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RESEARCH ARTICLE

A note on holomorphic sectional curvature of a Hermitian manifold*

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Abstract
As is well known, the holomorphic sectional curvature is just half of the sectional curvature in a holomorphic plane section on a Kähler manifold (Zheng, *Complex differential geometry* (2000)). In this article, we prove that if the holomorphic sectional curvature is half of the sectional curvature in a holomorphic plane section on a Hermitian manifold then the Hermitian metric is Kähler.

1. Introduction

Let M be an n-dimensional complex manifold. Suppose h is a Hermitian metric on M with $g = \operatorname{Re} h$ the background Riemannian metric. There are two canonical connections associated to h and g, the Hermitian (or Chern) connection D and the Riemannian (or Levi-Civita) connection ∇. As is well known, the Chern connection D coincides with the Levi-Civita ∇ if and only if h is Kähler [2]. Hence, curvatures associated with these two canonical connections are tightly related on Kähler manifolds.

Suppose that $z = (z^1, \cdots, z^n)$ is a local holomorphic coordinate system on M. Denote by $h = h_{\alpha\bar{\beta}}(z)dz^\alpha \otimes d\bar{z}^\beta$ a Hermitian metric on M, where $h_{\alpha\bar{\beta}}(z)$ are smooth functions and $H = (h_{\alpha\bar{\beta}}(z))$ is an $n \times n$ positive definite Hermitian matrix. Let D denote the Chern connection of the Hermitian manifold (M, h). Its connection 1-forms are given by

$$\theta^\alpha_\beta = \Gamma^\alpha_{\beta\gamma} dz^\gamma,$$

where

$$\Gamma^\alpha_{\beta\gamma} = h^{\bar{\delta}\alpha} \frac{\partial h_{\beta\bar{\delta}}}{\partial z^\gamma}. \tag{1.1}$$

Since $\partial\theta^\alpha_\beta - \theta^\gamma_\beta \wedge \theta^\alpha_\gamma = 0$, the curvature form of the Chern connection D is

$$\bar{\partial}\theta^\alpha_\beta = \Theta^\alpha_{\beta\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu,$$

where

$$\Theta^\alpha_{\beta\mu\bar{\nu}} = -\frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial \bar{z}^\nu} = -h^{\bar{\delta}\alpha} \frac{\partial^2 h_{\beta\bar{\delta}}}{\partial z^\mu \partial \bar{z}^\nu} + h^{\bar{\delta}\alpha} \frac{\partial h_{\beta\bar{\epsilon}}}{\partial z^\mu} h^{\bar{\epsilon}\gamma} \frac{\partial h_{\gamma\bar{\delta}}}{\partial \bar{z}^\nu}. \tag{1.2}$$

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The holomorphic sectional curvature tensor is defined by

$$\Theta_{\alpha\bar{\beta}\mu\bar{\nu}} = h_{\gamma\bar{\beta}}\Theta^{\gamma}_{\alpha\mu\bar{\nu}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^\mu \partial \bar{z}^\nu} + \frac{\partial h_{\alpha\bar{\lambda}}}{\partial z^\mu} h^{\bar{\lambda}\kappa} \frac{\partial h_{\kappa\bar{\beta}}}{\partial \bar{z}^\nu}. \tag{1.3}$$

The holomorphic bisectional curvature in two directions $v = v^\alpha \frac{\partial}{\partial z^\alpha}$, $w = w^\alpha \frac{\partial}{\partial z^\alpha} \in T^{1,0}_z M$ and the holomorphic sectional curvature in the direction $v = v^\alpha \frac{\partial}{\partial z^\alpha} \in T^{1,0}_z M$ are, respectively, defined by

$$\mathrm{HBSC}(z; v, w) = \frac{\Theta_{\alpha\bar{\beta}\mu\bar{\nu}} v^\alpha \bar{v}^\beta w^\mu \bar{w}^\nu}{h_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta \cdot h_{\mu\bar{\nu}} w^\mu \bar{w}^\nu}, \tag{1.4}$$

$$\mathrm{HSC}(z; v) = \frac{\Theta_{\alpha\bar{\beta}\mu\bar{\nu}} v^\alpha \bar{v}^\beta v^\mu \bar{v}^\nu}{\left(h_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta\right)^2}. \tag{1.5}$$

