Proceedings of the Edinburgh Mathematical Society (2020) **63**, 169–192
doi:10.1017/S001309151900021X

WEIGHTED ESTIMATES FOR THE CALDERÓN COMMUTATOR

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(Received 17 January 2018; first published online 23 September 2019)

Abstract In this paper the authors consider the weighted estimates for the Calderón commutator defined by

$$\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)=\text{p.v.}\int_{\mathbb{R}}\frac{P_2(A;x,y)\prod_{j=1}^{m}(A_j(x)-A_j(y))}{(x-y)^{m+2}}f(y)\mathrm{d}y,$$

with $P_2(A;x,y)=A(x)-A(y)-A'(y)(x-y)$ and $A'\in\mathrm{BMO}(\mathbb{R})$. Dominating this operator by multi(sub)linear sparse operators, the authors establish the weighted bounds from $L^{p_1}(\mathbb{R},w_1)\times\cdots\times L^{p_{m+1}}(\mathbb{R},w_{m+1})$ to $L^p(\mathbb{R},\nu_{\vec{w}})$, with $p_1,\dots,p_{m+1}\in(1,\infty)$, $1/p=1/p_1+\cdots+1/p_{m+1}$, and $\vec{w}=(w_1,\dots,w_{m+1})\in A_{\vec{P}}(\mathbb{R}^{m+1})$. The authors also obtain the weighted weak type endpoint estimates for $\mathcal{C}_{m+1,A}$.

Keywords: Calderón commutator; weighted inequality; multilinear singular integral operator; sparse operator; multiple weight

2010 *Mathematics subject classification:* Primary 42B20

1. Introduction

As is well known, the Calderón commutator arose in the study of the $L^2(\mathbb{R})$ boundedness for the Cauchy integral along Lipschitz curves. Let $A_1,\dots,A_m$ be functions defined on $\mathbb{R}$ such that $a_j=A_j'\in L^{q_j}(\mathbb{R})$. Define the mth-order commutator of Calderón by

$$\mathcal{C}_{m+1}(a_1,\dots,a_m;f)(x)=\int_{\mathbb{R}}\frac{\prod_{j=1}^{m}(A_j(x)-A_j(y))}{(x-y)^{m+1}}f(y)\mathrm{d}y. \tag{1.1}$$

By the $T(1)$ theorem and the Calderón–Zygmund theory, we know that for all $p\in(1,\infty)$,

$$\|\mathcal{C}_{m+1}(a_1,\dots,a_m;f)\|_{L^p(\mathbb{R})}\lesssim\prod_{j=1}^{m}\|a_j\|_{L^\infty(\mathbb{R})}\|f\|_{L^p(\mathbb{R})},$$

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and $\mathcal{C}_{m+1}$ is bounded from $L^\infty(\mathbb{R})\times\cdots\times L^\infty(\mathbb{R})\times L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. For the case of $m=1$, it is known that $\mathcal{C}_2$ is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$ provided that $p,q\in(1,\infty)$ and $r\in(1/2,\infty)$ with $1/r=1/p+1/q$; moreover, it is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ to $L^{r,\infty}(\mathbb{R})$ if $\min\{p,q\}=1$; see [2, 3] for details. By establishing the weak type endpoint estimates for multilinear singular integral operators with non-smooth kernels, and reducing the operator $\mathcal{C}_{m+1}$ to a suitable multilinear singular integral with non-smooth kernel, Duong *et al.* [7] proved the following theorem.

Theorem 1.1. *Let* $m\in\mathbb{N}$, $p_1,\dots,p_{m+1}\in[1,\infty)$ *and* $p\in[1/(m+1),\infty)$ *with* $1/p=1/p_1+\cdots+1/p_{m+1}$. *Then*

$$\|\mathcal{C}_{m+1}(a_1,\dots,a_m;f)\|_{L^{p,\infty}(\mathbb{R})}\lesssim\prod_{j=1}^m\|a_j\|_{L^{p_j}(\mathbb{R})}\|f\|_{L^{p_{m+1}}(\mathbb{R})}.$$

Moreover, if $\min_{1\le j\le m+1}p_j>1$, *then*

$$\|\mathcal{C}_{m+1}(a_1,\dots,a_m;f)\|_{L^{p}(\mathbb{R})}\lesssim\prod_{j=1}^m\|a_j\|_{L^{p_j}(\mathbb{R})}\|f\|_{L^{p_{m+1}}(\mathbb{R})}.$$

Considerable attention has also been paid to the weighted estimates for $\mathcal{C}_{m+1}$. Duong *et al.* [6] proved that if $p_1,\dots,p_{m+1}\in(1,\infty)$, $p\in(1/(m+1),\infty)$ with $1/p=1/p_1+\cdots+1/p_{m+1}$, then for $w\in A_p(\mathbb{R})$, $\mathcal{C}_{m+1}$ is bounded from $L^{p_1}(\mathbb{R},w)\times\cdots\times L^{p_{m+1}}(\mathbb{R},w)$ to $L^p(\mathbb{R},w)$; where $A_p(\mathbb{R}^n)$ denotes the weight function class of Muckenhoupt; see [9] for definitions and properties of $A_p(\mathbb{R}^n)$. Grafakos *et al.* [10] considered the weighted estimates with the following multiple $A_{\vec{P}}$ weights, introduced by Lerner *et al.* [23].

Definition 1.2. Let $m\in\mathbb{N}$, $w_1,\dots,w_m$ be weights, $p_1,\dots,p_m\in[1,\infty)$, $p\in[1/m,\infty)$ with $1/p=1/p_1+\cdots+1/p_m$. Set $\vec{w}=(w_1,\dots,w_m)$, $\vec{P}=(p_1,\dots,p_m)$ and $\nu_{\vec{w}}=\prod_{k=1}^m w_k^{p/p_k}$. We say that $\vec{w}\in A_{\vec{P}}(\mathbb{R}^{mn})$ if the $A_{\vec{P}}(\mathbb{R}^{mn})$ constant of $\vec{w}$, defined by

$$[\vec{w}]_{A_{\vec{P}}}=\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}(x)\mathrm{d}x\right)\prod_{k=1}^m\left(\frac{1}{|Q|}\int_Q w_k^{-1/(p_k-1)}(x)\,\mathrm{d}x\right)^{p/p_k'},$$

is finite, where, for $r\in[1,\infty)$, $r'=r/(r-1)$; when $p_k=1$, $(1/|Q|\int_Q w_k^{-1/(p_k-1)})^{1/p_k'}$ is understood as $(\inf_Q w_k)^{-1}$.

Grafakos *et al.* [10] proved that if $p_1,\dots,\ p_{m+1}\in[1,\infty)$ and $p\in[1/(m+1),\infty)$ with $1/p=1/p_1+\cdots+1/p_{m+1}$, $\vec{w}=(w_1,\dots,w_m,w_{m+1})\in A_{\vec{P}}(\mathbb{R}^{m+1})$, then $\mathcal{C}_{m+1}$ is bounded from $L^{p_1}(\mathbb{R},w_1)\times\cdots\times L^{p_{m+1}}(\mathbb{R},w_{m+1})$ to $L^{p,\infty}(\mathbb{R},\nu_{\vec{w}})$, and if $\min_{1\le j\le m+1}p_j>1$, $\mathcal{C}_{m+1}$ is bounded from $L^{p_1}(\mathbb{R},w_1)\times\cdots\times L^{p_{m+1}}(\mathbb{R},w_{m+1})$ to $L^p(\mathbb{R},\nu_{\vec{w}})$. Fairly recently, by dominating multilinear singular integral operators by sparse operators, Chen and Hu [4] improved the result of Grafakos *et al.* in [10], and obtained the following quantitative weighted bounds for $\mathcal{C}_{m+1}$.

https://doi.org/10.1017/S001309151900021X Published online by Cambridge University Press

Theorem 1.3. *Let* $m \in \mathbb{N}$, $p_1, \dots, p_{m+1} \in (1, \infty)$ *and* $p \in (1/(m+1), \infty)$ *with* $1/p = 1/p_1 + \cdots + 1/p_{m+1}$, $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$. *Then*

$$\begin{aligned}\|\mathcal{C}_{m+1}(a_1, \dots, a_m; f)\|_{L^p(\mathbb{R},\nu_{\vec{w}})} \\ \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1,p_1'/p,\dots,p_{m+1}'/p\}} \prod_{j=1}^{m} \|a_j\|_{L^{p_j}(\mathbb{R},w_j)} \|f\|_{L^{p_{m+1}}(\mathbb{R},w_{m+1})}.\end{aligned} \tag{1.2}$$

We remark that the study of quantitative weighted bounds for classical operators in harmonic analysis was begun by Buckley [**1**] and then continued by many other authors; see [**17**–**19**, **21**, **22**, **24**, **26**, **27**] and references therein.

