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POINTWISE TOPOLOGICAL STABILITY

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Abstract We decompose the topological stability (in the sense of P. Walters) into the corresponding notion for points. Indeed, we define a *topologically stable point* of a homeomorphism f as a point x such that for any C^0-perturbation g of f there is a continuous semiconjugation *defined on the g-orbit closure of x* which tends to the identity as g tends to f. We obtain some properties of the topologically stable points, including preservation under conjugacy, vanishing for minimal homeomorphisms on compact manifolds, the fact that topologically stable chain recurrent points belong to the periodic point closure, and that the chain recurrent set coincides with the closure of the periodic points when all points are topologically stable. Next, we show that the topologically stable points of an expansive homeomorphism of a compact manifold are precisely the shadowable ones. Moreover, an expansive homeomorphism of a compact manifold is topologically stable if and only if every point is topologically stable. Afterwards, we prove that a pointwise recurrent homeomorphism of a compact manifold has no topologically stable points. Finally, we prove that every chain transitive homeomorphism with a topologically stable point of a compact manifold has the pseudo-orbit tracing property. Therefore, a chain transitive expansive homeomorphism of a compact manifold is topologically stable if and only if it has a topologically stable point.

Keywords: topologically stable point; shadowable point; homeomorphism; metric space

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1. Introduction

There are concepts in dynamics admitting pointwise counterparts. This is the case for the equicontinuous, expansive, distal and persistent homeomorphisms having *equicontinuous*, *expansive* [**5**, **11**], *distal* [**3**] and *persistent points* [**8**] as such counterparts. More recently, the *shadowable points* appeared as the pointwise counterpart for the pseudo-orbit tracing property (POTP) [**9**]. A further example is the *entropy point* corresponding to the notion of topological entropy [**15**]. In light of these examples, it is natural to believe that pointwise versions of other dynamical notions can be found.

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In this paper, we suggest a definition of topologically stable point for homeomorphisms on metric spaces. Roughly, we will say that $x \in X$ is *topologically stable* for a homeomorphism $f : X \to X$ of a metric space X if for any C^0-perturbation g of f there is a continuous semiconjugation defined in the g-orbit closure of x which tends to the identity as g tends to f. We prove that the set of topologically stable points is f-invariant and preserved under conjugation. Moreover, such a set is empty for minimal homeomorphisms on compact manifolds. The chain recurrent points which are topologically stable belong to the periodic point closure. In particular, the chain recurrent set coincides with the closure of the periodic points when all points are topologically stable. Next, we show that the topologically stable points of an expansive homeomorphism of a compact manifold are precisely the shadowable ones. Moreover, an expansive homeomorphism of a compact manifold is topologically stable if and only if every point is topologically stable. Afterwards, we prove that a pointwise recurrent homeomorphism of a compact manifold has no topologically stable points. Finally, we prove that every chain transitive homeomorphism with a topologically stable point of a compact manifold has the POTP. Therefore, a chain transitive expansive homeomorphism of a compact manifold is topologically stable if and only if it has a topologically stable point.

This paper is organized as follows. In § 2 we state the definition of a topologically stable point, along with some examples and basic properties. In § 3 we state the main results of this paper. These results relate the topologically stable points to the shadowable points and the expansive systems.

2. Topologically stable points: definition, examples and properties

Denote by X a metric space and by $f : X \to X$ a homeomorphism. The C^0-*distance* between maps $l, r : A \subseteq X \to X$ is defined by

$$d_{C^0}(l, r) = \sup_{x \in A} d(l(x), r(x)).$$

Denote by I_X the identity of X.

Definition 2.1. We say that f is *topologically stable* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \leq \delta$ there is a continuous map $k : X \to X$ such that $d_{C^0}(k, I_X) \leq \epsilon$ and $f \circ k = k \circ g$.

The main purpose of this paper is to introduce a pointwise version of this definition. For this we recall some basic concepts. Let $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ be the orbit of $x \in X$ under f, $\overline{B}$ the closure of B, $\overline{O_f(x)}$ the orbit closure and i_B the inclusion $B \hookrightarrow X$ of a subset $B \subseteq X$.

