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Standing waves of modified Schrödinger equations coupled with the Chern–Simons gauge theory

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We are interested in standing waves of a modified Schrödinger equation coupled with the Chern–Simons gauge theory. By applying a constraint minimization of Nehari-Pohozaev type, we prove the existence of radial ground state solutions. We also investigate the nonexistence for nontrivial solutions.

Keywords: ground state solution; gauged Schrödinger equation; variational method

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1. Introduction

In this paper, we consider the following nonlocal quasilinear elliptic problem

$$\begin{aligned} &-\Delta u + \omega u - \mu u \Delta u^2 + q \frac{h_u^2(|x|)}{|x|^2}(1+\mu u^2)u \\ &+ q\left(\int_{|x|}^{\infty} \frac{h_u(s)}{s}\big(2+\mu u^2(s)\big)u^2(s)\,\mathrm{d}s\right)u = \lambda |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$ is radially symmetric, ω, μ, q, λ are positive constants, $p > 1$ and

$$h_u(s) = \int_0^s r u^2(r)\,\mathrm{d}r, \quad s \geqslant 0.$$

Equation (1.1) appears in the study of standing waves for a modified Schrödinger equation coupled with the Chern–Simons gauge theory. For the reader's convenience, we will give the derivation of (1.1) in § 2.

Our aim of this paper is to study existence and nonexistence of positive radial solutions and radial ground state solutions of (1.1).

When $q = 0$, (1.1) is reduced to the following quasilinear elliptic problem

$$-\Delta u + \omega u - \mu u \Delta u^2 = \lambda |u|^{p-1} u, \tag{1.2}$$

which is obtained by the *modified Schrödinger equation*

$$\mathrm{i}\psi_t + \Delta\psi + \mu\psi\Delta|\psi|^2 + \lambda|\psi|^{p-1}\psi = 0, \tag{1.3}$$

looking for standing waves of the form $\psi(t, x) = \exp(i\omega t)u(x)$. In the last decades, a considerable attention has been devoted to the study of solutions to the quasilinear Schrödinger equation (1.3) that arises in various fields of Physics (see [**3**–**5**, **18**, **19**]). This model is known to be more accurate in many physical phenomena compared with the classical semi-linear Schrödinger equation $\mathrm{i}\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0$. In (1.3) $\psi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ and λ is a constant representing the strength of self-interaction potential. Moreover, the *additional term* $\mu\psi\Delta|\psi|^2$ appears in various physical models and arises due to:

- the nonlocality of the nonlinear interaction for electron (see [**4**]),
- the weak nonlocal limit for nonlocal nonlinear Kerr media [**18**],
- the surface term for superfluid film (see [**19**]),

and the parameter μ represents the strength of each effect and may not be small.

The existence and properties of ground states of (1.2) as well as stability of standing wave solutions have also been studied widely, see e.g. [**1**, **8**, **9**, **22**, **28**, **29**] and references therein.

On the other hand if $\mu = 0$ in (1.1), one obtains the following nonlocal elliptic problem

$$-\Delta u + \omega u + q\frac{h_u^2(|x|)}{|x|^2}u + 2q\left(\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s)\,\mathrm{d}s\right)u = \lambda|u|^{p-1}u. \tag{1.4}$$

Equation (1.4) appears in the study of nonlinear Schrödinger equations coupled with the Chern–Simons gauge fields. Recently, a lot of works concerning (1.4) has been done, see [**6**, **7**, **10**, **13**, **14**, **17**, **24**–**26**, **33**, **34**]. Here we briefly introduce some known results on (1.4). In [**6**], the existence of a positive radial solution of (1.4) was shown in the case $p > 3$ by using a suitable constraint minimization argument. The authors in [**6**] also investigated the case $1 < p \leqslant 3$. They obtained existence and nonexistence results depending on λ for the case $p = 3$, and the existence of positive radial solutions as minimizers under L^2-constraint in the case $1 < p < 3$ (ω appears as a Lagrangian multiplier). When $p > 3$, the existence of a positive solution in the nonradial setting has been also obtained in [**33**]. In [**25**], a detailed study for the case $1 < p < 3$ has been performed. The authors in [**25**] investigated the geometry of the functional associated with (1.4) and obtained an explicit threshold value for ω. They also showed the multiple existence of positive radial solutions for ω in some range. We mention that in [**26**] the case of a bounded domain for $1 < p < 3$ is considered and some results on boundary concentration of solutions has been proved. In [**10**] the authors studied (1.4) with general nonlinearities of the Berestycki-Lions type, and obtained a multiplicity result when q is sufficiently small. We also recall that the multiple existence of normalized solutions of (1.4) has been studied in [**34**]. Finally, we refer to [**2**, **21**, **23**] for results on Cauchy problem associated with (1.4). To summarize, the existence and the nonexistence of solutions of (1.4) heavily depends on ω, q, λ and p, and the solution set of (1.4) has a rich structure depending on the parameters and the exponent p.

The purpose of this paper is to investigate the structure of the solutions set for (1.1), which seems to be more complicated due to the presence of the quasilinear term.

To state our main result, let us define the metric space

$$\mathcal{X} := \{u \in H_r^1(\mathbb{R}^2) : u^2 \in H^1(\mathbb{R}^2)\},$$

endowed with the distance

$$d_{\mathcal{X}}(u,v) := \|u - v\|_{H^1} + \|\nabla(u^2) - \nabla(v^2)\|_2.$$

We recall that

$$H_r^1(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : u \text{ is radially symmetric}\}.$$

Then $u \in \mathcal{X}$ is called a *weak solution* of (1.1) if u satisfies

$$\begin{aligned}\int_{\mathbb{R}^2} \Bigg\{ &(1+2\mu u^2)\nabla u \cdot \nabla\varphi + 2\mu u|\nabla u|^2\varphi + \omega u\varphi - \lambda|u|^{p-1}u\varphi + q\frac{h_u^2(|x|)}{|x|^2}(1+\mu u^2)u\varphi \\ &+ q\left(\int_{|x|}^{\infty} \frac{h_u(s)}{s}(2+\mu u^2(s))u^2(s)\,\mathrm{d}s\right)u\varphi\Bigg\}\,\mathrm{d}x = 0, \quad \text{for all } \varphi \in C_{0,r}^{\infty}(\mathbb{R}^2),\end{aligned} \tag{1.5}$$

where $C_{0,r}^{\infty}(\mathbb{R}^2) := \{u \in C_0^{\infty}(\mathbb{R}^2) : u \text{ is radially symmetric}\}$.

At least formally, weak solutions of (1.1) can be obtained as critical points of the following functional defined on $\mathcal{X}$

$$\begin{aligned}I(u) = &\frac{1}{2}\int_{\mathbb{R}^2} \left[(1+2\mu u^2)|\nabla u|^2 + \omega u^2\right]\,\mathrm{d}x + \frac{q}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\,\mathrm{d}s\right)^2\,\mathrm{d}x \\ &+ \frac{q\mu}{4}\int_{\mathbb{R}^2} \frac{u^4(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\,\mathrm{d}s\right)^2\,\mathrm{d}x - \frac{\lambda}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\,\mathrm{d}x,\end{aligned} \tag{1.6}$$

but $\mathcal{X}$ is not a vector space, because it is not close with respect to the sum. So we cannot speak about *critical points* of I in the usual way, since the functional is not differentiable. However, as we will see in § 3, one can show that I is well-defined and continuous on $\mathcal{X}$. Moreover, since, for every given $u \in \mathcal{X}$ and $\varphi \in C_{0,r}^{\infty}(\mathbb{R}^2)$, we

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have $u + \varphi \in \mathcal{X}$, we can evaluate the Gateaux derivative

$$I'(u)[\varphi] = \int_{\mathbb{R}^2} \left\{ (1 + 2\mu u^2)\nabla u \cdot \nabla\varphi + 2\mu u|\nabla u|^2\varphi + \omega u\varphi - \lambda|u|^{p-1}u\varphi \right.$$
$$\left. + q\frac{h_u^2(|x|)}{|x|^2}(1 + \mu u^2)u\varphi + q\left(\int_{|x|}^{\infty} \frac{h_u(s)}{s}\big(2 + \mu u^2(s)\big)u^2(s)\,\mathrm{d}s\right)u\varphi \right\} \mathrm{d}x.$$

Then $u \in \mathcal{X}$ is a weak solution of (1.1) if and only if the Gateaux derivative of I in every direction $\varphi \in C_{0,r}^{\infty}(\mathbb{R}^2)$ is zero (see lemma 3.3 below).

Our main results are the followings.

