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DIOPHANTINE TRANSFERENCE PRINCIPLE OVER FUNCTION FIELDS

SOURAV DAS and ARIJIT GANGULY

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Abstract

We study the Diophantine transference principle over function fields. By adapting the approach of Beresnevich and Velani [‘An inhomogeneous transference principle and Diophantine approximation’, *Proc. Lond. Math. Soc. (3)* **101** (2010), 821–851] to function fields, we extend many results from homogeneous to inhomogeneous Diophantine approximation. This also yields the inhomogeneous Baker–Sprindžuk conjecture over function fields and upper bounds for the general nonextremal scenario.

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1. Introduction

The theory of metric Diophantine approximation on manifolds began with Mahler's conjecture which states that almost every point on the Veronese curve $\mathcal{V}_n := \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$ is not very well approximable. Sprindžuk [20] proved this conjecture in 1960 and conjectured [21] that the conclusion of Mahler's conjecture is true for any nondegenerate analytic submanifold of $\mathbb{R}^n$. This was strengthened by Baker and is commonly referred to as the Baker–Sprindžuk conjecture. After some partial results, the conjecture in its full generality was finally resolved in 1998 by Kleinbock and Margulis [15] using techniques from homogeneous dynamics. Beresnevich and Velani and others [1–4] observed that, sometimes, the results of homogeneous Diophantine approximation can be transferred to the inhomogeneous context by a transference principle. In [4], Beresnevich and Velani proved many results of inhomogeneous Diophantine approximation, including the inhomogeneous Baker–Sprindžuk conjecture in this way. Our aim is to explore this theme in the context of function fields.

Consider the function field $\mathbb{F}_q(X)$, where $q = p^d$, p is a prime and $d \in \mathbb{N}$. Define a nonarchimedean absolute value $|\cdot|$ on $\mathbb{F}_q(X)$ by

$$|0| := 0 \quad \text{and} \quad \left|\frac{f}{g}\right| = e^{\deg f - \deg g} \quad \text{for } f, g \in \mathbb{F}_q[X] \setminus \{0\}.$$

By $\mathbb{F}_q((X^{-1}))$, we denote the field of Laurent series in X^{-1} over the finite field $\mathbb{F}_q$. Note that $\mathbb{F}_q((X^{-1}))$ is the completion of $\mathbb{F}_q(X)$ with respect to the absolute value $|\cdot|$ and the absolute value of $\mathbb{F}_q(X)$ extends to an absolute value of $\mathbb{F}_q((X^{-1}))$ as follows. For $f \in \mathbb{F}_q((X^{-1})) \setminus \{0\}$, we can write

$$f = \sum_{\ell \le \ell_0} \alpha_\ell X^\ell \quad \text{where } \ell_0 \in \mathbb{Z},\ \alpha_\ell \in \mathbb{F}_q \text{ and } \alpha_{\ell_0} \neq 0.$$

We define $|f| := e^{\ell_0}$. With respect to this absolute value, $\mathbb{F}_q((X^{-1}))$ is an ultrametric, complete and separable metric space. Every local field of positive characteristic is isomorphic to some $\mathbb{F}_q((X^{-1}))$.

From now on, $\mathbb{F}_q[X]$ and $\mathbb{F}_q((X^{-1}))$ will be denoted by Λ and F, respectively. We equip F^n with the supremum norm:

$$\|\mathbf{y}\| := \max_{1 \le i \le n} |y_i| \quad \text{for } \mathbf{y} = (y_1, y_2, \ldots, y_n) \in F^n.$$

In the topology induced by the sup norm, Λ^n is discrete in F^n and the local compactness of F implies that F^n is locally compact. By λ, we denote the Haar measure on F^n such that $\lambda(\{\mathbf{x} \in F^n : \|\mathbf{x}\| \le 1\}) = 1$.

Diophantine approximation over function fields has been studied extensively since Mahler developed the geometry of numbers in this context [18]. We refer to the survey in [17] and to [7, 13, 19] for recent developments. Let us first recall Dirichlet's theorem over function fields.

THEOREM 1.1 [9, Theorem 2.1]. *Let $m, n \in \mathbb{N}$, $\ell = m + n$ and*

$$\alpha^+ := \left\{\mathbf{t} := (t_1, \ldots, t_l) \in \mathbb{Z}_+^\ell : \sum_{i=1}^m t_i = \sum_{i=1}^n t_{m+i}\right\}.$$

Let Z be an $m \times n$ matrix over F. Then for any $\mathbf{t} \in \alpha^+$, there exist solutions $\mathbf{q} = (q_1, \ldots, q_n) \in \Lambda^n \setminus \{0\}$ and $\mathbf{p} = (p_1, \ldots, p_m) \in \Lambda^m$ of the system of inequalities:

$$\begin{cases} |Z_i \mathbf{q} + p_i| < e^{-t_i} & \text{for } i = 1, 2, \ldots, m, \\ |q_j| < e^{t_{m+j}} & \text{for } j = 1, 2, \ldots, n, \end{cases}$$

where $Z_1, \ldots, Z_m$ are the row vectors of Z.

Here, $\mathbb{Z}_+$ denotes the set of nonnegative integers. We will not pursue this much generality, but only consider unweighted Diophantine approximation. We denote the set of all $m \times n$ matrices over F by $F^{m \times n}$.

DEFINITION 1.2 [10]. Given $Y \in F^{m\times n}$ and $\boldsymbol{\theta} \in F^m$, we define the *inhomogeneous exponent* $\omega(Y, \boldsymbol{\theta})$ of Y as the supremum of real numbers $\omega \geq 0$ such that there exists a solution $(\mathbf{p}, \mathbf{q}) \in \Lambda^m \times (\Lambda^n \setminus \{0\})$ to the system of inequalities

$$\|Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}\|^m < e^{-\omega T} \quad \text{and} \quad \|\mathbf{q}\|^n < e^T$$

for arbitrarily large $T \geq 1$.

DEFINITION 1.3. Given $Y \in F^{m\times n}$ and $\boldsymbol{\theta} \in F^m$, we define the *inhomogeneous uniform exponent* $\hat{\omega}(Y, \boldsymbol{\theta})$ of Y as the supremum of real numbers $\hat{\omega} \geq 0$ such that there exists a solution $(\mathbf{p}, \mathbf{q}) \in \Lambda^m \times (\Lambda^n \setminus \{0\})$ to the system of inequalities

$$\|Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}\|^m < e^{-\hat{\omega} T} \quad \text{and} \quad \|\mathbf{q}\|^n < e^T$$

for all sufficiently large $T \geq 1$.

DEFINITION 1.4. Given $Y \in F^{m\times n}$ and $\boldsymbol{\theta} \in F^m$, we define the *multiplicative inhomogeneous exponent* $\omega^\times(Y, \boldsymbol{\theta})$ of Y as the supremum of real numbers $\omega \geq 0$ such that there exists a solution $(\mathbf{p}, \mathbf{q}) \in \Lambda^m \times (\Lambda^n \setminus \{0\})$ to the system of inequalities

$$\prod(Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}) < e^{-\omega T} \quad \text{and} \quad \prod\nolimits_+(\mathbf{q}) < e^T$$

for arbitrarily large $T \geq 1$, where

$$\prod(\mathbf{y}) := \prod_{j=1}^{m} |y_j| \quad \text{and} \quad \prod\nolimits_+(\mathbf{q}) := \prod_{i=1}^{n} \max\{1, |q_i|\}$$

for $\mathbf{y} = (y_1, \ldots, y_m) \in F^m$ and $\mathbf{q} = (q_1, \ldots, q_n) \in \Lambda^n$.