Our definition of a topologically stable point starts with the following simple remark.

Remark 2.2. A necessary condition for a homeomorphism $f : X \to X$ of a metric space X to be topologically stable is that every $x \in X$ satisfies the following property.

(P) For every $\epsilon > 0$ there is $\delta_x > 0$ such that for every homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) \leq \delta_x$ there is a continuous map $h : \overline{O_g(x)} \to X$ such that $d_{C^0}(h, i_{\overline{O_g(x)}}) \leq \epsilon$ and $f \circ h = h \circ g$.

Proof. Fix $x \in X$, $\epsilon > 0$ and take $\delta_x = \delta$ where $\delta > 0$ is given by the topological stability of f for this ϵ. If $g : X \to X$ is a homeomorphism with $d_{C^0}(f, g) \leq \delta_x$, then there is $k : X \to X$ continuous such that $d_{C^0}(k, I_X) \leq \epsilon$ and $f \circ k = k \circ g$. By taking h as the restriction $k|_{\overline{O_g(x)}}$ of k to $\overline{O_g(x)}$, we obtain (P). $\square$

This remark motivates the following definition.

Definition 2.3. We say that $x \in X$ is a *topologically stable point* of a homeomorphism $f : X \to X$ of a metric space X if x satisfies (P). Hereafter, $T(f)$ will denote the set of topologically stable points of f.

Let us present some related examples.

Example 2.4. By Remark 2.2, if f is topologically stable, then every point is. We do not know whether the converse holds. Corollary 3.5 gives some insight into this question.

Example 2.5. For every metric space X, one has that $T(I_X)$ consists of those points $x \in X$ satisfying the following property: for every $\epsilon > 0$ there is $\delta > 0$ such that $d(x, g^n(x)) \leq \epsilon$ for every $n \in \mathbb{Z}$ for every homeomorphism $g : X \to X$ with $d_{C^0}(I_X, g) \leq \delta$. This clearly implies that $T(I_X)$ contains the isolated points of X. It also implies that $T(I_X) = \emptyset$ when X is a manifold. The latter assertion will be generalized in Theorem 3.6.

Example 2.6. Suppose that X is a metric space and that $x_0 \in X$ is a fixed point for every homeomorphism $f : X \to X$. Then $x_0 \in T(f)$ for every homeomorphism $f : X \to X$. By applying this to the union X of two circles with a common point x_0 we have that $T(I_X) = \{x_0\}$. Hence, there are homeomorphisms on uncountable compact metric spaces exhibiting a unique topologically stable point.

By the previous example, the set of topologically stable points may not be open. The following example shows that such a set may also not be closed.

Example 2.7. There is a compact metric space X and a homeomorphism $f : X \to X$ such that $T(f)$ is not closed.

Proof. Define $X = S(1) \cup (\bigcup_{n \in \mathbb{N}^+} S(1 + (1/n)))$ where $S(r)$ is a circle of radius r of $\mathbb{R}^2$ centred at $(0, 0)$. Define $f : X \to X$ by setting $f_n = f|_{S(1+(1/n))}$ to be a Morse–Smale diffeomorphism with $2 + 4(n - 1)$ alternating hyperbolic fixed points and $f|_{S(1)} = I_{S(1)}$. Equipping X with the Euclidean metric, we obtain that X is a compact metric space and that $f : X \to X$ is a homeomorphism. Now, for every $x \in S(1 + (1/n))$ and $n \in \mathbb{N}^+$ there is $\delta_x > 0$ such that every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta_x$ satisfies $g(S(1 + (1/n))) = S(1 + (1/n))$. Since the Morse–Smale diffeomorphisms are topologically stable [14], this implies that $x \in T(f)$. On the other hand, by slightly perturbing f we get a homeomorphism $g : X \to X$ close to f such that $g|_{S(1)}$ is an irrational rotation. From this we get $S(1) \cap T(f) = \emptyset$, proving $T(f) = \bigcup_{n \in \mathbb{N}^+} S(1 + (1/n))$. Therefore, every $x \in S(1)$ lies in $Cl(T(f)) \setminus T(f)$ and so $T(f)$ is not closed. $\square$