THEOREM 1.1. *Assume that $p > 5$. Then for any ω, μ, q and $\lambda > 0$, (1.1) has a positive radial solution $u \in \mathcal{X} \cap C^2(\mathbb{R}^2)$. Moreover u is a radial ground state of (1.1), that is, u has least energy among any nontrivial radial weak solutions of (1.1).*

THEOREM 1.2. *Assume that $1 < p < 5$. Then, for any μ, q and $\lambda > 0$, there exists $\bar{\omega} > 0$ such that for $\omega \geqslant \bar{\omega}$, (1.1) has no nontrivial solution.*

We will also study the dependence of $\bar{\omega}$ with respect to μ and q in remark 5.3 below.

To prove theorem 1.1, we use a constraint minimization argument which is a combination of the Nehari manifold and the Pohozaev manifold, as performed in [**6**, **33**]. However we must pay attention to apply this approach in our case, since the functional I associated with (1.1) is only Gateaux differentiable and only in some directions. We will overcome this difficulty by establishing the regularity of weak solutions of (1.1). Once we could show that any weak solution of (1.1) satisfies the Nehari identity and the Pohozaev identity, we next aim to prove that the constraint minimizer is actually a ground state solution. For this purpose, we apply an argument performed in [**22**, **28**], which enables us to avoid considering complicated algebraic equations as in [**6**, **33**].

The proof of theorem 1.2 can be done similarly as in [**6**, **25**]. To this end, we will obtain a new inequality of Sobolev type for $u \in \mathcal{X}$. As shown in [**25**] for the case $\mu = 0$, the existence and the nonexistence of positive solutions of (1.1) in the case $1 < p < 5$ heavily depends on ω, μ, q and λ. We expect to obtain the (multiple) existence of positive solutions when ω is small. But we postpone this question to a future work.

This paper is organized as follows. In § 2, we introduce the derivation of (1.1) and the role of physical constants ω, μ, q and λ. We formulate (1.1) as a variational problem in § 3. The regularity property of weak solutions of (1.1), which enables us to apply the Pohozaev identity and plays a central role for the existence of ground state solutions, is also established here. In § 4, we will obtain the existence result (theorem 1.1) by applying the constraint minimization technique described before. Finally, by establishing a new inequality of Sobolev type for $u \in \mathcal{X}$, we prove theorem 1.2 in § 5.

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2. Derivation of the model

In this section, we introduce the derivation of equation (1.1) together with physical backgrounds. Let us consider the Lagrangian density $\mathcal{L}_{\mathrm{MNLS}}$ for a modified nonlinear Schrödinger equation, which is given by

$$\mathcal{L}_{\mathrm{MNLS}} = \frac{1}{2}\Im(\psi\bar{\psi}_t) - \frac{1}{2}|\nabla\psi|^2 + \frac{\lambda}{p+1}|\psi|^{p+1} - \frac{\mu}{4}\big|\nabla|\psi|^2\big|^2. \tag{2.1}$$

We are interested in the situation where the Schrödinger wave function ψ is, for instance, a charged particle and interacts with the gauge potential $(\phi, \mathbf{A})$ for the electro-magnetic field in the Chern–Simons theory. Here $\phi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{A} = (A^1, A^2) : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ are the electric potential and the magnetic potential respectively. Then the interaction between ψ and $(\phi, \mathbf{A})$ is obtained by replacing the usual derivatives with the covariant ones, namely

$$\partial_t \longmapsto \partial_t + ie\phi, \quad \nabla \longmapsto \nabla - ie\mathbf{A}, \tag{2.2}$$

where e denotes the strength of the interaction with the electro-magnetic field (see [**12**] for details). Substituting (2.2) in (2.1), one has the following Lagrangian

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathrm{MNLS}}(\psi,\phi,\mathbf{A}) = \frac{1}{2}\Im(\psi\bar{\psi}_t) - \frac{e}{2}\phi|\psi|^2 - \frac{1}{2}|\nabla\psi - ie\mathbf{A}\psi|^2 + \frac{\lambda}{p+1}|\psi|^{p+1}\\ - \frac{\mu}{4}\big|\nabla|\psi|^2 - ie\mathbf{A}|\psi|^2\big|^2.\end{aligned}$$

We have to consider also the Lagrangian density for the electro-magnetic field, which, in the Chern–Simons theory, is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{MCS}}(\phi,\mathbf{A}) &= -\frac{1}{8}F^{\alpha\beta}F_{\alpha\beta} + \frac{\kappa}{8}\varepsilon^{\nu\alpha\beta}A_\nu F_{\alpha\beta},\\ F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha, \quad \alpha,\beta,\nu \in \{0,1,2\},\end{aligned}$$

where the first term in $\mathcal{L}_{\mathrm{MCS}}$ is the usual Maxwell term and the second term is the so-called Chern–Simons term (see [**15**, **16**] for details). Here ε is the Levi-Civita tensor, $\kappa \in \mathbb{R}$ is a parameter which controls the Chern–Simons term, the Lorentz metric tensor is $\mathrm{diag}(1,-1,-1)$, and the coordinates are $x^\nu = (t, x_1, x_2)$. Moreover we have $A^0 = A_0 = \phi$ and $A^j = -A_j$, for $j = 1, 2$. At large distances and low energies, the lower derivatives of the Chern–Simons term dominates the higher derivative Maxwell term, and hence we may replace the Lagrangian density by

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathrm{MCS}}(\phi,\mathbf{A}) = \frac{\kappa}{8}\varepsilon^{\nu\alpha\beta}A_\nu F_{\alpha\beta} = \frac{\kappa}{4}\big(\phi(\partial_2 A^1 - \partial_1 A^2) - A^1(\partial_2\phi + \partial_t A^2)\\ + A^2(\partial_t A^1 + \partial_1\phi)\big).\end{aligned}$$

So, the total Lagrangian $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L}(\psi,\phi,\mathbf{A}) &= \tilde{\mathcal{L}}_{\mathrm{MNLS}}(\psi,\phi,\mathbf{A}) + \tilde{\mathcal{L}}_{\mathrm{MCS}}(\phi,\mathbf{A})\\ &= \frac{1}{2}\Im(\psi\bar{\psi}_t) - \frac{e}{2}\phi|\psi|^2 - \frac{1}{2}|\nabla\psi - ie\mathbf{A}\psi|^2 + \frac{\lambda}{p+1}|\psi|^{p+1} - \frac{\mu}{4}\big|\nabla|\psi|^2\\ &\quad - ie\mathbf{A}|\psi|^2\big|^2 + \frac{\kappa}{4}\big(\phi(\partial_2 A^1 - \partial_1 A^2) - A^1(\partial_2\phi + \partial_t A^2) + A^2(\partial_t A^1 + \partial_1\phi)\big).\end{aligned}$$

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Then the Euler-Lagrange equations for the total action

$$\mathcal{S} = \mathcal{S}(\psi, \phi, \mathbf{A}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{L}(\psi, \phi, \mathbf{A}) \, \mathrm{d}t \, \mathrm{d}x$$

are given by

$$\begin{cases} i\psi_t - e\phi\psi + (\nabla - ie\mathbf{A})^2\psi + \lambda|\psi|^{p-1}\psi + \mu\psi\Delta|\psi|^2 - e^2\mu|\mathbf{A}|^2|\psi|^2\psi = 0 \\ \kappa(\partial_2 A^1 - \partial_1 A^2) = e|\psi|^2 \\ \kappa(\partial_2\phi + \partial_t A^2) + e^2\mu|\psi|^4 A^1 = 2e\Im\Big(\bar{\psi}(\partial_1\psi - ieA^1\psi)\Big) \\ -\kappa(\partial_1\phi + \partial_t A^1) + e^2\mu|\psi|^4 A^2 = 2e\Im\Big(\bar{\psi}(\partial_2\psi - ieA^2\psi)\Big). \end{cases}$$

If we consider standing waves $\psi(t, x) = \exp(iS(t, x))u(t, x)$ with $u, S : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, the total action depends on $(u, S, \phi, \mathbf{A})$ and the Euler-Lagrange equations become

$$\begin{cases} -\Delta u + (S_t + e\phi + |\nabla S - e\mathbf{A}|^2)u - \mu u\Delta u^2 + e^2\mu|\mathbf{A}|^2 u^3 = \lambda|u|^{p-1}u \\ \partial_t(u^2) + 2\,\mathrm{div}\left((\nabla S - e\mathbf{A})u^2\right) = 0 \\ \kappa(\partial_2 A^1 - \partial_1 A^2) = eu^2 \\ \kappa(\partial_2\phi + \partial_t A^2) + e^2\mu u^4 A^1 = 2e(\partial_1 S - eA^1)u^2 \\ -\kappa(\partial_1\phi + \partial_t A^1) + e^2\mu u^4 A^2 = 2e(\partial_2 S - eA^2)u^2. \end{cases}$$

