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A MEASURE OF NON-IMMERSABILITY OF THE GRASSMANN MANIFOLDS IN SOME EUCLIDEAN SPACES

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Let $G_{k,n}$ be the Grassmann manifold consisting in all non-oriented k-dimensional vector subspaces of the space $\mathbf{R}^{k+n}$. In this paper we will show that any differentiable mapping $f : G_{k,n} \to \mathbf{R}^m$, has infinitely many critical points for suitable choices of the numbers m, n, k.

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1. Introduction

Recall that $G_{k,n}$ is a compact manifold of dimension kn and that the manifold $G_{1,n}$ is just the real projective space $P_n(\mathbf{R})$.

In the paper [4] it is proved that the Grassmann manifolds $G_{2,n}$ and $G_{2,s-1}$, where $s = 2^r$ is such that $2^{r-1} \leq n < 2^r$, cannot be immersed in the euclidean spaces $\mathbf{R}^{2s-3}$ and $\mathbf{R}^{3s-3}$ respectively. This means that any differentiable mapping $f : G_{2,n} \to \mathbf{R}^{2s-3}$ or $g : G_{2,s-1} \to \mathbf{R}^{3s-3}$, has one critical point at least. This observation justifies the investigations on the cardinal number

$$\varphi(M, N) = \min\{|C(f)| : f \in C^\infty(M, N)\},$$

called the φ-category of the pair (M, N) of the differentiable manifolds M and N. The φ-category of the pair (M, N) represents a measure of non-immersability of the manifold M into the manifold N if $dim\, M < dim\, N$, and it is a measure of the distance of the pair (M, N) from a fibration of the manifold M over N, if $dim\, M \geq dim\, N$ and M, N are compact manifolds. If $|C(f)|$ is infinite for all $f \in C^\infty(M, N)$, we shall use the notation $\varphi(M, N) = \infty$. In the present paper the φ-category of the pairs $(G_{2,n}, \mathbf{R}^m)$, $(G_{3,n}, \mathbf{R}^m)$ and $(P_n(\mathbf{R}), \mathbf{R}^m)$ will be studied.

2. Preliminary results

The following theorem is the principal result of the paper.

Theorem 2.1. *Let M^m, N^n be smooth manifolds such that $m < n$ and $f : M \to N$ be* *an immersion. If $y \in Im f$ is such that $f^{-1}(y)$ is finite, then there exists an immersion $g : M \to N \setminus \{y\}$.*

Proof. Supposing that $f^{-1}(y) = \{x_1, \ldots, x_p\}$, there exists the local charts (U_i, φ_i), (V_i, ψ_i), $i \in \{1, 2, \ldots, p\}$ and the real positive number r, such that

(i) $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$;

(ii) $y \in \bigcap_{i=1}^{p} V_i$, $x_i \in U_i$, $\varphi(x_i) = 0$, $\psi_i(y) = 0$ $(\forall)$ $i \in \{1, 2, \ldots, p\}$;

(iii) If D_φ^s denotes the pre-image of the open disk $D = \{x \in \mathbf{R}^k \mid \|x\| < s\}$ $(k \in \{m, n\})$ by a coordinate mapping $\varphi : U \to \mathbf{R}^k$ with $\varphi(0) = 0$ and $D \subseteq \varphi(U)$, then $\bar{D}_{\varphi_i}^{2r} \subseteq U_i$ and $\bar{D}_{\varphi_i}^{2r} \subseteq \bigcap_{i=1}^{p} V_i$, $(\forall)$ $i \in \{1, 2, \ldots, p\}$;

(iv) $(\psi_i \circ f \circ \varphi_i^{-1})(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m \text{ times}})$ $(\forall)$ $i \in \{1, 2, \ldots, p\}$.