To state properties of $T(f)$ we establish some short definitions. Let $f : X \to X$ be a homeomorphism of a metric space X. We say that f is *minimal* if $O_f(x)$ is dense in X

for every $x \in X$. We say that $x \in X$ is *periodic* if $f^n(x) = x$ for some positive integer n. Moreover, x is *closable* (or satisfies the C^0-closing lemma [7]) if for every $\delta > 0$ there is a homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta$ such that x is a periodic point of g. Denote by $CL(f)$ the set of closable points and by $Per(f)$ the set of periodic points. The interior of a subset B is denoted by $Int(B)$. We say that B is *invariant* if $f(B) = B$. Given $\epsilon > 0$, a finite sequence $\{x_0, \ldots, x_k\}$ is an *ϵ-chain* from x to y if $x_0 = x$, $x_k = y$ and $d(f(x_n), x_{n+1}) \leq \epsilon$ for every $0 \leq n \leq k$. We say that x is *chain recurrent* if for every ϵ there is an ϵ-chain from x to itself. The *chain recurrent set* is the set of chain recurrent points denoted by $CR(f)$.

The main result of this section collects some basic properties of $T(f)$.

Theorem 2.8. *The following properties hold for every homeomorphism $f : X \to X$ of every compact metric space X.*

1. *$T(f^{-1}) = T(f)$.*
2. *If $H : X \to Y$ is a homeomorphism of metric spaces, then $T(H \circ f \circ H^{-1}) = H(T(f))$.*
3. *$T(f^k)$ is an invariant set of f, $\forall k \in \mathbb{Z}$. In particular, the set of topologically stable points is invariant.*
4. *$CL(f) \cap T(f) \subseteq \overline{Per(f)}$. In particular, if X is infinite and f minimal, then $CL(f) \cap T(f) = \emptyset$. Consequently, a minimal homeomorphism of a compact manifold has no topologically stable points.*
5. *If X is infinite and f can be C^0 approximated by minimal homeomorphisms, then $Int(Per(f)) \cap T(f) = \emptyset$.*
6. *If X is a compact boundaryless manifold, then $CR(f) \cap T(f) \subseteq \overline{Per(f)}$. In particular, $CR(f) = \overline{Per(f)}$ if every point is topologically stable.*

Proof. To prove item (1) take $x \in T(f)$ and $\epsilon > 0$. For this ϵ we let $\delta'_x > 0$ be given by the topological stability of x with respect to f. Since f continuous and X compact, we have that f is uniformly continuous. It follows that there is $\delta_x > 0$ such that

$$d(a, b) \leq \delta_x \Longrightarrow d(f(a), f(b)) \leq \delta'_x.$$

Take a homeomorphism $g : X \to X$ with $d_{C^0}(f^{-1}, g) \leq \delta_x$. Then the choice of δ_x implies $d_{C^0}(I_X, f \circ g) \leq \delta'_x$ and so $d_{C^0}(f, g^{-1}) \leq \delta'_x$. So, the choice of δ'_x yields $h : \overline{O_{g^{-1}}(x)} \to X$ continuous such that $d_{C^0}(h, i_{\overline{O_{g^{-1}}(x)}}) \leq \epsilon$ and $f \circ h = h \circ g^{-1}$. But clearly $\overline{O_{g^{-1}}(x)} = \overline{O_g(x)}$ and so we really have a continuous map $h : \overline{O_g(x)} \to X$ with $d_{C^0}(h, i_{\overline{O_g(x)}}) \leq \epsilon$. As $f \circ h = h \circ g^{-1}$, we also obtain $f^{-1} \circ h = h \circ g$. Therefore, $x \in T(f^{-1})$ proving $T(f) \subseteq T(f^{-1})$. By replacing f by f^{-1} we also get $T(f) \supseteq T(f^{-1})$, yielding item (1).