Now we suppose that $u = u(x)$ and $S = \omega t$. Moreover we consider the static case: $\phi = \phi(x)$ and $A^i = A^i(x)$. Then we get

$$\begin{cases} -\Delta u + \omega u - \mu u\Delta u^2 + e\phi u + e^2|\mathbf{A}|^2(1 + \mu u^2)u = \lambda|u|^{p-1}u, \\ \mathrm{div}(\mathbf{A}u^2) = 0, \\ \kappa(\partial_2 A^1 - \partial_1 A^2) = eu^2, \\ -\kappa\partial_2\phi = e^2(2 + \mu u^2)u^2 A^1, \\ \kappa\partial_1\phi = e^2(2 + \mu u^2)u^2 A^2. \end{cases}$$

Finally in the Coulomb gauge $\mathrm{div}\,\mathbf{A} = 0$, it follows that $\mathrm{div}(\mathbf{A}u^2) = \mathbf{A} \cdot \nabla u^2$ and hence

$$\begin{cases} -\Delta u + \omega u - \mu u\Delta u^2 + e\phi u + e^2|\mathbf{A}|^2(1 + \mu u^2)u = \lambda|u|^{p-1}u, \\ \mathbf{A} \cdot \nabla u^2 = 0, \\ \kappa(\partial_2 A^1 - \partial_1 A^2) = eu^2, \\ -\kappa\partial_2\phi = e^2(2 + \mu u^2)u^2 A^1, \\ \kappa\partial_1\phi = e^2(2 + \mu u^2)u^2 A^2. \end{cases} \tag{2.3}$$

Observe that the second equation in (2.3) implies that, up to the 'trivial cases', the function u is radial if and only if $\mathbf{A}$ is a *tangential* vector field, i.e.

$$\mathbf{A} = \frac{e}{\kappa} h_u(x)\mathbf{t}, \quad \text{where} \quad \mathbf{t} = (x_2/|x|^2, -x_1/|x|^2).$$

Moreover, since the problem is invariant by translations, to avoid the related difficulties, we look for radial solutions u. Thus, from this choice, arguing as in [**13**,

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lemma 3.3], it follows that $\mathbf{A}$ has to be invariant for the group action:

$$\mathrm{T}_g\mathbf{A}(x) = g^{-1}\cdot\mathbf{A}(g(x)), \quad g\in O(2),$$

and this readily implies that h_u has to be a radial function. So, whenever u is radial, the magnetic potential $\mathbf{A}$ has to be necessarily written as

$$A^1(x) = \frac{e}{\kappa}\frac{x_2}{|x|^2}h_u(|x|), \quad A^2(x) = -\frac{e}{\kappa}\frac{x_1}{|x|^2}h_u(|x|).$$

Moreover, by the last two equations in system (2.3), one finds that

$$\begin{aligned}\nabla\phi &= \frac{e^2}{\kappa}\left(A^2, -A^1\right)(2+\mu u^2)u^2 = -\frac{e^3}{\kappa^2}\frac{x}{|x|^2}h_u(|x|)(2+\mu u^2)u^2\\ &= -\frac{e^3}{\kappa^2}h_u(|x|)(2+\mu u^2)u^2\mathbf{n}\end{aligned}$$

where $\mathbf{n} = (x^1/|x|^2, x^2/|x|^2)$. Thus it follows that the electric potential ϕ is radial. Assuming that $\lim_{|x|\to+\infty}\phi(|x|) = 0$, we have

$$\phi(|x|) = \frac{e^3}{\kappa^2}\int_{|x|}^{\infty}\frac{h_u(s)}{s}\big(2+\mu u^2(s)\big)u^2(s)\,\mathrm{d}s.$$

Finally, using the third equation in system (2.3) and assuming $h_u(0) = 0$, which is necessary to have $\mathbf{A}$ smooth, we have

$$h_u(|x|) = \int_0^{|x|} su^2(s)\,\mathrm{d}s.$$

In this way we have solved ϕ and $\mathbf{A}$ in terms of u and so, in order to solve the (2.3), we need to study only the first equation of the system which, now, can be written as

$$\begin{aligned}&-\Delta u + \omega u - \mu u\Delta u^2 + \frac{e^4}{\kappa^2}\frac{h_u^2(|x|)}{|x|^2}(1+\mu u^2)u\\ &+\frac{e^4}{\kappa^2}\left(\int_{|x|}^{\infty}\frac{h_u(s)}{s}\big(2+\mu u^2(s)\big)u^2(s)\,\mathrm{d}s\right)u = \lambda|u|^{p-1}u \quad \text{in } \mathbb{R}^2.\end{aligned}$$

Putting $q := e^4/\kappa^2$, we arrive at (1.1).

3. Variational settings and preliminaries

In this section, we formulate (1.1) as a variational problem and prepare some preliminary results. Now we observe that if $u\in\mathcal{X}$ is a solution of (1.1), then it solves

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$L(u) = 0$ where

$$L(u) = \operatorname{div} A(u, \nabla u) + B(x, u, \nabla u),$$

with

$$\begin{aligned} A(\sigma, \mathbf{p}) &= (1 + 2\mu\sigma^2)\mathbf{p}, \\ B(x, \sigma, \mathbf{p}) &= -\big(2\mu|\mathbf{p}|^2 + \omega + qV_1(x)(1 + \mu u^2) + qV_2(x)\big)\sigma + \lambda|\sigma|^{p-1}\sigma, \end{aligned} \tag{3.1}$$

and

$$V_1(x) = \begin{cases} \dfrac{h_u^2(|x|)}{|x|^2} & x \neq 0, \\ 0 & x = 0, \end{cases} \qquad V_2(x) = \int_{|x|}^{\infty} \frac{h_u(s)}{s} \big(2 + \mu u^2(s)\big) u^2(s) \, \mathrm{d}s.$$

From (3.1), we find that (1.1) is a quasilinear elliptic equation with principal part in divergence form and the structure conditions in [**20**] are all fulfilled (see [**20**, Chapter 4] or [**31**]).

First we establish that any weak solutions of (1.1) are classical ones. To this end, we begin with the following lemma.

LEMMA 3.1. *Let us fix $u \in \mathcal{X}$. We have:*

(i) *V_1, V_2 are nonnegative and bounded;*

(ii) *if we suppose further that $u \in C(\mathbb{R}^2)$, then V_1 and V_2 belong to the class $C^1(\mathbb{R}^2)$.*

Proof. We argue as in [**6**, proposition 2.1, 2.2]. First by the definition, we see that V_1, V_2 are nonnegative. Next by the Schwarz inequality, one finds that

$$h_u(s) = \frac{1}{2\pi} \int_{B_s(0)} u^2(y) \, \mathrm{d}y \leqslant \frac{1}{2\pi} |B_s(0)|^{1/2} \|u\|_4^2 \leqslant Cs\|u\|_4^2, \quad \text{for } s \geqslant 0. \tag{3.2}$$

Thus by the definition and from (3.2), we get

$$V_1(x) \leqslant C\|u\|_4^4, \quad \text{for all } x \in \mathbb{R}^2.$$

Moreover, observing that, for all $x \in \mathbb{R}^2$, $0 \leqslant V_2(x) \leqslant V_2(0)$, we need to estimate only

$$\begin{aligned} V_2(0) &= \int_0^1 \frac{h_u(s)}{s} \big(2 + \mu u^2(s)\big) u^2(s) \, \mathrm{d}s + \int_1^\infty \frac{h_u(s)}{s} \big(2 + \mu u^2(s)\big) u^2(s) \, \mathrm{d}s \\ &\leqslant C\|u\|_4^2 \left[\int_0^1 (2 + \mu u^2(s)) u^2(s) \, \mathrm{d}s + \int_1^\infty (2 + \mu u^2(s)) u^2(s) \, \mathrm{d}s \right] \end{aligned}$$

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$$\leqslant C\|u\|_4^2 \left[\left(\left(\int_0^1 s^{-(1/2)}\,\mathrm{d}s \right)^{2/3} \left(\int_0^1 (2+\mu u^2(s))^3 u^6(s) s\,\mathrm{d}s \right)^{1/3} + \int_1^\infty (2+\mu u^2(s)) u^2(s) s\,\mathrm{d}s \right]$$

$$\leqslant C\|u\|_4^2(\|u\|_6^2 + \|u\|_{12}^4 + \|u\|_2^2 + \|u\|_4^4).$$

This completes the proof of (3.1).
To prove (3.1), we observe that V_1, $V_2 \in C^1(\mathbb{R}^2 \setminus \{0\})$ if $u \in C(\mathbb{R}^2)$. Moreover since $u \in C(\mathbb{R}^2)$, it follows that

$$\frac{h_u(|x|)}{|x|^2} = \frac{1}{2\pi|x|^2}\int_{B_{|x|}(0)} u^2(y)\,\mathrm{d}y \to \frac{1}{2}u^2(0) \text{ as } |x| \to 0.$$

This implies that $h_u(|x|) = O(|x|^2)$ as $|x| \to 0$. Thus one has $V_1(x) = O(|x|^2)$ and, for $i = 1, 2$,

$$\frac{\partial V_1}{\partial x_i}(0) = 0,$$

and

$$\frac{\partial V_1}{\partial x_i}(x) = \frac{2x_i h_u(|x|)}{|x|^4}\big(|x|^2u^2(x) - h_u(|x|)\big) = O(|x|) \text{ as } |x| \to 0,$$

from which we conclude that $V_1 \in C^1(\mathbb{R}^2)$. In a similar way, it follows that $V_2 \in C^1(\mathbb{R}^2)$. □

Now we are ready to prove the following regularity result.