Observe that (1.2) also holds if $\max_{1\le j\le m} p_j = \infty$ but $p \in (1/(m+1), \infty)$ (in this case, $\|a_j\|_{L^\infty(\mathbb{R},w_j)}$ should be replaced by $\|a_j\|_{L^\infty(\mathbb{R})}$ and w_j should be replaced by 1 if $p_j = \infty$). A natural question is whether a result similar to (1.2) holds true when $a_j \in \mathrm{BMO}(\mathbb{R})$ for some $1 \le j \le m$. In this paper, we consider the operator defined by

$$\begin{aligned}&\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)(x) \\ &= \mathrm{p.v.} \int_{\mathbb{R}} \frac{P_2(A; x, y) \prod_{j=1}^{m} (A_j(x) - A_j(y))}{(x-y)^{m+2}} f(y)\mathrm{d}y,\end{aligned} \tag{1.3}$$

with $P_2(A; x, y) = A(x) - A(y) - A'(y)(x - y)$ and $A' \in \mathrm{BMO}(\mathbb{R})$. If $a = A' \in L^q(\mathbb{R})$ for some $q \in [1, \infty]$, then

$$\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)(x) = \mathcal{C}_{m+2}(a_1, \dots, a_m, a; f)(x) - \mathcal{C}_{m+1}(a_1, \dots, a_m, af)(x).$$

When $a_1, \dots, a_m \in L^\infty(\mathbb{R})$, it is obvious that $\prod_{j=1}^{m}(A_j(x) - A_j(y))(x - y)^{-m-1}$ is a Calderón–Zygmund kernel. Repeating the argument in [**5**], we know that for any $p \in (1, \infty)$,

$$\|\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)\|_{L^p(\mathbb{R})} \lesssim \|A'\|_{\mathrm{BMO}(\mathbb{R})} \prod_{j=1}^{m} \|a_j\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})}. \tag{1.4}$$

Moreover, the results in [**14**] imply that for each $\lambda > 0$,

$$|\{x \in \mathbb{R} : \mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)(x) > \lambda\}| \lesssim_{a_1,\dots,a_m} \int_{\mathbb{R}} \frac{|f(x)|}{\lambda} \log\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right)\mathrm{d}x.$$

Operators like $\mathcal{C}_{m+1,A}$ with $a_j \in L^\infty(\mathbb{R})$ were introduced by Cohen [**5**], and then considered by Hofmann [**11**] and other authors; see also [**12**–**14**] and the related references therein.

Our main purpose in this paper is to establish the weighted bound similar to (1.2) for the operator $\mathcal{C}_{m+1,A}$ in (1.3). For a weight $u \in A_\infty(\mathbb{R}^n) = \cup_{p\ge1} A_p(\mathbb{R}^n)$, $[u]_{A_\infty}$, the A_∞ constant of u, is defined by

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x)\mathrm{d}x.$$

Recall that for $p_1, \dots, p_m \in [1, \infty)$, $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ if and only if $\nu_{\vec{w}} \in A_{mp}(\mathbb{R}^n)$ and $w_j^{-1/(p_j-1)} \in A_{mp_j'}(\mathbb{R}^n)$; see [**23**] for details. Our main result can be stated as follows.

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Theorem 1.4. *Let* $m \in \mathbb{N}$, $p_1, \dots, p_{m+1} \in [1, \infty)$, $p \in [1/(m+1), \infty)$ *with* $1/p = 1/p_1 + \cdots + 1/p_{m+1}$, $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$, $A' \in \mathrm{BMO}(\mathbb{R})$ *with* $\|A'\|_{\mathrm{BMO}(\mathbb{R})} = 1$.

(i) *If* $\min_{1 \le j \le m+1} p_j > 1$, *then*

$$\|\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)\|_{L^p(\mathbb{R}, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, p_1'/p, \dots, p_{m+1}'/p\}} [w_{m+1}^{-1/p_{m+1}-1}]_{A_\infty} \times \|f\|_{L^{p_{m+1}}(\mathbb{R}, w_{m+1})} \prod_{j=1}^{m} \|a_j\|_{L^{p_j}(\mathbb{R}, w_j)}.$$

(ii) *If* $p_1 = \cdots = p_{m+1} = 1$, *then for each* $\lambda > 0$,

$$\nu_{\vec{w}}(\{x \in \mathbb{R} : |\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f)(x)| > \lambda\}) \lesssim \left(\prod_{j=1}^{m} \int_{\mathbb{R}} \frac{|a_j(y_j)|}{\lambda^{1/(m+1)}} \log \left(\mathrm{e} + \frac{|a_j(y_j)|}{\lambda^{1/(m+1)}} \right) w_j(y_j) \mathrm{d}y_j \right)^{1/(m+1)} \times \left(\int_{\mathbb{R}} \frac{|f(y)|}{\lambda^{1/(m+1)}} \log(\mathrm{e} + \frac{|f(y)|}{\lambda^{1/(m+1)}}) w_{m+1}(y) \mathrm{d}y \right)^{1/(m+1)}.$$

Remark 1.5. To prove Theorem 1.4, we will employ a suitable variant of the ideas of Lerner [**21**] (see also [**4**, **25**] in the case of multilinear operators), to dominate $\mathcal{C}_{m+1,A}$ by multilinear sparse operators. This argument needs certain weak type endpoint estimates for the grand maximal operator of $\mathcal{C}_{m+1,A}$. Although $K_A(x; y_1, \dots, y_{m+1})$, the kernel of the multilinear singular integral operator $\mathcal{C}_{m+1,A}$, satisfies the non-smooth kernel conditions on the variable $y_1, \dots, y_m$ as in [**7**], we do not know if $K_A(x; y_1, \dots, y_{m+1})$ enjoys a similar condition on the variable y_{m+1}. Our argument is a modification of the proof of [**7**, Theorem 1.1], based on a local estimate (see Lemma 2.5 below), and involves the combination of sharp function estimates and the argument used in [**7**].

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We write $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. Furthermore, we write $A \lesssim_p B$ to denote that there exists a positive constant C depending only on p such that $A \le CB$. Subscripted constants such as C_1 do not change in different occurrences. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ (interval $I \subset \mathbb{R}$) and $\lambda \in (0, \infty)$, we use λQ to denote the cube with the same centre as Q and whose side length is λ times that of Q. For a local function f on $\mathbb{R}$ and an interval I, we use $\langle f \rangle_I$ to denote the mean value of f on I, that is, $\langle f \rangle_I = |I|^{-1} \int_I f(y)\, \mathrm{d}y$.

2. An endpoint estimate

This section is devoted to an endpoint estimate for $\mathcal{C}_{m+1,A}$. We begin with a preliminary lemma.

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Lemma 2.1. *Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some $q \in (n, \infty]$. Then*

$$|A(x) - A(y)| \lesssim |x - y| \left(\frac{1}{|I_x^y|} \int_{I_x^y} |\nabla A(z)|^q \, \mathrm{d}z \right)^{1/q},$$

where I_x^y is the cube centred at x and having side length $2|x - y|$.

For the proof of Lemma 2.1, see [5].

For $\gamma \in [0, \infty)$ and a cube $Q \subset \mathbb{R}^n$, let $\|\cdot\|_{L(\log L)^\gamma, Q}$ be the Luxemburg norm defined by

$$\|f\|_{L(\log L)^\gamma, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{\lambda} \log^\gamma \left(\mathrm{e} + \frac{|f(y)|}{\lambda} \right) \mathrm{d}y \le 1 \right\}.$$

Define the maximal operator $M_{L(\log L)^\gamma}$ by

$$M_{L(\log L)^\gamma} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^\gamma, Q}.$$

Obviously, $M_{L(\log L)^0}$ is just the Hardy–Littlewood maximal operator M. It is well known that $M_{L(\log L)^\gamma}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$, and for $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_{L(\log L)^\gamma} f(x) > \lambda\}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^\gamma \left(\mathrm{e} + \frac{|f(x)|}{\lambda} \right) \mathrm{d}x. \tag{2.1}$$

Let $s \in (0, 1/2)$ and $M_{0,s}^\sharp$ be the John–Strömberg sharp maximal operator defined by

$$M_{0,s}^\sharp f(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \inf\{t > 0 : |\{y \in Q : |f(y) - c| > t\}| < s|Q|\},$$

where the supremum is taken over all cubes containing x. This operator was introduced by John [20] and recovered by Strömberg in [30].

Lemma 2.2. *Let Φ be a increasing function on $[0, \infty)$ which satisfies the doubling condition that*

$$\Phi(2t) \le C\Phi(t), \quad t \in [0, \infty).$$

Then there exists a constant $s_0 \in (0, 1/2)$, such that for any $s \in (0, s_0]$,

$$\sup_{\lambda > 0} \Phi(\lambda) |\{x \in \mathbb{R}^n : |h(x)| > \lambda\}| \lesssim \sup_{\lambda > 0} \Phi(\lambda) |\{x \in \mathbb{R}^n : M_{0,s}^\sharp h(x) > \lambda\}|,$$

provided that

$$\sup_{\lambda > 0} \Phi(\lambda) |\{x \in \mathbb{R}^n : |h(x)| > \lambda\}| < \infty.$$

This lemma can be proved by repeating the proof of [15, Theorem 2.1]. We omit the details for brevity.

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Lemma 2.3. *Let $R > 1$. There exists a constant $C(n, R)$ such that for all open sets $\Omega \subset \mathbb{R}^n$, Ω can be decomposed as $\Omega = \cup_j Q_j$, where $\{Q_j\}$ is a sequence of cubes with disjoint interiors, and*

(i)

$$5R \leq \frac{\mathrm{dist}(Q_j, \mathbb{R}^n \backslash \Omega)}{\mathrm{diam} Q_j} \leq 15R,$$

(ii) $\sum_j \chi_{RQ_j}(x) \leq C_{n,R}\chi_\Omega(x)$.

For the proof of Lemma 2.3, see [29, p. 256].