Now we prove item (2). Fix $x \in T(f)$ and $\epsilon > 0$. Since X is compact, H is uniformly continuous and so there is $\epsilon' > 0$ such that $d(a, b) \leq \epsilon' \implies d(H(a), H(b)) \leq \epsilon$. For this ϵ' we let δ'_x be given by $x \in T(f)$. Again H^{-1} is uniformly continuous, so there is $\delta_x > 0$ such that $d(a, b) \leq \delta_x \implies d(H^{-1}(a), H^{-1}(b)) \leq \delta'_x$. Take a homeomorphism $g : Y \to Y$ such

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that $d_{C^0}(H\circ f\circ H^{-1},g)\leq\delta_x$. Then, $d_{C^0}(f\circ H^{-1},H^{-1}\circ g)\leq\delta'_x$ by the choice of δ_x and so $d_{C^0}(f,h^{-1}\circ g\circ H)\leq\delta'_x$ too. Then, the choice of δ'_x provides a continuous map $h':\overline{O_{H^{-1}\circ g\circ H}(x)}\to X$ satisfying $d_{C^0}(h',i_{\overline{O_{H^{-1}\circ g\circ H}(x)}})\leq\epsilon'$ and $f\circ h'=h'\circ H^{-1}\circ g\circ H$. On the other hand, it is clear that $H^{-1}(O_g(H(x)))=O_{H^{-1}\circ g\circ H}(x)$. Since H^{-1} is continuous and $O_g(H(x))$ dense in $\overline{O_g(H(x))}$, we get $H^{-1}(\overline{O_g(H(x))})=\overline{O_{H^{-1}\circ g\circ H}(x)}$. From this we obtain the map $h:\overline{O_g(H(x))}\to X$ by $h=H\circ h'\circ H^{-1}$. It is clear that h is continuous. Moreover,

$$\begin{aligned}(H\circ f\circ H^{-1})\circ h&=H\circ f\circ H^{-1}\circ H\circ h'\circ H^{-1}\\&=H\circ f\circ h\circ H^{-1}\\&=H\circ h'\circ H^{-1}\circ g\circ H\circ H^{-1}\\&=H\circ h'\circ H^{-1}\circ g\\&=h\circ g\end{aligned}$$

yielding $(H\circ f\circ H^{-1})\circ h=h\circ g$. Since $d_{C^0}(h',i_{\overline{O_{H^{-1}\circ g\circ H}(x)}})\leq\epsilon'$, one has

$$d(h'(H^{-1}(w)),H^{-1}(w))\leq\epsilon',\quad\forall w\in\overline{O_g(H(x))}.$$

Then, the choice of ϵ' implies $d(H(h'(H^{-1}(w))),w)\leq\epsilon$ for every $w\in\overline{O_g(H(x))}$, proving $d_{C^0}(h,i_{\overline{O_g(H(x))}})\leq\epsilon$. All this together proves

$$H(T(f))\subseteq T(H\circ f\circ H^{-1}).$$

Replacing H by H^{-1} and f by $H\circ f\circ H^{-1}$ in this inclusion, we obtain $H^{-1}(T(H\circ f\circ H^{-1}))\subseteq T(H^{-1}\circ H\circ f\circ H^{-1}\circ H)=T(f)$. Then,

$$H(T(f))\supseteq T(H\circ f\circ H^{-1}).$$

This proves item (2).

By taking $H=f$ in item (2) we obtain $f(T(f^k))=T(f\circ f^k\circ f^{-1})=T(f^k)$, proving item (3).

To prove item (4), fix $x\in CL(f)\cap T(f)$ and $\epsilon>0$. Take δ_x from the topological stability of x for this ϵ. Since $x\in CL(f)$, there is a homeomorphism $g:X\to X$ with $d_{C^0}(f,g)\leq\delta_x$ such that $x\in Per(g)$. Since $d_{C^0}(f,g)\leq\delta_x$, there is a continuous map $h:\overline{O_g(x)}\to X$ such that $d(h,i_{\overline{O_g(x)}})\leq\epsilon$ and $f\circ h=h\circ g$. As $x\in Per(g)$, we have $g^n(x)=x$ for some $n\in\mathbb{N}^+$ and so $f^n(h(x))=h(g^n(x))=h(x)$ yielding $h(x)\in Per(f)$. Since $d(x,h(x))\leq d_{C^0}(h,i_{\overline{O_g(x)}})\leq\epsilon$, we get a periodic point of f within ϵ from x. As ϵ is arbitrary, $x\in\overline{Per(f)}$, proving the first part of item (4). For the second part, if X is infinite and f minimal we have $Per(f)=\emptyset$ so $\overline{Per(f)}=\emptyset$, and thus $CL(f)\cap T(f)=\emptyset$ by the first part. The last part follows from the second and the fact that $CL(f)=X$ for every minimal homeomorphism $f:X\to X$ of a compact manifold X (see [7, p. 173]). This proves item (4).