PROPOSITION 3.2. *Let $u \in \mathcal{X}$ be a weak solution of (1.1). Then $u \in C^2(\mathbb{R}^2)$ and decays exponentially up to second derivatives.*

Proof. The proof consists of two steps.

Step 1: We claim that $u \in L^\infty(\mathbb{R}^2)$ and $u(x) \to 0$, as $|x| \to \infty$.
For this purpose, we perform the De Giorgi iteration as in [**20**, theorem 7.1], [**22**, appendix 6]. Let $y \in \mathbb{R}^2$, $R > 0$ and $\sigma \in (0, 1)$ be arbitrarily given. Choose a cut-off function $\xi \in C_0^\infty(\mathbb{R}^2)$ with $\xi = 1$ on $B_{\sigma R}(y)$, $\xi = 0$ on $B_R^c(y)$, $0 \leqslant \xi \leqslant 1$ and $|\nabla\xi| \leqslant \frac{C}{(1-\sigma)R}$. Finally we set $\varphi = \xi^2(u-k)_+$ with $k \geqslant 0$.
By a density argument, one can use φ as a test function in (1.5) to obtain

$$\int_{\mathbb{R}^2} \xi^2(1+2\mu u^2)|\nabla(u-k)_+|^2 + \xi^2\big(2\mu|\nabla u|^2 + \omega + qV_1(x)(1+\mu u^2) + qV_2(x)\big)u(u-k)_+\,\mathrm{d}x$$

$$= -2\int_{\mathbb{R}^2} \xi(1+2\mu u^2)(u-k)_+\nabla u\cdot\nabla\xi\,\mathrm{d}x + \lambda\int_{\mathbb{R}^2}\xi^2|u|^{p-1}u(u-k)_+\,\mathrm{d}x.$$

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Since V_1, $V_2 \geqslant 0$, $(u-k)_+ = 0$ on $\{u \leqslant k\}$ and $0 \leqslant (u-k)_+ \leqslant u$ on $\{u > k\}$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^2} \xi^2\big(2\mu|\nabla u|^2 + \omega + qV_1(x)(1+\mu u^2) + qV_2(x)\big)u(u-k)_+ \,\mathrm{d}x \\
&\quad\geqslant 2\mu \int_{\{u>k\}} \xi^2 u(u-k)_+|\nabla u|^2 \,\mathrm{d}x \\
&\quad\geqslant 2\mu \int_{\{u>k\}} \xi^2 (u-k)_+^2|\nabla u|^2 \,\mathrm{d}x \\
&\quad= 2\mu \int_{\mathbb{R}^2} \xi^2 (u-k)_+^2|\nabla u|^2 \,\mathrm{d}x.
\end{aligned}$$

On the other hand by $\nabla(u-k)_+ = \nabla u$ on the set $\{u > k\}$, the Hölder inequality and the Young inequality, we also have

$$\begin{aligned}
&-2\int_{\mathbb{R}^2} \xi(1+2\mu u^2)(u-k)_+ \nabla u\cdot\nabla\xi \,\mathrm{d}x \\
&\quad\leqslant 2\int_{\{u>k\}} \xi(1+2\mu u^2)(u-k)_+|\nabla u||\nabla\xi| \,\mathrm{d}x \\
&\quad\leqslant 2\left(\int_{\{u>k\}} \xi^2(1+2\mu u^2)|\nabla u|^2 \,\mathrm{d}x\right)^{1/2}\left(\int_{\{u>k\}} (1+2\mu u^2)(u-k)_+^2|\nabla\xi|^2 \,\mathrm{d}x\right)^{1/2} \\
&\quad\leqslant \frac{1}{2}\int_{\{u>k\}} \xi^2(1+2\mu u^2)|\nabla u|^2 dx + 2\int_{\{u>k\}} (1+2\mu u^2)(u-k)_+^2|\nabla\xi|^2 \,\mathrm{d}x \\
&\quad= \frac{1}{2}\int_{\mathbb{R}^2} \xi^2(1+2\mu u^2)|\nabla (u-k)_+|^2 dx + 2\int_{\{u>k\}} (1+2\mu u^2)(u-k)_+^2|\nabla\xi|^2 \,\mathrm{d}x.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2} \xi^2(1+2\mu u^2)|\nabla(u-k)_+|^2 \,\mathrm{d}x + 2\mu\int_{\mathbb{R}^2} \xi^2(u-k)_+^2|\nabla u|^2 \,\mathrm{d}x \\
&\quad\leqslant 2\int_{\{u>k\}} (1+2\mu u^2)(u-k)_+^2|\nabla\xi|^2 \,\mathrm{d}x + \lambda\int_{\{u>k\}} \xi^2|u|^{p+1} \,\mathrm{d}x.
\end{aligned} \tag{3.3}$$

Now we put $v := \sqrt{1+2\mu u^2}(u-k)_+ + k$ so that $(1+2\mu u^2)(u-k)_+^2 = (v-k)_+^2$. Then it follows that

$$u > k \;\Leftrightarrow\; v > k \quad \text{and} \quad u \leqslant v \;\text{ a.e. } x \in \mathbb{R}^2. \tag{3.4}$$

Moreover by the Young inequality, one has

$$\begin{aligned}
|\nabla(v-k)_+|^2 &= (1+2\mu u^2)|\nabla(u-k)_+|^2 + 4\mu u(u-k)_+\nabla u\cdot\nabla(u-k)_+ \\
&\quad + \frac{4\mu^2u^2}{1+2\mu u^2}(u-k)_+^2|\nabla u|^2 \\
&\leqslant C\Big((1+2\mu u^2)|\nabla(u-k)_+|^2 + (u-k)_+^2|\nabla u|^2\Big).
\end{aligned}$$

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Thus from (3.3) and (3.4), we get

$$\int_{\mathbb{R}^2} \xi^2 |\nabla (v-k)_+|^2 \,\mathrm{d}x \leqslant C \left\{ \int_{\{v>k\}} (v-k)_+^2 |\nabla \xi|^2 \,\mathrm{d}x + \int_{\{v>k\}} \xi^2 |u|^{p+1} \,\mathrm{d}x \right\}. \tag{3.5}$$

Next by the Hölder inequality, one has

$$\begin{aligned} \int_{\{v>k\}} \xi^2 |u|^{p+1} \,\mathrm{d}x &\leqslant \left(\int_{\{v>k\}} \xi^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\{v>k\}} \xi^2 |u|^{2p+2} \,\mathrm{d}x \right)^{1/2} \\ &\leqslant \left(\int_{B_R(y)} |u|^{2p+2} \,\mathrm{d}x \right)^{1/2} |A_{k,R}^+|^{1/2}, \end{aligned}$$

where $A_{k,R}^+ := \{x \in B_R(y) : \; v(x) > k\}$. From (3.5), we find that

$$\begin{aligned} &\int_{B_{\sigma R}(y)} |\nabla (v-k)_+|^2 \,\mathrm{d}x \\ &\quad \leqslant C \left\{ \frac{1}{(1-\sigma)^2 R^2} \int_{B_R(y)} (v-k)_+^2 \,\mathrm{d}x + \|u\|_{L^{2p+2}(B_R(y))}^{p+1} |A_{k,R}^+|^{1/2} \right\}, \end{aligned}$$

for any $\sigma \in (0,1)$ and $k \geqslant 0$. This implies that v belongs to the De Giorgi class DG^+ and hence, by [**11**], we have

$$\sup_{x \in B_{\sigma R}(y)} v_+(x) \leqslant C \left(\|v\|_{L^2(B_R(y))} + \|u\|_{L^{2p+2}(B_R(y))}^{(p+1)/2} \right),$$

and so $v_+ \in L^\infty(\mathbb{R}^2)$. Since $u \leqslant v$, we deduce that $u_+ \in L^\infty(\mathbb{R}^2)$, too. Arguing similarly, one can show that u_- is bounded from above. This yields that $u \in L^\infty(\mathbb{R}^2)$. Finally since $u \in H_r^1(\mathbb{R}^2)$, by the radial lemma due to [**30**], u decays to zero at infinity.