Consider the smooth positive functions $\theta_i : N \to \mathbf{R}$ which has the properties $\theta_i^{-1}(0) = N \setminus D_{\varphi_i}^r$ and the smooth vector fields $X_1, X_2, \ldots, X_p$ which are defined on N by

$$X_i(z) = \begin{cases} \theta_i(z) \frac{\partial \psi_i}{\partial x_n}\big|_z & \text{if } z \in D_{\psi_i}^{2r}, \\ 0 & \text{if } N \setminus D_{\psi_i}^r. \end{cases}$$

Obviously the norms $\|X_1\|, \ldots, \|X_p\|$ of the fields $X_1, X_2, \ldots, X_p$ are bounded with respect to any Riemannian metric on N, namely they are completely integrable (see **[5**, pp. 183]). Denote by α_t^i the global flow induced by X_i and consider the projection $\beta : \mathbf{R}^n \to \mathbf{R}$, $\beta(x_1, \ldots, x_n) = x_n$. Observe that

$$(\beta \circ \psi_i \circ f \circ \varphi_i^{-1})(x_1, \ldots, x_m) = 0 \ (\forall) \ x = (x_1, \ldots, x_m) \in \varphi_i(U_i).$$

One can therefore say that

$$(\beta \circ \psi_i \circ f)(x) = 0 \ (\forall) x \in D_{\varphi_i}^{2r}.$$

Define the mapping g in the following way:

$$g(x) = \begin{cases} \alpha_1^1(f(x)) & \text{if } x \in D_{\varphi_1}^{2r} \\ \vdots & \vdots \\ \alpha_1^p(f(x)) & \text{if } x \in D_{\varphi_p}^{2r} \\ f(x) & \text{if } x \in M \setminus \bigcup_{i=1}^{p} D_{\varphi_i}^r \end{cases}$$

Because $\alpha_2^1, \ldots, \alpha_p^1$ are diffeomorphisms and f is an immersion, it follows that g is also an immersion. It remains only to show that $y \notin Im g$, that is, $\beta(\psi_i(g(x))) > 0$ $(\forall)$ $x \in D_{\varphi_i}^{2r}$ and $(\forall)$ $i \in \{1, 2, \ldots, p\}$. Further on, we have successively

$$\frac{d}{dt}[\psi_i(\alpha_t^i(y))] = (d\psi_i)_{\alpha_t^i(y)}\left(\frac{d}{dt}\alpha_t^i(y)\right) = (d\psi_i)_{\alpha_t^i(y)}(X_i(\alpha_t^i(y))) = (d\psi_i)_{\alpha_t^i(y)}\left(\theta_i(\alpha_t^i(y))\frac{\partial^{\psi_i}}{\partial x_n}\bigg|_{\alpha_t^i(y)}\right)$$
$$= \theta_i(\alpha_t^i(y))(d\psi_i)_{\alpha_t^i(x)}\left(\frac{\partial^{\psi_i}}{\partial x_n}|_{\alpha_t^i(y)}\right) = \theta_i(\alpha_t^i(y))e_n = (0, \ldots, 0, \theta_i(\alpha_t^i(y))).$$

Hence for $x \in D^r_{\varphi_i}$ we have

$$\beta(\psi_i(g(x))) = \beta(\psi_i(\alpha_1^i(f(x)))) = \int_0^1 \theta_i(\alpha_s^i(f(x)))ds > 0. \qquad \square$$

Remark The mapping g constructed above is homotopic to f relative to the set $M \setminus \bigcup_{i=1}^k D^r_{\varphi_i}$. More precisely we have the relation

$$f \simeq_H g\left(\text{rel } M \setminus \bigcup_{i=1}^k D^r_{\varphi_i}\right)$$

where $H : [0, 1] \times M \to N$ is given by

$$H(t, x) = \begin{cases} \alpha_t^1(f(x)) & \text{if } x \in D^{2r}_{\varphi_1} \\ \vdots & \vdots \\ \alpha_t^p(f(x)) & \text{if } x \in D^{2r}_{\varphi_k} \\ f(x) & \text{if } x \in M \setminus \bigcup_{i=1}^p D^r_{\varphi_k}. \end{cases}$$

Corollary 2.2. *Let M^m, N^n be smooth manifolds such that M is compact and $m < n$. If $f : M \to N$ is an immersion and $y_1, \ldots, y_l \in N$ are values of f, then there exists an immersion $g : M \to N \setminus \{y_1, \ldots, y_l\}$ such that $f \simeq g$.*

We close this section recalling a useful result proved in [**1**].