We return to $\mathcal{C}_{m+1}$. As was proved in [7], $\mathcal{C}_{m+1}$ can be rewritten as the multilinear singular integral operator

$$\begin{aligned}\mathcal{C}_{m+1}&(a_1, \dots, a_m; f)(x)\\ &= \int_{\mathbb{R}^{m+1}} K(x; y_1, \dots, y_{m+1}) \prod_{j=1}^m a_j(y_j) f(y_{m+1}) \mathrm{d}y_1 \cdots \mathrm{d}y_{m+1},\end{aligned}$$

where

$$K(x; y_1, \dots, y_{m+1}) = \frac{(-1)^{me(y_{m+1}-x)}}{(x - y_{m+1})^{m+1}} \prod_{j=1}^m \chi_{(x \wedge y_{m+1}, x \vee y_{m+1})}(y_j), \tag{2.2}$$

e is the characteristic function of $[0, \infty)$, $x \wedge y_{m+1} = \min\{x, y_{m+1}\}$ and $x \vee y_{m+1} = \max\{x, y_{m+1}\}$. Obviously, for $x, y_1, \dots, y_{m+1} \in \mathbb{R}$,

$$|K(x; y_1, \dots, y_{m+1})| \lesssim \frac{1}{\left(\sum_{j=1}^{m+1} |x - y_j|\right)^{m+1}}. \tag{2.3}$$

Lemma 2.4. *Let K be the same as in (2.2). Then for $x, x', y_1, \dots, y_{m+1} \in \mathbb{R}$ with $12|x - x'| < \min_{1 \leq j \leq m+1} |x - y_j|$,*

$$|K(x; y_1, \dots, y_{m+1}) - K(x'; y_1, \dots, y_{m+1})| \lesssim \frac{|x - x'|}{\left(\sum_{j=1}^{m+1} |x - y_j|\right)^{m+2}}.$$

For the proof of Lemma 2.4, see [16].

Lemma 2.5. *Let A be a function on $\mathbb{R}$ such that $A' \in \mathrm{BMO}(\mathbb{R})$, $a_1, \dots, a_m \in L^1(\mathbb{R})$. Then for $\tau \in (0, 1/(m+2))$ and any interval $I \subset \mathbb{R}$,*

$$\left(\frac{1}{|I|} \int_I \left|\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f\chi_I)(y)\right|^\tau \mathrm{d}y\right)^{1/\tau} \lesssim \|f\|_{L \log L, 4I} \prod_{j=1}^m \langle |a_j| \rangle_{4I}. \tag{2.4}$$

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Proof. For a fixed interval $I \subset \mathbb{R}$, let $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \le \varphi(y) \le 1$, $\varphi(y) \equiv 1$ for $y \in I$, $\mathrm{supp}\varphi \subset 2I$ and $\|\varphi'\|_{L^\infty(\mathbb{R})} \lesssim |I|^{-1}$. Set

$$A_I(y) = A(y) - \langle A' \rangle_I y, A^\varphi(y) = (A_I(y) - A_I(y_0))\varphi(y)$$

with $y_0 \in 3I \backslash 2I$, and let $a^\varphi(y) = (A^\varphi)'(y)$. Applying Lemma 2.1, we know that

$$|A_I(y) - A_I(y_0)| \lesssim |I|.$$

Thus for $y \in I$,

$$\begin{aligned} |a^\varphi(y)| &\lesssim \left(\frac{1}{|I|} |A_I(y) - A_I(y_0)| + |A'(y) - \langle A' \rangle_I| \right) \chi_{2I}(y) \\ &\lesssim (1 + |A'(y) - \langle A' \rangle_I|) \chi_{2I}(y). \end{aligned}$$

This in turn implies that

$$\|a^\varphi\|_{L^1(\mathbb{R})} \lesssim \|A'\|_{\mathrm{BMO}(\mathbb{R})} |I|,$$

and by the generalization of Hölder's inequality (see [**28**, p. 64]),

$$\|a^\varphi f \chi_I\|_{L^1(\mathbb{R})} \lesssim |I| \|f\|_{L \log L, I}.$$

For $j = 1, \dots, m$, let $A_j^\varphi(z) = (A_j(z) - A_j(y_0))\varphi(z)$ and $a_j^\varphi(z) = (A_j^\varphi)'(z)$. It then follows that

$$\|a_j^\varphi\|_{L^1(\mathbb{R})} \lesssim \int_{4I} |a_j(z)| \, \mathrm{d}z.$$

For $y \in I$, write

$$\begin{aligned} &\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f\chi_I)(y) \\ &\quad = \int_{\mathbb{R}} \frac{\prod_{j=1}^m (A_j^\varphi(y) - A_j^\varphi(z))(A^\varphi(y) - A^\varphi(z))}{(y-z)^{m+2}} f(z)\chi_I(z) \, \mathrm{d}z \\ &\qquad + \int_{\mathbb{R}} \frac{\prod_{j=1}^m (A_j^\varphi(y) - A_j^\varphi(z))}{(y-z)^{m+1}} a^\varphi(z) f(z) \chi_I(z) \, \mathrm{d}z \\ &\quad = \mathcal{C}_{m+2}(a_1^\varphi, \dots, a_m^\varphi, a^\varphi; f\chi_I)(y) + \mathcal{C}_{m+1}(a_1^\varphi, \dots, a_m^\varphi; a^\varphi f \chi_I)(y). \end{aligned}$$

Theorem 1.1 tells us that $\mathcal{C}_{m+2}$ is bounded from $L^1(\mathbb{R}) \times \cdots \times L^1(\mathbb{R})$ to $L^{1/(m+2),\infty}(\mathbb{R})$. As in the proof of Kolmogorov's inequality, we can deduce that for $\tau \in (0, 1/(m+2))$,

$$\begin{aligned} &\left(\frac{1}{|I|} \int_I \left| \mathcal{C}_{m+2}(a_1^\varphi, \dots, a_m^\varphi, a^\varphi; f\chi_I)(y) \right|^\tau \mathrm{d}y \right)^{1/\tau} \\ &\quad \lesssim |I|^{-m-2} \prod_{j=1}^m \|a_j^\varphi\|_{L^1(\mathbb{R})} \|f\chi_I\|_{L^1(\mathbb{R})} \|a^\varphi\|_{L^1(\mathbb{R})} \lesssim \langle |f| \rangle_I \prod_{j=1}^m \langle |a_j| \rangle_{4I}. \end{aligned}$$

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On the other hand, since $\mathcal{C}_{m+1}$ is bounded from $L^1(\mathbb{R})\times\ldots\times L^1(\mathbb{R})$ to $L^{1/(m+1),\infty}(\mathbb{R})$, we then know that for $\varsigma\in(0,1/(m+1))$,

$$\left(\frac{1}{|I|}\int_I\left|\mathcal{C}_{m+1}(a_1^{\varphi},\ldots,a_m^{\varphi};a^{\varphi}f\chi_I)(y)\right|^{\varsigma}\mathrm{d}y\right)^{1/\varsigma}\lesssim|I|^{-m-1}\prod_{j=1}^m\|a_j^{\varphi}\|_{L^1(\mathbb{R})}\|a^{\varphi}f\chi_I\|_{L^1(\mathbb{R})}$$
$$\lesssim\|f\|_{L\log L,I}\prod_{j=1}^m\langle|a_j|\rangle_{4I}.$$

Combining the last two estimates yields (2.4). □

We now rewrite $\mathcal{C}_{m+1,A}$ as the multilinear singular integral operator

$$\mathcal{C}_{m+1,A}(a_1,\ldots,a_m;f)(x)$$
$$=\int_{\mathbb{R}^{m+1}}K_A(x;y_1,\ldots,y_{m+1})\prod_{j=1}^m a_j(y_j)f(y_{m+1})\mathrm{d}y_1\ldots\mathrm{d}y_{m+1},$$

where

$$K_A(x;y_1,\ldots,y_{m+1})=K(x;y_1,\ldots,y_{m+1})\frac{P_2(A;x,y_{m+1})}{(x-y_{m+1})},\tag{2.5}$$

with $K(x;y_1,\ldots,y_{m+1})$ defined by (2.2). Obviously,

$$|K_A(x;y_1,\ldots,y_{m+1})|\lesssim\frac{1}{\left(\sum_{j=1}^{m+1}|x-y_j|\right)^{m+2}}|P_2(A;x,y_{m+1})|.\tag{2.6}$$

Lemma 2.6. *Let $\phi\in C^{\infty}(\mathbb{R})$ be even, $0\le\phi\le1$, $\phi(0)=0$ and $\mathrm{supp}\phi\subset[-1,1]$. Set $\Phi(t)=\phi'(t)$, $\Phi_t(y)=t^{-1}\Phi(x/t)$ and $k_t(x,y)=\Phi_t(x-y)\chi_{(x,\infty)}(y)$. For $j=1,\ldots,m$, set*

$$K_{A,t}^j(x;y_1,\ldots,y_m)=\int_{\mathbb{R}^n}K_A(x;y_1,\ldots,y_{j-1},z,y_{j+1},\ldots,y_{m+1})k_t(z,y_j)\,\mathrm{d}z.$$

Then for $j=1,\ldots,m$, $x,y_1,\ldots,y_{m+1}\in\mathbb{R}$ and $t>0$ with $2t\le|x-y_j|$,

$$|K_A(x;y_1,\ldots,y_{m+1})-K_{A,t}^j(x;y_1,\ldots,y_{m+1})|$$
$$\lesssim\frac{|P_2(A,x,y_{m+1})|}{\left(\sum_{k=1}^{m+1}|x-y_k|\right)^{m+2}}\phi\left(\frac{|y_{m+1}-y_j|}{t}\right).$$