Step 2: We claim that $u \in C^2(\mathbb{R}^2)$ and decays exponentially up to second derivatives.
By Step 1, we know that $u \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Although we only have V_1, $V_2 \in L^\infty(\mathbb{R}^2)$ at this stage, we find that $u \in C^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$ by applying the regularity result due to [**31**]. Then by (3.1) of lemma 3.1 and the Schauder estimate, we conclude that $u \in C^{2,\alpha}(\mathbb{R}^2)$. Finally the exponential decay follows by applying suitable comparison argument (see e.g. [**27**, theorem 4.1]). This completes the proof. □

Arguing as in [**6**], standard computations show that

LEMMA 3.3. *The functional I in (1.6) is well-defined and continuous in $\mathcal{X}$. Moreover, if the Gateaux derivative of I evaluated in $u \in \mathcal{X}$ is zero in every direction $\varphi \in C_{0,r}^\infty(\mathbb{R}^2)$, then u is a weak solution of (1.1).*

We conclude this section with the following

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LEMMA 3.4. *Any weak solution u of (1.1) satisfies the Nehari identity $N(u)=0$ and the Pohozaev identity $P(u)=0$, where*

$$N(u)=\int_{\mathbb{R}^2}\left\{(1+4\mu u^2)|\nabla u|^2+\omega u^2-\lambda|u|^{p+1}+q\frac{h_u^2(|x|)}{|x|^2}(3+2\mu u^2)u^2\right\}\mathrm{d}x, \tag{3.6}$$

$$P(u)=\int_{\mathbb{R}^2}\left\{\omega u^2-\frac{2\lambda}{p+1}|u|^{p+1}+q\frac{h_u^2(|x|)}{|x|^2}(2+\mu u^2)u^2\right\}\mathrm{d}x. \tag{3.7}$$

Proof. First by a density argument, one can use $u\in\mathcal{X}$ as a test function in (1.5). Then we see that the identity $N(u)=0$ holds.
Next let $u\in\mathcal{X}\cap C^2(\mathbb{R}^2)$ be a solution of (1.1). Then multiplying by $\nabla u\cdot x$ and integrating by parts on B_R, arguing as in [**6**, proposition 2.3], we have

$$\begin{aligned}
\int_{B_R}\Delta u(\nabla u\cdot x)\,\mathrm{d}x&=\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma=:\mathrm{I},\\
\int_{B_R}u\Delta u^2(\nabla u\cdot x)\,\mathrm{d}x&=\frac{1}{2}\int_{B_R}\Delta u^2(\nabla u^2\cdot x)\,\mathrm{d}x=\frac{R}{4}\int_{\partial B_R}|\nabla u^2|^2\,\mathrm{d}\sigma=:\mathrm{II},\\
\int_{B_R}u(\nabla u\cdot x)\,\mathrm{d}x&=-\int_{B_R}u^2\,\mathrm{d}x+\frac{R}{2}\int_{\partial B_R}u^2\,\mathrm{d}\sigma=-\int_{B_R}u^2\,\mathrm{d}x+\mathrm{III},\\
\int_{B_R}|u|^{p-1}u(\nabla u\cdot x)\,\mathrm{d}x&=-\frac{2}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x+\frac{R}{p+1}\int_{\partial B_R}|u|^{p+1}\,\mathrm{d}\sigma\\
&=-\frac{2}{p+1}\int_{B_R}|u|^{p+1}\,\mathrm{d}x+\mathrm{IV}.
\end{aligned}$$

Since $u\in\mathcal{X}$ and so $u^2\in H^1(\mathbb{R}^2)$, one can take $R_n\to\infty$ such that the terms I, II, III, IV with R_n replacing R converge to 0 as $n\to\infty$. Moreover, for $\alpha=2$ or $\alpha=4$, we have

$$\begin{aligned}
&\frac{4}{\alpha}\int_{B_{R_n}}\left(\int_{|x|}^{\infty}\frac{h_u(s)}{s}u^\alpha(s)\,\mathrm{d}s\right)u(\nabla u\cdot x)\,\mathrm{d}x+\int_{B_{R_n}}\frac{h_u^2(|x|)}{|x|^2}u^{\alpha-1}(\nabla u\cdot x)\,\mathrm{d}x\\
&=\int_{B_{R_n}}\frac{h_u^2(|x|)}{|x|^2}u^{\alpha-1}(\nabla u\cdot x)\,\mathrm{d}x+\frac{4}{\alpha}\int_{B_{R_n}}\frac{u^\alpha(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\,\mathrm{d}s\right)\\
&\quad\times\left(\int_0^{|x|}s^2u(s)u'(s)\,\mathrm{d}s\right)\mathrm{d}x\\
&\quad-\frac{4}{\alpha}\int_{B_{R_n}}\frac{u^\alpha(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\,\mathrm{d}s\right)\left(\int_0^{|x|}s^2u(s)u'(s)\,\mathrm{d}s\right)\mathrm{d}x\\
&\quad+\frac{4}{\alpha}\int_{B_{R_n}}\left(\int_{|x|}^{\infty}\frac{h_u(s)}{s}u^\alpha(s)\,\mathrm{d}s\right)u(\nabla u\cdot x)\,\mathrm{d}x
\end{aligned}$$

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$$= \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \int_{B_{R_n}} \frac{u^\alpha(tx)}{|x|^2} \left(\int_0^{|x|} su^2(ts)\,\mathrm{d}s \right)^2 \mathrm{d}x$$

$$- \frac{4}{\alpha} \int_{B_{R_n}} \frac{u^\alpha(x)}{|x|^2} \left(\int_0^{|x|} su^2(s)\,\mathrm{d}s \right) \left(\int_0^{|x|} s^2 u(s)u'(s)\,\mathrm{d}s \right) \mathrm{d}x$$

$$+ \frac{4}{\alpha} \int_{B_{R_n}} \left(\int_{|x|}^\infty \frac{h_u(s)}{s} u^\alpha(s)\,\mathrm{d}s \right) u(\nabla u \cdot x)\,\mathrm{d}x$$

$$= -\frac{4}{\alpha} \int_{B_{R_n}} \frac{u^\alpha(x)}{|x|^2} \left(\int_0^{|x|} su^2(s)\,\mathrm{d}s \right)^2 \mathrm{d}x + o_n(1).$$

Thus from (1.1), one has

$$\int_{B_{R_n}} \left\{ \omega u^2 - \frac{2\lambda}{p+1}|u|^{p+1} + q\frac{h_u^2(|x|)}{|x|^2}(2+\mu u^2)u^2 \right\} \mathrm{d}x + o_n(1) = 0,$$

from which we deduce that $P(u) = 0$. □

4. Proof of theorem 1.1

Throughout this section, we suppose that $p > 5$. In the following, for any $u \in \mathcal{X}$, we denote

$$A(u) = \int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x, \quad B(u) = \int_{\mathbb{R}^2} u^2\,\mathrm{d}x, \quad C(u) = \int_{\mathbb{R}^2} u^2|\nabla u|^2\,\mathrm{d}x,$$

$$D(u) = \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(\int_0^{|x|} su^2(s)\,\mathrm{d}s \right)^2 \mathrm{d}x,$$

$$E(u) = \int_{\mathbb{R}^2} \frac{u^4(x)}{|x|^2} \left(\int_0^{|x|} su^2(s)\,\mathrm{d}s \right)^2 \mathrm{d}x, \quad F(u) = \int_{\mathbb{R}^2} |u|^{p+1}\,\mathrm{d}x.$$

First we recall the following properties of $D(u)$.

LEMMA 4.1 [6, lemma 3.2]. *Suppose that a sequence $\{u_n\}$ converges weakly to a function u in $H_r^1(\mathbb{R}^2)$ as $n \to +\infty$. Then for each $\varphi \in H_r^1(\mathbb{R}^2)$, $\{D(u_n)\}$, $\{D'(u_n)[\varphi]\}$ and $\{D'(u_n)[u_n]\}$ converge up to a subsequence to $D(u)$, $D'(u)[\varphi]$ and $D'(u)[u]$, respectively, as $n \to +\infty$.*

Next we show that analogous properties hold for $E(u)$.