Theorem 2.3. *Let M^m be a compact differentiable manifold and let k be an integer with $m \geq k \geq 2$. Then the relation $\varphi(M, \mathbf{R}^k) = \aleph_1$ is satisfied.*

3. On the φ-category of the pairs $(G_{2,n}, \mathbf{R}^m)$ and $(G_{3,n}, \mathbf{R}^m)$

Theorem 3.1. (i) *If the natural number n is not a power of 2, then we have*

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$$\varphi(G_{2,n}, \mathbf{R}^m) = \begin{cases} \geq 2^{p+1} - 1 & \text{if } m = 1 \text{ and } n = 2^p - 1 \\ \aleph_1 & \text{if } 2 \leq m \leq 2n \\ \infty & \text{if } 2n < m \leq 2s - 3 \\ ? & \text{if } 2s - 3 < m < 4n - 1 \\ 0 & \text{if } m \geq 4n - 1 \end{cases}$$

where $s = 2^r$ *is such that* $2^{r-1} \leq n < 2^r$.

(ii) *If* n *is a power of* 2, *then we have*

$$\varphi(G_{2,n}, \mathbf{R}^m) = \begin{cases} \aleph_1 & \text{if } 2 \leq m \leq 2n \\ \infty & \text{if } 2n < m \leq 3n - 3 \\ ? & \text{if } 3n - 3 < m < 4n - 1 \\ 0 & \text{if } m \geq 4n - 1. \end{cases}$$

Proof. (i) The inequality $\varphi(G_{2,2^p-1}, \mathbf{R}) \geq 2^{p+1} - 1$ follows from the inequality $\varphi(M, \mathbf{R}) \geq cat\, M$ and from [**2**, Theorem 1.2]. The fact that $\varphi(G_{2,n}, \mathbf{R}^m) = \aleph_1$ for $2 \leq m \leq 2n = dim\, G_{2,n}$ follows from Theorem 2.3. For the proof of the fact that $\varphi(G_{2,n}, \mathbf{R}^m) = \infty$ under the conditions $2n < m < 2s - 3$, suppose that there exists a smooth mapping $f : G_{2,n} \to R^{2s-3}$ with a finite number of critical points $x_1, x_2, \ldots, x_l$. Consider the usual embedding $i : G_{2,n-1} \hookrightarrow G_{2,n}$ and, according to Corollary 2.2, an immersion $g : G_{2,n-1} \to G_{2,n} \setminus \{x_1, \ldots, x_l\}$ homotopic to i. Then the application $f \circ g : G_{2,n-1} \to \mathbf{R}^{2s-3}$ is an immersion, that is a contradiction with the fact that there is not any immersion from $G_{2,n-1}$ to $\mathbf{R}^{2s-3}$ proved in [**4**, Theorem 1. (*i*)]. The fact that $\varphi(G_{2,n}, \mathbf{R}^m) = 0$ for $m \geq 4n - 1$, follows from Whitney's embedding theorem.

The proof of the second statement can be made in an analogous manner, using the Corollary 2.2 and [**4**, Theorem 1. (*ii*)]. □

Theorem 3.2. *Let* $s = 2^r$ *be the natural number satisfying the condition* $2^{r+1} < 3n < 2^{r+2}$, *with* $n \geq 3$.