Now we prove item (5). If the conclusion is not true, there is $x\in Int(Per(f))\cap T(f)$. Fix $\epsilon>0$ such that $y\in Per(f)$ for every $y\in X$ with $d(x,y)\leq\epsilon$. Now take $\delta_x>0$ for this ϵ from the topological stability of x. By hypothesis, there is a minimal homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta_x$. Hence, there is $h : \overline{O_g(x)} \to X$ continuous such that $d_{C^0}(h, i_{\overline{O_g(x)}}) \leq \epsilon$ and $f \circ h = h \circ g$. Again the latter identity implies $O_f(h(x)) = h(O_g(x))$. As g is minimal, $O_g(x)$ is dense in X and so is $h(O_g(x))$ by the continuity of h. It follows that $O_f(h(x))$ is dense in X. However, $d(h(x), x) \leq \epsilon$ so $h(x) \in Per(f)$ by the choice of ϵ, thus $O_f(h(x))$ is a finite set. Hence, X has a finite dense subset, which contradicts that X is infinite. This ends the proof of item (5).

Finally, we prove item (6). We have that $CR(f) = CL(f)$ by the theorem in [**7**, p. 173], and so $CL(f) \cap T(f) \subseteq \overline{Per(f)}$ by item (4). This proves the first part of item (6). The last part of this item follows from the first part. □

A first application of this result is as follows.

Example 2.9. Notice that $T(f) = \emptyset$ for the circle rotations f. Indeed, the irrational ones satisfy it by Theorem 2.8(4), since they are minimal and C^0-approximated by homeomorphisms with all points periodic (the rational rotations). The rational ones satisfy it by Theorem 2.8(5).

3. Topologically stable and shadowable points for expansive and chain transitive systems

In this section we will present the main results of this paper. They deal with the topologically stable points for expansive homeomorphisms on compact manifolds. The relationship between the topologically stable and shadowable points will be also explored.

3.1. Definitions and statement of the results

We say that a homeomorphism $f : X \to X$ of a metric space X is *expansive* [**12**] if there is $e > 0$ (called the *expansivity constant*) such that $x = y$ whenever $x, y \in X$ satisfy $d(f^n(x), f^n(y)) \leq e$ for every $n \in \mathbb{Z}$. Given $\delta > 0$, a bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ of X is a *δ-pseudo-orbit* of f if $d(f(x_n), x_{n+1}) \leq \delta$ for all $n \in \mathbb{Z}$. We say that the sequence can be *δ-shadowed* if there is $x \in X$ such that $d(f^n(x), x_n) \leq \delta$ for all $n \in \mathbb{Z}$.

Definition 3.1 (see [9]). A point $x \in X$ is *shadowable* if for every $\epsilon > 0$ there is $\delta_x > 0$ such that every δ_x-pseudo-orbit $(x_n)_{n\in\mathbb{Z}}$ with $x_0 = x$ can be ϵ-shadowed.

With these definitions, we have the following result.

Theorem 3.2. *The topologically stable points of an expansive homeomorphism of a compact manifold are precisely the shadowable ones.*

The following definition is a natural subproduct of Example 2.4.

Definition 3.3. We call a homeomorphism $f : X \to X$ *pointwise topologically stable* if $T(f) = X$ (i.e. if every point is topologically stable).

Example 3.4. As reported in Example 2.4, every topologically stable homeomorphism is pointwise topologically stable, but we do not know whether the converse holds.

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On the other hand, Theorem 2.8(6) implies that every pointwise topologically stable homeomorphism of a compact boundaryless manifold has a periodic point.

By Theorem 3.2 we obtain the following corollary.

