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Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super–sublinear case

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We study the periodic and Neumann boundary-value problems associated with the second-order nonlinear differential equation

$$u'' + cu' + \lambda a(t)g(u) = 0,$$

where $g\colon [0,+\infty[\to [0,+\infty[$ is a sublinear function at infinity having superlinear growth at zero. We prove the existence of two positive solutions when

$$\int_0^T a(t)\,\mathrm{d}t < 0$$

and $\lambda > 0$ is sufficiently large. Our approach is based on Mawhin's coincidence degree theory and index computations.

Keywords: boundary-value problems; positive solutions; indefinite weight; multiplicity results; coincidence degree

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1. Introduction

This paper deals with the periodic boundary-value problem (BVP) associated with the nonlinear second-order ordinary differential equation

$$u'' + cu' + \lambda a(t)g(u) = 0. \tag{1.1}$$

Let $\mathbb{R}^+ := [0, +\infty[$ denote the set of non-negative real numbers. We suppose that $a\colon \mathbb{R} \to \mathbb{R}$ is a locally integrable T-periodic function and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and such that

(g_*) $g(0) = 0$, $g(s) > 0$ for $s > 0$.

The real constant c is arbitrary and results will be given depending on the parameter $\lambda > 0$.

We are interested in finding *positive and T-periodic* solutions to (1.1), namely we look for $u(t)$ satisfying (1.1) in the Carathéodory sense (see [17]) and such that $u(t + T) = u(t) > 0$ for all $t \in \mathbb{R}$.

As main assumptions on the nonlinearity we require that $g(s)$ tends to zero for $s \to 0^+$ faster than linearly and it has a sublinear growth at infinity, that is

(g_0) $\lim_{s\to 0^+} g(s)/s = 0$,

(g_∞) $\lim_{s\to +\infty} g(s)/s = 0$.

Under the above hypotheses, the search for positive solutions of (1.1) satisfying the two-point boundary condition $u(0) = u(T) = 0$ has received much attention. Note that in this case its is not restrictive to suppose that $c = 0$, since one can always reduce the problem to this situation via a standard change of variables. Typical theorems guarantee the existence of at least two (positive) solutions when $a(t) \geqslant 0$ for all t and $\lambda > 0$ is sufficiently large (cf. [11]). These proofs have been obtained by using different techniques, such as the theory of fixed points for positive operators or critical point theory. Under additional technical assumptions, similar results can also be given for the Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda a(x) g(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad\ \text{on } \partial\Omega \end{aligned}$$

(see, for example, [2, 18, 26]). In the recent paper [8] a dynamical system approach was proposed in order to obtain pairs of positive solutions even when $a(t)$ may change its sign.

Concerning the periodic BVP, analogous results on pairs of positive solutions have been provided in [15] for equations of the form

$$u'' - ku + \lambda a(t) g(u) = 0,$$

with $k > 0$. However, fewer results seem to be available when $k = 0$. One of the peculiar aspects of the periodic BVP associated with (1.1) is the fact that the differential operator has a non-trivial kernel (which is made by the constant functions). A second feature to take into account concerns the fact that we have to impose additional conditions on the weight function. Indeed, if $u(t) > 0$ is a T-periodic solution of (1.1), then (after integrating the equation on $[0, T]$) one has that

$$\int_0^T a(t) g(u(t))\, \mathrm{d}t = 0,$$

with $g(u(t)) > 0$ for every t. Hence, $a(t)$ cannot be of constant sign. These two facts make it unclear how to apply the methods based on the theory of positive operators for cones in Banach spaces.

A first contribution in the periodic problem for (1.1) was obtained in [6] in the case $c = 0$. More precisely, by taking advantage of the variational (Hamiltonian) structure of the equation

$$u'' + \lambda a(t)g(u) = 0, \tag{1.2}$$

critical point theory for the action functional

$$J_\lambda(u) := \int_0^T [\tfrac{1}{2}(u')^2 - \lambda a(t)G(u)]\,\mathrm{d}t$$

was used to prove the existence of at least two positive T-periodic solutions for (1.2), with λ positive and large, by assuming $a^+ \not\equiv 0$ on some interval and

(a_*) $\int_0^T a(t)\,\mathrm{d}t < 0$.

Roughly speaking, condition (a_*) guarantees both that the functional J_λ is coercive and bounded from below and that the origin is a strict local minimum. When $\lambda > 0$ is sufficiently large (so that $\inf J_\lambda < 0$) one gets two non-trivial critical points: a global minimum and a second one from a mountain pass geometry. To perform the technical estimates, in [6] some further conditions on $g(s)$ and

$$G(s) := \int_0^s g(\xi)\,\mathrm{d}\xi$$

(implying (g_0) and (g_∞)) were imposed. For example, the superlinearity assumption at zero is expressed by

(g_α) $\lim_{s\to 0^+} g(s)/s^\alpha = \ell_\alpha > 0$

for some $\alpha > 1$. Note that assumptions of this kind have also been used in previous works dealing with indefinite superlinear problems, such as [1, 4].

As observed in [6] (and first also in [3], in the context of the Neumann BVP), condition (a_*) becomes necessary when $g(s)$ is continuously differentiable with $g'(s) > 0$ for all $s > 0$. By repeating the argument in [6, proposition 2.1], one can check that the same necessary condition is valid for (1.1) with an arbitrary $c \in \mathbb{R}$.

Unlike the two-point (Dirichlet) BVP, where it is easy to enter into a variational formulation of Sturm–Liouville type for an arbitrary $c \in \mathbb{R}$, for the periodic problem this formulation is no longer guaranteed. Indeed, for $c \neq 0$, we lose the Hamiltonian structure if we pass to the natural equivalent system in the phase plane

$$u' = y, \qquad y' = -cy - \lambda a(t)g(u).$$

On the other hand, we can consider an equivalent first-order system of Hamiltonian type, as

$$u' = \mathrm{e}^{-ct}y, \qquad y' = -\lambda \mathrm{e}^{ct}a(t)g(u),$$

but its T-periodic solutions do not correspond to the T-periodic solutions of (1.1).

The main contribution of our paper is to provide an existence result for pairs of positive T-periodic solutions to (1.1) in the possibly non-variational setting (when $c \neq 0$). Towards this aim, we introduce a topological approach that may have some independent interest even for the case $c = 0$. Our proof is reminiscent of the classical approach in the case of positive operators in ordered Banach spaces, which consists in proving that the fixed-point index of the associated operator is 1 on small balls $B(0, r)$ as well as on large balls $B(0, R)$. Moreover, when $\lambda > 0$ is sufficiently large, one can find an intermediate ball $B(0, \rho)$ (with $r < \rho < R$), where the fixed-point index is 0. Thus, there exist a non-trivial (positive) solution in $P \cap (B(0, \rho) \setminus B[0, r])$ and another in $P \cap (B(0, R) \setminus B[0, \rho])$, where P is the positive cone. In our setting we do not have a positive operator, but, using a maximum-principle-type argument, we can work directly with the topological degree in the Banach space of continuous T-periodic functions and then prove that the two non-trivial solutions that we reach are indeed positive. Actually, the situation is even more complicated because (1.1) is a *coincidence equation* of the form

$$Lu = N_\lambda u,$$

with L a non-invertible differential operator. In this case Mawhin's coincidence degree theory (see [20]), adapted to the case of locally compact operators (cf. [25]), is the appropriate tool for our purposes. In the recent paper [12] a similar approach was adopted for the study of positive solutions when the nonlinearity is superlinear both at zero and at infinity. In such a situation the existence of at least one positive solution is guaranteed.

The advantage of using an approach based on degree theory lies in the fact that the existence results are stable with respect to small perturbations of the differential equation. Hence, we can also provide pairs of positive T-periodic solutions for equations of the form

$$u'' + cu' + \varepsilon u + \lambda a(t) g(u) = 0$$

for ε small. This also gives an interesting result in the variational case (when $c = 0$).

The technical assumptions on $g(s)$ that we have to impose at zero (as well as at infinity) allow us to slightly improve (g_α), by using a condition of *regular oscillation* type. Let $\mathbb{R}^+_0 :=]0, +\infty[$ and let $h \colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be a continuous function. We say that h is *regularly oscillating at zero* if

$$\lim_{\substack{s \to 0^+ \\ \omega \to 1}} \frac{h(\omega s)}{h(s)} = 1.$$

Analogously, we say that h is *regularly oscillating at infinity* if

$$\lim_{\substack{s \to +\infty \\ \omega \to 1}} \frac{h(\omega s)}{h(s)} = 1.$$

The concept of a regularly oscillating function (usually referring to the case at infinity) is related to classical conditions of Karamata type that have been developed and studied by several authors due to their significance in different areas of real analysis and probability (cf. [5, 27]). For the specific definition considered in our paper, as well as for some historical remarks, see [10] and the references therein. Observe that any function $h(s)$ such that $h(s) \sim K s^p$, $K, p > 0$, is regularly oscillating both at zero and at infinity. However, the class of regularly oscillating functions is quite broad. For instance, functions such as

$$h(s) = s^p \exp\biggl(\int_1^s \frac{b(t)}{t}\,\mathrm{d}t\biggr),$$

with $b(t)$ continuous and bounded, are regularly oscillating at infinity.

Now we are in a position to state our main result.