(i) *If* $\frac{2}{3} < n \leq s - 3$, *then we have*

$$\varphi(G_{3,n+1}, \mathbf{R}^m) = \begin{cases} \aleph_1 & \text{if } 2 \leq m \leq 3n + 3 \\ \infty & \text{if } 3n + 3 < m \leq 3s - 4 \\ ? & \text{if } 3s - 4 < m < 6n + 4 \\ 0 & \text{if } m \geq 6n + 5. \end{cases}$$

(ii) *If* $s \geq 8$, *then we have*

$$\varphi(G_{3,s-1}, \mathbf{R}^m) = \begin{cases} \aleph_1 & \text{if } 2 \le m \le 3s-3 \\ \infty & \text{if } 3s-3 < m \le 4s-4 \\ ? & \text{if } 4s-4 < m < 6s-7 \\ 0 & \text{if } m \ge 6s-7 \end{cases}$$

and

$$\varphi(G_{3,s}, \mathbf{R}^m) = \begin{cases} \aleph_1 & \text{if } 2 \le m \le 3s \\ \infty & \text{if } 3s < m \le 5s-4 \\ ? & \text{if } 5s-4 < m < 6s-1 \\ 0 & \text{if } m \ge 6s-1. \end{cases}$$

(iii) *If $s < n < \frac{4}{3}s$, then we have*

$$\varphi(G_{3,n}, \mathbf{R}^m) = \begin{cases} \aleph_1 & \text{if } 2 \le m \le 3n \\ \infty & \text{if } 3n < m \le 6s-4 \\ ? & \text{if } 6s-4 < m < 6n-1 \\ 0 & \text{if } m \ge 6n-1. \end{cases}$$

Proof. (i) Theorem 2.3 ensures us that $\varphi(G_{3,n+1}, \mathbf{R}^m) = \aleph_1$ if $2 \le m \le 3n+3 = \dim G_{3,n+1}$, while $\varphi(G_{3,n+1}, \mathbf{R}^m) = 0$ for $m \ge 6n+5$, follows from Whitney's embedding theorem. It remains only to show that $\varphi(G_{3,n+1}, \mathbf{R}^m) = \infty$ for $3n+3 < m \le 3s-4$, that is, any differentiable mapping from $G_{3,n+1}$ to $\mathbf{R}^{3s-4}$, has a finite number of critical points. Assume that there exists a mapping $f : G_{3,n+1} \to \mathbf{R}^{3s-4}$ having a finite number of critical points $\{x_1, x_2, \ldots, x_l\}$ and consider the standard inclusion $j : G_{3,n} \hookrightarrow G_{3,n+1}$. Let $h : G_{3,n} \hookrightarrow G_{3,n+1} \setminus \{x_1, x_2, \ldots, x_l\}$ be the immersion (which is homotopic with j) ensured by the Corollary 2.2. Obviously $f \circ h : G_{3,n} \to \mathbf{R}^{3s-4}$ is an immersion and we can consider the associated $3(s-n)-4$-normal fibre bundle ν. Taking into account the fact that $w_{3(s-n)-3}(\nu) = \bar{w}_{3(s-n)-3}(G_{3,n})$, the relation $\bar{w}_{3(s-n)-3}(G_{3,n}) \ne 0$ proved in [4, Theorem 2 (*i*)] finishes the proof of the statement (i). The statements (ii) and (iii) can be proved analogously using the relations $\bar{w}_{s+3}(G_{3,s-2}) \ne 0$, $\bar{w}_{2s}(G_{3,s-1}) \ne 0$ and $\bar{w}_{3(2s-n+1)-3} \ne 0$ respectively, which are also proved in [4, Theorem 2 (*ii*)] and [4, Theorem 2 (*iii*)] respectively. □

4. On the φ-category of the pair $(P_n(\mathbf{R}), \mathbf{R}^m)$

In this section the case of the pair $(P_n(\mathbf{R}), \mathbf{R}^m)$ will be treated. For this purpose we need some helpful results.