In a similar manner, given any $Y \in F^{m\times n}$ and $\boldsymbol{\theta} \in F^m$, we define the *multiplicative inhomogeneous uniform exponent* $\hat{\omega}^\times(Y, \boldsymbol{\theta})$ of Y. Note that the analogous homogeneous exponents are provided by the special case of $\boldsymbol{\theta} = \mathbf{0}$, where $\mathbf{0} = (0, \ldots, 0) \in F^m$, that is, $\omega(Y) := \omega(Y, \mathbf{0})$, $\hat{\omega}(Y) := \hat{\omega}(Y, \mathbf{0})$, $\omega^\times(Y) := \omega^\times(Y, \mathbf{0})$ and $\hat{\omega}^\times(Y) := \hat{\omega}^\times(Y, \mathbf{0})$. Also

$$\omega^\times(Y, \boldsymbol{\theta}) \geq \omega(Y, \boldsymbol{\theta}) \quad \text{for all } Y \in F^{m\times n}, \boldsymbol{\theta} \in F^m. \tag{1.1}$$

By Dirichlet's theorem, $\omega(Y) \geq 1$ for all $Y \in F^{m\times n}$. Hence, $\omega^\times(Y) \geq 1$. An easy application of the Borel–Cantelli lemma shows that $\omega(Y) = 1$ and $\omega^\times(Y) = 1$ for λ-almost every $Y \in F^{m\times n}$. We say that $Y \in F^{m\times n}$ is *very well approximable* (VWA) if $\omega(Y) > 1$ and $Y \in F^{m\times n}$ is *very well multiplicatively approximable* (VWMA) if $\omega^\times(Y) > 1$.

The function field analogue of the Baker–Sprindžuk conjecture states that almost every point on an analytic nondegenerate submanifold $\mathcal{M}$ of F^n, identified with either columns $F^{n\times 1}$ (in simultaneous Diophantine approximation) or rows $F^{1\times n}$ (in dual Diophantine approximation), is not VWMA with respect to the natural measure on $\mathcal{M}$. In the language of exponents, if $\mathcal{M}$ is any analytic nondegenerate submanifold of F^n, then

$$\omega^\times(Y) = 1 \quad \text{for almost every } Y \in \mathcal{M}, \tag{1.2}$$

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and this clearly implies that

$$\omega(Y) = 1 \quad \text{for almost every } Y \in \mathcal{M}. \tag{1.3}$$

This was settled by Ghosh in [11]. The manifolds of F^n which satisfy (1.2) and (1.3) are referred to as *strongly extremal* and *extremal*, respectively. In view of the above discussion, we have the following theorem.

THEOREM 1.5 [11, Theorem 3.7]. *Let $\mathcal{M}$ be an analytic nondegenerate submanifold of F^n. Then $\mathcal{M}$ is strongly extremal.*

Kleinbock, Lindenstrauss and Weiss introduced the idea of measures being extremal rather than sets. Following their terminology, a measure μ supported on a subset of $F^{m\times n}$ is *extremal* (*strongly extremal*) if $\omega(Y) = 1$ ($\omega^{\times}(Y) = 1$) for μ-almost every point $Y \in F^{m\times n}$. Due to Khintchine's transference principle, we say that a measure μ on F^n is (strongly) extremal if it is (strongly) extremal as a measure on $F^{1\times n}$ or $F^{n\times 1}$.

THEOREM 1.6. *Let μ be a friendly measure on F^n. Then μ is strongly extremal.*

We will define the notion of *friendly measure* in Section 3. Friendly measures provide us with a large class of measures on F^n that includes the natural measure on a nondegenerate analytic manifold. The proof of Theorem 1.6 is analogous to that of Theorem 11.1 and Corollary 11.2 of [16]. Hence, Theorem 1.6 implies Theorem 1.5.

DEFINITION 1.7. A measure μ supported on a subset of $F^{m\times n}$ is said to be *inhomogeneously extremal* if, for all $\boldsymbol{\theta} \in F^m$, we have $\omega(Y, \boldsymbol{\theta}) = 1$ for μ-almost every $Y \in F^{m\times n}$. We call μ *inhomogeneously strongly extremal* if, for all $\boldsymbol{\theta} \in F^m$, we have $\omega^{\times}(Y, \boldsymbol{\theta}) = 1$ for μ-almost every $Y \in F^{m\times n}$.

It is easy to see that this naturally generalises the notion of extremality in the homogeneous case ($\boldsymbol{\theta} = \mathbf{0}$). In the homogeneous case, strong extremality implies extremality but it is not at all clear why an inhomogeneously strong extremal measure would be inhomogeneously extremal. The following proposition, proved in Section 4, answers this question.

PROPOSITION 1.8. *Suppose that μ be a measure supported on a subset of $F^{m\times n}$. Then*

$$\mu \textit{ is inhomogeneously strongly extremal} \implies \mu \textit{ is inhomogeneously extremal.}$$

As mentioned earlier, in the homogeneous case, both the simultaneous and dual forms of Diophantine approximation lead to the same notion of extremality due to Khintchine's transference principle. As there is no Khitchine's transference principle in the inhomogeneous case, one has to deal with simultaneous and dual forms of Diophantine approximation separately.

DEFINITION 1.9. A measure μ supported on a subset of F^n is said to be *dually inhomogeneously (strongly) extremal* if μ is inhomogeneously (strongly) extremal on $F^{1\times n}$. We call μ *simultaneously inhomogeneously (strongly) extremal* if μ is inhomogeneously (strongly) extremal on $F^{n\times 1}$. Furthermore, we say that μ is *inhomogeneously (strongly) extremal* if μ is both dually and simultaneously inhomogeneously (strongly) extremal.

In [10], Ganguly and Ghosh proved the inhomogeneous Sprindžuk conjecture over a field of positive characteristic.

THEOREM 1.10 [10, Theorem 1.1]. *Let $\mathcal{M}$ be an analytic nondegenerate submanifold of F^n. Then $\mathcal{M}$ is inhomogeneously extremal.*

2. Main results

In this section, we describe the main results of this paper.

THEOREM 2.1. (A) *Let μ be an almost everywhere contracting measure on $F^{m\times n}$. Then*

$$\mu \text{ is extremal} \iff \mu \text{ is inhomogeneously extremal.}$$

(B) *Let μ be an almost everywhere strongly contracting measure on $F^{m\times n}$. Then*

$$\mu \text{ is strongly extremal} \iff \mu \text{ is inhomogeneously strongly extremal.}$$

We will define a *(strongly) contracting measure* in the next section. Strongly contracting measures form a class of measures containing friendly measures. Hence, in view of Theorem 2.1 and Corollary 3.5, we have the following inhomogeneous version of Theorem 1.6.

THEOREM 2.2. *Let μ be a friendly measure on F^n. Then μ is inhomogeneously strongly extremal.*

It is well known that the natural measure supported on an analytic nondegenerate manifold is friendly. Hence, as an immediate consequence of Theorems 2.1, 1.5 and Corollary 3.5, we get the inhomogeneous Baker–Sprindžuk conjecture.

THEOREM 2.3. *Let $\mathcal{M}$ be an analytic nondegenerate submanifold of F^n. Then $\mathcal{M}$ is inhomogeneously strongly extremal.*

Furthermore, we prove the following upper bounds for the nonextremal case.

THEOREM 2.4. (I) *Let μ be a measure on $F^{m\times n}$ which is contracting almost everywhere. Suppose that $\omega(Y) = \eta$ for μ-almost every $Y \in F^{m\times n}$. Then, for all $\boldsymbol{\theta} \in F^m$,*

$$\omega(Y, \boldsymbol{\theta}) \leq \eta \quad \text{for } \mu\text{-almost every } Y \in F^{m\times n}.$$

(II) *Let μ be a measure on $F^{m\times n}$ which is strongly contracting almost everywhere. Suppose that $\omega^\times(Y) = \eta$ for μ-almost every $Y \in F^{m\times n}$. Then, for all $\boldsymbol{\theta} \in F^m$,*

$$\omega^\times(Y, \boldsymbol{\theta}) \leq \eta \quad \text{for } \mu\text{-almost every } Y \in F^{m\times n}.$$

The above theorem is the function field analogue of [12, Theorem 2.2].