THEOREM 1.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that g is regularly oscillating at zero and at infinity and satisfies (g_0) and (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition (a_*). Furthermore, suppose that there exists an interval $I \subseteq [0, T]$ such that $a(t) \geqslant 0$ for almost every (a.e.) $t \in I$ and*

$$\int_I a(t)\,\mathrm{d}t > 0.$$

Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ (1.1) has at least two positive T-periodic solutions.

As will become clear from the proof, the constant λ^* can be chosen depending (as well as on c and $g(s)$) only on the behaviour of $a(t)$ on the interval I. This remark allows us to obtain the following corollary for the related two-parameter equation

$$u'' + cu' + (\lambda a^+(t) - \mu a^-(t))g(u) = 0, \tag{1.3}$$

with $\lambda, \mu > 0$, where, as usual, we have set

$$a^+(t) := \frac{a(t) + |a(t)|}{2}, \qquad a^-(t) := \frac{-a(t) + |a(t)|}{2}.$$

Equation (1.3), for $c = 0$, has been considered in [7], with the aim of investigating multiplicity results and complex dynamics when $\mu \gg 0$ (see also [13] and the references therein for related results in the superlinear case).

COROLLARY 1.2. *Let $g(s)$ be as above and let $a(t)$ be a T-periodic function with $a^\pm \in L^1([0, T])$ and $a^- \not\equiv 0$. Suppose also that there exists an interval $I \subseteq [0, T]$ such that*

$$\int_I a^-(t)\,\mathrm{d}t = 0 < \int_I a^+(t)\,\mathrm{d}t.$$

Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for each

$$\mu > \lambda \frac{\int_0^T a^+(t)\,\mathrm{d}t}{\int_0^T a^-(t)\,\mathrm{d}t}$$

equation (1.3) has at least two positive T-periodic solutions.

Our results are sharp in the sense that there are examples of functions $g(s)$ satisfying all the assumptions of theorem 1.1 or of corollary 1.2 and such that there are no positive T-periodic solutions if $\lambda > 0$ is small or if (a_*) is not satisfied (see [6, § 2], where the assertions are proved in the case $c = 0$). One can easily check that those results can be extended to the case of an arbitrary $c \in \mathbb{R}$ (see also § 4.4).

Another sharp result can be given when $g(s)$ is smooth. Indeed, initially we produce a variant of theorem 1.1 (see theorem 4.4) by replacing the hypothesis of regular oscillation of g at zero (respectively, at infinity), with the condition of continuous differentiability of $g(s)$ in a neighbourhood of $s = 0$ (respectively, near infinity). Next, in the smooth case and further assuming that $|g'(s)|$ is bounded on $\mathbb{R}^+_0$, we can also provide a non-existence result for $\lambda > 0$ small (see theorem 4.6). As a consequence of these results, the following variant of theorem 1.1 can be stated. We define $g'(\infty) = \lim_{s\to+\infty} g'(s)$.

THEOREM 1.3. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*) and such that $g'(0) = 0$ and $g'(\infty) = 0$. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition (a_*). Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geqslant 0$ for a.e. $t \in I$ and $\int_I a(t)\,\mathrm{d}t > 0$. Then there exists $\lambda_* > 0$ such that for each $0 < \lambda < \lambda_*$ (1.1) has no positive T-periodic solution. Moreover, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ (1.1) has at least two positive T-periodic solutions. Condition (a_*) is also necessary if $g'(s) > 0$ for $s > 0$.*

To show a simple example of the applicability of theorem 1.3, we consider the T-periodic BVP

$$\left.\begin{aligned} u'' + cu' + \lambda(\sin(t) + k)g(u) = 0,\\ u(2\pi) - u(0) = u'(2\pi) - u'(0) = 0,\end{aligned}\right\} \tag{1.4}$$

where $k \in \mathbb{R}$ and

$$g(s) = \arctan(s^\alpha) \quad \text{with } \alpha > 1$$

(other examples of functions $g(s)$ can easily be produced). Since $g'(s) > 0$ for all $s > 0$, we know that there are positive T-periodic solutions only if $-1 < k < 0$. Moreover, for any fixed $k \in \,]-1,0[$ there exist two constants $0 < \lambda_{*,k} \leqslant \lambda^{*,k}$ such that for $0 < \lambda < \lambda_{*,k}$ there are no positive solutions for (1.4), while for $\lambda > \lambda^{*,k}$ there are at least two positive solutions. Estimates for $\lambda_{*,k}$ and $\lambda^{*,k}$ can be given for any specific equation.

The plan of the paper is as follows. In § 2 we recall some basic facts about Mawhin's coincidence degree and we present two lemmas for the computation of the degree (see lemmas 2.1 and 2.2); next we show the general scheme followed in the proof of theorem 1.1, which is performed in § 3. In § 4 we present some consequences and variants of the main theorem (including the existence of small (large) solutions using only conditions for $g(s)$ near zero (near infinity)). We also deal with the smooth case and give a non-existence result. In § 5 we briefly describe how all the results can be adapted to the Neumann problem, including a final application to radially symmetric solutions on annular domains.

2. The abstract setting

Let $X := \mathcal{C}_T$ be the Banach space of continuous and T-periodic functions $u\colon \mathbb{R} \to \mathbb{R}$, endowed with the norm

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)| = \max_{t\in\mathbb{R}} |u(t)|,$$

and let $Z := L^1_T$ be the Banach space of measurable and T-periodic functions $v\colon \mathbb{R} \to \mathbb{R}$ that are integrable on $[0, T]$, endowed with the norm

$$\|v\|_{L^1_T} := \int_0^T |v(t)| \, \mathrm{d}t.$$

The linear differential operator

$$L\colon u \mapsto -u'' - cu'$$

is a (linear) Fredholm map of index zero defined on $\operatorname{dom} L := W^{2,1}_T \subseteq X$, with range

$$\operatorname{Im} L = \left\{ v \in Z\colon \int_0^T v(t) \, \mathrm{d}t = 0 \right\}.$$

Associated with L we have the projectors

$$P\colon X \to \ker L \cong \mathbb{R}, \qquad Q\colon Z \to \operatorname{coker} L \cong Z/\operatorname{Im} L \cong \mathbb{R},$$

which, in our situation, can be chosen as the average operators

$$Pu = Qu := \frac{1}{T} \int_0^T u(t) \, \mathrm{d}t.$$

Finally, let

$$K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$$

be the right inverse of L, that is, for any $v \in L^1_T$ with $\int_0^T v(t) \, \mathrm{d}t = 0$, $K_P v = u$ is the unique T-periodic solution u of

$$u'' + cu' + v(t) = 0 \quad \text{with} \quad \int_0^T u(t) \, \mathrm{d}t = 0.$$

Next, we define the L^1-Carathéodory function

$$f_\lambda(t, s) := \begin{cases} -s & \text{if } s \leqslant 0, \\ \lambda a(t) g(s) & \text{if } s \geqslant 0, \end{cases}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a T-periodic and locally integrable function, $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $g(0) = 0$ and $\lambda > 0$ is a fixed parameter. Let us denote by $N_\lambda\colon X \to Z$ the Nemytskii operator induced by the function f_λ, that is

$$(N_\lambda u)(t) := f_\lambda(t, u(t)), \quad t \in \mathbb{R}.$$

By coincidence degree theory we know that the equation

$$Lu = N_\lambda u, \quad u \in \operatorname{dom} L, \tag{2.1}$$

is equivalent to the fixed-point problem

$$u = \Phi_\lambda u := Pu + QN_\lambda u + K_P(\mathrm{Id} - Q)N_\lambda u, \quad u \in X.$$

Technically, the term $QN_\lambda u$ in the above formula should be more correctly written as $JQN_\lambda u$, where J is a linear (orientation-preserving) isomorphism from $\operatorname{coker} L$

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to $\ker L$. However, in our situation, we can take as J the identity on $\mathbb{R}$, having identified $\operatorname{coker} L$, as well as $\ker L$, with $\mathbb{R}$. It is standard to verify that $\Phi_\lambda \colon X \to X$ is a completely continuous operator. In such a situation, we usually say that N_λ is *L-completely continuous* (see [20], where the treatment has been given for the most general cases).

If $\mathcal{O} \subseteq X$ is an open and *bounded* set such that

$$Lu \neq N_\lambda u \quad \forall u \in \partial\mathcal{O} \cap \operatorname{dom} L,$$

the *coincidence degree* $D_L(L - N_\lambda, \mathcal{O})$ (*of L and N_λ in $\mathcal{O}$*) is defined as

$$D_L(L - N_\lambda, \mathcal{O}) := \deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \mathcal{O}, 0),$$

where '$\deg_{\mathrm{LS}}$' denotes the Leray–Schauder degree.