An n-dimensional complex manifold M is also a $2n$-dimensional real manifold. Set

$$x = (\mathrm{Re}\,(z), \mathrm{Im}\,(z)) \in \mathbb{R}^{2n},$$

and

$$A(x) = \mathrm{Re}\,(H(z)), \quad B(x) = \mathrm{Im}\,(H(z)), \tag{1.6}$$

then

$$G(x) = \left(g_{ij}(x)\right)_{2n\times 2n} = \begin{pmatrix} A(x) & B(x) \\ -B(x) & A(x) \end{pmatrix} \tag{1.7}$$

is a positive definite real symmetric matrix and

$$G(x)^{-1} = \left(g^{ij}(x)\right)_{2n\times 2n} = \begin{pmatrix} A_1(x) & B_1(x) \\ -B_1(x) & A_1(x) \end{pmatrix}, \tag{1.8}$$

where

$$A_1(x) = \mathrm{Re}\,(H^{-1}(z)), \quad B_1(x) = \mathrm{Im}\,(H^{-1}(z)). \tag{1.9}$$

Therefore, $g = \mathrm{Re}\, h = g_{ij}(x) dx^i \otimes dx^j$ is a Riemannian metric on M, where

$$dx^\alpha = \frac{1}{2}\left(dz^\alpha + d\bar{z}^\alpha\right), \quad dx^{\alpha+n} = -\frac{i}{2}\left(dz^\alpha - d\bar{z}^\alpha\right). \tag{1.10}$$

Let us denote by ∇ the Levi-Civita connection, then its connection 1-forms are given by

$$\omega_i^k = \gamma_{ij}^k dx^j,$$

where

$$\gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right). \tag{1.11}$$

The curvature form of the Levi-Civita connection ∇ is

$$d\omega_k^l - \omega_k^h \wedge \omega_h^l = \frac{1}{2} R^l_{kij} dx^i \wedge dx^j,$$

where

$$R^l_{kij} = \frac{\partial \gamma^l_{kj}}{\partial x^i} - \frac{\partial \gamma^l_{ki}}{\partial x^j} + \gamma^h_{kj}\gamma^l_{hi} - \gamma^h_{ki}\gamma^l_{hj}. \tag{1.12}$$

In local coordinates, we denote by $\mathcal{R} = R^i_{jkl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$. The sectional curvature tensor is defined by

$$R_{ijkl} = g_{ih} R^h_{jkl}.$$

In local coordinates,

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) + g^{st}\left([jk, s][il, t] - [jl, s][ik, t]\right), \tag{1.13}$$

where

$$[jk, s] = \frac{1}{2}\left(\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x_s}\right). \tag{1.14}$$

The sectional curvature of the 2-plane $\Pi(u, y)$ spanned by two linearly independent tangent vectors $u = u^i \frac{\partial}{\partial x^i}$, $y = y^i \frac{\partial}{\partial x^i} \in T_xM$ of the Riemannian metric g is defined by

$$K(x; u, y) = \frac{g(\mathcal{R}(u, y)y, u)}{g(u, y)g(y, y) - g(u, y)^2} = \frac{R_{ijkl}u^i y^j u^k y^l}{(g_{ij}g_{kl} - g_{il}g_{jk})u^i u^j y^k y^l}. \tag{1.15}$$

For a Hermitian manifold (M, h), a 2-plane $\Pi(u, Ju)$ is called a holomorphic plane section in [4], where $u \in TM$ and J is the complex structure on M.

Assume h is a Kähler metric, then the holomorphic sectional curvature is just half of the sectional curvature in a holomorphic plane section [4], i.e.,

$$\mathrm{HSC}(z; v) = \frac{1}{2}K(x; u, Ju), \quad \forall v \in T_z^{1,0}M, \quad u = v + \bar{v} \in T_xM. \tag{1.16}$$

A nature question arises, when a Hermitian metric h satisfies (1.16), must it be Kähler? In this article, we give a positive answer.