Lemma 4.1. *If $A \subseteq S^n$, $(n \geq 2)$ is a finite set, then there exists $x \in S^n$ such that $\langle x\rangle^\perp \cap A = \emptyset$ where $\langle x\rangle^\perp$ denotes the orthogonal complement of x with respect to the usual scalar product from $\mathbf{R}^{n+1}$.*

Proof. The proof will be made by induction with respect to $k = |A|$. If $k = 1$, then $A = \{a\}$ and we can choose $x = a$. Suppose that $|A| = k + 1$ and choose $a \in A$. From the induction hypothesis it follows that there exists $x' \in S^n$ such that $\langle x'\rangle^\perp \cap (A \setminus \{a\}) = \emptyset$. If $a \notin \langle x'\rangle^\perp$ choose $x = x'$, else we choose $\theta \in (0, m)$ where

$$m = \min\left\{\left|arctg\frac{\langle a, x'\rangle}{\langle a, a'\rangle}\right| : a' \in A\setminus\{a\}\right\},$$

with $m = \frac{\pi}{2}$ if $\langle a, a'\rangle = 0$ $(\forall)$ $a' \in A\setminus\{a\}$, and $x = cos\theta\, x' + sin\theta\, a \in S^n$. Obviously $\langle a, x\rangle = sin\theta > 0$, that is $a \notin \langle x\rangle^\perp$ and since $\langle a', x\rangle = cos\theta\,\langle a', x'\rangle + sin\theta\,\langle a, a'\rangle \neq 0$ $(\forall)$ $a' \in A\setminus\{a\}$, it implies that $(A\setminus\{a\}) \cap \langle x\rangle^\perp = \emptyset$ which together with $a \notin \langle x\rangle^\perp$ leads to the conclusion that $\langle x\rangle^\perp \cap A = \emptyset$. □

Proposition 4.2. *If $A \subseteq S^n, n \geq 2$ is a finite set $\mathbf{Z}_2$-invariant (symmetric), then there exists a $\mathbf{Z}_2$-equivariant (odd) embedding $f : S^{n-1} \to S^n\setminus A$.*

Proof. Let us consider $x \in S^n$ such that $\langle x\rangle^\perp \cap A = \emptyset$. Because the orthogonal group $O(n)$ acts transitively on S^n, it follows that there exists $T \in O(n)$ such that $T(e_{n+1}) = x$ where $e_{n+1} = (0, \ldots, 0, 1) \in \mathbf{R}^{n+1}$. But since $\langle e_{n+1}\rangle^\perp = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_{n+1} = 0\} \simeq \mathbf{R}^n$ and T is an orthogonal diffeomorphism which leaves invariant the sphere S^n, it implies that $T(\mathbf{R}^n) = \langle x\rangle^\perp$. Choose $f = T|_{S^{n-1}}$.

Corollary 4.3. *If $A \subseteq P_n(\mathbf{R}^n)$, $(n \geq 2)$ is a finite subset, then there exists an immersion $g : P_{n-1}(\mathbf{R}) \to P_n(\mathbf{R})\setminus A$.*

Proof. Let $f : S^{n-1} \to S^n \setminus p_n^{-1}(A)$, where $p_n : S^n \to P_n(\mathbf{R})$ is the canonical projection, be the embedding ensured by Proposition 4.2. g will be chosen as being the mapping which makes commutative the following diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & S^n \setminus p_n^{-1}(A) \\ p_{n-1}\downarrow & & \downarrow p_n|_{S^n\setminus p_n^{-1}(A)} \\ P_{n-1}(\mathbf{R}) & \xrightarrow{g} & P_n(\mathbf{R})\setminus A. \end{array}$$

□

Let A be a finite subset of $P_n(\mathbf{R})$ and $E(\gamma_n^1(A))$ be the subset of $(P_n(\mathbf{R})\setminus A) \times \mathbf{R}^{n+1}$ consisting in all pairs $(\{\pm x\}, v)$ such that v is a multiple of x. Define $\pi_A : E(\gamma_n^1(A)) \to P_n(\mathbf{R})\setminus A$ by $\pi_A(\{\pm x\}, v) = \{\pm x\}$. Hence every fibre $\pi_A^{-1}(\{\pm x\})$ can be identified with the straight line through x and $-x$ from $\mathbf{R}^{n+1}$. The resultant fibre bundle $\gamma_n^1(A)$ will be called *the canonical line bundle over* $P_n(\mathbf{R})\backslash A$. Note that $\gamma_n^1(\emptyset)$ is even the canonical line bundle γ_n^1 (over $P_n(\mathbf{R})$) defined in [3, pp. 16].

