.3 can be interpreted in terms of epoch times of nonhomogeneous Poisson processes.

4.3. Progressive Type II Censored Order Statistics

In a progressive type II censoring scheme, N units are placed on a lifetime test. The failure times are described by i.i.d. random variables with a common distribution F. A number n ($n \leq N$) of units are observed to fail. A predetermined number R_i of surviving units at the time of the ith failure are randomly selected and removed from further testing. Thus, $\sum_{i=1}^n R_i$ units are progressively censored; hence, $N = n \sum_{i=1}^n R_i$. The n observed failure times are called progressive type II censored order statistics based on

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F, denoted by $X^{\mathbf{R}}_{1:n,N} \le X^{\mathbf{R}}_{2:n,N} \le \cdots \le X^{\mathbf{R}}_{n:n,N}$, where $\mathbf{R} = (R_1, \ldots, R_n)$. Progressive type II censored order statistics based on F correspond to the GOSs based on F with parameters $k = R_n + 1$ and $m_i = R_i$ for $i = 1, \ldots, n-1$. For details on the model of progressive type II censoring, we refer to Balakrishnan and Aggarwala [2] and Cramer and Kamps [5].

For progressive type II censored order statistics, we have the following corollary.

Corollary 4.4: Let $\{X^{\mathbf{R}}_{1:n,N}, \ldots, X^{\mathbf{R}}_{n:n,N}\}$ *and* $\{Y^{\mathbf{R}}_{1:n,N}, \ldots, Y^{\mathbf{R}}_{n:n,N}\}$ *be respective progressive type II censored order statistics based on* F and G *with a common censoring policy* $\mathbf{R}$ *with* $R_1 = \cdots = R_{n-1}$.

(*a*) *If* $F \le_{\mathrm{hr}} G$, $s > r$ *and* $s' - s \ge r' - r \ge 0$, *then*

$$[X^{\mathbf{R}}_{s:n,N} - y \mid X^{\mathbf{R}}_{r:n,N} > y] \le_{\mathrm{st}} [Y^{\mathbf{R}}_{s':n,N} - y \mid Y^{\mathbf{R}}_{r':n,N} > y] \quad \text{for } y \in \Re.$$

(*b*) *If* $F \le_{\mathrm{lr}} G$ *and* $r' \ge r$, *then*

$$[X^{\mathbf{R}}_{r+1:n,N} - y \mid X^{\mathbf{R}}_{r:n,N} > y] \le_{\mathrm{lr}} [Y^{\mathbf{R}}_{r'+1:n,N} - y \mid Y^{\mathbf{R}}_{r':n,N} > y] \quad \text{for } y \in \Re.$$

(*c*) *If* F *is absolutely continuous and* $r' > r$, *then*

$$[X^{\mathbf{R}}_{r+p:n,N} - y \mid X^{\mathbf{R}}_{r:n,N} > y] \le_{\mathrm{lr}} [X^{\mathbf{R}}_{r'+p:n,N} - y \mid Y^{\mathbf{R}}_{r':n,N} > y] \quad \text{for } y \in \Re.$$

4.4. Bounds on Mean Residual Life Functions

First, we state a lemma that will be used in the sequel.

Lemma 4.1: *Let* $\{Z^{\lambda}_{(i,n,m,k)}, i = 1, \ldots, n\}$ *be GOSs based on exponential distribution with hazard rate* λ. *Then, for* $1 \le r < s \le n$ *and* $m \ne -1$,

$$\mathbb{E}[Z^{\lambda}_{(s,n,m,k)} - y \mid Z^{\lambda}_{(r,n,m,k)} > y] = \frac{1}{\lambda} \sum_{i=1}^{s-r} \frac{1}{\gamma_{i,n-r}} + \frac{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-2} e^{-\lambda \gamma_{i,n} y}}{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-1} e^{-\lambda \gamma_{i,n} y}}, \tag{4.1}$$

where $\gamma_{i,n}$ *is defined in* (2.3) *and*

$$a_{i,r} = \prod_{j=1, j \ne i}^{r} \frac{1}{\gamma_{j,n} - \gamma_{i,n}} \quad \text{for } i = 1, \ldots, r. \tag{4.2}$$

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Proof: From (3.2), it follows that

$$\begin{aligned}
&\mathbb{E}[Z^{\lambda}_{(s,n,m,k)} - y | Z^{\lambda}_{(r,n,m,k)} > y] \\
&= \int_0^{\infty} \mathbb{P}[Z^{\lambda}_{(s,n,m,k)} - y > x | Z^{\lambda}_{(r,n,m,k)} > y]\, dx \\
&= \int_y^{\infty} \left(\int_0^{\infty} \mathbb{P}[Z^{\lambda}_{(s-r,n-r,m,k)} + u - y > x]\, dx \right) \frac{f_{Z^{\lambda}_{(r,n,m,k)}}(u)}{\overline{F}_{Z^{\lambda}_{(r,n,m,k)}}(y)}\, du \\
&= \int_y^{\infty} \left(\mathbb{E}[Z^{\lambda}_{(s-r,n-r,m,k)}] + u - y \right) \frac{f_{Z^{\lambda}_{(r,n,m,k)}}(u)}{\overline{F}_{Z^{\lambda}_{(r,n,m,k)}}(y)}\, du \\
&= \mathbb{E}[Z^{\lambda}_{(s-r,n-r,m,k)}] + \mathbb{E}[Z^{\lambda}_{(r,n,m,k)} - y | Z^{\lambda}_{(r,n,m,k)} > y], \qquad \textbf{(4.3)}
\end{aligned}$$