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3. Strongly contracting and friendly measures

We retain the notation and terminologies of Beresnevich and Velani from [4]. Let X be a metric space and $B \subseteq X$ be a ball. For $a > 0, aB$ denotes the ball with the same centre as B and radius a times the radius of B. We say that a measure μ on X is *nonatomic* if $\mu(\{x\}) = 0$ for any $x \in X$. The *support* of a measure is defined to be the smallest closed set S such that $\mu(X \setminus S) = 0$. We say that μ is *doubling* if there exists a constant $c > 0$ such that for any ball B with centre in S, we have $\mu(2B) \leq c\mu(B)$.

Consider the plane $\mathcal{L}_{\mathbf{b},\mathbf{c}} := \{Y \in F^{m\times n} : Y\mathbf{b} + \mathbf{c} = 0\}$ for $\mathbf{b} \in F^n$ with $\|\mathbf{b}\| = 1$ and $\mathbf{c} \in F^m$. For $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in (0,\infty)^m$, we define the $\boldsymbol{\varepsilon}$-neighbourhood of the plane $\mathcal{L}_{\mathbf{b},\mathbf{c}}$ by

$$\mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\boldsymbol{\varepsilon})} := \{Y \in F^{m\times n} : |Y_i\mathbf{b} + c_i| < \varepsilon_i \text{ for all } i = 1, \ldots, m\}, \tag{3.1}$$

where Y_i is the ith row of Y. If $\varepsilon_1 = \cdots = \varepsilon_m = \varepsilon$, we simply denote it by $\mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\varepsilon)}$.

DEFINITION 3.1. Let μ be a finite, nonatomic and doubling Borel measure on $F^{m\times n}$. Then μ is said to be *strongly contracting* if there exists $C, \alpha, r_0 > 0$ such that for any plane $\mathcal{L}_{\mathbf{b},\mathbf{c}}$, any $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in (0,\infty)^m$ with $\min\{\varepsilon_j : 1 \leq j \leq m\} < r_0$ and $0 < \delta < 1$, the following property is satisfied: for all $Y \in \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\delta\boldsymbol{\varepsilon})} \cap S$, there is an open ball B centred at Y such that

$$B \cap S \subset \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\boldsymbol{\varepsilon})} \tag{3.2}$$

and

$$\mu(5B \cap \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\delta\boldsymbol{\varepsilon})}) \leq C\delta^{\alpha}\mu(5B). \tag{3.3}$$

We call the measure μ contracting if the same property holds with $\varepsilon_1 = \cdots = \varepsilon_m = \varepsilon$. Further, μ is (strongly) contracting almost everywhere if for μ-almost every point $Y_0 \in F^{m\times n}$, there is a neighbourhood V of Y_0 such that the restriction $\mu|_V$ of μ to V is (strongly) contracting.

First, note that if a measure μ is strongly contracting then it is contracting. Next, we recall the notion of friendly measures from [14, 16]. For this, we first need to define nonplanarity and (C,α)-decaying for a measure μ. Let μ be a Borel measure on F^n and $S :=$ the support of μ. We say μ is nonplanar if $\mu(\mathcal{L}) = 0$ for any hyperplane $\mathcal{L}$. Given a hyperplane $\mathcal{L}$ and a ball B of F^n with $\mu(B) > 0$, we define $\|d_{\mathcal{L}}\|_{\mu,B}$ to be the supremum of $\operatorname{dist}(\mathbf{y}, \mathcal{L})$ over $\mathbf{y} \in S \cap B$, where $\operatorname{dist}(\mathbf{y}, \mathcal{L}) = \inf\{\|\mathbf{y} - \mathbf{l}\| : \mathbf{l} \in \mathcal{L}\}$. Given an open set V of F^n and $C, \alpha > 0$, we say that the measure μ is (C,α)-decaying on V if for any nonempty open ball $B \subset V$ centred in S, any affine hyperplane $\mathcal{L}$ of F^n and any $\varepsilon > 0$,

$$\mu(B \cap \mathcal{L}^{(\varepsilon)}) \leq C\Big(\frac{\varepsilon}{\|d_{\mathcal{L}}\|_{\mu,B}}\Big)^{\alpha}\mu(B),$$

where $\mathcal{L}^{(\varepsilon)}$ is the ε-neighbourhood of $\mathcal{L}$.

Definition 3.2. Let μ be a nonatomic Borel measure on F^n. We say that the measure μ is friendly if for μ-almost every point $\mathbf{y}_0 \in F^n$, there exists a neighbourhood V of $\mathbf{y}_0$ such that the restriction $\mu|_V$ of μ to V is finite, doubling, nonplanar and (C, α)-decaying for some $C, \alpha > 0$.

Following [4], the notion of d-contracting measure on F^n generalises the notion of contracting measure on F^n (identified with $F^{1\times n}$ or $F^{n\times 1}$).

Definition 3.3. Let $d \in \mathbb{N}$ with $0 \le d \le n-1$. A finite, nonatomic and doubling Borel measure on F^n is said to be d-contracting if it satisfies the conditions (3.2) and (3.3) for a contracting measure with the plane $\mathcal{L}_{\mathbf{b},\mathbf{c}}$ replaced by any d-dimensional plane $\mathcal{L}$.

It is easy to see that a contracting measure on $F^{1\times n}$ is essentially the same as a 0-contracting measure on F^n and a contracting measure on $F^{n\times 1}$ is essentially the same as an $(n-1)$-contracting measure on F^n.

Theorem 3.4. *Let $d \in \mathbb{N}$ with $0 \le d \le n-1$. Then any friendly measure on F^n is d-contracting almost everywhere.*

Proof. Let $d \in \mathbb{N}$, $0 \le d \le n-1$, and μ be a friendly measure on F^n. Then for μ-almost every point $\mathbf{y}_0 \in F^n$, there exists a neighbourhood V of $\mathbf{y}_0$ such that the restriction $\mu|_V$ of μ to V is nonplanar, finite, doubling and (C, α)-decaying on V for some $C, \alpha > 0$. For the sake of simplicity, without loss of generality, we may assume that $\mu = \mu|_V$. We need to show that there exists $C', \alpha', r_0 > 0$ such that for any d-dimensional plane $\mathcal{L}$ of $F^n, 0 < \varepsilon < r_0$ and $0 < \delta < 1$, the following property is satisfied: for all $\mathbf{y} \in \mathcal{L}^{(\delta\varepsilon)} \cap S$, there is an open ball B centred at $\mathbf{y}$ such that

$$B \cap S \subset \mathcal{L}^{(\varepsilon)} \tag{3.4}$$

and

$$\mu(5B \cap \mathcal{L}^{(\delta\varepsilon)}) \le C'\delta^{\alpha'}\mu(5B), \tag{3.5}$$

where S is the support of μ.

First we observe that any d-dimensional plane of F^n is of the form

$$\mathcal{L} = \{\mathbf{y} = (y_1, \ldots, y_n) \in F^n : b_{i1}y_1 + \cdots + b_{in}y_n + a_i = 0 \text{ for } i = 1, \ldots, s\},$$

where $\mathbf{b}_i = (b_{i1}, \ldots, b_{in}) \in F^n$ with $\|\mathbf{b}_i\| = 1$ and $s \in \{1, \ldots, n\}$. Then

$$\mathcal{L}^{(\varepsilon)} = \{\mathbf{y} = (y_1, \ldots, y_n) \in F^n : |b_{i1}y_1 + \cdots + b_{in}y_n + a_i| < \varepsilon \text{ for } i = 1, \ldots, s\}.$$

Since μ is nonplanar, the support S of μ contains n linearly independent points $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of F^n. Hence, we can find $r_0 > 0$ such that whenever $0 < \varepsilon < r_0$, the ε-neighbourhood of any d-dimensional plane $\mathcal{L}$ cannot contain all the points $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Therefore, for any d-dimensional plane $\mathcal{L}$ and $0 < \varepsilon < r_0$,

$$S \not\subset \mathcal{L}^{(\varepsilon)}. \tag{3.6}$$

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Fix $0 < \delta < 1$. If $S \cap \mathcal{L}^{(\delta\varepsilon)} = \emptyset$, then there is nothing to show. So we now assume that $S \cap \mathcal{L}^{(\delta\varepsilon)} \neq \emptyset$ and let $\mathbf{y} \in S \cap \mathcal{L}^{(\delta\varepsilon)}$. Our goal is to find a ball B centred at $\mathbf{y}$ such that (3.4) and (3.5) hold. Since $\mathcal{L}^{(\delta\varepsilon)}$ is an open set, we can find a ball B' centred at $\mathbf{y}$ such that

$$B' \subset \mathcal{L}^{(\varepsilon)}. \tag{3.7}$$

In view of (3.6) and (3.7), there exists a real number $\gamma \geq 1$ such that

$$5\gamma B' \cap S \not\subset \mathcal{L}^{(\varepsilon)} \quad \text{and} \quad \gamma B' \cap S \subset \mathcal{L}^{(\varepsilon)}.$$