Main Theorem *Let (M, h) be a Hermitian manifold. If the holomorphic sectional curvature is just half of the sectional curvature in a holomorphic plane section, i.e., (1.16) holds, then h is a Kähler metric.*

2. Proof of main theorem

In this section, we consider a Hermitian manifold such that the holomorphic sectional curvature is half of the sectional curvature in a holomorphic plane section.

For simplicity, we introduce some notations. Denote by

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{2n}}\right), \quad \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\right), \quad \frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\right),$$

$$dx = \left(dx^1, \cdots, dx^{2n}\right), \quad dz = \left(dz^1, \cdots, dz^n\right), \quad d\bar{z} = \left(d\bar{z}^1, \cdots, d\bar{z}^n\right).$$

Then

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)F, \quad dx = (dz, d\bar{z})\left(F^t\right)^{-1}, \tag{2.1}$$

where

$$F = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}, \quad \left(F^t\right)^{-1} = \frac{1}{2}\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix},$$

I is the $n \times n$ unit matrix, and F^t means transposition of F. We will denote by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

the $2n \times 2n$ matrix associated with the complex structure J when no confusion can rise. Then

$$FF^t = 2\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad F\bar{F}^t = 2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.2}$$

$$FJF^t = -2iJ, \quad FJF^{-1} = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.3}$$

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For any $v = v^\alpha \frac{\partial}{\partial z^\alpha} \in T_z^{1,0}M$, we set $u = u^i \frac{\partial}{\partial x^i}$, $\tilde{u} = \tilde{u}^i \frac{\partial}{\partial x^i} \in T_x M$ such that

$$u^\alpha = \tilde{u}^{\alpha+n} = \frac{1}{2}\left(v^\alpha + \bar{v}^\alpha\right), \quad u^{\alpha+n} = -\tilde{u}^\alpha = -\frac{i}{2}\left(v^\alpha - \bar{v}^\alpha\right).$$

Then

$$u = v + \bar{v}, \quad \tilde{u} = -i(v - \bar{v}) = -Ju. \tag{2.4}$$

Set $\mathbb{v} = (v^1, \ldots, v^n)$, $\mathbb{u} = (u^1, \ldots, u^{2n})$, $\tilde{\mathbb{u}} = (\tilde{u}^1, \ldots, \tilde{u}^{2n})$, we have

$$\mathbb{u} = (\mathbb{v}, \bar{\mathbb{v}})(F^t)^{-1}, \quad \tilde{\mathbb{u}} = \mathbb{u}J. \tag{2.5}$$

It is easy to check that $JG = GJ$, $JGJ^t = G$, and

$$FGF^t = 2\begin{pmatrix} 0 & \bar{H} \\ H & 0 \end{pmatrix}, \quad (F^t)^{-1}GF^{-1} = \frac{1}{2}\begin{pmatrix} 0 & H \\ \bar{H} & 0 \end{pmatrix}, \tag{2.6}$$

$$FG^{-1}F^t = 2\begin{pmatrix} 0 & \bar{H}^{-1} \\ H^{-1} & 0 \end{pmatrix}, \quad (F^t)^{-1}G^{-1}F^{-1} = \frac{1}{2}\begin{pmatrix} 0 & H^{-1} \\ \bar{H}^{-1} & 0 \end{pmatrix}, \tag{2.7}$$

$$\mathbb{u}G = \left(\mathbb{v}H, \bar{\mathbb{v}}\bar{H}\right)\left(F^t\right)^{-1}, \quad \tilde{\mathbb{u}}G = -i\left(\mathbb{v}H, -\bar{\mathbb{v}}\bar{H}\right)\left(F^t\right)^{-1}. \tag{2.8}$$