Corollary 3.5. *An expansive homeomorphism of a compact manifold is topologically stable if and only if it is pointwise topologically stable.*

Define the *omega-limit set* of $x \in X$ with respect to a homeomorphism $f : X \to X$ by $\omega(x) = \{y \in X : y = \lim_{k\to\infty} f^{n_k}(x)$ for some sequence $n_k \to \infty\}$. We say that f is *pointwise recurrent* if $x \in \omega(x)$ for every $x \in X$. Examples of such homeomorphisms are the distal ones [**3**]. For these homeomorphisms we obtain the following result.

Theorem 3.6. *A pointwise recurrent homeomorphism of a compact manifold has no topologically stable points.*

Given a homeomorphism $f : X \to X$, a finite sequence of points $(x_n)_{n=0}^k$ in X is called a δ-*chain* if $d(f(x_n), x_{n+1}) \leq \delta$ for every $0 \leq n \leq k-1$. Then we say that f is *chain transitive* if, for any $x, y \in X$ and $\delta > 0$, there is a δ-chain $(x_n)_{n=0}^k$ such that $x_0 = x$ and $x_k = y$. Recall that a homeomorphism $f : X \to X$ of a metric space X has the POTP if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit can be ϵ-shadowed [**10**].

Theorem 3.7. *Every chain transitive homeomorphism with a topologically stable point of a compact manifold has the POTP.*

Therefore, we obtain the following corollary.

Corollary 3.8. *A chain transitive expansive homeomorphism of a compact manifold is topologically stable if and only if it has a topologically stable point.*

3.2. Preliminaries

We introduce an auxiliary definition.

Definition 3.9. Let $f : X \to X$ be a homeomorphism of a metric space X. We say that $x \in X$ is *finitely shadowable* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every finite set $\{x_0, \ldots, x_k\}$ satisfying $x_0 = x$ and $d(f(x_n), x_{n+1}) \leq \delta$ for every $0 \leq n \leq k-1$ there is $y \in X$ such that $dist(f^n(y), x_n) \leq \epsilon$ for every $0 \leq n \leq k-1$.

Every shadowable point is clearly finitely shadowable. The converse is true on compact metric spaces, as in [**13**, Lemma 8]. More precisely, we obtain the following result.

Lemma 3.10. *Every finitely shadowable point of a homeomorphism of a compact metric space is shadowable.*

With this lemma, we obtain the following one. Hereafter, we denote by $Sh(f)$ the set of shadowable points of a homeomorphism f.

Lemma 3.11. *Every topologically stable point of a homeomorphism of a compact manifold of dim ≥ 2 is shadowable.*

Proof. By Lemma 3.10, it suffices to prove that every topologically stable point of a homeomorphism of a compact manifold of dim ≥ 2 is finitely shadowable.

Let $f : X \to X$ be a homeomorphism of a compact manifold X of dim ≥ 2, and let $x \in X$ be a topologically stable point. Fix $\epsilon > 0$ and let $\delta_x > 0$ be given by the topological stability of x for $\epsilon/2$. Take $\{x_0, \ldots, x_k\}$ satisfying $x_0 = x$ and $d(f(x_n), x_{n+1}) \leq (\delta_x/4\pi)$ for every $0 \leq n \leq k-1$.

By [**13**, Lemma 9] there are points $\{x'_0, \ldots, x'_k\}$ such that $x'_0 = x$, $d(x_n, x'_n) < (\epsilon/2)$ $(0 \leq n \leq k)$, $d(f(x'_n), x'_{n+1}) < \delta_x/2\pi$ $(0 \leq n \leq k-1)$, $x'_n \neq x'_m$ (and so $f(x'_n) \neq f(x'_m)$) if $n \neq m$ $(0 \leq n \leq k,\ 0 \leq m \leq k)$.

Then, by [**13**, Lemma 10], there is a diffeomorphism $s : X \to X$ with $d(s, I_X) < \delta_x$ and $s(f(x'_n)) = x'_{n+1}$ $(0 \leq n \leq k-1)$.