In our applications we need to consider a slight extension of coincidence degree to open (not necessarily bounded) sets. Towards this aim, we just follow the standard approach used to define the Leray–Schauder degree for locally compact maps defined on open sets, which is classical in the theory of fixed-point index (cf. [16, 22, 24, 25]). More precisely, let $\Omega \subseteq X$ be an open set and suppose that the solution set

$$\operatorname{Fix}(\Phi_\lambda, \Omega) := \{u \in \Omega \colon u = \Phi_\lambda u\} = \{u \in \Omega \cap \operatorname{dom} L \colon Lu = N_\lambda u\}$$

is compact. The extension of the Leray–Schauder degree to open (not necessarily bounded) sets allows us to define

$$\deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \Omega, 0) := \deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \mathcal{V}, 0),$$

where $\mathcal{V}$ is an open and bounded set with

$$\operatorname{Fix}(\Phi_\lambda, \Omega) \subseteq \mathcal{V} \subseteq \bar{\mathcal{V}} \subseteq \Omega. \tag{2.2}$$

One can check that the definition is independent of the choice of $\mathcal{V}$. Accordingly, we define the *coincidence degree* $D_L(L - N_\lambda, \Omega)$ (*of L and N_λ in Ω*) as

$$D_L(L - N_\lambda, \Omega) := D_L(L - N_\lambda, \mathcal{V}) = \deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \mathcal{V}, 0),$$

with $\mathcal{V}$ as above. In the special case when Ω is an open and *bounded* set such that

$$Lu \neq N_\lambda u \quad \forall u \in \partial\Omega \cap \operatorname{dom} L, \tag{2.3}$$

it is easy to verify that the above definition is exactly the usual definition of coincidence degree, according to Mawhin. Indeed, if (2.3) holds with Ω open and bounded, then, by the excision property of the Leray–Schauder degree, we have $\deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \mathcal{V}, 0) = \deg_{\mathrm{LS}}(\mathrm{Id} - \Phi_\lambda, \Omega, 0)$ for each open and bounded set $\mathcal{V}$ satisfying (2.2). We refer the reader to [23] for an analogous introduction from a different point of view.

Combining the properties of coincidence degree from [20, ch. II] with the fixed-point index theory for locally compact operators (cf. [24, 25]), it is possible to derive the following versions of the main properties of the degree.

- *Additivity.* Let Ω_1 and Ω_2 be open and disjoint subsets of Ω such that $\operatorname{Fix}(\Phi_\lambda, \Omega) \subseteq \Omega_1 \cup \Omega_2$. Then

$$D_L(L - N_\lambda, \Omega) = D_L(L - N_\lambda, \Omega_1) + D_L(L - N_\lambda, \Omega_2).$$

- *Excision.* Let Ω_0 be an open subset of Ω such that $\mathrm{Fix}(\Phi_\lambda, \Omega) \subseteq \Omega_0$. Then
$$D_L(L - N_\lambda, \Omega) = D_L(L - N_\lambda, \Omega_0).$$
- *Existence theorem.* If $D_L(L - N_\lambda, \Omega) \neq 0$, then $\mathrm{Fix}(\Phi_\lambda, \Omega) \neq \emptyset$; hence, there exists $u \in \Omega \cap \mathrm{dom}\, L$ such that $Lu = N_\lambda u$.
- *Homotopic invariance.* Let $H\colon [0,1] \times \Omega \to X$, $H_\vartheta(u) := H(\vartheta, u)$, be a continuous homotopy such that
$$\mathcal{S} := \bigcup_{\vartheta \in [0,1]} \{u \in \Omega \cap \mathrm{dom}\, L\colon Lu = H_\vartheta u\}$$
is a compact set and there exists an open neighbourhood $\mathcal{W}$ of $\mathcal{S}$ such that $\bar{\mathcal{W}} \subseteq \Omega$ and $(K_P(\mathrm{Id} - Q)H)|_{[0,1]\times\bar{\mathcal{W}}}$ is a compact map. Then the map $\vartheta \mapsto D_L(L - H_\vartheta, \Omega)$ is constant on $[0,1]$.

For more details, proofs and applications, we refer to [14,20,21] and the references therein.

Later we shall apply this general setting as follows. We consider an L-completely continuous operator $\mathcal{N}$ and an open (not necessarily bounded) set $\mathcal{A}$ such that the solution set $\{u \in \bar{\mathcal{A}} \cap \mathrm{dom}\, L\colon Lu = \mathcal{N}u\}$ is compact and disjoint from $\partial\mathcal{A}$. Therefore, $D_L(L - \mathcal{N}, \mathcal{A})$ is well defined. We shall proceed analogously when dealing with homotopies.

2.1. Auxiliary lemmas

Within the framework introduced above, we now present two auxiliary semi-abstract results that are useful for the computation of the coincidence degree. In the following, we denote by $B(0,d)$ and $B[0,d]$ respectively the open and closed balls of centre at the origin and radius $d > 0$ in X. For lemmas 2.1 and 2.2 we do not require all the assumptions on $a(t)$ and $g(s)$ stated in theorem 1.1. In this way we hope that the two results may have independent interest beyond that of providing a proof of theorem 1.1.

LEMMA 2.1. *Let $\lambda > 0$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $g(0) = 0$ and let $a \in L^1_T$. Assume that there exists a constant $d > 0$ and a compact interval $\mathcal{I} \subseteq [0,T]$ such that the following properties hold.*

$(A_{d,\mathcal{I}})$ *If $\alpha \geqslant 0$, then any non-negative T-periodic solution $u(t)$ of*
$$u'' + cu' + \lambda a(t) g(u) + \alpha = 0 \tag{2.4}$$
satisfies $\max_{t\in\mathcal{I}} u(t) \neq d$.

$(B_{d,\mathcal{I}})$ *For every $\beta \geqslant 0$ there exists a constant $D_\beta \geqslant d$ such that if $\alpha \in [0,\beta]$ and $u(t)$ is any non-negative T-periodic solution of (2.4) with $\max_{t\in\mathcal{I}} u(t) \leqslant d$, then $\max_{t\in[0,T]} u(t) \leqslant D_\beta$.*

$(C_{d,\mathcal{I}})$ *There exists $\alpha^* \geqslant 0$ such that (2.4), with $\alpha = \alpha^*$, does not possess any non-negative T-periodic solution $u(t)$ with $\max_{t\in\mathcal{I}} u(t) \leqslant d$.*

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Then

$$D_L(L - N_\lambda, \Omega_{d,\mathcal{I}}) = 0,$$

where

$$\Omega_{d,\mathcal{I}} := \Big\{ u \in X \colon \max_{t \in \mathcal{I}} |u(t)| < d \Big\}.$$

Note that $\Omega_{d,\mathcal{I}}$ is open but not bounded (unless $\mathcal{I} = [0,T]$).

Proof. For a fixed constant $d > 0$ and a compact interval $\mathcal{I} \subseteq [0,T]$ as in the statement, let us consider the open set $\Omega_{d,\mathcal{I}}$ defined above. We study the equation

$$u'' + cu' + f_\lambda(t,u) + \alpha = 0 \tag{2.5}$$

for $\alpha \geqslant 0$, which can be written as a coincidence equation in the space X:

$$Lu = N_\lambda u + \alpha \mathbf{1}, \quad u \in \operatorname{dom} L,$$

where $\mathbf{1} \in X$ is the constant function $\mathbf{1}(t) \equiv 1$.

As a first step, we check that the coincidence degree $D_L(L - N_\lambda - \alpha\mathbf{1}, \Omega_{d,\mathcal{I}})$ is well defined for any $\alpha \geqslant 0$. Towards this aim, suppose that $\alpha \geqslant 0$ is fixed and consider the set

$$\begin{aligned} \mathcal{R}_\alpha &:= \{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \cap \operatorname{dom} L \colon Lu = N_\lambda u + \alpha\mathbf{1}\} \\ &= \{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \colon u = \Phi_\lambda u + \alpha\mathbf{1}\}. \end{aligned}$$

We have that $u \in \mathcal{R}_\alpha$ if and only if $u(t)$ is a T-periodic solution of (2.5) such that $|u(t)| \leqslant d$ for every $t \in \mathcal{I}$. By a standard application of the maximum principle, we find that $u(t) \geqslant 0$ for all $t \in \mathbb{R}$ and, indeed, $u(t)$ solves (2.4), with $\max_{t\in\mathcal{I}} u(t) \leqslant d$. Condition $(B_{d,\mathcal{I}})$ gives a constant D_α such that $\|u\|_\infty \leqslant D_\alpha$ and so $\mathcal{R}_\alpha$ is bounded. The complete continuity of the operator Φ_λ ensures the compactness of $\mathcal{R}_\alpha$. Moreover, condition $(A_{d,\mathcal{I}})$ guarantees that $|u(t)| < d$ for all $t \in \mathcal{I}$ and then we conclude that $\mathcal{R}_\alpha \subseteq \Omega_{d,\mathcal{I}}$. In this manner we have proved that the coincidence degree $D_L(L - N_\lambda - \alpha\mathbf{1}, \Omega_{d,\mathcal{I}})$ is well defined for any $\alpha \geqslant 0$.