Hence, there exists a point $\mathbf{y}' = (y_1', \dots, y_n') \in (5\gamma B' \cap S) \setminus \mathcal{L}^{(\varepsilon)}$ and this implies that

$$|b_{l1}y_1' + \cdots + b_{ln}y_n' + a_l| \geq \varepsilon \quad \text{for some } l \in \{1, \dots, s\}. \tag{3.8}$$

Consider the hyperplane $\mathcal{L}_0 = \{\mathbf{y} = (y_1, \dots, y_n) \in F^n : b_{l1}y_1 + \cdots + b_{ln}y_n + a_l = 0\}$. In view of (3.8) and using the ultrametric triangle inequality, $\mathrm{dist}(\mathbf{y}', \mathcal{L}_0) \geq \varepsilon$.

Take $B = 5\gamma B'$. From the above discussion, it is clear that $\|d_{\mathcal{L}_0}\|_{\mu,B} \geq \varepsilon$. Since μ is (C, α)-decaying with $\delta\varepsilon$ in place of ε,

$$\mu(5\gamma B' \cap \mathcal{L}^{(\delta\varepsilon)}) \leq \mu(5\gamma B' \cap \mathcal{L}_0^{(\delta\varepsilon)}) \leq C\Big(\frac{\delta\varepsilon}{\varepsilon}\Big)^{\alpha} \mu(5\gamma B') = C\delta^{\alpha}\mu(5\gamma B').$$

Therefore, the ball $\gamma B'$ satisfies conditions (3.4) and (3.5) and μ is d-contracting. □

COROLLARY 3.5. (1) *Any friendly measure on $F^{m\times 1}$ is (strongly) contracting almost everywhere.*

(2) *Any friendly measure on $F^{1\times n}$ is (strongly) contracting almost everywhere.*

PROOF. From Theorem 3.4, it follows at once that any friendly measure on $F^{m\times 1}$ or $F^{1\times n}$ is contracting almost everywhere.

The proof that any friendly measure on $F^{m\times 1}$ or $F^{1\times n}$ is strongly contracting almost everywhere is the same as the proof of Theorem 3.4. □

4. Lower bounds for Diophantine exponents

For the nontrivial implication in Theorem 2.1(B), we need to show that a certain measure μ on $F^{m\times n}$ is inhomogeneously strongly extremal. This amounts to showing that the following two statements hold for all $\boldsymbol{\theta} \in F^m$:

$$\omega^{\times}(Y, \boldsymbol{\theta}) \leq 1 \quad \text{for } \mu\text{-almost every } Y \in F^{m\times n}, \tag{4.1}$$

$$\omega^{\times}(Y, \boldsymbol{\theta}) \geq 1 \quad \text{for } \mu\text{-almost every } Y \in F^{m\times n}. \tag{4.2}$$

Proving the nontrivial implication in Theorem 2.1(A) amounts to showing analogous upper and lower bounds for $\omega(Y, \boldsymbol{\theta})$. We devote this section to prove (4.2), that is, the lower bound for inhomogeneous strong extremality. As a by-product, we get the lower bound for inhomogeneous extremality. Before going to the proof of this result, we collect some results related to the Diophantine transference principle over function fields. The first is the function field analogue of a result by Bugeaud and Laurent [5].

THEOREM 4.1 (Bugeaud and Zhang, [6]). *Let $Y \in F^{m\times n}$. Then for all $\boldsymbol{\theta} \in F^m$,*

$$\omega(Y,\boldsymbol{\theta}) \geq \frac{1}{\hat{\omega}(Y^t)} \quad \textit{and} \quad \hat{\omega}(Y,\boldsymbol{\theta}) \geq \frac{1}{\omega(Y^t)}. \tag{4.3}$$

Furthermore, equality occurs in (4.3) for almost all $\boldsymbol{\theta} \in F^m$.

Next, we recall a positive characteristic version of Dyson's transference principle [8].

THEOREM 4.2 [10]. *For any $Y \in F^{m\times n}$, we have $\omega(Y) = 1$ if and only if $\omega(Y^t) = 1$.*

Now we are ready to prove the desired lower bound.

PROPOSITION 4.3. *Let μ be an extremal measure on $F^{m\times n}$. Then for every $\boldsymbol{\theta} \in F^m$, we have $\omega^\times(Y,\boldsymbol{\theta}) \geq \omega(Y,\boldsymbol{\theta}) \geq 1$ for μ-almost every $Y \in F^{m\times n}$.*

PROOF. It is clear from the definitions that

$$\omega^\times(Y,\boldsymbol{\theta}) \geq \omega(Y,\boldsymbol{\theta}) \tag{4.4}$$

for all $Y \in F^{m\times n}$ and $\boldsymbol{\theta} \in F^m$. From Dirichlet's theorem and the definition of uniform exponent,

$$\hat{\omega}(Y) \geq 1 \quad \text{and} \quad \omega(Y) \geq \hat{\omega}(Y) \tag{4.5}$$

for all $Y \in F^{m\times n}$. Since the given measure μ is extremal, $\omega(Y) = 1$ for μ-almost every $Y \in F^{m\times n}$. From Theorem 4.2, $\omega(Y^t) = 1$ for μ-almost every $Y \in F^{m\times n}$. Again from (4.5), $\hat{\omega}(Y^t) = 1$ for μ-almost every $Y \in F^{m\times n}$. Finally, using Theorem 4.1,

$$\omega(Y,\boldsymbol{\theta}) \geq \frac{1}{\hat{\omega}(Y^t)} \geq 1$$

for μ-almost every $Y \in F^{m\times n}$. This completes the proof in view of (4.4). □

PROOF OF PROPOSITION 1.8. Let μ be an inhomogeneously strongly extremal measure on $F^{m\times n}$. Given any $\boldsymbol{\theta} \in F^m$, we have $\omega^\times(Y,\boldsymbol{\theta}) = 1$ for μ-almost every $Y \in F^{m\times n}$. Therefore, $\omega(Y,\boldsymbol{\theta}) \leq 1$ for all $\boldsymbol{\theta} \in F^m$ and for μ-almost every $Y \in F^{m\times n}$ by (1.1).

To complete the proof, we need to show that $\omega(Y,\boldsymbol{\theta}) \geq 1$ for all $\boldsymbol{\theta} \in F^m$ and for μ-almost every $Y \in F^{m\times n}$. Since μ is inhomogeneously strongly extremal, it is trivially extremal. So the desired result follows from Proposition 4.3. □

5. Inhomogeneous transference principle

The main ingredient in the proof of Theorems 2.1 and 2.4 is the inhomogeneous transference principle [4, Section 5]. Let (X, d) be a locally compact metric space. Given two countable indexing sets $\mathcal{A}$ and $\mathbf{T}$, let H and I be two maps from $\mathbf{T} \times \mathcal{A} \times \mathbb{R}_+$ into the set of open subsets of X such that

$$H : (\mathbf{t}, \alpha, \lambda) \in \mathbf{T} \times \mathcal{A} \times \mathbb{R}_+ \mapsto H_{\mathbf{t}}(\alpha, \lambda)$$

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and

$$I : (\mathbf{t}, \alpha, \lambda) \in \mathbf{T} \times \mathcal{A} \times \mathbb{R}_+ \mapsto I_{\mathbf{t}}(\alpha, \lambda).$$

Also, let

$$H_{\mathbf{t}}(\lambda) \stackrel{\text{def}}{=} \bigcup_{\alpha \in \mathcal{A}} H_{\mathbf{t}}(\alpha, \lambda) \quad \text{and} \quad I_{\mathbf{t}}(\lambda) \stackrel{\text{def}}{=} \bigcup_{\alpha \in \mathcal{A}} I_{\mathbf{t}}(\alpha, \lambda).$$

Let Φ denote a set of functions $\phi : \mathbf{T} \to \mathbb{R}_+$. For $\phi \in \Phi$, consider the limsup sets

$$\Lambda_H(\phi) = \limsup_{\mathbf{t} \in \mathbf{T}} H_{\mathbf{t}}(\phi(\mathbf{t})) \quad \text{and} \quad \Lambda_I(\phi) = \limsup_{\mathbf{t} \in \mathbf{T}} I_{\mathbf{t}}(\phi(\mathbf{t})).$$

We call the sets associated with the maps H and I the homogeneous and inhomogeneous sets, respectively. Now we discuss two important properties, which are the key ingredients for the inhomogeneous transference principle.