where the second equality follows from the fact that $[Z - u | Z > u]$ and Z are identically distributed if Z is exponential random variable.

The first term in (4.3) can be written as

$$\mathbb{E}[Z^{\lambda}_{(s-r,n-r,m,k)}] = \sum_{j=1}^{s-r} \frac{1}{\lambda \gamma_{j,n-r}}, \qquad \textbf{(4.4)}$$

since $Z^{\lambda}_{(s-r,n-r,m,k)} \overset{\text{st}}{=} \sum_{j=1}^{s-r} Z'_j$, and Z'_j's are independent exponential random variables with Z'_j having hazard rate $\lambda\gamma_{j,n-r}$ (see Cramer and Kamps [6, Theorem 3.1] and Hu and Zhuang [7, Lemma 2.1]).

The second term in (4.3) can be written as

$$\begin{aligned}
\mathbb{E}[Z^{\lambda}_{(r,n,m,k)} - y | Z^{\lambda}_{(r,n,m,k)} > y] &= \int_0^{\infty} \frac{\mathbb{P}[Z^{\lambda}_{(r,n,m,k)} > x + y]}{\mathbb{P}[Z^{\lambda}_{(r,n,m,k)} > y]}\, dx \\
&= \int_0^{\infty} \frac{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-1} e^{-\lambda \gamma_{i,n}(x+y)}}{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-1} e^{-\lambda \gamma_{i,n} y}}\, dx \\
&= \frac{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-2} \lambda^{-1} e^{-\lambda \gamma_{i,n} y}}{\sum_{i=1}^{r} a_{i,r} \gamma_{i,n}^{-1} e^{-\lambda \gamma_{i,n} y}}, \qquad \textbf{(4.5)}
\end{aligned}$$

where $a_{i,r} = \prod_{j=1, j \neq i}^{r} (\gamma_{j,n} - \gamma_{i,n})^{-1}$ for $i = 1, \ldots, r$ (see Eq. (5) in Kamps and Cramer [12]).

Now, substituting (4.4) and (4.5) into (4.3), the required result follows. ■

If the hazard rate function $\lambda(x)$ of a distribution function F is bounded from above or from below, then the next result enables us to obtain a computable

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lower bound or upper bound on the mean residual life function of GOSs based on F.

THEOREM 4.1: *Let* $\{X_{(r,n,m,k)}, r = 1, \ldots, n\}$ *be the GOSs based on distribution function F with corresponding hazard rate* $\lambda(x)$, *where* $m > -1$. *Let* $\lambda_L \leq \lambda(x) \leq \lambda_U$ *for some positive constants* λ_L *and* λ_U, *and for all x. Then*

$$\frac{1}{\lambda_U}\sum_{i=1}^{s-r}\frac{1}{\gamma_{i,n-r}} + \frac{\sum_{i=1}^{r} a_{i,r}\gamma_{i,n}^{-2}\, e^{-\lambda_U\gamma_{i,n}y}}{\sum_{i=1}^{r} a_{i,r}\gamma_{i,n}^{-1} e^{-\lambda_U\gamma_{i,n}y}}$$

$$\leq \mathbb{E}[X^{\lambda}_{(s,n,m,k)} - y | X^{\lambda}_{(r,n,m,k)} > y]$$

$$\leq \frac{1}{\lambda_L}\sum_{i=1}^{s-r}\frac{1}{\gamma_{i,n-r}} + \frac{\sum_{i=1}^{r} a_{i,r}\gamma_{i,n}^{-2}\, e^{-\lambda_L\gamma_{i,n}y}}{\sum_{i=1}^{r} a_{i,r}\gamma_{i,n}^{-1} e^{-\lambda_L\gamma_{i,n}y}},$$

where $\gamma_{j,n}$ *and* $a_{i,r}$ *are defined in* (2.3) *and* (4.2), *respectively.*

PROOF: Let $\{Z^{\lambda_U}_{(i,n,m,k)},\ i = 1, \ldots, n\}$ and $\{Z^{\lambda_L}_{(i,n,m,k)},\ i = 1, \ldots, n\}$ be GOSs based on exponential distributions with hazard rates λ_U and λ_L, respectively. Then, using $\lambda_L \leq \lambda(x) \leq \lambda_U$ for all x, the desired result follows from Theorem 3.1 and Lemma 4.1. ■

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