Given $w = w^\alpha \frac{\partial}{\partial z^\alpha} \in T_z^{1,0}M$, we set $y = y^i \frac{\partial}{\partial x^i}$, $\tilde{y} = \tilde{y}^i \frac{\partial}{\partial x^i} \in T_x M$ such that

$$y^\alpha = \tilde{y}^{\alpha+n} = \frac{1}{2}\left(w^\alpha + \bar{w}^\alpha\right), \quad y^{\alpha+n} = -\tilde{y}^\alpha = -\frac{i}{2}\left(w^\alpha - \bar{w}^\alpha\right),$$

and $\mathbb{w} = (w^1, \ldots, w^n)$, $\mathbb{y} = (y^1, \ldots, y^{2n})$, $\tilde{\mathbb{y}} = (\tilde{y}^1, \ldots, \tilde{y}^{2n})$. Then

$$\mathbb{u}G\,\mathbb{y}^t = \tilde{\mathbb{u}}G\tilde{\mathbb{y}}^t = \frac{1}{2}\left(\mathbb{v}H\bar{\mathbb{w}}^t + \mathbb{w}H\bar{\mathbb{v}}^t\right), \tag{2.9}$$

$$\mathbb{u}G\,\tilde{\mathbb{y}}^t = -\,\tilde{\mathbb{u}}G\mathbb{y}^t = \frac{i}{2}\left(\mathbb{v}H\bar{\mathbb{w}}^t - \mathbb{w}H\bar{\mathbb{v}}^t\right). \tag{2.10}$$

Especially, $g(u, \tilde{u}) = \mathbb{u}G\,\tilde{\mathbb{u}}^t = -\tilde{\mathbb{u}}G\mathbb{u}^t = 0$.

Lemma 2.1. *Given a Hermitian manifold (M, h), then*

$$\begin{aligned}\frac{1}{2}R_{ijkl}u^i\tilde{u}^j y^k\tilde{y}^l = &-\frac{1}{2}\frac{\partial h_{\alpha\bar{\beta}}}{\partial z^\mu \partial \bar{z}^\nu}\left(v^\alpha\bar{w}^\beta w^\mu\bar{v}^\nu + w^\alpha\bar{v}^\beta v^\mu\bar{w}^\nu\right) \\ &+\frac{1}{4}\left(v^\alpha w^\mu + w^\alpha v^\mu\right)\frac{\partial h_{\alpha\bar{\lambda}}}{\partial z^\mu}h^{\bar{\lambda}\gamma}\frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^\nu}\left(\bar{w}^\nu\bar{v}^\beta + \bar{v}^\nu\bar{w}^\beta\right) \\ &-\frac{1}{4}v^\alpha\bar{w}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial \bar{z}^\nu} - \frac{\partial h_{\alpha\bar{\nu}}}{\partial \bar{z}^\lambda}\right)h^{\bar{\lambda}\gamma}\left(\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^\mu} - \frac{\partial h_{\mu\bar{\beta}}}{\partial z^\gamma}\right)w^\mu\bar{v}^\beta \\ &-\frac{1}{4}w^\alpha\bar{v}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial \bar{z}^\nu} - \frac{\partial h_{\alpha\bar{\nu}}}{\partial \bar{z}^\lambda}\right)h^{\bar{\lambda}\gamma}\left(\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^\mu} - \frac{\partial h_{\mu\bar{\beta}}}{\partial z^\gamma}\right)v^\mu\bar{w}^\beta.\end{aligned} \tag{2.11}$$

Proof. For simplicity, we denote by

$$L_{(u,y)} = u^i y^j \frac{\partial^2}{\partial x^i \partial x^j}, \quad L_{(v,w)} = v^\alpha w^\beta \frac{\partial^2}{\partial z^\alpha \partial z^\beta},$$

$$L_{(v,\bar{w})} = L_{(\bar{w},v)} = v^\alpha \bar{w}^\beta \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}, \quad L_{(\bar{v},\bar{w})} = \bar{v}^\alpha \bar{w}^\beta \frac{\partial^2}{\partial \bar{z}^\alpha \partial \bar{z}^\beta},$$

and

$$v(H) = \left(v^\gamma \frac{\partial h_{\alpha\bar{\beta}}}{\partial z^\gamma}\right)_{n\times n}, \quad u(G) = \left(u^k \frac{\partial g_{ij}}{\partial x^k}\right)_{2n\times 2n},$$