Proposition 4.4. *The total Stiefel-Whitney class of the canonical line fibre bundle $\gamma_n^1(A)$ over $P_n(\mathbf{R})\backslash A$ is given by*

$$\omega(\gamma_n^1(A)) = 1 + a_A$$

where $a_A \in H^1(P_n(\mathbf{R})\backslash A; \mathbf{Z}_2)$ is not zero.

Proof. Let $j' : S^1 \to S^n\backslash p_n^{-1}(A)$ be a $\mathbf{Z}_2$-equivariant embedding. Obviously j' induces an immersion $j : P_1(\mathbf{R}) \to P_n(\mathbf{R})\backslash A$ covered by an application of fibrations from γ_1^1 to $\gamma_n^1(A)$. Therefore denoting by a_A the Stiefel-Whitney class $\omega_1(\gamma_1(A))$, one can say that $j^*(a_A) = \omega_1(\gamma_1^1) \neq 0$ which shows that $a_A \neq 0$. □

Remark. If $n \geq 2$, then $a_A^k \neq 0$, $(\forall)k \in \{1, 2, \ldots, n-1\}$. Indeed if $k : P_1(\mathbf{R}) \to P_{n-1}(\mathbf{R})$ denotes the usual inclusion, which can be obviously covered by an application of fibrations from γ_1^1 to γ_{n-1}^1 and $j : P_{n-1}(\mathbf{R}) \to P_n(\mathbf{R})\backslash A$ the immersion ensured by Corollary 4.3, which can be also covered by an application of fibrations from γ_{n-1}^1 to $\gamma_n^1(A)$, then from the second axiom of the Stiefel-Whitney classes, it follows that $k^*(j^*(a_A)) = \omega_1(\gamma_1^1) \neq 0$, and therefore $j^*(a_A) = a \in H^1(P_{n-1}(\mathbf{R}); \mathbf{Z}_2)$ is the generator (obviously non zero) of $H^1(P_{n-1}(\mathbf{R}); \mathbf{Z}_2)$. But since $a^k = j^*(a_A^k)$ is the generator (obviously non zero) of $H^k(P_{n-1}(\mathbf{R}); \mathbf{Z}_2)$ for any $k \in \{1, 2, \ldots, n-1\}$, it implies that $a_A^k \neq 0$, for each $k \in \{1, 2, \ldots, n-1\}$.

Using a similar judgement with that from [3, Theorem 4.5, p. 45] one can show that the manifold $P_n(\mathbf{R})\backslash A$ has the total Stiefel-Whitney class

$$\omega(P_n(\mathbf{R})\backslash A) = (1 + a_A)^{n+1} = 1 + \binom{n+1}{1}a_A + \binom{n+1}{2}a_A^2 + \cdots + \binom{n+1}{n}a_A^n.$$

For $n = 2^r$ we get

$$\omega(P_{2^r}(\mathbf{R})\backslash A) = (1 + a_A)^{2^r+1} = 1 + a_A + a_A^{2^r}$$

and also

$$\bar{\omega}(P_{2^r}(\mathbf{R})\backslash A) = 1 + a_A + a_A^2 + \cdots + a_A^{2^r-1}.$$