LEMMA 4.2. *Suppose that a sequence $\{u_n\}$ converges weakly to a function u in $H_r^1(\mathbb{R}^2)$ as $n \to +\infty$. Then for each $\varphi \in H_r^1(\mathbb{R}^2)$, $\{E(u_n)\}$, $\{E'(u_n)[\varphi]\}$ and $\{E'(u_n)[u_n]\}$ converge up to a subsequence to $E(u)$, $E'(u)[\varphi]$ and $E'(u)[u]$, respectively, as $n \to +\infty$.*

Proof. First we prove that $E(u_n) \to E(u)$ as $n \to +\infty$. Now one has

$$\begin{aligned}(2\pi)^2|E(u_n) - E(u)| &\leqslant \int_{\mathbb{R}^2} |u_n^4(x) - u^4(x)| \left(\frac{1}{|x|}\int_{B_{|x|}} u_n^2(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} u^4(x) \left| \left(\frac{1}{|x|}\int_{B_{|x|}} u_n^2(y)\,\mathrm{d}y\right)^2 \right. \\ &\quad \left. - \left(\frac{1}{|x|}\int_{B_{|x|}} u^2(y)\,\mathrm{d}y\right)^2 \right| \mathrm{d}x \\ &\leqslant \|u_n^4 - u^4\|_2 \left\| \frac{1}{|\cdot|}\int_{B_{|\cdot|}} u_n^2(y)\,\mathrm{d}y \right\|_4^2 \\ &\quad + \|u\|_8^4 \left\| \left(\frac{1}{|\cdot|}\int_{B_{|\cdot|}} u_n^2(y)\,\mathrm{d}y\right)^2 - \left(\frac{1}{|\cdot|}\int_{B_{|\cdot|}} u^2(y)\,\mathrm{d}y\right)^2 \right\|_2 .\end{aligned}$$

Since $H_r^1(\mathbb{R}^2)$ is compactly embedded into $L^q(\mathbb{R}^2)$ for all $q > 2$, it follows that $u_n^4 \to u^4$ in $L^2(\mathbb{R}^2)$. Moreover as shown in [**6**, lemma 3.2], we also have

$$\frac{1}{|x|}\int_{B_{|x|}} u_n^2(y)\,\mathrm{d}y \to \frac{1}{|x|}\int_{B_{|x|}} u^2(y)\,\mathrm{d}y \quad \text{in} \quad L^q(\mathbb{R}^2) \quad \text{for} \quad q > 2, \quad \text{as} \quad n \to \infty,$$

from which we conclude that $E(u_n) \to E(u)$, as $n \to +\infty$. Analogously one can show that $E'(u_n)[\varphi] \to E'(u)[\varphi]$ for any $\varphi \in H_r^1(\mathbb{R}^2)$ and $E'(u_n)[u_n] \to E'(u)[u]$, as $n \to \infty$. □

For any $u \in \mathcal{X}$ and $\alpha > 0$, we hereafter consider the map

$$t \in \mathbb{R}^+ \longmapsto u_t \in \mathcal{X}, \quad u_t(x) = t^\alpha u(tx).$$

By direct calculations we have $D(u_t) = t^{6\alpha-4} D(u)$ and $E(u_t) = t^{8\alpha-4} E(u)$. Thus we get

$$\begin{aligned}I(u_t) = \frac{t^{2\alpha}}{2} A(u) + \frac{t^{2\alpha-2}}{2}\omega B(u) + t^{4\alpha}\mu C(u) + \frac{t^{6\alpha-4}}{2} q D(u) + \frac{t^{8\alpha-4}}{4} q\mu E(u) \\ - \frac{t^{(p+1)\alpha-2}}{p+1}\lambda F(u).\end{aligned}$$

Let

$$\alpha \in \begin{cases} (1, +\infty) & \text{if } p \geqslant 7, \\ \left(1, \frac{2}{7-p}\right) & \text{if } 5 < p < 7. \end{cases} \tag{4.1}$$

We observe that, fixed $u \in \mathcal{X} \setminus \{0\}$, using (4.1), the dominant term near $t \sim 0$ of $I(u_t)$ is $t^{2\alpha-2}$, which implies that $I(u_t)$ is strictly positive for small $t > 0$ and any $u \in \mathcal{X} \setminus \{0\}$. Furthermore the dominant term near $t \sim +\infty$ among all positive

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terms of $I(u_t)$ is $t^{8\alpha-4}$. Thus under the assumption (4.1), we see that $8\alpha - 4 < (p+1)\alpha - 2$ and hence $I(u_t) \to -\infty$ as $t \to +\infty$ for any $u \in \mathcal{X} \setminus \{0\}$. These facts imply that the map

$$\gamma_u := t \in (0, +\infty) \longmapsto I(u_t)$$

has a maximum point at a positive level. The next lemma shows that this maximum point is the unique critical point.

LEMMA 4.3. *Let $u \in \mathcal{X} \setminus \{0\}$. Then the map γ_u attains its maximum at exactly one point $t(u) > 0$. Moreover γ_u is positive and increasing on $(0, t(u))$, and decreasing for $t > t(u)$.*

Proof. Let $u \in \mathcal{X} \setminus \{0\}$. A simple computation yields that

$$\begin{aligned}\gamma_u'(t) &= \alpha A(u)t^{2\alpha-1} + (\alpha-1)\omega B(u)t^{2\alpha-3} + 4\alpha\mu C(u)t^{4\alpha-1} + (3\alpha-2)qD(u)t^{6\alpha-5} \\ &\quad + (2\alpha-1)q\mu E(u)t^{8\alpha-5} - \frac{(p+1)\alpha-2}{p+1}\lambda F(u)t^{(p+1)\alpha-3} \\ &= t^{8\alpha-5}\left(\frac{\alpha A(u)}{t^{6\alpha-4}} + \frac{(\alpha-1)\omega B(u)}{t^{6\alpha-2}} + \frac{4\alpha\mu C(u)}{t^{4\alpha-4}} + \frac{(3\alpha-2)qD(u)}{t^{2\alpha}}\right. \\ &\quad \left. +(2\alpha-1)q\mu E(u) - \frac{(p+1)\alpha-2}{p+1}\lambda F(u)t^{(p-7)\alpha+2}\right) \\ &=: t^{8\alpha-5}g(t).\end{aligned}$$

From (4.1), it is clear that $\gamma_u'(t) > 0$ for small $t > 0$ and $\gamma_u'(t) < 0$ for large $t > 0$. Then, there exists $t_0 > 0$ such that $\gamma_u'(t_0) = 0$. Moreover, from the choice of α, the function $g(t)$ is strictly decreasing for all $t > 0$. Thus since $\{t > 0 : \gamma_u'(t) = 0\} = \{t > 0 : g(t) = 0\}$, the critical point of $\gamma_u(t)$ is unique. □

Let us define

$$\begin{aligned}\Gamma(u) : = \gamma_u'(1) &= \alpha A(u) + (\alpha-1)\omega B(u) + 4\alpha\mu C(u) + (3\alpha-2)qD(u) \\ &\quad + (2\alpha-1)q\mu E(u) - \frac{(p+1)\alpha-2}{p+1}\lambda F(u)\end{aligned}$$

and

$$\mathcal{M} := \{u \in \mathcal{X} \setminus \{0\} : \Gamma(u) = 0\}.$$

REMARK 4.4. From (3.6) and (3.7), we readily see that $\Gamma(u) = \alpha N(u) - P(u)$ and hence by lemma 3.4, any weak solution of (1.1) belongs to $\mathcal{M}$.