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$$L_{(u,y)}(G) = \left(u^i y^j \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} \right)_{2n\times 2n}, \quad L_{(v,\bar{w})}(H) = \left(v^\alpha \bar{w}^\beta \frac{\partial^2 h_{\mu\bar{\nu}}}{\partial z^\alpha \partial \bar{z}^\beta} \right)_{n\times n}.$$

In addition, we set

$$a = \mathbb{v}H\bar{\mathbb{w}}^t - \mathbb{w}H\bar{\mathbb{v}}^t, \quad b = \mathbb{v}H\bar{\mathbb{w}}^t + \mathbb{w}H\bar{\mathbb{v}}^t,$$

$$\begin{aligned}
\mathscr{A} &= \mathbb{v}(w(H)) + \mathbb{w}(v(H)) + \mathbb{v}(\bar{w}(H)) - \mathbb{w}(\bar{v}(H)) - \frac{\partial a}{\partial \bar{z}},\\
\mathscr{B} &= \mathbb{w}(v(H)) + \mathbb{v}(w(H)) + \mathbb{w}(\bar{v}(H)) - \mathbb{v}(\bar{w}(H)) + \frac{\partial a}{\partial \bar{z}},\\
\mathscr{C} &= \mathbb{v}(w(H)) + \mathbb{w}(v(H)) - \mathbb{v}(\bar{w}(H)) - \mathbb{w}(\bar{v}(H)) + \frac{\partial b}{\partial \bar{z}},\\
\mathscr{D} &= \mathbb{w}(v(H)) + \mathbb{v}(w(H)) + \mathbb{w}(\bar{v}(H)) + \mathbb{v}(\bar{w}(H)) - \frac{\partial b}{\partial \bar{z}}.
\end{aligned}$$

A direct computation shows

$$L_{(u,y)} + L_{(\tilde{u},\tilde{y})} = 2\left(L_{(v,\bar{w})} + L_{(\bar{v},w)} \right), \tag{2.12}$$

$$L_{(\tilde{u},y)} - L_{(u,\tilde{y})} = 2i\left(L_{(\bar{v},w)} - L_{(v,\bar{w})} \right). \tag{2.13}$$

Hence,

$$\begin{aligned}
&\frac{1}{4}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} \right) u^i\, \tilde{u}^j\, y^k\, \tilde{y}^l\\
=&\frac{1}{4}\mathbb{u}\left(L_{(\tilde{u},y)}(G) - L_{(u,\tilde{y})}(G) \right) \tilde{\mathbb{y}}^t - \frac{1}{4}\mathbb{u}\left(L_{(\tilde{u},\tilde{y})}(G) + L_{(u,y)}(G) \right) \mathbb{y}^t\\
=&-\frac{1}{2}\left[\mathbb{v}\left(L_{(\bar{v},w)}(H) \right) \bar{\mathbb{w}}^t + \mathbb{w}\left(L_{(v,\bar{w})}(H) \right) \bar{\mathbb{v}}^t \right]\\
=&-\frac{1}{2}\left[\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^\mu \partial \bar{z}^\nu} w^\alpha \bar{v}^\beta v^\mu \bar{w}^\nu + \frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^\mu \partial \bar{z}^\nu} v^\alpha \bar{w}^\beta w^\mu \bar{v}^\nu \right].
\end{aligned}$$