Now, condition $(C_{d,\mathcal{I}})$, together with the property of existence of solutions when the degree D_L is non-zero, implies that there exists $\alpha^* \geqslant 0$ such that

$$D_L(L - N_\lambda - \alpha^*\mathbf{1}, \Omega_{d,\mathcal{I}}) = 0.$$

On the other hand, from condition $(B_{d,\mathcal{I}})$ applied on the interval $[0,\beta] := [0,\alpha^*]$, by repeating the argument in the first step above, we find that the set

$$\begin{aligned} \mathcal{S} &:= \bigcup_{\alpha \in [0,\alpha^*]} \mathcal{R}_\alpha \\ &= \bigcup_{\alpha \in [0,\alpha^*]} \{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \cap \operatorname{dom} L \colon Lu = N_\lambda u + \alpha\mathbf{1}\} \\ &= \bigcup_{\alpha \in [0,\alpha^*]} \{u \in \operatorname{cl}(\Omega_{d,\mathcal{I}}) \colon u = \Phi_\lambda u + \alpha\mathbf{1}\} \end{aligned}$$

is a compact subset of $\Omega_{d,\mathcal{I}}$. Hence, by the homotopic invariance of the coincidence degree, we have that

$$D_L(L - N_\lambda, \Omega_{d,\mathcal{I}}) = D_L(L - N_\lambda - \alpha^* \mathbf{1}, \Omega_{d,\mathcal{I}}) = 0.$$

This concludes the proof. □

LEMMA 2.2. *Let $\lambda > 0$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $g(0) = 0$. Suppose $a \in L^1_T$ with*

$$\int_0^T a(t)\,\mathrm{d}t < 0.$$

Assume that there exists a constant $d > 0$ such that $g(d) > 0$ and the following property holds.

(H_d) *If $\vartheta \in \,]0, 1]$ and $u(t)$ is any non-negative T-periodic solution of*

$$u'' + cu' + \vartheta\lambda a(t) g(u) = 0, \tag{2.6}$$

then $\max_{t\in[0,T]} u(t) \neq d$.

Then

$$D_L(L - N_\lambda, B(0, d)) = 1.$$

Proof. First, we claim that there are no solutions to the parametrized coincidence equation

$$Lu = \vartheta N_\lambda u, \quad u \in \partial B(0, d) \cap \operatorname{dom} L,\ 0 < \vartheta \leqslant 1.$$

Indeed, if any such a solution u exists, it is a T-periodic solution of

$$u'' + cu' + \vartheta f_\lambda(t, u) = 0$$

with $\|u\|_\infty = d$. By the definition of $f_\lambda(t, s)$ and a standard application of the maximum principle, we easily get that $u(t) \geqslant 0$ for every $t \in \mathbb{R}$. Therefore, $u(t)$ is a non-negative T-periodic solution of (2.6) with $\max_{t\in[0,T]} u(t) = d$. This contradicts property (H_d) and the claim is thus proved.

As a second step, we consider $QN_\lambda u$ for $u \in \ker L$. Since $\ker L \cong \mathbb{R}$, we have

$$QN_\lambda u = \frac{1}{T} \int_0^T f_\lambda(t, s)\,\mathrm{d}t \quad \text{for } u \equiv \text{const.} = s \in \mathbb{R}.$$

For notational convenience, we set

$$f^{\#}_\lambda(s) := \frac{1}{T} \int_0^T f_\lambda(t, s)\,\mathrm{d}t = \begin{cases} -s & \text{if } s \leqslant 0, \\ \lambda\biggl(\dfrac{1}{T}\displaystyle\int_0^T a(t)\,\mathrm{d}t\biggr) g(s) & \text{if } s \geqslant 0. \end{cases}$$

Note that $s f^{\#}_\lambda(s) < 0$ for each $s \neq 0$. As a consequence, we find that $QN_\lambda u \neq 0$ for each $u \in \partial B(0, d) \cap \ker L$.

An important result from Mawhin's continuation theorem (see [21, theorem 2.4] and also [19], where the result was previously given in the context of the periodic problem for ODEs) guarantees that

$$D_L(L - N_\lambda, B(0, d)) = d_{\mathrm{B}}(-QN_\lambda|_{\ker L}, B(0, d) \cap \ker L, 0) = d_{\mathrm{B}}(-f^{\#}_\lambda,\ ]-d, d[, 0),$$

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where d_{B} denotes the Brouwer degree; the latter is clearly equal to 1 as

$$-f_{\lambda}^{\#}(-d) = -d < 0 < \lambda\biggl(-\frac{1}{T}\int_0^T a(t)\,\mathrm{d}t\biggr)g(d) = -f_{\lambda}^{\#}(d).$$

This concludes the proof. □

2.2. Proof of theorem 1.1: the general strategy

With the aid of the two lemmas just proved, we can give a proof of theorem 1.1, as follows.

We fix a constant $\rho > 0$ and consider, for $\mathcal{I} := I$, the open set

$$\Omega_{\rho,I} := \Bigl\{u \in X\colon \max_{t\in I}|u(t)| < \rho\Bigr\}.$$

First, we show that condition $(A_{\rho,I})$ is satisfied provided that $\lambda > 0$ is sufficiently large, say $\lambda > \lambda^* := \lambda^*_{\rho,I}$. Such a lower bound for λ does not depend on α. Then, we fix an arbitrary $\lambda > \lambda^*$ and show that conditions $(B_{\rho,I})$ and $(C_{\rho,I})$ are satisfied as well. In particular, for $\beta = 0$, we find a constant $D_0 = D_0(\rho, I, \lambda) \geqslant \rho$ such that any possible solution of

$$Lu = N_\lambda u, \quad u \in \mathrm{cl}(\Omega_{\rho,I}) \cap \mathrm{dom}\,L,$$

satisfies

$$\|u\|_\infty \leqslant D_0.$$

In this manner, we have that

$$B(0,\rho) \subseteq \Omega_{\rho,I} \quad \text{and} \quad \mathrm{Fix}(\Phi_\lambda, \Omega_{\rho,I}) \subseteq B(0,R) \quad \forall R > D_0.$$

Moreover,

$$D_L(L - N_\lambda, \Omega_{\rho,I}) = D_L(L - N_\lambda, \Omega_{\rho,I} \cap B(0,R)) = 0 \quad \forall R > D_0.$$

As a next step, using (g_0) and the regular oscillation of $g(s)$ at zero, we find a positive constant $r_0 < \rho$ such that for each $r \in \,]0, r_0]$ condition (H_r) (of lemma 2.2) is satisfied and therefore

$$D_L(L - N_\lambda, B(0,r)) = 1 \quad \forall 0 < r \leqslant r_0.$$

With a similar argument, using (g_∞) and the regular oscillation of $g(s)$ at infinity, we find a positive constant $R_0 > D_0$ such that for each $R \geqslant R_0$ condition (H_R) is also satisfied, and therefore

$$D_L(L - N_\lambda, B(0,R)) = 1 \quad \forall R \geqslant R_0.$$

By the additivity property of the coincidence degree we obtain

$$D_L(L - N_\lambda, \Omega_{\rho,I} \setminus B[0,r]) = -1 \quad \forall 0 < r \leqslant r_0 \tag{2.7}$$

and

$$D_L(L - N_\lambda, B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))) = 1 \quad \forall R > R_0. \tag{2.8}$$

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Thus, in conclusion, we find a first solution $\underline{u}$ of (2.1) with $\underline{u} \in \Omega_{\rho,I} \setminus B[0,r]$ (using (2.7) for a fixed $r \in \,]0, r_0])$ and a second solution $\bar{u}$ of (2.1) with $\bar{u} \in B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))$ (using (2.8) for a fixed $R > R_0$). Both $\underline{u}(t)$ and $\bar{u}(t)$ are non-trivial T-periodic solutions of

$$u'' + cu' + f_\lambda(t,u) = 0$$

and, by the maximum principle, they are actually *non-negative* solutions of (1.1). Finally, since by condition (g_0) we know that $a(t)g(s)/s$ is L^1-bounded in a right neighbourhood of $s = 0$, it is immediate to prove (by an elementary form of the strong maximum principle) that such solutions are in fact *strictly positive*.

3. Proof of theorem 1.1: the technical details

In this section we give a proof of theorem 1.1 by following the steps described in § 2.2. Towards this aim, it is sufficient to check separately the validity of the assumptions in lemma 2.1, for $\mathcal{I} := I$ and $d = \rho > 0$ a fixed number, and those in lemma 2.2, for $d = r > 0$ small ($0 < r \leqslant r_0$) and for $d = R > 0$ large ($R \geqslant R_0$). Note that r_0 and R_0 are chosen after both ρ and $\lambda > 0$ have been fixed.

Throughout the section, for simplicity, we suppose the validity of all the assumptions in theorem 1.1. However, from a careful checking of the proofs below, one can see that not all of them are needed for the verification of each individual lemma.

3.1. Checking the assumptions of lemma 2.1 for λ large

Let $\rho > 0$ be fixed. Let $I := [\sigma,\tau] \subseteq [0,T]$ be such that $a(t) \geqslant 0$ for a.e. $t \in I$ and $\int_I a(t)\,\mathrm{d}t > 0$. We fix $\varepsilon > 0$ such that for

$$\sigma_0 := \sigma + \varepsilon < \tau - \varepsilon =: \tau_0$$

it holds that

$$\int_{\sigma_0}^{\tau_0} a(t)\,\mathrm{d}t > 0.$$

Let us consider the non-negative solutions of (2.4) for $t \in I$. Such an equation takes the form

$$u'' + cu' + h(t,u) = 0, \tag{3.1}$$

where we have set (for notational convenience)

$$h(t,s) = h_{\lambda,\alpha}(t,s) := \lambda a(t)g(s) + \alpha,$$

where $\lambda > 0$ and $\alpha \geqslant 0$. Note that $h(t,s) \geqslant 0$ for a.e. $t \in I$ and for all $s \geqslant 0$.