The intersection property. The triple (H, I, Φ) satisfies the *intersection property* if for any $\phi \in \Phi$, there exists $\phi^* \in \Phi$ such that for all but finitely many $\mathbf{t} \in T$, and all distinct α and α' in $\mathcal{A}$, we have $I_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \cap I_{\mathbf{t}}(\alpha', \phi(\mathbf{t})) \subseteq H_{\mathbf{t}}(\phi^*(\mathbf{t}))$.

The contraction property. Let μ be a nonatomic finite doubling measure supported on a bounded subset $\mathbf{S}$ of X. We say μ is *contracting with respect* to (I, Φ) if for any $\phi \in \Phi$, there exists $\phi^+ \in \Phi$ and a sequence of positive numbers $\{k_{\mathbf{t}}\}_{\mathbf{t} \in \mathbf{T}}$ satisfying $\sum_{\mathbf{t} \in \mathbf{T}} k_{\mathbf{t}} < \infty$, and such that for all but finitely many $\mathbf{t} \in \mathbf{T}$ and all $\alpha \in \mathcal{A}$, there exists a collection $C_{\mathbf{t},\alpha}$ of balls B centred in $\mathbf{S}$ satisfying the following three conditions:

(C.1) $\mathbf{S} \cap I_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \subseteq \bigcup_{B \in C_{\mathbf{t},\alpha}} B$;
(C.2) $\mathbf{S} \cap \bigcup_{B \in C_{\mathbf{t},\alpha}} B \subseteq I_{\mathbf{t}}(\alpha, \phi^+(\mathbf{t}))$; and
(C.3) $\mu(5B \cap I_{\mathbf{t}}(\alpha, \phi(\mathbf{t}))) \leq k_{\mathbf{t}} \mu(5B)$.

THEOREM 5.1 [4, Theorem 5]. *If the triple* (H, I, Φ) *satisfies the intersection property and* μ *is contracting with respect to* (I, Φ)*, then*

$$\mu(\Lambda_H(\phi)) = 0 \quad \textit{for all } \phi \in \Phi \Longrightarrow \mu(\Lambda_I(\phi)) = 0 \quad \textit{for all } \phi \in \Phi.$$

6. Upper bounds for Diophantine exponents

In this section, we will prove Theorem 2.1(B) (the proof of Theorem 2.1(A) is similar). Clearly, one direction is trivial. For the other direction, let μ be a measure on $F^{m \times n}$, which is strongly contracting almost everywhere. We want to show that

$$\mu \text{ is strongly extremal} \Longrightarrow \mu \text{ is inhomogeneously strongly extremal.}$$

Earlier we observed that this amounts to showing the upper and lower bounds, (4.1) and (4.2), respectively. We have already proved the lower bound in the last section. We prove the upper bound in this section using the inhomogeneous transference principle. In fact, we will prove the upper bound for the general nonextremal case (that is, Theorem 2.4(II)). We follow the strategy of Beresnevich and Velani [4].

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Let μ be a measure on $F^{m\times n}$, which is strongly contracting almost everywhere. Suppose that $\omega^{\times}(Y) = \eta$ for μ-almost every $Y \in F^{m\times n}$. We want to prove that, for all $\boldsymbol{\theta} \in F^m$, $\omega^{\times}(Y, \boldsymbol{\theta}) \leq \eta$ for μ-almost every $Y \in F^{m\times n}$. Define

$$\mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta) := \{Y \in F^{m\times n} : \omega^{\times}(Y, \boldsymbol{\theta}) > \eta\}.$$

Observe that Theorem 2.4(II) reduces to proving

$$\mu(\mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta)) = 0 \quad \text{for all } \boldsymbol{\theta} \in F^m. \tag{6.1}$$

Consider $\mathbf{T} = \mathbb{Z}^{m+n}$. For each $\mathbf{t} = (t_1, \ldots, t_{m+n}) \in \mathbf{T}$, let

$$a_{\mathbf{t}} := \operatorname{diag}\{X^{t_1}, \ldots, X^{t_m}, X^{-t_{m+1}}, \ldots, X^{-t_{m+n}}\} \in F^{m\times n}.$$

For any matrix $Y \in F^{m\times n}$, let

$$U_Y := \begin{bmatrix} I_m & Y \\ 0 & I_n \end{bmatrix},$$

where I_k denotes the identity matrix of size $k \times k$. We can view U_Y as a linear operator on F^{m+n}. Now given any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in F^m$, we define the affine transformation $U^{\boldsymbol{\theta}}_Y$ on F^{m+n} by

$$U^{\boldsymbol{\theta}}_Y(\mathbf{a}) := U^{\boldsymbol{\theta}}_Y \mathbf{a} := U_Y(\mathbf{a}) + \Theta \quad \text{for all } \mathbf{a} \in F^{m+n},$$

where $\Theta = (\theta_1, \ldots, \theta_m, 0, \ldots, 0)^t \in F^{m+n}$. Let $\mathcal{A} = \Lambda^m \times (\Lambda^n \setminus \{0\})$. For $\varepsilon > 0, \mathbf{t} \in T$ and $\alpha \in \mathcal{A}$, let

$$\Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\alpha, \varepsilon) := \{Y \in F^{m\times n} : \|a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha\| < \varepsilon\} \tag{6.2}$$

and

$$\Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\varepsilon) := \bigcup_{\alpha\in\mathcal{A}} \Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\alpha, \varepsilon) = \{Y \in F^{m\times n} : \inf_{\alpha\in\mathcal{A}} \|a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha\| < \varepsilon\}.$$

For $\tau > 0$, consider the function

$$\phi^{\tau} : \mathbf{T} \to \mathbb{R}_+; \ \mathbf{t} \mapsto \phi^{\tau}_{\mathbf{t}} := e^{-\tau\sigma(\mathbf{t})}, \tag{6.3}$$

where $\sigma(\mathbf{t}) := t_1 + \cdots + t_{m+n}$, and the set

$$\Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi^{\tau}) := \limsup_{\mathbf{t}\in\mathbf{T}} \Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\phi^{\tau}). \tag{6.4}$$

For $\boldsymbol{\theta} = \mathbf{0}$, that is, the homogeneous case, we denote $\Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi^{\tau})$ by $\Delta_{\mathbf{T}}(\phi^{\tau})$. The following proposition allows us to reformulate the set $\mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta)$ in terms of these limsup sets.