Theorem 4.5. *If n is a natural number such that $n + 1$ and $n + 2$ are not powers of 2, then the φ-category of the pair $(P_n(\mathbf{R}), \mathbf{R}^m)$ is given by:*

$$\varphi(P_n(\mathbf{R}),\mathbf{R}^m)=\begin{cases} n+1 & \textit{if } m=1\\ \aleph_1 & \textit{if } 2\le m\le n\\ \infty & \textit{if } n<m\le 2^{[\log_2 n]+1}-2\\ ? & \textit{if } 2^{[\log_2]+1}-1\le m\le 2n-2\\ 0 & \textit{if } m\ge 2n-1.\end{cases}$$

Proof. The case $m=1$ is justified in [6]. The fact that $\varphi(P_n(\mathbf{R}),\mathbf{R}^m)=\aleph_1$ for $2\le m\le n$ follows from Theorem 2.3. Consider firstly the case when n is a power of 2, that is, $n=2^{[\log_2 n]}$. Assume that $n<m\le 2^{[\log_2 n]+1}-2$ and that there exists $f:P_{2^{[\log_2 n]}}(\mathbf{R})\to\mathbf{R}^m$ such that $C(f)$ is finite. If ν is the associated normal fibre bundle (over $P_{2^{[\log_2 n]}}(\mathbf{R})\setminus C(f)$) to the immersion $f|_{P_{2^{[\log_2 n]}}(\mathbf{R})\setminus C(f)}$ then

$$\omega(\nu)=\bar{\omega}(P_{2^{[\log_2 n]}}(\mathbf{R})\setminus C(f))=1+a_{C(f)}+a^2_{C(f)}+\cdots+a^{2^{[\log_2 n]-1}}_{C(f)}.$$

But since ν is a $m-2^{[\log_2 n]}$-vector fibre bundle and $a^{2^{[\log_2 n]-1}}_{C(f)}\ne 0$ it follows that $m-2^{[\log_2 n]}\ge 2^{[\log_2 n]}-1$ which means that $m\ge 2^{[\log_2 n]+1}-1>2^{[\log_2 n]+1}-2$ that is a contradiction. If n is not a power of 2, then the hypothesis of the theorem ensures that $2^{[\log_2 n]}+1\le n\le 2^{[\log_2 n]+1}-3$. Assume that $n<m\le 2^{[\log_2 n]+1}-2$ and that there exists a differentiable application $g:P_n(\mathbf{R})\to\mathbf{R}^m$ such that $C(g)$ is finite. If $h:P_{2^{[\log_2 n]}}(\mathbf{R})\to P_n(\mathbf{R})\setminus C(g)$ is the immersion ensured by Corollary 4.3, then obviously $g\circ h:P_{2^{[\log_2 n]}}(\mathbf{R})\to\mathbf{R}^m$ is an immersion. If ν' is the associated normal fibre bundle (over $P_{2^{[\log_2 n]}}(\mathbf{R})$) of the immersion $g\circ h$, then $w(\nu')=\bar{w}(P_{2^{[\log_2 n]}}(\mathbf{R}))=1+a+a^2+\cdots+a^{2^{[\log_2 n]}-1}$. But since ν' is a $m-2^{2^{[\log_2 n]}}$-vector fibre bundle and $a^{2^{[\log_2 n]}-1}\ne 0$ it follows that $m-2^{[\log_2 n]}\ge 2^{[\log_2 n]}-1$ which means that $m\ge 2^{[\log_2 n]+1}-1>2^{[\log_2 n]+1}-2$ that is a contradiction. The fact that $\varphi(P_n(\mathbf{R}),\mathbf{R}^m)=0$ for $m\ge 2n-1$ follows from Whitney's embedding theorem. □

Corollary 4.6. *If m and n are natural numbers such that $n+1$ and $n+2$ are not powers of 2 and $2\le m\le 2^{2^{[\log_2 n]+1}}-2$, then any smooth $\mathbf{Z}_2$-invariant (even) mapping $f:S^n\to\mathbf{R}^m$ has an infinite number of critical orbits, that is, there exists infinitely many points $x\in S^n$ such that x and $-x$ are critical points of f.*

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