Defining $g = s \circ f$ we obtain a homeomorphism $g : X \to X$ satisfying $d_{C^0}(f, g) < \delta_x$ and $g(x'_n) = x'_{n+1}$ $(0 \leq n \leq k-1)$.

In particular, $g^n(x) = g^n(x'_0) = x'_n$ $(0 \leq n \leq k)$.

Since $d_{C^0}(f, g) < \delta_x$, the choice of δ_x yields $h : \overline{O_g(x)} \to X$ such that $d(h, i_{\overline{O_g(x)}}) \leq \epsilon/2$ and $f \circ h = h \circ g$. Then,

$$\begin{aligned} d(f^n(h(x)), x_n) &= d(h(g^n(x'_0)), x_n) \\ &= d(h(x'_n), x_n) \\ &\leq d(h(x'_n), x'_n) + d(x'_n, x_n) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall 0 \leq n \leq k-1. \end{aligned}$$

Taking $y = h(x)$ we have $d(f^n(y), x_n) \leq \epsilon$ for every $0 \leq n \leq k-1$. It follows that x is finitely shadowable. □

Walters Shadowing Lemma [**13**] establishes that every topologically stable homeomorphism of a compact manifold of dim ≥ 2 has the POTP. The following result is a pointwise version of this result.

Corollary 3.12. *Let $f : X \to X$ be a homeomorphism of a compact manifold X of dim ≥ 2. If every point $x \in X$ is topologically stable, then f has the POTP.*

Proof. If every point of X is topologically stable, then $T(f) = X$ and so $Sh(f) = X$ by Lemma 3.11. Therefore, f has the POTP by [**9**, Theorem 1.3]. □

Now we introduce another auxiliary definition.

Definition 3.13. Let $f : X \to X$ be a homeomorphism of a metric space X. We say that $x \in X$ is *α-persistent* if for every $\epsilon > 0$ there is $\delta_x > 0$ such that for every homeomorphism $g : X \to X$ with $d_{C^0}(f, g) \leq \delta_x$ there is $y \in X$ such that $d(f^n(y), g^n(x)) \leq \epsilon$ for every $n \in \mathbb{Z}$.

In other words, x is α-persistent if and only if the g-orbit of x can be seen in f for every C^0-perturbation g of f (see [**8**, p. 8]). This is the pointwise counterpart of the

α-persistence homeomorphisms of [4]. The following elementary lemma gives examples of such points.

Lemma 3.14. *Both the shadowable and topologically stable points are α-persistent.*

Proof. Let $f : X \to X$ be a homeomorphism of a metric space X and $x \in X$. First, suppose that x is shadowable. Fix $\epsilon > 0$ and let $\delta_x > 0$ be given by the shadowableness of x for this ϵ. Take a homeomorphism $g : X \to X$ such that $d_{C^0}(f, g) \le \delta_x$. It follows that

$$d(f(g^n(x)), g^{n+1}(x)) = d(f(g^n(x)), g(g^n(x))) \le d_{C^0}(f, g) \le \delta_x, \quad \forall n \in \mathbb{Z}.$$

Hence $(g^n(x))_{n\in\mathbb{Z}}$ is a δ_x-pseudo orbit of f. It follows that there exists $y \in X$ satisfying $d(f^n(y), g^n(x)) \le \epsilon$ for every $n \in \mathbb{Z}$. From this we conclude that x is α-persistent.

Now suppose that x is topologically stable. Fix $\epsilon > 0$ and let $\delta_x > 0$ be given by the topological stability of x. Then, if $g : X \to X$ is a homeomorphism with $d_{C^0}(f, g) \le \delta_x$, there is a map $h : \overline{O_g(x)} \to X$ such that $d_{C^0}(h, i_{\overline{O_g(x)}}) \le \epsilon$ and $f \circ h = h \circ g$. Since $x \in \overline{O_g(x)}$, we can take $y = h(x)$, yielding

$$\begin{aligned} d(f^n(y), g^n(x)) &= d(f^n(h(x)), g^n(x)) \\ &= d(h(g^n(x)), g^n(x)) \le d_{C^0}(h, i_{\overline{O_g(x)}}) \le \epsilon, \quad \forall n \in \mathbb{Z}. \end{aligned}$$

Therefore, x is α-persistent and the proof follows. □

The question now is whether every α-persistent point is topologically stable.