Proposition 6.1. *There exists a subset* $\mathbf{T}$ *of* $\mathbb{Z}^{m+n}$ *such that*

$$\sum_{\mathbf{t}\in\mathbf{T}} e^{-\tau\sigma(\mathbf{t})} < \infty \quad \text{for all } \tau > 0 \tag{6.5}$$

and

$$\mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta) = \bigcup_{\tau>0} \Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi^{\tau}) \quad \text{for all } \boldsymbol{\theta} \in F^m.$$

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PROOF. First, we define the required set $\mathbf{T} \subset \mathbb{Z}^{m+n}$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}_+^m$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}_+^n$, we define

$$\sigma(\mathbf{u}) := \sum_{j=1}^{m} u_j, \quad \sigma(\mathbf{v}) := \sum_{i=1}^{n} v_i \quad \text{and} \quad \xi := \xi(\mathbf{u}, \mathbf{v}) = \frac{\sigma(\mathbf{u}) - \eta\sigma(\mathbf{v})}{m + \eta n},$$

where $\mathbb{Z}_+ = \{s \in \mathbb{Z} : s \geq 0\}$. Given $\mathbf{u}$ and $\mathbf{v}$ as above, define the $(m+n)$-tuple $\mathbf{t} = (t_1, \ldots, t_{m+n})$ as

$$\mathbf{t} := (u_1 - [\xi], \ldots, u_m - [\xi], v_1 + [\xi], \ldots, v_n + [\xi]), \tag{6.6}$$

where for any $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not greater than x. Finally,

$$\mathbf{T} := \{\mathbf{t} \in \mathbb{Z}^{m+n} \text{ given by (6.6)} : \mathbf{u} \in \mathbb{Z}_+^m,\ \mathbf{v} \in \mathbb{Z}_+^n \text{ with } \sigma(\mathbf{u}) \geq \eta\sigma(\mathbf{v})\}.$$

We show that $\mathbf{T}$ has the properties required in Proposition 6.1. First, let us record a few inequalities which will be essential later. Note that

$$\sigma(\mathbf{t}) := \sum_{i=1}^{m+n} t_i = \sigma(\mathbf{u}) - m[\xi] + \sigma(\mathbf{v}) + n[\xi] = \frac{\eta+1}{\eta}\sigma(\mathbf{u}) - m\Big(\frac{\xi}{\eta} + [\xi]\Big) + n([\xi] - \xi) \tag{6.7}$$

and also

$$\sigma(\mathbf{t}) = (\eta+1)\sigma(\mathbf{v}) + m(\xi - [\xi]) + n([\xi] + \eta\xi). \tag{6.8}$$

From (6.7) and (6.8) respectively

$$\begin{aligned}
\frac{\eta}{\eta+1}\sigma(\mathbf{t}) &= \sigma(\mathbf{u}) - \frac{m\eta}{\eta+1}\Big(\frac{\xi}{\eta} + [\xi]\Big) + \frac{n\eta}{\eta+1}([\xi] - \xi) \\
&\leq \sigma(\mathbf{u}) - \frac{m\eta}{\eta+1}\Big(\frac{\xi}{\eta} + [\xi]\Big), \quad \text{since } \xi - 1 < [\xi] \leq \xi \\
&= \sigma(\mathbf{u}) - \frac{m\eta}{\eta+1}\Big(\frac{[\xi]}{\eta} + [\xi]\Big) + \frac{m\eta}{\eta+1}\Big(\frac{[\xi]}{\eta} - \frac{\xi}{\eta}\Big) \\
&\leq \sigma(\mathbf{u}) - m[\xi], \quad \text{since } \xi - 1 < [\xi] \leq \xi
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
\frac{1}{\eta+1}\sigma(\mathbf{t}) &= \sigma(\mathbf{v}) + \frac{m}{\eta+1}(\xi - [\xi]) + \frac{n}{\eta+1}([\xi] + \eta\xi) \\
&\geq \sigma(\mathbf{v}) + \frac{n}{\eta+1}([\xi] + \eta\xi), \quad \text{since } \xi - 1 < [\xi] \leq \xi \\
&= \sigma(\mathbf{v}) + \frac{n}{\eta+1}([\xi] + \eta[\xi]) + \frac{n}{\eta+1}(\eta\xi - \eta[\xi]) \\
&\geq \sigma(\mathbf{v}) + n[\xi], \quad \text{as } \xi - 1 < [\xi] \leq \xi.
\end{aligned} \tag{6.10}$$

Since $\xi \geq 0$, in view of (6.9) and (6.10),

$$(\eta+1)\sigma(\mathbf{v}) \leq \sigma(\mathbf{t}) \leq \frac{\eta+1}{\eta}\sigma(\mathbf{u}). \tag{6.11}$$

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From (6.9), (6.10) and using $\xi - 1 < [\xi] \le \xi$,

$$\begin{aligned}\sigma(\mathbf{t}) = \frac{\eta}{\eta+1}\sigma(\mathbf{t}) + \frac{1}{\eta+1}\sigma(\mathbf{t}) &= \sigma(\mathbf{u}) + \sigma(\mathbf{v}) - (m-n)[\xi]\\ &\ge \sigma(\mathbf{u}) + \sigma(\mathbf{v}) - (m-n)\xi\\ &= \sigma(\mathbf{u}) + \sigma(\mathbf{v}) - \frac{(m-n)}{m+\eta n}(\sigma(\mathbf{u}) - \eta\sigma(\mathbf{v}))\\ &= \frac{\eta+1}{m+\eta n}(n\sigma(\mathbf{u}) + m\sigma(\mathbf{v})). \qquad (6.12)\end{aligned}$$

From (6.12), we will conclude that the series (6.5) is convergent and, for any $r \in \mathbb{R}_+$,

$$\#\{\mathbf{t} \in \mathbf{T} : \sigma(\mathbf{t}) < r\} < \infty. \qquad (6.13)$$

First, we show that

$$\mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta) \subseteq \bigcup_{\tau>0} \Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi^{\tau}). \qquad (6.14)$$

Let $Y \in \mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta)$. Note that $Y \in \mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta)$ if and only if there exists $\varepsilon > 0$ such that for arbitrarily large $T \ge 1$, there exists $\alpha = (\mathbf{p}, \mathbf{q}) \in \mathcal{A} = \Lambda^m \times (\Lambda^n \setminus \{0\})$ satisfying $\|Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}\| \le 1/e$ such that

$$\prod(Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}) < e^{-(\eta+\varepsilon)T} \quad \text{and} \quad \prod\nolimits_+(\mathbf{q}) < e^{T}. \qquad (6.15)$$

Since $Y \in \mathcal{U}^{\boldsymbol{\theta}}_{m,n}(\eta)$, (6.15) is satisfied for infinitely many $T \in \mathbb{N}$. For any such T, there is a unique $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^m_+$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n_+$ such that

$$e^{-u_j} \le \max\{|Y_j\mathbf{q} + p_j + \theta_j|, e^{-(\eta+\varepsilon)T}\} < e^{-u_j+1} \quad \text{for } 1 \le j \le m \qquad (6.16)$$

and

$$e^{v_i} \le \max\{1, |q_i|\} < e^{v_i+1} \quad \text{for } 1 \le i \le n, \qquad (6.17)$$

where Y_j denotes the jth row of $Y \in F^{m\times n}$. From (6.16) and (6.17),

$$e^{-\sigma(\mathbf{u})} < \max\Big\{\prod(Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}), e^{-(\eta+\varepsilon)T}\Big\} \quad \text{and} \quad e^{\sigma(\mathbf{v})} \le \prod\nolimits_+(\mathbf{q}). \qquad (6.18)$$

Now (6.15) and (6.18) imply that $e^{-\sigma(\mathbf{u})} < e^{-\sigma(\mathbf{v})(\eta+\varepsilon)}$. Therefore,

$$\sigma(\mathbf{u}) - \eta\sigma(\mathbf{v}) > \varepsilon\sigma(\mathbf{v}) \ge 0. \qquad (6.19)$$

Hence, $\mathbf{t}$ given by (6.6) with $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ satisfying (6.16) and (6.17), respectively, is in $\mathbf{T}$. If $\sigma(\mathbf{u}) \le 2\eta\sigma(\mathbf{v})$, then from (6.11) and (6.19),

$$\xi = \frac{\sigma(\mathbf{u}) - \eta\sigma(\mathbf{v})}{m+\eta n} \ge \frac{\varepsilon\sigma(\mathbf{v})}{m+\eta n} \ge \frac{\varepsilon\sigma(\mathbf{u})}{2\eta(m+n)} > \frac{\varepsilon\sigma(\mathbf{t})}{2(\eta+1)(m+n)}.$$