Proof. We only consider $j=1$. Write

$$K_A(x;y_1,\ldots,y_{m+1})-K_{A,t}^1(x;y_1,\ldots,y_{m+1})$$
$$=\frac{(-1)^{me(y_{m+1}-x)}}{(x-y_{m+1})^{m+1}}\frac{P_2(A;x,y_{m+1})}{(x-y_{m+1})}\prod_{j=2}^m\chi_{(x\wedge y_{m+1},x\vee y_{m+1})}(y_j)$$
$$\times\left(\chi_{(x\wedge y_{m+1},x\vee y_{m+1})}(y_1)-\int_{-\infty}^{y_1}\chi_{(x\wedge y_{m+1},x\vee y_{m+1})}(z)k_t(z-y)\,\mathrm{d}z\right).$$

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From the proof of [7, Theorem 4.1], we find that when $|x-y_1|>2t$,

$$\begin{aligned}\Big|\chi_{(x\wedge y_{m+1},x\vee y_{m+1})}(y_1)-\int_{-\infty}^{y_1}\chi_{(x\wedge y_{m+1},x\vee y_{m+1})}(z)k_t(z-y)\,\mathrm{d}z\Big|\\ \lesssim \phi\bigg(\frac{|y_{m+1}-y_1|}{t}\bigg).\end{aligned}$$

Note that

$$|K_A(x;y_1,\dots,y_{m+1})-K^1_{A,t}(x;y_1,\dots,y_{m+1})|\neq 0$$

only if $|x-y_{m+1}|>\max_{1\le k\le m}|x-y_k|$. Our desired conclusion then follows directly. □

Remark 2.7. We do not know if $K_A(x;y_1,\dots,y_{m+1})$ enjoys the properties in Lemma 2.6 concerning the variable y_{m+1}.

We now recall the approximation to the identity introduced by Duong and McIntosh [8].

Definition 2.8. A family of operators $\{D_t\}_{t>0}$ is said to be an approximation to the identity in $\mathbb{R}$ if, for every $t>0$, D_t can be represented by the kernel a_t in the following sense: for every function $u\in L^p(\mathbb{R})$ with $p\in[1,\infty]$ and almost every $x\in\mathbb{R}$,

$$D_tu(x)=\int_{\mathbb{R}}a_t(x,y)u(y)\,\mathrm{d}y,$$

and the kernel a_t satisfies that for all $x,y\in\mathbb{R}$ and $t>0$,

$$|a_t(x,y)|\le h_t(x,y)=t^{-1/s}h\bigg(\frac{|x-y|}{t^{1/s}}\bigg),$$

where $s>0$ is a constant and h is a positive, bounded and decreasing function such that for some constant $\eta>0$,

$$\lim_{r\to\infty}r^{1+\eta}h(r)=0.$$

Lemma 2.9. *Let A be a function on $\mathbb{R}$ such that $A'\in\mathrm{BMO}(\mathbb{R})$, $q_1,\dots,q_{m+1}\in[1,\infty)$. Suppose that for some $\beta\in[0,\infty)$, $\mathcal{C}_{m+1,A}$ satisfies the estimate that*

$$\begin{aligned}&|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>1\}|\\ &\quad\lesssim\sum_{j=1}^m\|a_j\|^{q_j}_{L^{q_j}(\mathbb{R})}+\int_{\mathbb{R}}|f(x)|^{q_{m+1}}\log^{\beta}(\mathrm{e}+|f(x)|)\,\mathrm{d}x.\end{aligned}$$

Then for $p_j\in[1,q_j)$, $j=1,\dots,m$,

$$\begin{aligned}&|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>1\}|\\ &\quad\lesssim\sum_{j=1}^m\|a_j\|^{p_j}_{L^{p_j}(\mathbb{R})}+\int_{\mathbb{R}}|f(x)|^{q_{m+1}}\log^{\beta_{q_{m+1}}}(\mathrm{e}+|f(x)|)\,\mathrm{d}x,\end{aligned}$$

where $\beta_{q_{m+1}}=\beta$ if $q_{m+1}\in(1,\infty)$ and $\beta_{q_{m+1}}=\max\{1,\beta\}$ if $q_{m+1}=1$.

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Proof. We employ the ideas in [7], together with some modifications. First, we prove that

$$|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>1\}| \lesssim \|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}+\sum_{j=2}^m\|a_j\|_{L^{q_j}(\mathbb{R}^n)}^{q_j}+\int_{\mathbb{R}}|f(x)|^{q_{m+1}}\log^{\beta}(\mathrm{e}+|f(x)|)\,\mathrm{d}x. \tag{2.7}$$

To do this, we apply Lemma 2.3 to the set

$$\Omega=\{x\in\mathbb{R}:M(|a_1|^{p_1})(x)>1\},$$

and obtain a sequence of intervals $\{I_l\}$ with disjoint interiors, such that $\Omega=\cup_l I_l$,

$$\frac{1}{|I_l|}\int_{I_l}|a_1(x)|^{p_1}\,\mathrm{d}x\lesssim 1,$$

and $\sum_l\chi_{4I_l}(x)\lesssim\chi_\Omega(x)$. Let D_t be the integral operator defined by

$$D_th(x)=\int_{\mathbb{R}}k_t(x,y)h(y)\,\mathrm{d}y,$$

with k_t the same as in Lemma 2.6. Then $\{D_t\}_{t>0}$ is an approximation to the identity in the sense of Definition 2.8. Set

$$a_1^1(x)=a_1(x)\chi_{\mathbb{R}^n\setminus\Omega}(x),\quad a_1^2(x)=\sum_l D_{|I_l|}b_1^l(x)$$

and

$$a_1^3(x)=\sum_l(b_1^l(x)-D_{|I_l|}b_1^l(x)),$$

with $b_1^l(y)=a_1(y)\chi_{I_l}(y)$. Obviously, $\|a_1^1\|_{L^\infty(\mathbb{R})}\lesssim 1$. Our hypothesis states that

$$|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1^1,\dots,a_m;f)(x)|>1\}| \lesssim \|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}+\sum_{j=2}^m\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j}+\int_{\mathbb{R}^n}|f(x)|^{q_{m+1}}\log^{\beta}(\mathrm{e}+|f(x)|)\,\mathrm{d}x.$$

On the other hand, as was pointed out in [8, p. 241], we know that

$$\|a_1^2\|_{L^{q_1}(\mathbb{R})}\lesssim\Big\|\sum_l\chi_{I_l}\Big\|_{L^{q_1}(\mathbb{R})}\lesssim\Big(\sum_l|Q_l|\Big)^{1/q_1}\lesssim\|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1/q_1}.$$

Thus,

$$\begin{aligned}|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1^2,\dots,a_m;f)(x)|>1\}| &\lesssim \|a_1^2\|_{L^{q_1}(\mathbb{R})}^{q_1}+\sum_{j=2}^m\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j}+\int_{\mathbb{R}}|f_{m+1}(x)|^{q_{m+1}}\log^{\beta}(\mathrm{e}+|f_{m+1}(x)|)\,\mathrm{d}x\\ &\lesssim \|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}+\sum_{j=2}^m\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j}+\int_{\mathbb{R}}|f_{m+1}(x)|^{q_{m+1}}\log^{\beta}(\mathrm{e}+|f_{m+1}(x)|)\,\mathrm{d}x.\end{aligned}$$

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Our proof for (2.7) is now reduced to proving

$$
\begin{aligned}
|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1^3,\ldots,a_m,f)(x)|>1\}| &\lesssim \|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}\\
+\sum_{j=2}^{m}\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j} &+ \int_{\mathbb{R}^n}|f_{m+1}(x)|^{q_{m+1}}\log^{\tilde\beta_{q_{m+1}}}(\mathrm{e}+|f_{m+1}(x)|)\,\mathrm{d}x, \qquad (2.8)
\end{aligned}
$$

where $\tilde\beta_{q_{m+1}}=0$ if $q_{m+1}\in(1,\infty)$ and $\tilde\beta_{q_{m+1}}=1$ if $q_{m+1}=1$.

We now prove (2.8). Let $\widetilde\Omega=\cup_l 16I_l$. It is obvious that

$$
|\widetilde\Omega|\lesssim\|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}.
$$

For each $x\in\mathbb{R}\backslash\widetilde\Omega$, by Lemma 2.6, we can write

$$
\begin{aligned}
&|\mathcal{C}_{m+1,A}(a_1^3,a_2,\ldots,a_m,f)(x)|\\
&\quad\lesssim\sum_l\int_{\mathbb{R}^{m+1}}\frac{|P_2(A,x,y_{m+1})|}{\left(\sum_{k=1}^{m+1}|x-y_k|\right)^{m+2}}\phi\left(\frac{|y_{m+1}-y_1|}{|I_l|}\right)|b_1^l(y_1)|\\
&\qquad\times\prod_{j=2}^{m}|a_j(y_j)||f(y_{m+1})|\,\mathrm{d}y_1\ldots\mathrm{d}y_{m+1}.
\end{aligned}
$$

Observe that

$$
\int_{I_l}|b_1^l(y_1)|\,\mathrm{d}y_1\lesssim|I_l|,
$$

and for $x\in\mathbb{R}\backslash\widetilde\Omega$,

$$
\begin{aligned}
&\int_{\mathbb{R}^{m-1}}\frac{1}{\left(\sum_{k=1}^{m+1}|x-y_k|\right)^{m+2}}\prod_{j=2}^{m}|a_j(y_j)|\,\mathrm{d}y_2\ldots\mathrm{d}y_m\\
&\quad\lesssim\frac{1}{|x-y_{m+1}|^3}\prod_{j=2}^{m}Ma_j(x).
\end{aligned}
$$