To complete the proof of theorem 1.1, we prepare several lemmas. The first one is a direct consequence of lemma 4.3 since

$$\Gamma(u_t) = t\gamma_u'(t). \tag{4.2}$$

LEMMA 4.5. *For any $u \in \mathcal{X} \setminus \{0\}$ there exists a unique $t(u) > 0$ such that $u_{t(u)} \in \mathcal{M}$.*

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Next we establish, in the next lemmas, that the functional I is strictly positive on $\mathcal{M}$. Indeed, as a first step we have

Lemma 4.6. *There exists $c > 0$ such that for any $u \in \mathcal{M}$*

$$I(u) \geqslant c \int_{\mathbb{R}^2} [|\nabla u|^2 + u^2 + u^2|\nabla u|^2] \, \mathrm{d}x.$$

Proof. Let $u \in \mathcal{M}$. Then we have

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{(p+1)\alpha - 2}\Gamma(u) \\
&= \left(\frac{1}{2} - \frac{\alpha}{(p+1)\alpha - 2}\right) A(u) + \left(\frac{1}{2} - \frac{\alpha - 1}{(p+1)\alpha - 2}\right) \omega B(u) \\
&\quad + \left(1 - \frac{4\alpha}{(p+1)\alpha - 2}\right) \mu C(u) \\
&\quad + \left(\frac{1}{2} - \frac{3\alpha - 2}{(p+1)\alpha - 2}\right) qD(u) + \left(\frac{1}{4} - \frac{2\alpha - 1}{(p+1)\alpha - 2}\right) q\mu E(u) \\
&= \frac{(p-1)\alpha - 2}{2\big((p+1)\alpha - 2\big)} A(u) + \frac{(p-1)\alpha}{2\big((p+1)\alpha - 2\big)} \omega B(u) + \frac{(p-3)\alpha - 2}{(p+1)\alpha - 2} \mu C(u) \\
&\quad + \frac{(p-5)\alpha + 2}{2\big((p+1)\alpha - 2\big)} qD(u) + \frac{(p-7)\alpha + 2}{4\big((p+1)\alpha - 2\big)} q\mu E(u).
\end{aligned}$$

By (4.1), all coefficients are positive and we conclude. □

Lemma 4.7. *There exist $c_1, c_2 > 0$ such that for any $u \in \mathcal{M}$*

$$\|u\|_{p+1}^{p+1} \geqslant c_1 \int_{\mathbb{R}^2} [|\nabla u|^2 + u^2 + u^2|\nabla u|^2] \, \mathrm{d}x \geqslant c_2.$$

Proof. Since $D(u)$ and $E(u)$ are nonnegative it follows that

$$(\alpha - 1) \int_{\mathbb{R}^2} [|\nabla u|^2 + \omega u^2 + \mu u^2 |\nabla u|^2] \, \mathrm{d}x - \frac{(p+1)\alpha - 2}{p+1} \lambda \int_{\mathbb{R}^2} |u|^{p+1} \, \mathrm{d}x \leqslant \Gamma(u) = 0,$$

for all $u \in \mathcal{M}$, and we have the first inequality.
Moreover, by the Sobolev inequality, one gets

$$\begin{aligned}
\int_{\mathbb{R}^2} [|\nabla u|^2 + u^2 + u^2|\nabla u|^2] \, \mathrm{d}x &\leqslant C_1 \|u\|_{p+1}^{p+1} \leqslant C_2 \|u\|_{H^1}^{p+1} \\
&\leqslant C_2 \left(\int_{\mathbb{R}^2} [|\nabla u|^2 + u^2 + u^2|\nabla u|^2] \, \mathrm{d}x \right)^{(p+1)/2}.
\end{aligned}$$

This completes the proof. □

Combining lemmas 4.6 and 4.7, we have

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LEMMA 4.8. *There exists $c > 0$ such that $I(u) \geqslant c$, for any $u \in \mathcal{M}$.*

Let us define

$$\sigma := \inf_{u\in\mathcal{M}} I(u). \tag{4.3}$$

Then by lemma 4.8, we infer that $\sigma > 0$. Moreover by lemmas 4.3, 4.5 and from (4.3), it follows that

$$\sigma = \inf_{u\in\mathcal{X}\setminus\{0\}} \max_{t>0} I(u_t). \tag{4.4}$$

Finally we establish the following result.

PROPOSITION 4.9. *Let $u \in \mathcal{X}$ be a minimizer of $I(u)$ under the constraint $\mathcal{M}$. Then u is a radial ground state solution of (1.1). Moreover, any radial ground state solution of (1.1) is positive.*

Proof. We argue as in [**22**, lemma 2.5] or [**28**, theorem 2.2].
Let $u \in \mathcal{M}$ be a minimizer of the functional $I|_{\mathcal{M}}$. Then from (4.4), one has

$$I(u) = \inf_{v\in\mathcal{X}\setminus\{0\}} \max_{t>0} I(v_t) = \inf_{v\in\mathcal{M}} I(v) = \sigma. \tag{4.5}$$

Suppose by contradiction that u is not a weak solution of (1.1). Then one can find $\varphi \in C^\infty_{0,r}(\mathbb{R}^2)$ such that

$$I'(u)[\varphi] < -1.$$

We choose small $\varepsilon > 0$ so that

$$I'(u_t + \tau\varphi)[\varphi] \leqslant -\frac{1}{2} \quad \text{for } |t-1| + |\tau| \leqslant \varepsilon. \tag{4.6}$$

Finally let $\xi \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $0 \leqslant \xi \leqslant 1$, $\xi(t) = 1$ for $|t-1| \leqslant \varepsilon/2$ and $\xi(t) = 0$ for $|t-1| \geqslant \varepsilon$.
For $t \geqslant 0$, we construct a path $\eta : \mathbb{R}_+ \to \mathcal{X}$ defined by

$$\eta(t) = \begin{cases} u_t & \text{if } |t-1| \geqslant \varepsilon \\ u_t + \varepsilon\xi(t)\varphi & \text{if } |t-1| < \varepsilon. \end{cases}$$

Then η is continuous on the metric space $(\mathcal{X}, d_{\mathcal{X}})$. Moreover, choosing ε smaller if necessary, it follows that $d_{\mathcal{X}}(\eta(t), 0) > 0$, for $|t-1| < \varepsilon$. Next we claim that

$$\sup_{t\geqslant 0} I(\eta(t)) < \sigma. \tag{4.7}$$

If $|t-1| \geqslant \varepsilon$, one has

$$I(\eta(t)) = I(u_t) < I(u) = \sigma,$$

because the function $t \mapsto I(u_t)$ attains its maximum at $t = 1$ for $u \in \mathcal{M}$.
If $|t-1| < \varepsilon$, we get

$$I(u_t + \varepsilon\xi(t)\varphi) = I(u_t) + \int_0^\varepsilon I'\big(u_t + \tau\xi(t)\varphi\big)[\xi(t)\varphi]\,\mathrm{d}\tau.$$

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Then from (4.6), we obtain

$$I(\eta(t)) \leqslant I(u_t) - \frac{1}{2}\varepsilon\xi(t) < \sigma,$$

yielding that (4.7) holds.
Now by (4.2) and arguing as in lemma 4.3, it follows that $\Gamma(\eta(1-\varepsilon)) > 0$ and $\Gamma(\eta(1+\varepsilon)) < 0$. By the continuity of the map $t \mapsto \Gamma(\eta(t))$, there exists $t_0 \in (1-\varepsilon, 1+\varepsilon)$ such that $\Gamma(\eta(t_0)) = 0$. This implies that $\eta(t_0) = u_{t_0} + \varepsilon\xi(t_0)\varphi \in \mathcal{M}$ and $I(\eta(t_0)) < \sigma$ by (4.7). This contradicts (4.5), and hence u is a weak solution of (1.1). By remark 4.4, since any weak solution of (1.1) belongs to $\mathcal{M}$, we conclude that u is a radial ground state solution.
Finally, if u is a minimizer of $I|_{\mathcal{M}}$, then one finds that $|u|$ is also a minimizer. Thus we may assume that $u \geqslant 0$. Then, by proposition 3.2, we know that $u \in C^2(\mathbb{R}^2)$ and hence we can apply the Harnack inequality [**32**] to conclude that $u > 0$. □

Proof of theorem 1.1. Let $\{u_n\}$ be a minimizing sequence for $I|_{\mathcal{M}}$, namely $\{u_n\} \subset \mathcal{M}$ and $I(u_n) \to \sigma$ as $n \to +\infty$. By lemma 4.6, the sequences $\{u_n\}$ and $\{u_n^2\}$ are bounded in $H_r^1(\mathbb{R}^2)$. Therefore, there exists $\bar{u} \in \mathcal{X}$ such that, by the compactness result due to [**30**], up to a subsequence

$$\begin{aligned} &u_n \rightharpoonup \bar{u} \text{ weakly in } H^1(\mathbb{R}^2),\\ &u_n^2 \rightharpoonup \bar{u}^2 \text{ weakly in } H^1(\mathbb{R}^2),\\ &u_n \to \bar{u} \text{ in } L^q(\mathbb{R}^2) \text{ for any } q > 2. \end{aligned}$$

Then, by lemma 4.7, we infer that $\bar{u} \neq 0$.
Next, by lemma 4.5, let us consider $\bar{t} = t(\bar{u}) > 0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}$. Since $[u_n]_{\bar{t}} \rightharpoonup \bar{u}_{\bar{t}}$ and $([u_n]_{\bar{t}})^2 \rightharpoonup \bar{u}_{\bar{t}}^2$ weakly in $H^1(\mathbb{R}^2)$ as $n \to +\infty$, by lemmas 4.1 and 4.2, we have

$$\sigma \leqslant I(\bar{u}_{\bar{t}}) \leqslant \liminf_{n\to\infty} I([u_n]_{\bar{t}}).$$

On the other hand, since $\{u_n\} \subset \mathcal{M}$, the function $t \mapsto I([u_n]_t)$ reaches its maximum at $t = 1$ for all $n \in \mathbb{N}$. This implies that

$$\liminf_{n\to\infty} I([u_n]_{\bar{t}}) \leqslant \liminf_{n\to\infty} I(u_n) = \sigma.$$

Therefore, $\bar{u}_{\bar{t}}$ is minimizer of I on $\mathcal{M}$. Finally by proposition 4.9, we conclude that, actually, $\bar{u}_{\bar{t}}$ is a radial ground state solution of (1.1). □

5. Proof of theorem 1.2

In this section, we prove the nonexistence result for (1.1) when $1 < p < 5$. First we state the following inequality which was obtained in [**6**].