If $\sigma(\mathbf{u}) > 2\eta\sigma(\mathbf{v})$, then from (6.11),

$$\xi = \frac{\sigma(\mathbf{u}) - \eta\sigma(\mathbf{v})}{m+\eta n} = \frac{2\sigma(\mathbf{u}) - 2\eta\sigma(\mathbf{v})}{2(m+\eta n)} > \frac{\sigma(\mathbf{u})}{2(m+\eta n)} > \frac{\eta\sigma(\mathbf{t})}{2(\eta+1)(m+\eta n)}.$$

In view of these two inequalities,

$$\xi > \tau_0\sigma(\mathbf{t}), \quad \text{where } \tau_0 := \frac{\min\{\varepsilon, \eta\}}{2(\eta+1)(m+\eta n)}. \tag{6.20}$$

Now we take

$$a_{\mathbf{t}} = X^{-([\xi]+1)}\mathrm{diag}\{X^{u_1}, \ldots, X^{u_m}, X^{-v_1}, \ldots, X^{-v_n}\}.$$

From (6.16) and (6.17),

$$\inf_{\alpha\in\mathcal{A}} \|a_{\mathbf{t}} U_Y^{\boldsymbol{\theta}} \alpha\| < e \cdot e^{-([\xi]+1)} \le e \cdot e^{-\xi}. \tag{6.21}$$

Combining (6.20) and (6.21), we conclude that for $0 < \tau < \tau_0$,

$$\inf_{\alpha\in\mathcal{A}} \|a_{\mathbf{t}} U_Y^{\boldsymbol{\theta}} \alpha\| < e^{-\tau\sigma(\mathbf{t})} \tag{6.22}$$

for all sufficiently large $\sigma(\mathbf{t})$. Together, (6.15) and (6.16) imply that $\sigma(\mathbf{u}) \to \infty$ as $T \to \infty$. Hence, in view of (6.12) and the fact that (6.15) holds for arbitrarily large $T \in \mathbb{N}$, we conclude that (6.22) holds for infinitely many $\mathbf{t} \in \mathbf{T}$. Therefore, $Y \in \Delta_{\mathbf{T}}^{\boldsymbol{\theta}}(\phi^\tau)$ for any $\tau \in (0, \tau_0)$. This completes the proof of (6.14).

Finally, to complete the proof of Proposition 6.1, we show that

$$\mathcal{U}_{m,n}^{\boldsymbol{\theta}}(\eta) \supseteq \bigcup_{\tau>0} \Delta_{\mathbf{T}}^{\boldsymbol{\theta}}(\phi^\tau).$$

Let $Y \in \Delta_{\mathbf{T}}^{\boldsymbol{\theta}}(\phi^\tau)$ for some $\tau > 0$. By definition, $\inf_{\alpha\in\mathcal{A}} \|a_{\mathbf{t}} U_Y^{\boldsymbol{\theta}} \alpha\| < e^{-\tau\sigma(\mathbf{t})}$ for infinitely many $\mathbf{t} \in \mathbf{T}$. For any such $\mathbf{t}$, there exists $\alpha = (\mathbf{p}, \mathbf{q}) \in \mathcal{A}$ such that $\|a_{\mathbf{t}} U_Y^{\boldsymbol{\theta}} \alpha\| < e^{-\tau\sigma(\mathbf{t})}$. By taking the product of the first m coordinates and the last n nonzero coordinates of $a_{\mathbf{t}} U_Y^{\boldsymbol{\theta}} \alpha$, respectively we get

$$\prod_{j=1}^{m} e^{t_j} |Y_j \mathbf{q} + p_j + \theta_j| < e^{-m\tau\sigma(\mathbf{t})} \quad \text{and} \quad \prod_{1\le i\le n,\, q_i\ne 0} e^{-t_{m+i}} |q_i| < e^{-n\tau\sigma(\mathbf{t})}.$$

From (6.11) and the fact that $\sigma(\mathbf{t}) \ge 0$ (by (6.10)),

$$\prod (Y\mathbf{q} + \mathbf{p} + \boldsymbol{\theta}) < e^{-m\tau\sigma(\mathbf{t})} \cdot e^{-\eta\sigma(\mathbf{t})/(\eta+1)} = e^{-(\eta+m\tau(\eta+1))\sigma(\mathbf{t})/(\eta+1)} \tag{6.23}$$

and

$$\prod\nolimits_{+} (\mathbf{q}) < e^{-n\tau\sigma(\mathbf{t})} e^{\sigma(\mathbf{t})/(\eta+1)} < e^{\sigma(\mathbf{t})/(\eta+1)}. \tag{6.24}$$

If we take $T = \sigma(\mathbf{t})/(\eta+1)$ and $\varepsilon := m\tau(\eta+1)$, then (6.23) and (6.24) hold for arbitrarily large T. Therefore, $Y \in \mathcal{U}_{m,n}^{\boldsymbol{\theta}}(\eta)$, completing the proof of Proposition 6.1. □

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PROOF OF THEOREM 2.4. In view of (6.1) and Proposition 6.1, we first note that to prove Theorem 2.4(II), it is enough to show that

$$\mu(\Delta_{\mathbf{T}}(\phi^\tau)) = 0 \quad \text{for all } \tau > 0 \implies \mu(\Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi^\tau)) \quad \text{for all } \tau > 0. \tag{6.25}$$

We use the inhomogeneous transference principle to prove (6.25). From now on, $\boldsymbol{\theta} \in F^m$ is fixed and, without loss of generality, we assume that μ is strongly contracting on $F^{m\times n}$. Let $X := F^{m\times n}$, $\mathcal{A} = \Lambda^m \times (\Lambda^n \setminus \{0\})$, $\mathbf{T}$ be as in Proposition 6.1, and the maps H and I be given by

$$H_{\mathbf{t}}(\alpha, \varepsilon) := \Delta_{\mathbf{t}}(\alpha, \varepsilon) = \Delta^{\mathbf{0}}_{\mathbf{t}}(\alpha, \varepsilon) \quad \text{and} \quad I_{\mathbf{t}}(\alpha, \varepsilon) := \Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\alpha, \varepsilon),$$

where $\varepsilon > 0, \mathbf{t} \in \mathbf{T}$, $\alpha \in \mathcal{A}$ and $\Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\alpha, \varepsilon)$ is given by (6.2). From these definitions, it follows readily that $H_{\mathbf{t}}(\varepsilon) = \Delta^{\mathbf{0}}_{\mathbf{t}}(\varepsilon)$ and $I_{\mathbf{t}}(\varepsilon) = \Delta^{\boldsymbol{\theta}}_{\mathbf{t}}(\varepsilon)$. Let Φ be the collection of functions given by (6.3). Then

$$\Lambda_H(\phi) = \Delta_{\mathbf{T}}(\phi) := \Delta^{\mathbf{0}}_{\mathbf{T}}(\phi) \quad \text{and} \quad \Lambda_I(\phi) = \Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi),$$

where $\Delta^{\boldsymbol{\theta}}_{\mathbf{T}}(\phi)$ is given by (6.4). Now (6.25) will immediately follow by applying the inhomogeneous transference principle if we can verify that the triple (H, I, Φ) satisfies the intersection property and the measure μ is contracting with respect to (I, Φ).