Writing (3.1) as

$$(\mathrm{e}^{ct}u')' + \mathrm{e}^{ct}h(t,u) = 0,$$

we find that $(\mathrm{e}^{ct}u'(t))' \leqslant 0$ for almost every $t \in I$, so that the map $t \mapsto \mathrm{e}^{ct}u'(t)$ is non-increasing on I.

We split the proof into different steps.

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Step 1 (a general estimate). For every non-negative solution $u(t)$ of (3.1) the following estimate holds:

$$|u'(t)| \leqslant \frac{u(t)}{\varepsilon} \mathrm{e}^{|c|T} \quad \forall t \in [\sigma_0, \tau_0]. \tag{3.2}$$

To prove this, let us fix $t \in [\sigma_0, \tau_0]$. The result is trivially true if $u'(t) = 0$. Suppose that $u'(t) > 0$ and consider the function $u(t)$ on the interval $[\sigma, t]$. Since $\xi \mapsto \mathrm{e}^{c\xi} u'(\xi)$ is non-increasing on $[\sigma, t]$, we have

$$u'(\xi) \geqslant u'(t)\mathrm{e}^{c(t-\xi)} \quad \forall \xi \in [\sigma, t].$$

Integrating on $[\sigma, t]$, we obtain

$$u(t) \geqslant u(t) - u(\sigma) \geqslant u'(t)\mathrm{e}^{-|c|(t-\sigma)}(t - \sigma) \geqslant u'(t)\mathrm{e}^{-|c|T}\varepsilon$$

and therefore (3.2) follows. If $u'(t) < 0$, we obtain the same result, after an integration on $[t, \tau]$. Hence, (3.2) is proved in any case. Observe that only a condition on the sign of $h(t, s)$ is used, and therefore the estimate is valid independently on $\lambda > 0$ and $\alpha \geqslant 0$.

Step 2 (verification of $(A_{\rho,I})$ for $\lambda > \lambda^*$, with λ^* depending on ρ and I but not on α). Suppose that $u(t)$ is a non-negative T-periodic solution of (2.4) with

$$\max_{t \in I} u(t) = \rho.$$

Let $t_0 \in I$ be such that $u(t_0) = \rho$ and observe that $u'(t_0) = 0$ if $\sigma < t_0 < \tau$, while $u'(t_0) \leqslant 0$ if $t_0 = \sigma$ and $u'(t_0) \geqslant 0$ if $t_0 = \tau$.

First, we prove the existence of a constant $\delta \in]0, 1[$ such that

$$\min_{t \in [\sigma_0, \tau_0]} u(t) \geqslant \delta\rho. \tag{3.3}$$

This follows from the estimate (3.2). Indeed, if $t_* \in [\sigma_0, \tau_0]$ is such that $u(t_*) = \min_{t \in [\sigma_0, \tau_0]} u(t)$, we obtain that

$$|u'(t_*)| \leqslant \frac{u(t_*)}{\varepsilon} \mathrm{e}^{|c|T}. \tag{3.4}$$

On the other hand, by the monotonicity of the function $t \mapsto \mathrm{e}^{ct} u'(t)$ in $[\sigma, \tau]$,

$$u'(\xi)\mathrm{e}^{c\xi} \geqslant u'(t_*)\mathrm{e}^{ct_*} \quad \forall \xi \in [\sigma, t_*] \tag{3.5}$$

and

$$u'(\xi)\mathrm{e}^{c\xi} \leqslant u'(t_*)\mathrm{e}^{ct_*} \quad \forall \xi \in [t_*, \tau]. \tag{3.6}$$

From the properties about $u'(t_0)$ listed above, we deduce that if $t_0 > t_*$, then $u'(t_0) \geqslant 0$, and therefore we must have $u'(t_*) \geqslant 0$. Similarly, if $t_0 < t_*$, then $u'(t_0) \leqslant 0$, and therefore we must have $u'(t_*) \leqslant 0$. The case $t_* = t_0$ can be handled trivially and we do not consider it here. Thus, we have that one of the following situations occurs: either

$$\sigma \leqslant t_0 < t_* \in [\sigma_0, \tau_0], \quad u(t_0) = \rho, \quad u'(\xi) \leqslant 0, \quad \forall \xi \in [t_0, t_*] \tag{3.7}$$

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or

$$\tau \geqslant t_0 > t_* \in [\sigma_0, \tau_0], \quad u(t_0) = \rho, \quad u'(\xi) \geqslant 0, \quad \forall \xi \in [t_*, t_0]. \tag{3.8}$$

Suppose that (3.7) holds. In this situation, from (3.5) we have

$$-u'(\xi) \leqslant -u'(t_*)\mathrm{e}^{c(t_*-\xi)} \quad \forall \xi \in [t_0, t_*]$$

and thus, integrating on $[t_0, t_*]$ and using (3.4), we obtain

$$\rho - u(t_*) \leqslant |u'(t_*)|\mathrm{e}^{|c|T}(t_* - t_0) \leqslant \frac{u(t_*)}{\varepsilon}\mathrm{e}^{2|c|T}T.$$

This gives (3.3) for

$$\delta := \frac{\varepsilon}{\varepsilon + \mathrm{e}^{2|c|T}T}.$$

We get exactly the same estimate in the case of (3.8), by using (3.6) and then integrating on $[t_*, t_0]$. Observe that the constant $\delta \in \,]0, 1[$ does not depend on λ or α.

Having found the constant δ, we now define

$$\eta = \eta(\rho) := \min\{g(s) \colon s \in [\delta\rho, \rho]\}.$$

Then, integrating (2.4) on $[\sigma_0, \tau_0]$ and using (3.2) (for $t = \sigma_0$ and $t = \tau_0$), we obtain

$$\begin{aligned}\lambda\eta \int_{\sigma_0}^{\tau_0} a(t)\,\mathrm{d}t &\leqslant \lambda \int_{\sigma_0}^{\tau_0} a(t)g(u(t))\,\mathrm{d}t \\ &= u'(\sigma_0) - u'(\tau_0) + c(u(\sigma_0) - u(\tau_0)) - \alpha(\tau_0 - \sigma_0) \\ &\leqslant 2\frac{\rho}{\varepsilon}\mathrm{e}^{|c|T} + 2|c|\rho.\end{aligned}$$

Now, we define

$$\lambda^* := \frac{2\rho(\varepsilon|c| + \mathrm{e}^{|c|T})}{\varepsilon\eta \int_{\sigma_0}^{\tau_0} a(t)\,\mathrm{d}t}. \tag{3.9}$$

Arguing by contradiction, we immediately conclude that there are no (non-negative) T-periodic solutions $u(t)$ of (2.4) with $\max_{t\in I} u(t) = \rho$ if $\lambda > \lambda^*$. Thus, condition $(A_{\rho,I})$ is proved.

STEP 3 (verification of $(B_{\rho,I})$). Let $u(t)$ be any non-negative T-periodic solution of (2.4) with $\max_{t\in I} u(t) \leqslant \rho$. Let us fix an instant $\hat{t} \in [\sigma_0, \tau_0]$. By (3.2), we know that

$$|u'(\hat{t})| \leqslant \frac{\rho}{\varepsilon}\mathrm{e}^{|c|T}.$$

Using the fact that

$$|h(t,s)| \leqslant M(t)|s| + N(t) \quad \text{for a.e. } t \in [0,T],\ \forall s \in \mathbb{R},\ \forall \alpha \in [0,\beta],$$

with suitable $M, N \in L^1_T$ (depending on β), from a standard application of the (generalized) Gronwall inequality (cf. [17]), we find a constant $D_\beta = D_\beta(\rho, \lambda)$ such that

$$\max_{t\in[0,T]} (|u(t)| + |u'(t)|) \leqslant D_\beta.$$

So condition $(B_{\rho,I})$ is verified.

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Step 4 (verification of $(C_{\rho,I})$). Let $u(t)$ be an arbitrary non-negative T-periodic solution of (2.4) with $\max_{t\in I} u(t) \leqslant \rho$. Integrating (2.4) on $[\sigma_0, \tau_0]$ and using (3.2) (for $t = \sigma_0$ and $t = \tau_0$), we obtain

$$\begin{aligned}\alpha(\tau_0 - \sigma_0) &= u'(\sigma_0) - u'(\tau_0) + c(u(\sigma_0) - u(\tau_0)) - \lambda\biggl(\int_{\sigma_0}^{\tau_0} a(t)g(u(t))\,\mathrm{d}t\biggr)\\ &\leqslant 2\frac{\rho}{\varepsilon}\mathrm{e}^{|c|T} + 2|c|\rho =: K = K(\rho,\varepsilon).\end{aligned}$$

This yields a contradiction if $\alpha > 0$ is sufficiently large. Hence, $(C_{\rho,I})$ is verified by taking $\alpha^* > K/(\tau_0 - \sigma_0)$.

In conclusion, all the assumptions of lemma 2.1 have been verified for a fixed $\rho > 0$ and for $\lambda > \lambda^*$.

Remark 3.1. Note that, of the assumptions of theorem 1.1, in this part of the proof we have used only the following: $g(s) > 0$ for all $s \in \,]0,\rho]$, $\limsup_{s\to+\infty} |g(s)|/s < +\infty$, $a \in L^1_T$ and $a(t) \geqslant 0$ for a.e. $t \in I$, with $\int_I a(t)\,\mathrm{d}t > 0$.