Let

$$
\mathrm{E}(x)=\sum_l|I_l|\left(\int_{4I_l}\frac{|P_2(A;x,y_{m+1})|}{|x-y_{m+1}|^3}|f(y_{m+1})|\,\mathrm{d}y_{m+1}\right).
$$

We then have

$$
|\mathcal{C}_{m+1,A}(a_1^3,a_2,\ldots,a_m,f)(x)|\lesssim\prod_{j=2}^{m}Ma_j(x)\mathrm{E}(x).
$$

Set

$$
A_{I_l}(y)=A(y)-\langle A'\rangle_{I_l}y. \qquad (2.9)
$$

It is easy to verify that for all $y,z\in\mathbb{R}$,

$$
P_2(A;y,z)=P_2(A_{I_l};y,z).
$$

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A straightforward computation involving Lemma 2.1 shows that for $y_{m+1}\in 4I_l$,

$$|A_{I_l}(x)-A_{I_l}(y_{m+1})|\lesssim |x-y_{m+1}|(1+|\langle A'\rangle_{I_l}-\langle A'\rangle_{I_x^{y_{m+1}}}|).$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}\backslash\widetilde{\Omega}}\frac{|P_2(A;x,y_{m+1})|}{|x-y_{m+1}|^3}\,\mathrm{d}x &\lesssim \sum_{k=2}^{\infty}\int_{2^kI_l}(k+|A'(y_{m+1})-\langle A'\rangle_{I_l}|)\,\frac{\mathrm{d}x}{|x-y_{m+1}|^2}\\ &\lesssim |I_l|^{-1}(1+|A'(y_{m+1})-\langle A'\rangle_{I_l}|).\end{aligned}$$

This, via the generalization of Hölder's inequality, yields

$$\begin{aligned}\int_{\mathbb{R}\backslash\widetilde{\Omega}}\int_{4I_l}\frac{|P_2(A;x,y)|}{|x-y|^3}|f(y)|\,\mathrm{d}y\,\mathrm{d}x &\lesssim |I_l|^{-1}\int_{4I_l}|f(y)||A'(y)-\langle A'\rangle_{I_l}|\,\mathrm{d}y\\ &\lesssim \|f\|_{L\log L,4I_l}.\end{aligned}$$

Combining the estimates above then yields

$$\int_{\mathbb{R}\backslash\widetilde{\Omega}}\mathrm{E}(x)\,\mathrm{d}x\lesssim\sum_l|I_l|\|f\|_{L\log L,4I_l}\lesssim\sum_l|I_l|+\int_{\mathbb{R}}|f(y)|\log(\mathrm{e}+|f(y)|)\,\mathrm{d}y,$$

since

$$\|f\|_{L\log L,4I_l}\lesssim 1+\frac{1}{|4I_l|}\int_{4I_l}|f(y)|\log(\mathrm{e}+|f(y)|)\,\mathrm{d}y;$$

see [**28**, p. 69]. Thus,

$$\begin{aligned}&|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1^3,a_2\dots,a_m,f)(x)|>1\}|\\ &\quad\lesssim|\widetilde{\Omega}|+\sum_{j=2}^{m}|\{x\in\mathbb{R}:Ma_j(x)>1\}|+|\{x\in\mathbb{R}\backslash\widetilde{\Omega}:\mathrm{E}(x)>1\}|\\ &\quad\lesssim\sum_{j=2}^{m}\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j}+\int_{\mathbb{R}^n\backslash\widetilde{\Omega}}\mathrm{E}(x)\,\mathrm{d}x\\ &\quad\lesssim\|a_1\|_{L^{p_1}(\mathbb{R})}^{p_1}+\sum_{j=2}^{m}\|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j}+\int_{\mathbb{R}}|f(x)|\log(\mathrm{e}+|f(x)|)\,\mathrm{d}x.\end{aligned}$$

This establishes (2.8) for the case of $q_{m+1}=1$. For the case of $q_{m+1}\in(1,\infty)$, it follows from Hölder's inequality that

$$\begin{aligned}\sum_l|I_l|\|f\|_{L\log L,4I_l}&\lesssim\sum_l|I_l|^{1-1/q_{m+1}}\left(\int_{4I_l}|f(y)|^{q_{m+1}}\,\mathrm{d}y\right)^{1/q_{m+1}}\\ &\lesssim\sum_l|I_l|+\sum_l\int_{4I_l}|f(y)|^{q_{m+1}}\,\mathrm{d}y.\end{aligned}$$

Thus, inequality (2.8) still holds for $q_{m+1}\in(1,\infty)$.

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With the estimate (2.7) in hand, applying the argument above to a_2 (fixing the exponents $p_1, q_3, \ldots, q_m, q_{m+1}$), we can prove that

$$\begin{aligned}&|\{x \in \mathbb{R} : |\mathcal{C}_{m+1,A}(a_1, \ldots, a_m; f)(x)| > 1\}| \\ &\quad \lesssim \sum_{j=1}^{2} \|a_j\|_{L^{p_j}(\mathbb{R})}^{p_j} + \sum_{j=3}^{m} \|a_j\|_{L^{q_j}(\mathbb{R})}^{q_j} + \int_{\mathbb{R}} |f(x)|^{q_{m+1}} \log^{\beta_{q_{m+1}}}(\mathrm{e} + |f(x)|)\, \mathrm{d}x.\end{aligned}$$

Repeating this procedure m times then leads to our desired conclusion. □

Lemma 2.10. *Let A be a function on $\mathbb{R}$ such that $A' \in \mathrm{BMO}(\mathbb{R})$. Then for $s \in (0, 1/2)$,*

$$M^{\sharp}_{0,s}(\mathcal{C}_{m+1,A}(a_1, \ldots, a_m; f))(x) \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x), \tag{2.10}$$

provided that $a_1, \ldots, a_j$ are bounded functions with compact supports.

Proof. Without loss of generality, we may assume that $\|A'\|_{\mathrm{BMO}(\mathbb{R})} = 1$. Let $x \in \mathbb{R}$, $I \subset \mathbb{R}$ be an interval containing x. Decompose f as

$$f(y) = f(y)\chi_{64I}(y) + f(y)\chi_{\mathbb{R}\backslash 64I}(y) := f^1(y) + f^2(y),$$

and for $j = 1, \ldots, m$,

$$a_j(y) = a_j(y)\chi_{64I}(y) + a_j(y)\chi_{\mathbb{R}\backslash 64I}(y) := a_j^1(y) + a_j^2(y).$$

By estimate (1.4), we know that $|\mathcal{C}_{m+1,A}(a_1, \ldots, a_m, f^2)(z)| < \infty$ for almost every $z \in \mathbb{R}$ and we can choose some $x_I \in 3I\backslash 2I$ such that $|\mathcal{C}_{m+1,A}(a_1, \ldots, a_m, f^2)(x_I)| < \infty$. For $\delta \in (0, 1)$, write

$$\begin{aligned}&\frac{1}{|I|}\int_I |\mathcal{C}_{m+1,A}(a_1, \ldots, a_m, f)(y) - \mathcal{C}_{m+1,A}(a_1, \ldots, a_m, f^2)(x_I)|^{\delta}\, \mathrm{d}y \\ &\quad \lesssim \frac{1}{|I|}\int_I |\mathcal{C}_{m+1,A}(a_1, \ldots, a_m; f^1)(y)|^{\delta}\, \mathrm{d}y \\ &\qquad + \sum_{\Lambda} \frac{1}{|I|}\int_I |\mathcal{C}_{m+1,A}(a_1^{i_1}, \ldots, a_m^{i_m}; f^2)(y)|^{\delta}\, \mathrm{d}y \\ &\qquad + \frac{1}{|I|}\int_I |\mathcal{C}_{m+1,A}(a_1^2, \ldots, a_m^2; f^2)(y) - \mathcal{C}_{m+1,A}(a_1^2, \ldots, a_m^2; f^2)(x_I)|^{\delta}\, \mathrm{d}y \\ &\quad := \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned}$$

where $\Lambda = \{(i_1, \ldots, i_m) : i_1, \ldots, i_m \in \{1, 2\}, \min_j i_j = 1\}$. It follows from Lemma 2.5 that

$$\mathrm{I}^{1/\delta} \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x).$$

We turn our attention to the term III. Let A_I be defined as in (2.9). Applying Lemma 2.1 and the John–Nirenberg inequality, we can verify that if $y \in I$, and

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$z \in 4^{l+1}I\backslash 4^l I$ with $l \in \mathbb{N}$, then

$$|P_2(A_I; y, z)| \lesssim (l + |A'(z) - \langle A'\rangle_I|)|y - z|. \tag{2.11}$$

This, along with another application of Lemma 2.1, gives us that for $y \in I$ and $z_{m+1} \in 4^{l+1}I\backslash 4^l I$,

$$\begin{aligned}
&\left|\frac{P_2(A_I; y, z_{m+1})}{|y - z_{m+1}|} - \frac{P_2(A_I; x_I, z_{m+1})}{|x_I - z_{m+1}|}\right| \\
&\quad \le \frac{|A_I(y) - A_I(x_I)|}{|y - z_{m+1}|} + |P_2(A_I; x_I, z_{m+1})|\left|\frac{1}{|x_I - z_{m+1}|} - \frac{1}{|y - z_{m+1}|}\right| \\
&\quad \le (l + A'(z_{m+1}) - \langle A'\rangle_I|)\frac{|y - x_I|}{|x_I - z_{m+1}|}.
\end{aligned} \tag{2.12}$$