The intersection property. Let $\alpha = (\mathbf{p}, \mathbf{q})$ and $\alpha' = (\mathbf{p}', \mathbf{q}')$ be two distinct elements in $\mathcal{A}$, where $\mathbf{p}, \mathbf{p}' \in \Lambda^m$ and $\mathbf{q}, \mathbf{q}' \in \Lambda^n \setminus \{0\}$. Also let $\phi \in \Phi$. Then $\phi(\mathbf{t}) = e^{-\tau\sigma(\mathbf{t})}$ for some $\tau > 0$. Consider any element Y from $I_{\mathbf{t}}(\alpha, \phi(\mathbf{t})) \cap I_{\mathbf{t}}(\alpha', \phi(\mathbf{t}))$. Then

$$\|a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha\| < \phi(\mathbf{t}) \quad \text{and} \quad \|a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha'\| < \phi(\mathbf{t}).$$

Since $\mathbf{T}$ satisfies (6.5) and (6.13),

$$\|a_{\mathbf{t}} U_Y(\alpha - \alpha')\| = \|a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha - a_{\mathbf{t}} U^{\boldsymbol{\theta}}_Y \alpha'\| < \phi(\mathbf{t}) \tag{6.26}$$

for all but finitely many $\mathbf{t} \in \mathbf{T}$. Let $\alpha'' := \alpha - \alpha' = (\mathbf{p}'', \mathbf{q}'')$, where $\mathbf{p}'' = \mathbf{p} - \mathbf{p}' \in \Lambda^m$ and $\mathbf{q}'' = \mathbf{q} - \mathbf{q}' \in \Lambda^n$. If $\mathbf{q}'' = 0$, from (6.26), we get $\|\mathbf{p}''\| < 1$ for all but finitely many $\mathbf{t} \in \mathbf{T}$. Then $\mathbf{p}'' = 0$ (since $\mathbf{p}'' \in \Lambda^m$), which is a contradiction as $\alpha \neq \alpha'$. Hence, $\mathbf{q}'' \neq 0$ and so $\alpha'' \in \mathcal{A}$. Therefore, $Y \in \Delta_{\mathbf{t}}(\alpha'', \phi(\mathbf{t})) \subset H_{\mathbf{t}}(\phi(\mathbf{t}))$. This completes the verification of the intersection property with $\phi^* = \phi \in \Phi$.

The contracting property. Recall that μ is a strongly contracting measure on $F^{m\times n}$. Without loss in generality, we may assume that the support S of μ is bounded. Note that μ is already doubling, finite and nonatomic. Hence, to show that μ is contracting with respect to (I, Φ), it only remains to verify conditions (C.1)–(C.3). Let $\phi \in \Phi$. Then $\phi(\mathbf{t}) = e^{-\tau\sigma(\mathbf{t})}$ for some constant $\tau > 0$. Define $\phi^+ := \sqrt{\phi} \in \Phi$. Let r_0 be as in the definition of strongly contracting measure. Since $\mathbf{T}$ satisfies (6.5) and (6.13),

$$\phi^+(\mathbf{t}) \le \min\{1, r_0\} \quad \text{and} \quad \sigma(\mathbf{t}) \ge 0 \tag{6.27}$$

for all but finitely many $\mathbf{t} \in \mathbf{T}$. If $\mathbf{t} = (t_1, \ldots, t_{m+n}) \in \mathbf{T}$ and $\alpha' = (\mathbf{p}', \mathbf{q}') \in \mathcal{A}$, the set $I_{\mathbf{t}}(\alpha', \phi(\mathbf{t}))$ is essentially the set of all $Y \in F^{m\times n}$ such that

$$|Y_j \mathbf{q}' + p'_j + \theta_j| < e^{-t_j}\phi(\mathbf{t}) \quad \text{and} \quad |q'_i| < e^{t_{m+i}}\phi(\mathbf{t}) \tag{6.28}$$

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for $1 \le j \le m$ and $1 \le i \le n$. In a similar fashion, we see that $I_{\mathbf{t}}(\alpha', \phi^+(\mathbf{t}))$ is the set of all $Y \in F^{m\times n}$ such that

$$|Y_j\mathbf{q}' + p_j' + \theta_j| < e^{-t_j}\phi^+(\mathbf{t}) \quad \text{and} \quad |q_i'| < e^{t_{m+i}}\phi^+(\mathbf{t}). \tag{6.29}$$

Now we define

$$\varepsilon_j = \varepsilon_{j,\mathbf{t}} := \frac{e^{-t_j}\phi^+(\mathbf{t})}{\|\mathbf{q}'\|} \quad \text{for } j = 1, \dots, m \quad \text{and} \quad \delta = \delta_{\mathbf{t}} := \phi^+(\mathbf{t}).$$

Equation (6.11) and $\sigma(\mathbf{t}) \ge 0$ imply that $\sum_{j=1}^m t_j \ge 0$. Hence, there exists some $l \in \{1, \dots, m\}$ such that $t_l \ge 0$. Therefore, since $\|\mathbf{q}'\| \ge 1$ and using (6.27),

$$\min_{1\le j\le m} \varepsilon_{j,\mathbf{t}} < r_0 \quad \text{and} \quad \delta_{\mathbf{t}} < 1.$$

Note that $\delta\varepsilon_j = e^{-t_j}\phi(\mathbf{t})/\|\mathbf{q}'\|$ for all $j = 1, \dots, m$. Let $\mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\boldsymbol{\varepsilon})}$ and $\mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\delta\boldsymbol{\varepsilon})}$ be given by (3.1) with $\mathbf{c} := X^{-(\max_{1\le i\le n}\deg q_i')}\mathbf{q}'$ and $\mathbf{b} := X^{-(\max_{1\le i\le n}\deg q_i')}(\mathbf{q}' + \boldsymbol{\theta})$, where $\mathbf{q}' = (q_1', \dots, q_n')$. Then in view of (6.28) and (6.29),

$$I_{\mathbf{t}}(\alpha', \phi(\mathbf{t})) = \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\delta\boldsymbol{\varepsilon})} \quad \text{and} \quad I_{\mathbf{t}}(\alpha', \phi^+(\mathbf{t})) = \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\boldsymbol{\varepsilon})}. \tag{6.30}$$

Since μ is strongly contracting, for all $Y \in \mathcal{L}_{\mathbf{b},\mathbf{c}}^{(\delta\boldsymbol{\varepsilon})} \cap S$, there is an open ball B_Y centred at Y satisfying (3.2) and (3.3). Following the notation from Section 5, define $C_{\mathbf{t},\alpha'}$ to be the collection of all such balls. Also, define

$$k_{\mathbf{t}} := C(\phi^+)^{\alpha},$$

where C and α are constants as in the definition of strongly contracting measure. By (6.5), $\sum_{\mathbf{t}\in\mathbf{T}} k_{\mathbf{t}} < \infty$. Observe that condition (C.1) follows from the definition of $C_{\mathbf{t},\alpha'}$. Finally, conditions (C.2) and (C.3) follow from (3.2), (3.3) and (6.30). This shows that μ is contracting with respect to (I, Φ). □

REMARK 6.2. To prove Theorem 2.4(I), we take $\mathcal{U}_{m,n}^{\boldsymbol{\theta}}(\eta) := \{Y \in F^{m\times n} : \omega(Y, \boldsymbol{\theta}) > \eta\}$ and show that $\mu(\mathcal{U}_{m,n}^{\boldsymbol{\theta}}(\eta)) = 0$ for all $\boldsymbol{\theta} \in F^m$. Proposition 6.1 remains unchanged and the only change needed in the proof of Proposition 6.1 is the definition of $\mathbf{T}$. For $u \in \mathbb{Z}_+$ and $v \in \mathbb{Z}_+$, let $\mathbf{u} := (u, \dots, u) \in \mathbb{Z}_+^m$, $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{Z}_+^n$ and $\xi = (mu - \eta nv)/(m + \eta n)$. Given u and v, define $\mathbf{t} = (t_1, \dots, t_{m+n})$ by

$$\mathbf{t} := (u - [\xi], \dots, u - [\xi], v + [\xi], \dots, v + [\xi]). \tag{6.31}$$

Define

$$\mathbf{T} := \{\mathbf{t} \in \mathbb{Z}^{m+n} \text{ given by (6.31)} : u, v \in \mathbb{Z}_+ \text{ with } mu \ge \eta nv\}.$$

The new version of Proposition 6.1 follows by appropriately modifying the arguments of the proof of Proposition 6.1. Once we have Proposition 6.1, we can apply the inhomogeneous transference principle to get our desired result. This also completes the proof of Theorem 2.1(A).

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SOURAV DAS, Department of Mathematics and Statistics,
Indian Institute of Technology Kanpur, Kanpur 208016, India
e-mail: iamsouravdas1@gmail.com

ARIJIT GANGULY, Department of Mathematics and Statistics,
Indian Institute of Technology Kanpur, Kanpur 208016, India
e-mail: arijit.ganguly1@gmail.com

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