3.2. Checking the assumptions of lemma 2.2 for r small

We prove that condition (H_d) of lemma 2.2 is satisfied for $d = r$ sufficiently small. Indeed, we claim that there exists $r_0 > 0$ such that there is no non-negative T-periodic solution $u(t)$ of (2.6) for some $\vartheta \in \,]0,1]$ with $\|u\|_\infty = r \in \,]0, r_0]$. Arguing by contradiction, we suppose that there exists a sequence of T-periodic functions $u_n(t)$ with $u_n(t) \geqslant 0$ for all $t \in \mathbb{R}$ and such that

$$u_n''(t) + cu_n'(t) + \vartheta_n\lambda a(t)g(u_n(t)) = 0 \tag{3.10}$$

for a.e. $t \in \mathbb{R}$ with $\vartheta_n \in \,]0,1]$, and also such that $\|u_n\|_\infty = r_n \to 0^+$. Let $t_n^* \in [0,T]$ be such that $u_n(t_n^*) = r_n$.

We define

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty} = \frac{u_n(t)}{r_n}$$

and observe that (3.10) can equivalently be written as

$$v_n''(t) + cv_n'(t) + \vartheta_n\lambda a(t)q(u_n(t))v_n(t) = 0, \tag{3.11}$$

where $q\colon \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $q(s) := g(s)/s$ for $s > 0$ and $q(0) = 0$. Note that q is continuous on $\mathbb{R}^+$ (by (g_0)). Moreover, $q(u_n(t)) \to 0$ uniformly in $\mathbb{R}$, as a consequence of $\|u_n\|_\infty \to 0$. Multiplying (3.11) by v_n and integrating on $[0,T]$, we find

$$\|v_n'\|^2_{L^2_T} = \int_0^T v_n'(t)^2\,\mathrm{d}t \leqslant \lambda\|a\|_{L^1_T} \sup_{t\in[0,T]} |q(u_n(t))| \to 0 \quad \text{as } n \to \infty.$$

As an easy consequence, $\|v_n - 1\|_\infty \to 0$ as $n \to \infty$.

Integrating (3.10) on $[0,T]$ and using the periodic boundary conditions, we have

$$0 = \int_0^T a(t)g(u_n(t))\,\mathrm{d}t = \int_0^T a(t)g(r_n)\,\mathrm{d}t + \int_0^T a(t)(g(r_nv_n(t)) - g(r_n))\,\mathrm{d}t$$

and hence, dividing by $g(r_n) > 0$, we obtain

$$0 < -\int_0^T a(t)\,\mathrm{d}t \leqslant \|a\|_{L^1_T} \sup_{t\in[0,T]} \left| \frac{g(r_n v_n(t))}{g(r_n)} - 1 \right|.$$

Using the fact that $g(s)$ is regularly oscillating at zero and $v_n(t) \to 1$ uniformly as $n \to \infty$, we find that the right-hand side of the above inequality tends to zero and thus we achieve a contradiction.

REMARK 3.2. Note that, among the assumptions of theorem 1.1, in this part of the proof we have used only the following ones (for verifying (H_r)): $g(s) > 0$ for all s in a right neighbourhood of $s = 0$, $g(s)$ regularly oscillating at zero and satisfying (g_0), $a \in L^1_T$ with $\int_0^T a(t)\,\mathrm{d}t < 0$.

3.3. Checking the assumptions of lemma 2.2 for R large

We shall check that condition (H_d) of lemma 2.2 is satisfied for $d = R$ sufficiently large. In other words, we claim that there exists $R_0 > 0$ such that there is no non-negative T-periodic solution $u(t)$ of (2.6) for some $\vartheta \in \,]0,1]$ with $\|u\|_\infty = R \geqslant R_0$. Arguing by contradiction, we suppose that there exists a sequence of T-periodic functions $u_n(t)$ with $u_n(t) \geqslant 0$ for all $t \in \mathbb{R}$ and such that

$$u_n''(t) + cu_n'(t) + \vartheta_n \lambda a(t) g(u_n(t)) = 0 \tag{3.12}$$

for a.e. $t \in \mathbb{R}$ with $\vartheta_n \in \,]0,1]$, and also such that $\|u_n\|_\infty = R_n \to +\infty$. Let $t_n^* \in [0,T]$ be such that $u_n(t_n^*) = R_n$.

First, we claim that $u_n(t) \to +\infty$ uniformly in t (as $n \to \infty$). Indeed, to be more precise, we have that $u_n(t) \geqslant \frac{1}{2}R_n$ for all t. To prove this assertion, let us suppose, by contradiction, that $\min u_n(t) < \frac{1}{2}R_n$. In this case, we can take a maximal compact interval $[\alpha_n, \beta_n]$ containing t_n^* and such that $u_n(t) \geqslant \frac{1}{2}R_n$ for all $t \in [\alpha_n, \beta_n]$. By the maximality of the interval, we also have that $u_n(\alpha_n) = u_n(\beta_n) = \frac{1}{2}R_n$ with $u_n'(\alpha_n) \geqslant 0 \geqslant u_n'(\beta_n)$.

We set

$$w_n(t) := u_n(t) - \tfrac{1}{2}R_n$$

and observe that $0 \leqslant w_n(t) \leqslant \frac{1}{2}R_n$ for all $t \in [\alpha_n, \beta_n]$. Equation (3.12) reads equivalently as

$$-w_n''(t) - cw_n'(t) = \vartheta_n \lambda a(t) g(u_n(t)).$$

Multiplying this equation by $w_n(t)$ and integrating on $[\alpha_n, \beta_n]$, we obtain

$$\int_{\alpha_n}^{\beta_n} w_n'(t)^2\,\mathrm{d}t \leqslant \lambda \|a\|_{L^1_T} \tfrac{1}{2}R_n \sup_{R_n/2 \leqslant s \leqslant R_n} |g(s)|.$$

From condition (g_∞), for any fixed $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that $|g(s)| \leqslant \varepsilon s$ for all $s \geqslant L_\varepsilon$. Thus, for n sufficiently large that $R_n \geqslant 2L_\varepsilon$, we find

$$\int_{\alpha_n}^{\beta_n} w_n'(t)^2\,\mathrm{d}t \leqslant \tfrac{1}{2}\lambda\varepsilon R_n^2 \|a\|_{L^1_T}.$$

By an elementary form of the Poincaré–Sobolev inequality, we conclude that

$$\tfrac{1}{4}R_n^2 = \max_{t\in[\alpha_n,\beta_n]} |w_n(t)|^2 \leqslant T\int_{\alpha_n}^{\beta_n} w_n'(t)^2\,\mathrm{d}t \leqslant \tfrac{1}{2}\lambda\varepsilon T R_n^2 \|a\|_{L^1_T}$$

and a contradiction is achieved if we take ε sufficiently small.

Consider now the auxiliary function

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty} = \frac{u_n(t)}{R_n}$$

and divide (3.12) by R_n. In this manner we again obtain (3.11). By (g_∞) and the fact that $u_n(t) \to +\infty$ uniformly in t, we conclude that $q(u_n(t)) = g(u_n(t))/u_n(t) \to 0$ uniformly (as $n \to \infty$). Hence, we are exactly in the same situation as in the case in § 3.2 for r small and we can end the proof in a similar way. More precisely, $\|v_n'\|_{L^2_T} \to 0$ as $n \to \infty$ (this follows by multiplying (3.11) by $v_n(t)$ and integrating on $[0,T]$) so that $\|v_n - 1\|_\infty \to 0$, as $n \to \infty$. Then, integrating (3.12) on $[0,T]$ and dividing by $g(R_n) > 0$, we obtain

$$0 < -\int_0^T a(t)\,\mathrm{d}t \leqslant \|a\|_{L^1_T} \sup_{t\in[0,T]} \left|\frac{g(R_n v_n(t))}{g(R_n)} - 1\right|.$$

Using the fact that $g(s)$ is regularly oscillating at infinity and $v_n(t) \to 1$ uniformly as $n \to \infty$, we find that the right-hand side of the above inequality tends to zero and thus we achieve a contradiction.

Remark 3.3. Note that, of the assumptions of theorem 1.1, in this part of the proof we have used only the following (for verifying (H_R)): $g(s) > 0$ for all s in a neighbourhood of infinity, $g(s)$ regularly oscillating at infinity and satisfying (g_∞), $a \in L^1_T$ with $\int_0^T a(t)\,\mathrm{d}t < 0$.

4. Related results

In this section we present some consequences and variants obtained from theorem 1.1. We also examine the cases of non-existence of solutions when the parameter λ is small.

4.1. Proof of corollary 1.2

In order to deduce corollary 1.2 from theorem 1.1, we stress the fact that the constant $\lambda^* > 0$ (defined in (3.9)) is produced along the proof of lemma 2.1 in dependence of an interval $I \subseteq [0,T]$ where $a(t) \geqslant 0$ and $\int_I a(t)\,\mathrm{d}t > 0$. For this step in the proof we do not need any information about the weight function on $[0,T] \setminus I$. As a consequence, when we apply our result to (1.3), we have that λ^* can be chosen independently on μ. On the other hand, for lemma 2.2 with r small as well as with R large, we do not need any special condition on λ (except that λ in (3.10) or in (3.12) is fixed) and we use only the fact that $\int_0^T a(t)\,\mathrm{d}t < 0$ (without requiring any other information on the sign of $a(t)$). Accordingly, once $\lambda > \lambda^*$ is fixed, to obtain a pair of positive T-periodic solutions we only need to check that the integral of the weight function on $[0,T]$ is negative. For (1.3) this condition is equivalent to

$$\frac{\mu}{\lambda} > \frac{\int_0^T a^+(t)\,\mathrm{d}t}{\int_0^T a^-(t)\,\mathrm{d}t}.$$

By using these remarks, we immediately deduce corollary 1.2 from theorem 1.1.