We now deduce from Lemma 2.4 and (2.11) that

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}} |K(y; z_1, \dots, z_{m+1}) - K(x_I; z_1, \dots, z_{m+1})| \\
&\qquad \times \frac{|P_2(A_I; y, z_{m+1})|}{|y - z_{m+1}|} \prod_{j=1}^{m} |a_j^2(z_j)||f(z_{m+1})|\, \mathrm{d}z_1 \dots \mathrm{d}z_{m+1} \\
&\quad \lesssim \sum_{l=3}^{\infty} l 2^{-\gamma l} \prod_{j=1}^{m} \left(\frac{1}{|4^l I|}\int_{4^l I} |a_j(z_j)|\, \mathrm{d}z_j\right) \\
&\qquad \times \left(\frac{1}{|4^l I|}\int_{4^l I} |A'(z_{m+1}) - \langle A'\rangle_I||f(z_{m+1})|\, \mathrm{d}z_{m+1}\right) \\
&\quad \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x).
\end{aligned}$$

On the other hand, we obtain from (2.12) and the size condition (2.3) that

$$\begin{aligned}
&\int_{\mathbb{R}^{m+1}} |K(x_I; z_1, \dots, z_{m+1})|\left|\frac{P_2(A_I; y, z_{m+1})}{|y - z_{m+1}|} - \frac{P_2(A_I; x_I, z_{m+1})}{|x_I - z_{m+1}|}\right| \\
&\quad \times \prod_{j=1}^{m} |a_j^2(z_j)||f^2(z_{m+1})|\, \mathrm{d}z_1 \cdots \mathrm{d}z_{m+1} \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x).
\end{aligned}$$

Therefore, for each $y \in I$,

$$\begin{aligned}
&|\mathcal{C}_{m+1,A}(a_1, \dots, a_m; f^2)(y) - \mathcal{C}_{m+1,A}(a_1, \dots, a_m; f^2)(x_I)| \\
&\quad \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x),
\end{aligned} \tag{2.13}$$

which shows that

$$\mathrm{III}^{1/\delta} \lesssim M_{L\log L} f(x) \prod_{j=1}^{m} M a_j(x).$$

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It remains to estimate II. For simplicity, we assume that for some $l_0 \in \mathbb{N}$, $i_1 = \cdots = i_{l_0} = 1$ and $l_{l_0+1} = \cdots = i_m = 2$. Observe that for $y \in I$,

$$\begin{aligned}\int_{\mathbb{R}\setminus 64I} &\frac{|P_2(A_I; y, z_{m+1})|}{|y - z_{m+1}|^{(m+1)/(m+1-l_0)+1}} |f(z_{m+1})| \,\mathrm{d}z_{m+1} \\ &\lesssim \sum_{k=3}^{\infty} \frac{1}{(4^k|I|)^{(l_0+1)/(m-l_0+1)}} \int_{4^k I} (k + \|A'(z_{m+1}) - \langle A' \rangle_I|) |f(z_{m+1})| \,\mathrm{d}z_{m+1} \\ &\lesssim |I|^{-l_0/(m+1-l_0)} M_{L \log L} f(x)\end{aligned}$$

and

$$\int_{\mathbb{R}\setminus 64I} \frac{1}{|y - z_j|^{(m+1)/(m+1-l_0)}} |a_j(z_j)| \,\mathrm{d}z_j \lesssim |I|^{-l_0/(m+1-l_0)} M a_j(x).$$

This in turn implies that for each $y \in I$,

$$\begin{aligned}|\mathcal{C}_{m+1,A}&(a_1^{i_1}, \ldots, a_m^{i_m}; f^2)(y)| \\ &\lesssim \prod_{j=1}^{l_0} \int_{64I} |a_j^1(z_j)| \,\mathrm{d}z_j \prod_{j=l_0+1}^{m} \int_{\mathbb{R}\setminus 64I} \frac{|a_j(z_j)|}{|y - z_j|^{(m+1)/(m+1-l_0)}} \,\mathrm{d}z_j \\ &\quad \times \int_{\mathbb{R}\setminus 64I} \frac{|P_2(A_I; y, z)|}{|y - z|^{(m+1)/(m+1-l_0)+1}} |f(z)| \,\mathrm{d}z \\ &\lesssim M_{L \log L} f(x) \prod_{j=1}^{m} M a_j(x). \qquad (2.14)\end{aligned}$$

Therefore,

$$\mathrm{II}^{1/\delta} \lesssim M_{L \log L} f(x) \prod_{j=1}^{m} M a_j(x).$$

Combining the estimates for I, II and III leads to (2.10). □

We are now ready to establish the main result in this section.

Theorem 2.11. *Let A be a function on $\mathbb{R}$ such that $A' \in \mathrm{BMO}(\mathbb{R})$. Then*

$$\begin{aligned}|\{x \in \mathbb{R} : |\mathcal{C}_{m+1,A}&(a_1, \ldots, a_m; f)(x)| > 1\}| \\ &\lesssim \sum_{j=1}^{m} \|a_j\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} |f(y)| \log(\mathrm{e} + |f(y)|) \,\mathrm{d}y. \qquad (2.15)\end{aligned}$$

Proof. Without loss of generality, we may assume that $a_1, \ldots, a_m$ are bounded functions with compact supports. Let $q_1, \ldots, q_{m+1}, q \in (1, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_{m+1}$. Recalling that $\mathcal{C}_{m+1,A}$ is bounded from $L^\infty(\mathbb{R}) \times \cdots \times L^\infty(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^q(\mathbb{R})$

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(see [14]), we then know that

$$\sup_{\lambda>0}\lambda^q|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>\lambda\}|\lesssim\|f\|^q_{L^q(\mathbb{R})}\prod_{j=1}^m\|a_j\|^q_{L^\infty(\mathbb{R})}<\infty.$$

This, along with Lemmas 2.2 and 2.10, leads to

$$\|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)\|_{L^{q,\infty}(\mathbb{R})}\lesssim\|f\|_{L^{q_{m+1}}(\mathbb{R})}\prod_{j=1}^m\|a_j\|_{L^{q_j}(\mathbb{R})}. \tag{2.16}$$

Now let $r_1\in[1,q_1),\dots,r_m\in[1,q_m)$ and $1/r=1/r_1+\cdots+1/r_m+1/q_{m+1}$. Invoking Lemma 2.9, we deduce from (2.16) that

$$|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>1\}|\lesssim\sum_{j=1}^m\|a_j\|^{r_j}_{L^{r_j}(\mathbb{R})}+\|f\|^{q_{m+1}}_{L^{q_{m+1}}(\mathbb{R})}.$$

This, via homogeneity, shows that $\mathcal{C}_{m+1,A}$ is bounded from $L^{r_1}(\mathbb{R})\times\cdots\times L^{r_m}(\mathbb{R})\times L^{q_{m+1}}(\mathbb{R})$ to $L^{r,\infty}(\mathbb{R})$.

We now prove that for $p_1,\dots,p_m\in(1,\infty)$, and $p\in(1/(m+1),1)$ such that $1/p=1/p_1+\cdots+1/p_m+1$,

$$\begin{aligned}&|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>1\}|\\&\quad\lesssim\sum_{j=1}^m\|a_j\|^{p_j}_{L^{p_j}(\mathbb{R})}+\int_{\mathbb{R}}|f(x)|\log(\mathrm{e}+|f(x)|)\,\mathrm{d}x.\end{aligned} \tag{2.17}$$

To this end, we choose $q_1,\dots q_{m+1}\in(1,\infty)$ such that $1/q=1/q_1+\cdots+1/q_{m+1}<1$, and $p_1^*\in[1,q_1),\dots,p_m^*\in[1,q_m)$, $p^*\in(0,1)$ such that $1/p^*=1/p_1^*+\cdots+1/p_m^*+1/q_{m+1}$ and $p^*<p$. Recall that $\mathcal{C}_{m+1,A}$ is bounded from $L^{p_1^*}(\mathbb{R})\times\cdots\times L^{p_m^*}(\mathbb{R})\times L^{q_{m+1}}(\mathbb{R})$ to $L^{p^*,\infty}(\mathbb{R})$. Thus,

$$\lambda^{p^*}|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>\lambda\}|\lesssim\prod_{j=1}^m\|a_j\|^{p^*}_{L^{p_j^*}(\mathbb{R})}\|f\|^{p^*}_{L^{q_{m+1}}(\mathbb{R})}.$$

Let $\psi(t)=t^p\log^{-1}(\mathrm{e}+t^{-p})$. A trivial computation gives us that

$$\begin{aligned}&\sup_{\lambda>0}\psi(\lambda)|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>\lambda\}|\\&\quad\lesssim\sup_{0<\lambda<1}\lambda^{p^*}|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>\lambda\}|\\&\qquad+\sup_{\lambda\ge1}\lambda^2|\{x\in\mathbb{R}:|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)|>\lambda\}|\\&\quad\lesssim\|f\|^{p^*}_{L^{q_{m+1}}(\mathbb{R})}\prod_{j=1}^m\|a_j\|^{p^*}_{L^{p_j^*}(\mathbb{R})}+\|f\|^2_{L^2(\mathbb{R})}\prod_{j=1}^m\|a_j\|^2_{L^\infty(\mathbb{R})}<\infty.\end{aligned}$$

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This, via Lemmas 2.2 and 2.10 and estimate (2.1), tells us that

$$
\begin{aligned}
&|\{x\in\mathbb{R} : |\mathcal{C}_{m+1,A}(a_1,\ldots,a_m;f)(x)|>1\}|\\
&\lesssim \sup_{\lambda>0}\psi(\lambda)\bigg(\sum_{j=1}^{m}|\{x\in\mathbb{R} : Ma_j(x)>\lambda^{p/p_j}\}|+|\{x\in\mathbb{R} : M_{L\log L}f(x)>\lambda^p\}|\bigg)\\
&\lesssim \sum_{j=1}^{m}\|a_j\|_{L^{p_j}(\mathbb{R})}^{p_j}+\int_{\mathbb{R}}|f(x)|\log(\mathrm{e}+|f(x)|)\,\mathrm{d}x,
\end{aligned}
$$

and then establishes (2.17).