By a direct computation, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\tilde{\mathbb{u}} G \mathbb{y}^t \right) &= -\frac{i}{2}\left(\frac{\partial a}{\partial z}, \frac{\partial a}{\partial \bar{z}} \right) F = -i\left(\frac{\partial a}{\partial \bar{z}}, \frac{\partial a}{\partial z} \right) \left(F^t \right)^{-1},\\
y\left(\tilde{\mathbb{u}}\, G \right) + \tilde{u}\left(\mathbb{y} G \right) - \frac{\partial}{\partial x}\left(\tilde{\mathbb{u}} G \mathbb{y}^t \right) &= -i\left(\mathscr{A}, -\bar{\mathscr{A}} \right) \left(F^t \right)^{-1},\\
u\left(\tilde{\mathbb{y}} G \right) + \tilde{y}\left(\mathbb{u} G \right) - \frac{\partial}{\partial x}\left(\tilde{\mathbb{y}} G \mathbb{u}^t \right) &= -i\left(\mathscr{B}, -\bar{\mathscr{B}} \right) \left(F^t \right)^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2} g^{st}[jk,s][il,t]\, u^i\, \tilde{u}^j\, y^k\, \tilde{y}^l &= \frac{1}{8}\left(\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right) \tilde{u}^j\, y^k\, g^{st}\left(\frac{\partial g_{it}}{\partial x^l} + \frac{\partial g_{lt}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^t} \right) u^i\, \tilde{y}^l\\
&= \frac{1}{8}\left(y\left(\tilde{\mathbb{u}} G \right) + \tilde{u}\left(\mathbb{y} G \right) - \frac{\partial}{\partial x}\left(\tilde{u} G \mathbb{y}^t \right) \right) G^{-1}\left(u\left(\tilde{\mathbb{y}} G \right) + \tilde{y}\left(\mathbb{u} G \right) - \frac{\partial}{\partial x}\left(\tilde{\mathbb{y}} G \mathbb{u}^t \right) \right)^t\\
&= -\frac{1}{8}\left(\mathscr{A}, -\bar{\mathscr{A}} \right) \left(F^t \right)^{-1} G^{-1} F^{-1} \left(\mathscr{B}, -\bar{\mathscr{B}} \right)^t\\
&= -\frac{1}{16}\left(\mathscr{A}, -\bar{\mathscr{A}} \right) \begin{pmatrix} 0 & H^{-1} \\ \bar{H}^{-1} & 0 \end{pmatrix} \left(\mathscr{B}, -\bar{\mathscr{B}} \right)^t\\
&= \frac{1}{16}\left(\mathscr{A} H^{-1} \bar{\mathscr{B}}^t + \mathscr{B} H^{-1} \bar{\mathscr{A}}^t \right).
\end{aligned}$$

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Similarly,

$$\frac{\partial}{\partial x}\left(\tilde{\mathbb{u}}G\tilde{\mathbb{y}}^t\right)=\frac{\partial}{\partial x}\left(\mathbb{u}G\mathbb{y}^t\right)=\left(\frac{\partial b}{\partial \bar{z}},\frac{\partial b}{\partial z}\right)\left(F^t\right)^{-1},$$

$$-\left(\tilde{u}\left(\tilde{\mathbb{y}}G\right)+\tilde{y}\left(\tilde{\mathbb{u}}G\right)-\frac{\partial}{\partial x}\left(\tilde{\mathbb{u}}G\tilde{\mathbb{y}}^t\right)\right)=\left(\mathscr{C},\bar{\mathscr{C}}\right)\left(F^t\right)^{-1},$$

$$u\left(\mathbb{y}G\right)+y\left(\mathbb{u}G\right)-\frac{\partial}{\partial x}\left(\mathbb{u}G\mathbb{y}^t\right)=\left(\mathscr{D},\bar{\mathscr{D}}\right)\left(F^t\right)^{-1},$$