4.2. Existence of small and large solutions

Theorem 1.1 guarantees the existence of at least two positive T-periodic solutions of (1.1). More precisely, we have found a first solution in $\Omega_{\rho,I} \setminus B[0,r]$ and a second one in $B(0,R) \setminus \mathrm{cl}(\Omega_{\rho,I} \cap B(0,R_0))$, verifying that the coincidence degree is non-zero in these sets (see (2.7) and (2.8)). The positivity of both solutions follows from maximum-principle arguments. A careful reading of the proof (cf. § 3) shows that weaker conditions on $g(s)$ are sufficient to repeat some of the steps in § 2.2 in order to prove (2.7) (respectively, (2.8)) and thus obtain the existence of a small (respectively, large) positive T-periodic solution of (1.1).

More precisely, taking into account remarks 3.1 and 3.2 we can state the following theorem, ensuring the existence of a small positive T-periodic solution.

THEOREM 4.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and*

$$\limsup_{s\to+\infty} \frac{g(s)}{s} < +\infty. \tag{4.1}$$

Suppose also that g is regularly oscillating at zero and satisfies (g_0). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition (a_*). Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geqslant 0$ for a.e. $t \in I$ and $\int_I a(t)\,\mathrm{d}t > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ (1.1) has at least one positive T-periodic solution.

On the other hand, in view of remarks 3.1 and 3.3 we have the following result, giving the existence of a large positive T-periodic solution.

THEOREM 4.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and*

$$\limsup_{s\to 0^+} \frac{g(s)}{s} < +\infty. \tag{4.2}$$

Suppose also that g is regularly oscillating at infinity and satisfies (g_∞). Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition (a_*). Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geqslant 0$ for a.e. $t \in I$ and $\int_I a(t)\,\mathrm{d}t > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ (1.1) has at least one positive T-periodic solution.

Note that the possibility of applying a strong maximum principle (in order to obtain positive solutions) is ensured by (g_0) in theorem 4.1, while it follows by (4.2) in theorem 4.2. The dual condition (4.1) in theorem 4.1 is, on the other hand, needed to apply Gronwall's inequality (checking the assumptions of lemma 2.1).

4.3. Smoothness versus regular oscillation

It can be observed that the assumptions of regular oscillation of $g(s)$ at zero (respectively, at infinity) can be replaced by suitable smoothness assumptions. Indeed, we can provide an alternative way to check the assumptions of lemma 2.2 for r small or R large, by assuming that $g(s)$ is smooth in a neighbourhood of zero (respectively, infinity). For this purpose, we present some preliminary considerations.

Let $u(t)$ be a positive and T-periodic solution of

$$u'' + cu' + \nu a(t)g(u) = 0, \tag{4.3}$$

where $\nu > 0$ is a given parameter (in the following, we shall take $\nu = \lambda$ or $\nu = \vartheta\lambda$). Suppose that the map $g(s)$ is continuously differentiable on an interval containing the range of $u(t)$. In such a situation, we can perform the change of variable

$$z(t) := \frac{u'(t)}{\nu g(u(t))} \tag{4.4}$$

and observe that $z(t)$ satisfies

$$z' + cz = -\nu g'(u(t))z^2 - a(t). \tag{4.5}$$

The function $z(t)$ is absolutely continuous, T-periodic with $\int_0^T z(t)\,\mathrm{d}t = 0$ and, moreover, there exists a $t^* \in [0, T]$ such that $z(t^*) = 0$.

This change of variables (recently also considered in [9]) is used to provide a non-existence result as well as *a priori* bounds for the solutions. We first state the following result.

Lemma 4.3. *Let $J \subseteq \mathbb{R}$ be an interval. Let $g\colon J \to \mathbb{R}_0^+$ be a continuously differentiable function with bounded derivative (on J). Let $a \in L_T^1$ satisfy (a_*). Then there exists $\omega_* > 0$ such that if*

$$\nu \sup_{s\in J} |g'(s)| < \omega_*,$$

there are no T-periodic solutions of (4.3) with $u(t) \in J$ for all $t \in \mathbb{R}$.

Proof. For notational convenience, let us set

$$D := \sup_{s\in J} |g'(s)|.$$

First, we fix a positive constant $M > \mathrm{e}^{|c|T}\|a\|_{L_T^1}$ and define

$$\omega_* := \min\left\{\frac{M - \mathrm{e}^{|c|T}\|a\|_{L_T^1}}{M^2 T \mathrm{e}^{|c|T}}, \frac{-\int_0^T a(t)\,\mathrm{d}t}{M^2 T}\right\}.$$

Note that ω_* does not depend on ν, J and D. We shall prove that if

$$0 < \nu D < \omega_*,$$

then (4.3) has no T-periodic solution $u(t)$ with range in J.

By contradiction we suppose that $u(t)$ is a solution of (4.3) with $u(t) \in J$ for all $t \in \mathbb{R}$. Setting $z(t)$ as in (4.4), we claim that

$$\|z\|_\infty \leqslant M. \tag{4.6}$$

Indeed, if by contradiction we suppose that (4.6) is not true, then, using the fact that $z(t)$ vanishes at some point of $[0,T]$, we can find a maximal interval $\mathcal{I}$ of the form $[t^*,\tau]$ or $[\tau,t^*]$ such that $|z(t)| \leqslant M$ for all $t \in \mathcal{I}$ and $|z(t)| > M$ for some $t \notin \mathcal{I}$. By the maximality of the interval $\mathcal{I}$, we also know that $|z(\tau)| = M$. Multiplying (4.5) by $\mathrm{e}^{c(t-\tau)}$ yields

$$(z(t)\mathrm{e}^{c(t-\tau)})' = (-\nu g'(u(t))z^2(t) - a(t))\mathrm{e}^{c(t-\tau)}.$$

Then, by integrating on $\mathcal{I}$ and passing to the absolute value, we obtain

$$\begin{aligned}
M = |z(\tau)| &= |z(\tau) - z(t^*)\mathrm{e}^{c(t^*-\tau)}| \\
&\leqslant \left| \int_{\mathcal{I}} \nu g'(u(t))z^2(t)\,\mathrm{d}t \right| \mathrm{e}^{|c|T} + \|a\|_{L^1_T}\mathrm{e}^{|c|T} \\
&\leqslant \nu D M^2 T\mathrm{e}^{|c|T} + \|a\|_{L^1_T}\mathrm{e}^{|c|T} \\
&< \omega_* M^2 T\mathrm{e}^{|c|T} + \|a\|_{L^1_T}\mathrm{e}^{|c|T} \\
&\leqslant M,
\end{aligned}$$

a contradiction. Thus, we have verified that (4.6) is true.

Now, integrating (4.5) on $[0,T]$ and using (4.6), we reach

$$0 < -\int_0^T a(t)\,\mathrm{d}t = \int_0^T \nu g'(u(t))z^2(t)\,\mathrm{d}t < \omega_* M^2 T \leqslant -\int_0^T a(t)\,\mathrm{d}t,$$

a contradiction. This concludes the proof. □

The same change of variable is employed to provide the following variant of theorem 1.1.

Theorem 4.4. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and such that $g(s)$ is continuously differentiable on a right neighbourhood of $s = 0$ and on a neighbourhood of infinity. Suppose also that (g_0) and*

$$(g'_\infty) \qquad g'(\infty) := \lim_{s\to+\infty} g'(s) = 0$$

hold. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying the average condition (a_*). Furthermore, suppose that there exists an interval $I \subseteq [0,T]$ such that $a(t) \geqslant 0$ for a.e. $t \in I$ and $\int_I a(t)\,\mathrm{d}t > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ (1.1) has at least two positive T-periodic solutions.

Proof. We follow the scheme described in § 2.2. The verification of the assumptions of lemma 2.1 for λ large is exactly the same as in § 3.1. We just describe the changes with respect to §§ 3.2 and 3.3. It is important to emphasize that $\lambda > \lambda^*$ is fixed from now on.

Verification of the assumption of lemma 2.2 for r small. Let $[0, \varepsilon_0[$ be a right neighbourhood of 0 where g is continuously differentiable. We claim that there exists $r_0 \in]0, \varepsilon_0[$ such that for all $0 < r \leqslant r_0$ and for all $\vartheta \in]0, 1]$ there are no non-negative T-periodic solutions $u(t)$ of (2.6) such that $\|u\|_\infty = r$.

First, we observe that any non-negative T-periodic solution $u(t)$ of (2.6), with $\|u\|_\infty = r$, is positive. This follows either by the uniqueness of the trivial solution (due to the smoothness of $g(s)$ in $[0, \varepsilon_0[$), or by an elementary form of the strong maximum principle. Thus, we have to prove that there are no T-periodic solutions $u(t)$ of (2.6) with range in the interval $]0, r]$ (for all $0 < r \leqslant r_0$).