Finally, by (2.17) and invoking Lemma 2.9 m times, we obtain the estimate (2.15). This completes the proof of Theorem 2.11. □

3. Proof of Theorem 1.4

Let $\mathcal{S}$ be a family of cubes and $\eta\in(0,1)$. We say that $\mathcal{S}$ is an η-sparse family if, for each fixed $Q\in\mathcal{S}$, there exists a measurable subset $E_Q\subset Q$ such that $|E_Q|\geq\eta|Q|$ and the sets E_Q are pairwise disjoint. A sparse family is called simply sparse if $\eta=1/2$. For a fixed cube Q, denote by $\mathcal{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathcal{D}(Q)$ are formed by repeated subdivision of Q and each of descendants into 2^n congruent subcubes.

For constants $\beta_1,\ldots,\beta_m\in[0,\infty)$, let $\vec{\beta}=(\beta_1,\ldots,\beta_m)$. Associated with the sparse family $\mathcal{S}$ and $\vec{\beta}$, we define sparse operator $\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}$ by

$$
\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\ldots,f_m)(x)=\sum_{Q\in\mathcal{S}}\prod_{j=1}^{m}\|f_j\|_{L(\log L)^{\beta_j},Q}\chi_Q(x).
$$

Lemma 3.1. *Let $p_1,\ldots,p_m\in(1,\infty)$, $p\in(0,\infty)$ such that $1/p=1/p_1+\cdots+1/p_m$, and $\vec{w}=(w_1,\ldots,w_m)\in A_{\vec{P}}(\mathbb{R}^{mn})$. Set $\sigma_i=w_i^{-1/(p_i-1)}$. Let $\mathcal{S}$ be a sparse family. Then for $\beta_1,\ldots,\beta_m\in[0,\infty)$,*

$$
\|\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\ldots,f_m)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\lesssim[\vec{w}]_{A_p}^{\max\{1,p_1'/p,\ldots,p_m'/p\}}\prod_{j=1}^{m}[\sigma_j]_{A_\infty}^{\beta_j}\|f_j\|_{L^{p_j}(\mathbb{R}^n,w_j)}.
$$

If $\vec{w}\in A_{1,\ldots,1}(\mathbb{R}^{mn})$, then

$$
\begin{aligned}
&\nu_{\vec{w}}(\{x\in\mathbb{R}^n : \mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1,\ldots,f_m)(x)>1\})\\
&\lesssim\prod_{j=1}^{m}\bigg(\int_{\mathbb{R}^n}|f_j(y_j)|\log^{|\beta|}(1+|f_j(y_j)|)w_j(y_j)\,\mathrm{d}y_j\bigg)^{1/m},
\end{aligned}
$$

with $|\beta|=\sum_{j=1}^{m}|\beta_j|$.

For the proof of Lemma 3.1, see [4].

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In the following, we say that U is an m-sublinear operator if U satisfies that for each i with $1 \le i \le m$,

$$U(f_1, \dots, f_i^1 + f_i^2, f_{i+1}, \dots, f_m)(x) \le U(f_1, \dots, f_i^1, f_{i+1}, \dots, f_m)(x) + U(f_1, \dots, f_i^2, f_{i+1}, \dots, f_m)(x),$$

and for any $t \in \mathbb{C}$,

$$|U(f_1, \dots, tf_i^1, f_{i+1}, \dots, f_m)(x)| = |t||U(f_1, \dots, f_i^1, f_{i+1}, \dots, f_m)(x)|.$$

For an m-sublinear operator U and $\kappa \in \mathbb{N}$, let $\mathcal{M}_U^\kappa$ be the corresponding grand maximal operator, defined by

$$\mathcal{M}_U^\kappa(f_1, \dots, f_m)(x) = \sup_{Q \ni x} \|U(f_1, \dots, f_m) - U(f_1\chi_{Q^\kappa}, \dots, f_m\chi_{Q^\kappa})\|_{L^\infty(Q)},$$

with $Q^\kappa = 3^\kappa Q$. This operator was introduced by Lerner [**21**] and plays an important role in the proof of weighted estimates for singular integral operators; see [**4**, **24**, **25**].

Lemma 3.2. *Let $m, \kappa \in \mathbb{N}$, U be an m-sublinear operator and $\mathcal{M}_U^\kappa$ the corresponding grand maximal operator. Suppose that U is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $q_1, \dots, q_m \in (1, \infty)$ and $q \in (1/m, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_m$. Then for bounded functions $f_1, \dots, f_m$, cube $Q_0 \subset \mathbb{R}^n$, and almost every $x \in Q_0$,*

$$|U(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa})(x)| \lesssim \prod_{j=1}^m |f_j(x)| + \mathcal{M}_U^\kappa(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa})(x).$$

For the proof of Lemma 3.2, see [**4**, **25**].

The following theorem is an extension of [**21**, Theorem 4.2], and will be useful in the proof of Theorem 1.4.

Theorem 3.3. *Let $\beta_1, \dots, \beta_m \in [0, \infty)$, $\kappa, m \in \mathbb{N}$, U be an m-sublinear operator and $\mathcal{M}_U^\kappa$ be the corresponding grand maximal operator. Suppose that U is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $q_1, \dots, q_m \in (1, \infty)$ and $q \in (1/m, \infty)$ with $1/q = 1/q_1 + \cdots + 1/q_m$, and satisfies that*

$$|\{x \in \mathbb{R}^n : \mathcal{M}_U^\kappa(f_1, \dots, f_m)(x) > 1\}| \le C_1 \sum_{j=1}^m \int_{\mathbb{R}^n} |f_j(y_j)| \log^{\beta_j}(\mathrm{e} + |f_j(y_j)|)\, \mathrm{d}y_j.$$

Then for bounded functions $f_1, \dots, f_m$ with compact supports, there exists a $1/(21 \cdot 3^{\kappa n})$-sparse family $\mathcal{S}$ such that for almost every $x \in \mathbb{R}^n$,

$$|U(f_1, \dots, f_m)(x)| \lesssim \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, Q}\chi_Q(x).$$

Proof. We employ the argument used in [**21**], together with suitable modifications; see also [**4**, **25**]. As in [**4**, **25**], it suffices to prove that for each cube $Q_0 \subset \mathbb{R}^n$, there exist

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pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and for almost every $x \in Q_0$,

$$\begin{aligned} &|U(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa})(x)|\chi_{Q_0}(x) \\ &\quad \leq C\prod_{i=1}^m \|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa} + \sum_j |U(f_1\chi_{P_j^\kappa}, \dots, f_m\chi_{P_j^\kappa})(x)|\chi_{P_j}(x). \end{aligned} \tag{3.1}$$

To prove this, let $C_2 > 1$ (to be chosen later) and

$$\begin{aligned} E = &\left\{x \in Q_0 : |f_1(x)\cdots f_m(x)| > C_2\prod_{i=1}^m \|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}\right\} \\ &\cup \left\{x \in Q_0 : \mathcal{M}_U^\kappa(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa})(x) > C_2\prod_{i=1}^m \|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}\right\}. \end{aligned}$$

Our assumption implies that

$$\begin{aligned} &\left|\left\{x \in Q_0 : \mathcal{M}_U^\kappa(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa})(x) > C_2\prod_{i=1}^m \|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}\right\}\right| \\ &\quad \leq \frac{C_1}{C_2}\sum_{i=1}^m \int_{Q_0^\kappa} \frac{|f_i(y_i)|}{\|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}} \log^{\beta_i}\left(\mathrm{e} + \frac{|f_i(y_i)|}{\|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}}\right) \mathrm{d}y_i \\ &\quad \leq \frac{C_1}{C_2}|Q_0|, \end{aligned}$$

since

$$\int_{Q_0^\kappa} \frac{|f_i(y_i)|}{\|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}} \log^{\beta_i}\left(\mathrm{e} + \frac{|f_i(y_i)|}{\|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}}\right) \mathrm{d}y_i \leq |Q_0^\kappa|.$$

If we choose C_2 large enough, our assumption then says that $|E| \leq |Q_0|/(2^{n+2})$. Applying the Calderón–Zygmund decomposition to χ_E on Q_0 at level $1/(2^{n+1})$, we then obtain a family of pairwise disjoint cubes $\{P_j\}$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and $|E\backslash \cup_j P_j| = 0$. It then follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap E^c \neq \emptyset$. Therefore,

$$\left\|U(f_1\chi_{Q_0^\kappa}, \dots, f_m\chi_{Q_0^\kappa}) - U(f_1\chi_{P_j^\kappa}, \dots, f_m\chi_{P_j^\kappa})\right\|_{L^\infty(P_j)} \leq C_2\prod_{i=1}^m \|f_i\|_{L(\log L)^{\beta_i}, Q_0^\kappa}. \tag{3.2}$$