and

$$\begin{aligned}
-\frac{1}{2}g^{st}[jl,s][ik,t]u^i\tilde{u}^jy^k\tilde{y}^l &= -\frac{1}{8}\left(\frac{\partial g_{js}}{\partial x^l}+\frac{\partial g_{ls}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^s}\right)\tilde{u}^j\tilde{y}^lg^{st}\left(\frac{\partial g_{it}}{\partial x^k}+\frac{\partial g_{kt}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^t}\right)u^iy^k\\
&= -\frac{1}{8}\Bigg(\tilde{u}\left(\tilde{\mathbb{y}}G\right)+\tilde{y}\left(\tilde{\mathbb{u}}G\right)\\
&\quad -\frac{\partial}{\partial x}\left(\tilde{\mathbb{u}}G\tilde{\mathbb{y}}^t\right)\Bigg)G^{-1}\left(u\left(\mathbb{y}G\right)+y\left(\mathbb{u}G\right)-\frac{\partial}{\partial x}\left(\mathbb{u}G\mathbb{y}^t\right)\right)^t\\
&= \frac{1}{8}\left(\mathscr{C},\bar{\mathscr{C}}\right)\left(F^t\right)^{-1}G^{-1}\left(F\right)^{-1}\left(\mathscr{D},\bar{\mathscr{D}}\right)^t\\
&= \frac{1}{16}\left(\mathscr{C},-\bar{\mathscr{C}}\right)\begin{pmatrix}0 & H^{-1}\\ \bar{H}^{-1} & 0\end{pmatrix}\left(\mathscr{D},-\bar{\mathscr{D}}\right)^t\\
&= \frac{1}{16}\left(\mathscr{C}H^{-1}\bar{\mathscr{D}}^t+\mathscr{D}H^{-1}\bar{\mathscr{C}}^t\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}g^{st}\left([jk,s][il,t]-[jl,s][ik,t]\right)u^i\tilde{u}^jy^k\tilde{y}^l\\
=&\frac{1}{4}\left[\mathbb{v}(w(H))+\mathbb{w}(v(H))\right]H^{-1}\overline{\left[\mathbb{v}(w(H))+\mathbb{w}(v(H))\right]}^t\\
&-\frac{1}{4}\left[\mathbb{v}(\bar{w}(H))-\frac{\partial\left(\mathbb{v}H\bar{\mathbb{w}}^t\right)}{\partial\bar{z}}\right]H^{-1}\overline{\left[\mathbb{v}(\bar{w}(H))-\frac{\partial\left(\mathbb{v}H\bar{\mathbb{w}}^t\right)}{\partial\bar{z}}\right]}^t\\
&-\frac{1}{4}\left[\mathbb{w}(\bar{v}(H))-\frac{\partial\left(\mathbb{w}H\bar{\mathbb{v}}^t\right)}{\partial\bar{z}}\right]H^{-1}\overline{\left[\mathbb{w}(\bar{v}(H))-\frac{\partial\left(\mathbb{w}H\bar{\mathbb{v}}^t\right)}{\partial\bar{z}}\right]}^t\\
=&\frac{1}{4}\left(v^\alpha w^\mu+w^\alpha v^\mu\right)\frac{\partial h_{\alpha\bar{\lambda}}}{\partial z^\mu}h^{\bar{\lambda}\gamma}\frac{\partial h_{\gamma\bar{\beta}}}{\partial\bar{z}^\nu}\left(\bar{w}^\nu\bar{v}^\beta+\bar{v}^\nu\bar{w}^\beta\right)\\
&-\frac{1}{4}v^\alpha\bar{w}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\nu}-\frac{\partial h_{\alpha\bar{\nu}}}{\partial\bar{z}^\lambda}\right)h^{\bar{\lambda}\gamma}\left(\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^\mu}-\frac{\partial h_{\mu\bar{\beta}}}{\partial z^\gamma}\right)w^\mu\bar{v}^\beta\\
&-\frac{1}{4}w^\alpha\bar{v}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\nu}-\frac{\partial h_{\alpha\bar{\nu}}}{\partial\bar{z}^\lambda}\right)h^{\bar{\lambda}\gamma}\left(\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^\mu}-\frac{\partial h_{\mu\bar{\beta}}}{\partial z^\gamma}\right)v^\mu\bar{w}^\beta.
\end{aligned}$$

This completes the proof. □

Now we prove Main Theorem.