We apply lemma 4.3 to the present situation with $\nu = \vartheta\lambda$ and $J =]0, r]$. There exists a constant $\omega_* > 0$ (independent of r) such that there are no T-periodic solutions with range in $]0, r]$ if

$$\sup_{0<s\leqslant r} |g'(s)| = \max_{0\leqslant s\leqslant r} |g'(s)| < \frac{\omega_*}{\lambda}$$

(recall that $0 < \vartheta \leqslant 1$). This latter condition is clearly satisfied for every $r \in]0, r_0]$, with $r_0 > 0$ suitably chosen using the continuity of $g'(s)$ at $s = 0^+$.

Verification of the assumption of lemma 2.2 for R large. Let $]N, +\infty[$ be a neighbourhood of infinity where g is continuously differentiable. As in § 3.3, we argue by contradiction. Suppose that there exists a sequence of non-negative T-periodic functions $u_n(t)$ satisfying (3.12) and such that $\|u_n\|_\infty = R_n \to +\infty$. By the argument developed in § 3.3, we find that $u_n(t) \geqslant \frac{1}{2}R_n$ for all $t \in \mathbb{R}$ (for n sufficiently large). Note that for this part of the proof we require condition (g_∞), but we do not need the hypothesis of regular oscillation at infinity. Clearly, (g_∞) is implied by (g'_∞).

For n sufficiently large (such that $R_n > 2N$), we apply lemma 4.3 to the present situation with $\nu = \nu_n := \vartheta_n\lambda$ and $J = J_n := [\frac{1}{2}R_n, R_n]$. There exists a constant $\omega_* > 0$ (independent of n) such that there are no T-periodic solutions with range in J_n if

$$\max_{R_n/2\leqslant s\leqslant R_n} |g'(s)| < \frac{\omega_*}{\lambda}$$

(recall that $0 < \vartheta_n \leqslant 1$). This latter condition is clearly satisfied for every n sufficiently large as a consequence of condition (g'_∞). The desired contradiction is thus achieved. □

Remark 4.5. Clearly, one can easily produce two further theorems by combining the assumptions of regular oscillation at zero (at infinity) with the smoothness condition at infinity (at zero).

4.4. Non-existence results

In the proof of theorem 4.4 we applied lemma 4.3 to intervals of the form $]0, r]$ or $[\frac{1}{2}R_n, R_n]$ in order to check the assumptions of lemma 2.2. Clearly, one could apply such a lemma to the whole interval $\mathbb{R}^+_0$ of positive real numbers. In this manner, we can easily provide a non-existence result of positive T-periodic solutions to (1.1) when $g'(s)$ is bounded in $\mathbb{R}^+_0$ and λ is small. In this respect, the following result holds.

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THEOREM 4.6. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*), (g_0) and (g'_∞). Let $a \in L^1_T$ satisfy (a_*). Then there exists $\lambda_* > 0$ such that for each $0 < \lambda < \lambda_*$ (1.1) has no positive T-periodic solution.*

Proof. First, we observe that g' is bounded on $\mathbb{R}^+_0$ (since $g(s)$ is continuously differentiable in $\mathbb{R}^+$ with $g'(0) = g'(\infty) = 0$). Accordingly, let us set

$$D := \max_{s \geqslant 0} |g'(s)|.$$

We now apply lemma 4.3 to (1.1) for $J = \mathbb{R}^+_0$. This lemma guarantees the existence of a constant $\omega_* > 0$ such that if $0 < \lambda < \omega_*/D$, (1.1) has no positive T-periodic solution. This ensures the existence of a suitable constant $\lambda_* \geqslant \omega_*/D$, as claimed in the statement of the theorem. □

At this point, theorem 1.3 is a straightforward consequence of theorems 4.4 and 4.6.

5. Neumann boundary conditions

In this final section we briefly describe how to obtain the preceding results for the Neumann BVP. For simplicity, we deal with the case $c = 0$. If $c \neq 0$, we can write (1.1) as

$$(u'\mathrm{e}^{ct})' + \lambda\tilde{a}(t)g(u) = 0 \quad \text{with } \tilde{a}(t) := a(t)\mathrm{e}^{ct},$$

and enter the setting of coincidence degree theory for the linear operator $L\colon u \mapsto -(u'\mathrm{e}^{ct})'$. Accordingly, we consider the BVP

$$\left.\begin{aligned} u'' + \lambda a(t)g(u) = 0,\\ u'(0) = u'(T) = 0, \end{aligned}\right\} \tag{5.1}$$

where $a\colon [0,T] \to \mathbb{R}$ and $g(s)$ satisfy the same conditions as in the previous sections. In this case, the abstract setting of § 2 can be reproduced almost verbatim with $X := \mathcal{C}([0,T])$, $Z := L^1([0,T])$ and $L\colon u \mapsto -u''$, by taking

$$\operatorname{dom} L := \{u \in W^{2,1}([0,T])\colon u'(0) = u'(T) = 0\}.$$

With the above positions $\ker L \cong \mathbb{R}$, $\operatorname{Im} L$ and the projectors P and Q are exactly the same as in § 2. All the results given up to § 4 can now be restated for problem (5.1). In particular, we again obtain theorems 1.1, 4.4 and 4.6, as well as their corollaries for (1.1) (with $c = 0$) and the Neumann boundary conditions.

We now present a consequence of these results in the study of a partial differential equation in an annular domain. In order to simplify the exposition of the next results, we assume the continuity of the weight function. In this way, the solutions we find are the 'classical' ones (at least twice continuously differentiable).

5.1. Radially symmetric solutions

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^N$ (for $N \geqslant 2$) and let

$$\Omega := B(0,R_2) \setminus B[0,R_1] = \{x \in \mathbb{R}^N\colon R_1 < \|x\| < R_2\}$$

be an open annular domain, with $0 < R_1 < R_2$.

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We deal with the Neumann BVP

$$\left.\begin{aligned} -\Delta u = \lambda q(x) g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.2}$$

where $q \colon \bar{\Omega} \to \mathbb{R}$ is a continuous function that is radially symmetric; namely, there exists a continuous scalar function $\mathcal{Q} \colon [R_1, R_2] \to \mathbb{R}$ such that

$$q(x) = \mathcal{Q}(\|x\|) \quad \forall x \in \bar{\Omega}.$$

We look for existence/non-existence and multiplicity of radially symmetric positive solutions of (5.2) that are classical solutions such that $u(x) > 0$ for all $x \in \Omega$ and also $u(x) = \mathcal{U}(\|x\|)$, where $\mathcal{U}$ is a scalar function defined on $[R_1, R_2]$.

Accordingly, our study can be reduced to the search for positive solutions of the Neumann BVP

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \lambda \mathcal{Q}(r) g(\mathcal{U}(r)) = 0, \qquad \mathcal{U}'(R_1) = \mathcal{U}'(R_2) = 0. \tag{5.3}$$

Using the standard change of variable

$$t = h(r) := \int_{R_1}^{r} \xi^{1-N} \, \mathrm{d}\xi$$

and defining

$$T := \int_{R_1}^{R_2} \xi^{1-N} \, \mathrm{d}\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = \mathcal{U}(r(t)),$$

we transform (5.3) into the equivalent problem

$$v'' + \lambda a(t) g(v) = 0, \qquad v'(0) = v'(T) = 0, \tag{5.4}$$

with

$$a(t) := r(t)^{2(N-1)} \mathcal{Q}(r(t)).$$

Consequently, the Neumann BVP (5.4) is in the same form as (5.1) and we can apply the previous results.

Note that condition (a_*) reads as

$$0 > \int_0^T r(t)^{2(N-1)} \mathcal{Q}(r(t)) \, \mathrm{d}t = \int_{R_1}^{R_2} r^{N-1} \mathcal{Q}(r) \, \mathrm{d}r.$$

Up to a multiplicative constant, the latter integral is the integral of $q(x)$ on Ω, using the change of variable formula for radially symmetric functions. Thus, $a(t)$ satisfies (a_*) if and only if

(q_*) $\displaystyle\int_{\Omega} q(x) \, \mathrm{d}x < 0.$

The analogue of theorem 1.1 for problem (5.2) now becomes the following.

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Theorem 5.1. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*). Suppose also that g is regularly oscillating at zero and at infinity and satisfies (g_0) and (g_∞). Let $q(x)$ be a continuous (radial) weight function as above satisfying (q_*) and such that $q(x_0) > 0$ for some $x_0 \in \Omega$. Then there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ problem (5.2) has at least two positive radially symmetric solutions.*

Similarly, if we replace the regularly oscillating conditions with the smoothness assumptions, by theorems 4.4 and 4.6 we obtain the next result.

Theorem 5.2. *Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuously differentiable function satisfying (g_*), (g_0) and (g'_∞). Let $q(x)$ be a continuous (radial) weight function, as above, satisfying (q_*) and such that $q(x_0) > 0$ for some $x_0 \in \Omega$. Then there exist two positive constants $\lambda_* \leqslant \lambda^*$ such that for each $0 < \lambda < \lambda_*$ there are no positive radially symmetric solutions for problem (5.2), while for each $\lambda > \lambda^*$ there exist at least two positive radially symmetric solutions. Moreover, if $g'(s) > 0$ for all $s > 0$, then condition (q_*) is also necessary.*

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