PROPOSITION 5.1 (**6**, proposition 2.4). *For any $u \in H_r^1(\mathbb{R}^2)$, the following inequality holds:*

$$\int_{\mathbb{R}^2} |u|^4 \,\mathrm{d}x \leqslant 2\|\nabla u\|_2 \left(\int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 \,\mathrm{d}x\right)^{1/2}. \tag{5.1}$$

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Next we establish the following inequality, which cannot be obtained by (5.1) directly.

PROPOSITION 5.2. *For any $u \in \mathcal{X}$, the following inequality holds:*

$$\int_{\mathbb{R}^2} |u|^6 \, \mathrm{d}x \leqslant 4 \left(\int_{\mathbb{R}^2} u^2 |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^4 \, \mathrm{d}x \right)^{1/2}. \tag{5.2}$$

Proof. The proof is the same as that of proposition 5.1. By the density, we may assume that $u \in C_0^\infty(\mathbb{R}^2)$. Then by the Fubini theorem and the Schwarz inequality, one has

$$\begin{aligned}
\int_{\mathbb{R}^2} |u|^6 \, \mathrm{d}x &= 2\pi \int_0^\infty r u^2(r) \left(\int_r^\infty -\left(u^4(s)\right)' \, \mathrm{d}s \right) \mathrm{d}r \\
&\leqslant 8\pi \int_0^\infty \int_0^\infty r u^2(r) |u(s)|^3 |u'(s)| \chi_{\{s>r\}} \, \mathrm{d}s \, \mathrm{d}r \\
&= 8\pi \int_0^\infty |u(s)|^3 |u'(s)| \left(\int_0^s r u^2(r) \, \mathrm{d}r \right) \mathrm{d}s \\
&= 4 \int_{\mathbb{R}^2} \frac{|u|^3 |\nabla u|}{|x|} h_u(|x|) \, \mathrm{d}x \\
&\leqslant 4 \left(\int_{\mathbb{R}^2} u^2 |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^4 \, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

□

Proof of theorem 1.2. Suppose that $1 < p < 5$ and let $u \in \mathcal{X}$ be a solution of (1.1). We distinguish three cases: $0 < q < 1/3$, $1/3 \leqslant q < 2$ and $q \geqslant 2$.
First we consider the case $0 < q < 1/3$. From (5.1), (5.2) and by the Young inequality, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u|^4 \, \mathrm{d}x \leqslant \|\nabla u\|_2^2 + \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 \, \mathrm{d}x, \\
&\int_{\mathbb{R}^2} |u|^6 \, \mathrm{d}x \leqslant 2 \int_{\mathbb{R}^2} u^2 |\nabla u|^2 \, \mathrm{d}x + 2 \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^4 \, \mathrm{d}x.
\end{aligned} \tag{5.3}$$

Then since $N(u) = 0$, we obtain

$$\begin{aligned}
0 &\geqslant (1 - 3q) \|\nabla u\|_2^2 + 2\mu(2 - q) \int_{\mathbb{R}^2} u^2 |\nabla u|^2 \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \left(\omega u^2 + 3q u^4 + q\mu u^6 - \lambda |u|^{p+1} \right) \mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^2} \left(\omega u^2 + 3q u^4 + q\mu u^6 - \lambda |u|^{p+1} \right) \mathrm{d}x.
\end{aligned}$$

In the case $\frac{1}{3} \leqslant q < 2$, we slightly modify the use of (5.1) to obtain

$$\int_{\mathbb{R}^2} |u|^4 \, \mathrm{d}x \leqslant \frac{1}{3q} \|\nabla u\|_2^2 + 3q \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 \, \mathrm{d}x.$$

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From this estimate, the Nehari identity and (5.3), one gets

$$0 \geqslant \left(1 - \frac{1}{3q}\right) \|\nabla u\|_2^2 + 2\mu(2-q) \int_{\mathbb{R}^2} u^2|\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} \left(\omega u^2 + u^4 + q\mu u^6 - \lambda |u|^{p+1}\right) \mathrm{d}x.$$

Finally in the case $q \geqslant 2$, we apply the following estimate which is derived from (5.2):

$$\int_{\mathbb{R}^2} |u|^6 \, \mathrm{d}x \leqslant \frac{4}{q} \int_{\mathbb{R}^2} u^2 |\nabla u|^2 \, \mathrm{d}x + q \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^4 \, \mathrm{d}x.$$

Then we have

$$0 \geqslant \left(1 - \frac{1}{3q}\right) \|\nabla u\|_2^2 + 4\mu \left(1 - \frac{2}{q}\right) \int_{\mathbb{R}^2} u^2|\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} \left(\omega u^2 + u^4 + 2\mu u^6 - \lambda |u|^{p+1}\right) \mathrm{d}x.$$

Now we define $g : \mathbb{R} \to \mathbb{R}$ where

$$g(t) := \begin{cases} \omega t^2 + 3qt^4 + q\mu t^6 - \lambda |t|^{p+1} & \text{if } 0 < q < \frac{1}{3}, \\ \omega t^2 + t^4 + q\mu t^6 - \lambda |t|^{p+1} & \text{if } \frac{1}{3} \leqslant q < 2, \\ \omega t^2 + t^4 + 2\mu t^6 - \lambda |t|^{p+1} & \text{if } q \geqslant 2. \end{cases}$$

Then one has

$$\int_{\mathbb{R}^2} g(u) \, \mathrm{d}x \leqslant 0. \tag{5.4}$$

We observe that for given q, μ and $\lambda > 0$, there exists $\bar{\omega} > 0$ such that $g(t) > 0$ for $\omega \geqslant \bar{\omega}$ and $t \neq 0$. Indeed for $0 < q < 1/3$, one has

$$g'(t) = t\Big(2\omega + 12qt^2 + 6q\mu t^4 - (p+1)\lambda |t|^{p-1}\Big).$$

Since $1 < p < 5$, we can apply the Young inequality to

$$(p+1)\lambda |t|^{p-1} = \left(\frac{24q\mu |t|^4}{p-1}\right)^{(p-1)/4} \cdot (p+1)\lambda \left(\frac{p-1}{24q\mu}\right)^{(p-1)/4}$$

and obtain

$$(p+1)\lambda |t|^{p-1} \leqslant 6q\mu t^4 + \frac{5-p}{4} \left((p+1)\lambda\right)^{4/(5-p)} \left(\frac{p-1}{24q\mu}\right)^{(p-1)/(5-p)}.$$

Thus it follows that

$$g'(t) \geqslant t \left(2\omega - \frac{5-p}{4} \left((p+1)\lambda\right)^{4/(5-p)} \left(\frac{p-1}{24q\mu}\right)^{(p-1)/(5-p)} + 12qt^2\right) \text{ for } t > 0.$$

Taking ω larger, we have $g'(t) > 0$ for $t > 0$. Other cases can be treated in the same way. Similarly one has $g'(t) < 0$ for $t < 0$ and hence $g(t) > 0$ for $t \neq 0$, as claimed. This and (5.4) imply that $u \equiv 0$ and hence the proof is complete. □

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Remark 5.3. It is easy to check that $\bar{\omega}$, defined in theorem 1.2, increases as q decreases. In other words, if we fix ω, we have to take q smaller in order to obtain nontrivial solutions of (1.1). This is exactly the situation studied in [**10**] for the case $\mu = 0$.
Analogously $\bar{\omega}$ increases as μ decreases. In particular, when $1 < p < 3$, we notice that $\bar{\omega}$ can be chosen independent of μ and this is consistent with the result obtained in [**25**]. On the other hand, if $3 \leqslant p < 5$, we have to choose larger $\bar{\omega}$ as μ becomes smaller. We expect, therefore, that, for fixed ω and $3 \leqslant p < 5$, we are able to find a nontrivial solution of (1.1) provided that μ is sufficiently large.

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