Theorem 2.2. (**Main Theorem**). *Let (M, h) be a Hermitian manifold. If the holomorphic sectional curvature is just half of the sectional curvature in a holomorphic plane section, i.e., (1.16) holds, then h is a Kähler metric.*

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Proof. Take $y = u$ in (2.11), then

$$\frac{1}{2}R_{ijkl}u^i\tilde{u}^ju^k\tilde{u}^l - \Theta_{\alpha\bar{\beta}\mu\bar{\nu}}v^\alpha\bar{v}^\beta v^\mu\bar{v}^\nu = -\frac{1}{2}v^\alpha\bar{v}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\nu} - \frac{\partial h_{\alpha\bar{\nu}}}{\partial\bar{z}^\lambda}\right)h^{\bar{\lambda}\gamma}\left(\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^\mu} - \frac{\partial h_{\mu\bar{\beta}}}{\partial z^\gamma}\right)v^\mu\bar{v}^\beta. \tag{2.14}$$

We can see

$$\frac{1}{2}R_{ijkl}u^i\tilde{u}^ju^k\tilde{u}^l = \Theta_{\alpha\bar{\beta}\mu\bar{\nu}}v^\alpha\bar{v}^\beta v^\mu\bar{v}^\nu,$$

if and only if

$$v^\alpha\bar{v}^\nu\left(\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\nu} - \frac{\partial h_{\alpha\bar{\nu}}}{\partial\bar{z}^\lambda}\right) = 0 \tag{2.15}$$

holds for any $v \in T_z^{1,0}M$. Take $\mathbb{v} = e_1 = (1, 0, \ldots, 0)$, then $\frac{\partial h_{1\bar{\lambda}}}{\partial\bar{z}^1} = \frac{\partial h_{1\bar{1}}}{\partial\bar{z}^\lambda}$, $1 \le \lambda \le n$. Take $\mathbb{v} = e_2 = (0, 1, \ldots, 0)$, then $\frac{\partial h_{2\bar{\lambda}}}{\partial\bar{z}^2} = \frac{\partial h_{2\bar{2}}}{\partial\bar{z}^\lambda}$, $1 \le \lambda \le n$. $\cdots$ Take $\mathbb{v} = e_n = (0, 0, \ldots, 1)$, then $\frac{\partial h_{n\bar{\lambda}}}{\partial\bar{z}^n} = \frac{\partial h_{n\bar{n}}}{\partial\bar{z}^\lambda}$, $1 \le \lambda \le n$. Take $\mathbb{v} = e_\alpha + e_\beta$, where $1 \le \alpha < \beta \le n$ or $1 \le \beta < \alpha \le n$, then $\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\beta} = \frac{\partial h_{\alpha\bar{\beta}}}{\partial\bar{z}^\lambda}$, $1 \le \lambda \le n$. Hence, if

$$\mathrm{HSC}(z; v) = \frac{1}{2}K(x; u, Ju),$$

then

$$\frac{\partial h_{\alpha\bar{\lambda}}}{\partial\bar{z}^\beta} = \frac{\partial h_{\alpha\bar{\beta}}}{\partial\bar{z}^\lambda}, \quad 1 \le \alpha, \beta, \lambda \le n,$$

i.e., h is a Kähler metric. □

As is well known, if h is a Kähler metric, then [1,3]

$$2R_{ijkl}u^iy^ju^ky^l = \Theta_{\alpha\bar{\beta}\mu\bar{\nu}}\left(v^\alpha\bar{w}^\beta - w^\alpha\bar{v}^\beta\right)\left(w^\mu\bar{v}^\nu - v^\mu\bar{w}^\nu\right). \tag{2.16}$$

Now we have the following result.

Proposition 2.3. *Let (M,h) be a Hermitian manifold such that (2.16) holds for any two directions u, $y \in T_xM$. Then h is a Kähler metric.*

Proof. Take $y = \tilde{u}$, it follows from (2.16) that $\frac{1}{2}R_{ijkl}u^i\tilde{u}^ju^k\tilde{u}^l - \Theta_{\alpha\bar{\beta}\mu\bar{\nu}}v^\alpha\bar{v}^\beta v^\mu\bar{v}^\nu = 0$ holds for any $u \in T_xM$. By the proof of Main Theorem, we complete the proof. □

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