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Note that

$$\begin{aligned}&|U(f_1\chi_{Q_0^\kappa},\dots,f_m\chi_{Q_0^\kappa})(x)|\chi_{Q_0}(x)\\&\quad\le |U(f_1\chi_{Q_0^\kappa},\dots,f_m\chi_{Q_0^\kappa})(x)|\chi_{Q_0\backslash\cup_j P_j}(x)+\sum_j |U(f_1\chi_{P_j^\kappa},\dots,f_m\chi_{P_j^\kappa})(x)|\chi_{P_j}(x)\\&\qquad+\sum_j \|U(f_1\chi_{Q_0^\kappa},\dots,f_m\chi_{Q_0^\kappa})-U(f_1\chi_{P_j^\kappa},\dots,f_m\chi_{P_j^\kappa})\|_{L^\infty(P_j)}\chi_{P_j}(x). \qquad (3.3)\end{aligned}$$

Inequality (3.1) now follows from (3.2), (3.3) and Lemma 3.2 immediately. This completes the proof of Theorem 3.3. □

For $s\in(0,\infty)$, let M_s be the maximal operator defined by

$$M_s f(x)=(M(|f|^s)(x))^{1/s}.$$

It was proved in [**13**, p. 651] that for $s\in(0,1)$ and $\lambda>0$,

$$|\{x\in\mathbb{R}^n: M_s h(x)>\lambda\}|\lesssim \lambda^{-1}\sup_{t\ge 2^{-1/s}\lambda} t|\{x\in\mathbb{R}^n: |h(x)|>t\}|. \qquad (3.4)$$

Proof of Theorem 1.4. By Lemma 3.1, Theorem 3.3 and (2.16), it suffices to prove that the grand maximal operator $\mathcal{M}^3_{\mathcal{C}_{m+1,A}}$ satisfies that

$$\begin{aligned}&|\{x\in\mathbb{R}: \mathcal{M}^3_{\mathcal{C}_{m+1,A}}(a_1,\dots,a_m;f)(x)>1\}|\\&\quad\lesssim \sum_{j=1}^m \|a_j\|_{L^1(\mathbb{R})}+\int_{\mathbb{R}}|f(y)|\log(\mathrm{e}+|f(y)|)\,\mathrm{d}y. \qquad (3.5)\end{aligned}$$

We assume that $\|A'\|_{\mathrm{BMO}(\mathbb{R})}=1$ for simplicity.

Let $x\in\mathbb{R}$ and I be an interval containing x. For $j=1,\dots,m$, set

$$a_j^1(y)=a_j(y)\chi_{27I}(y),\quad a_j^2(y)=a_j(y)\chi_{\mathbb{R}\backslash 27I}(y).$$

Also, let

$$f^1(y)=f(y)\chi_{27I}(y),\quad f^2(y)=f(y)\chi_{\mathbb{R}\backslash 27I}(y).$$

Set

$$\Lambda_1=\{(i_1,\dots,i_{m+1}): i_1,\dots,i_{m+1}\in\{1,2\},\ \max_{1\le j\le m+1} i_j=2,\ \min_{1\le j\le m+1} i_j=1\}.$$

Let $A_I(y)$ be the same as in (2.9). For each fixed $\xi\in I$ and $z\in 2I\backslash 3/2I$, write

$$\begin{aligned}&|\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(\xi)-\mathcal{C}_{m+1,A}(a_1\chi_{27I},\dots,a_m\chi_{27I};f\chi_{27I})(\xi)|\\&\quad\le |\mathcal{C}_{m+1,A_I}(a_1^2,\dots,a_m^2;f^2)(\xi)-\mathcal{C}_{m+1,A_I}(a_1^2,\dots,a_m^2;f^2)(z)|\\&\qquad+|\mathcal{C}_{m+1,A_I}(a_1^2,\dots,a_m^2;f^2)(z)|\\&\qquad+\sum_{(i_1,\dots,i_m)\in\Lambda_1}|\mathcal{C}_{m+1,A_I}(a_1^{i_1},\dots,a_m^{i_m};f^{i_{m+1}})(\xi)|\\&\quad=\mathrm{D}_1(\xi,z)+\mathrm{D}_2(z)+\mathrm{D}_3(\xi).\end{aligned}$$

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As in the estimate (2.13), we know that for each $z \in 2I\backslash 3/2I$,

$$\mathrm{D}_1(\xi, z) \lesssim M_{L\log L}f(x) \prod_{j=1}^{m} Ma_j(x).$$

We turn our attention to D_3. We claim that for each $y \in 2I$,

$$\sum_{(i_1,\dots,i_m)\in\Lambda_1} |\mathcal{C}_{m+1;A_I}(a_1^{i_1},\dots,a_m^{i_m}; f^{i_{m+1}})(y)| \lesssim M_{L\log L}f(x) \prod_{j=1}^{m} Ma_j(x). \tag{3.6}$$

To see this, we consider the following two cases.

Case 1. $i_{m+1} = 1$. In this case, $\max_{1\le k\le m} i_k = 2$. We only consider the case where $i_1 = \dots = i_{m-1} = 1$ and $i_m = 2$. It follows from the size condition (2.6) that in this case

$$\begin{aligned}|\mathcal{C}_{m+1;A_I}(a_1^{i_1},\dots,a_m^{i_m}, f^1)(y)| \lesssim \prod_{j=1}^{m-1}\int_{27I}|a_j(y_j)|\,\mathrm{d}y_j \int_{\mathbb{R}\backslash 27I}\frac{|a_m(z)|}{|x-z|^{m+2}}\,\mathrm{d}z\\ \times\int_{27I}|f(z)||P_2(A_I; y, z)|\,\mathrm{d}z.\end{aligned}$$

Let $q \in (1,\infty)$. Another application of Lemma 2.1 shows that for $y \in 2I$ and $z \in I$,

$$|A_I(z) - A_I(y)| \lesssim |z-y|\left(1 + \log\frac{|I|}{|z-y|}\right) \lesssim |I|,$$

and in this case,

$$|P_2(A_I; z, y)| \lesssim |I|(1 + |A' - \langle A'\rangle_I|).$$

We thus get

$$|\mathcal{C}_{m+1;A_I}(a_1^{i_1},\dots,a_m^{i_m}; f^1)(y)| \lesssim M_{L\log L}f(x) \prod_{j=1}^{m} Ma_j(x).$$

Case 2. $i_{m+1} = 2$. As in the estimates (2.14), we also have that

$$|\mathcal{C}_{m+1,A}(a_1^{i_1},\dots,a_m^{i_m}, f^2)(y)| \lesssim M_{L\log L}f(x) \prod_{j=1}^{m} Ma_j(x).$$

Our argument for the above three cases leads to (3.6).

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As to the term D_2, we have by inequality (3.6) that for each $z \in 2I$,

$$\begin{aligned}\mathrm{D}_2(z) &\le |\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(z)| + |\mathcal{C}_{m+1,A}(a_1^1,\dots,a_m^1;f^1)(z)| \\ &\quad + \sum_{(i_1,\dots,i_m)\in\Lambda_1} |\mathcal{C}_{m+1;A}(a_1^{i_1},\dots,a_m^{i_m};f^{i_{m+1}})(z)| \\ &\lesssim |\mathcal{C}_{m+1;A}(a_1^1,\dots,a_m^1;f^1)(z)| + |\mathcal{C}_{m+1;A}(a_1,\dots,a_m;f)(z)| \\ &\quad + M_{L\log L}f(x)\prod_{j=1}^m Ma_j(x).\end{aligned}$$

We can now conclude the proof of Theorem 1.4. Estimates for D_1, D_2 and D_3, via Lemma 2.5, tell us that for any $\tau \in (0, 1/(m+2))$,

$$\begin{aligned}&\sup_{\xi\in I} |\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(\xi) - \mathcal{C}_{m+1,A}(a_1^1,\dots,a_m^1;f^1)(\xi)| \\ &\quad\lesssim M_{L\log L}f(x)\prod_{j=1}^m Ma_j(x) + \left(\frac{1}{|2I|}\int_{2I} |\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(z)|^\tau \mathrm{d}z\right)^{1/\tau} \\ &\qquad + \left(\frac{1}{|2I|}\int_{2I} |\mathcal{C}_{m+1;A}(a_1^1,\dots,a_m^1;f^1)(z)|^\tau \mathrm{d}z\right)^{1/\tau} \\ &\quad\lesssim M_{L\log L}f(x)\prod_{j=1}^m Ma_j(x) + M_\tau(\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f))(x),\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{M}^3_{\mathcal{C}_{m+1,A}}(a_1,\dots,a_m;f)(x) &\lesssim M_{L\log L}f(x)\prod_{j=1}^m Ma_j(x) \\ &\quad + M_\tau(\mathcal{C}_{m+1;A}(a_1,\dots,a_m;f))(x). \qquad (3.7)\end{aligned}$$

Applying inequality (3.4) and Theorem 2.11, we obtain that

$$\begin{aligned}&|\{x\in\mathbb{R} : M_\tau(\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f))(x) > 1\}| \\ &\quad\lesssim \sup_{s\ge 2^{-1/(m+1)\tau}} s^{1/(m+1)}|\{x\in\mathbb{R} : |\mathcal{C}_{m+1,A}(a_1,\dots,a_m;f)(x)| > s\}| \\ &\quad\lesssim \sum_{j=1}^m \|a_j\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} |f(y)|\log(\mathrm{e}+|f(y)|)\,\mathrm{d}y. \qquad (3.8)\end{aligned}$$

Combining estimates (3.7) and (3.8) yields (3.5) and completes the proof of Theorem 1.4. □

Acknowledgements. The research of the first author was supported by the NNSF of China under grant no. 11671363, and the research of the second (corresponding) author was supported by the NNSF of China under grant no. 11871108.

The authors would like to thank the referee for his/her valuable suggestions and helpful comments.

https://doi.org/10.1017/S001309151900021X Published online by Cambridge University Press

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