A possible solution for this problem is as follows.

Let x be an α-persistent point of a homeomorphism $f : X \to X$. Take $\epsilon' > 0$ and let $\delta > 0$ be given by the α-persistence of x. Then, if $g : X \to X$ is a homeomorphism with $d_{C^0}(f, g) \le \delta$, we can select $y \in X$ satisfying $d(f^n(y), g^n(x)) \le \epsilon'$ for every $n \in \mathbb{Z}$.

Suppose for a while that the map

$$\begin{array}{rccc} h : & O_g(x) & \longrightarrow & X \\ & g^n(x) & \longmapsto & f^n(y) \end{array} \tag{3.1}$$

is well defined.

Since

$$(f \circ h)(g^n(x)) = f(f^n(y)) = f^{n+1}(y) = h(g^{n+1}(x)) = h(g(g^n(x))) = (h \circ g)(g^n(x))$$

and

$$d(h(g^n(x)), g^n(x)) = d(f^n(y), g^n(x)) \le \epsilon', \quad \forall n \in \mathbb{Z},$$

one would have

$$d_{C^0}(h, i_{O_g(x)}) \le \epsilon' \quad \text{and} \quad f \circ h = h \circ g \text{ in } O_g(x). \tag{3.2}$$

However, there is no guarantee that the map h will be well defined. Another problem is whether we can extend h continuously to the g-orbit closure $\overline{O_g(x)}$.

The following lemma, which can be seen as a pointwise version of [4, Theorem 2], tackles such problems in the expansive case.

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Lemma 3.15. *The following properties are equivalent for every expansive homeomorphism of a compact metric space $f : X \to X$ and every $x \in X$:*

(1) *x is α-persistent;*

(2) *x is topologically stable.*

Proof. By Lemma 3.14 it suffices to prove that every α-persistent point is topologically stable.

Let x be an α-persistent point of f. Fix $\epsilon > 0$ and let δ be given by the α-persistence of x for $\epsilon' = ((\min\{\epsilon, e\})/4)$, where e is an expansivity constant of f. Take a homeomorphism $g : X \to X$ such that $d_{C^0}(f, g) \leq \delta$. Then there is $y \in X$ such that

$$d(f^n(y), g^n(x)) \leq \epsilon', \quad \forall n \in \mathbb{Z}.$$

Define $h : O_g(x) \to X$ as in (3.1).

Let us use the expansivity of f to prove that h is well defined. Indeed, suppose that $g^n(x) = g^m(x)$ for $n, m \in \mathbb{Z}$. Then $g^{r+n}(x) = g^{r+m}(x)$ for every $r \in \mathbb{Z}$, and so

$$\begin{aligned} d(f^r(f^n(y)), f^r(f^m(y))) &\leq d(f^{r+n}(y), g^{r+n}(x)) + d(g^{r+n}(x), g^{r+m}(x)) \\ &\quad + d(f^{r+m}(y), g^{r+m}(x)) \\ &= d(f^{r+n}(y), g^{r+n}(x)) + d(f^{r+m}(y), g^{r+m}(x)) \\ &\leq 2\epsilon' < e, \quad \forall r \in \mathbb{Z}. \end{aligned}$$

Since e is an expansivity constant, we obtain $f^n(y) = f^m(y)$, proving the assertion. As previously remarked, we have that h satisfies (3.2).

Let us use the expansivity once more to prove that h is uniformly continuous. Fix $\Delta > 0$. Since e is an expansivity constant and X is compact, we have from [**13**, Lemma 2] that there is $N \in \mathbb{N}$ such that $d(a, b) \leq \Delta$ whenever $a, b \in X$ satisfy $d(f^n(a), f^n(b)) \leq e$ for every $-N \leq n \leq N$. Since g is continuous and X is compact, we have that g is uniformly continuous. Hence, there is $\rho > 0$ such that $d(g^n(a), g^n(b)) \leq (e/2)$ for all $-N \leq n \leq N$ whenever $a, b \in X$ satisfy $d(a, b) \leq \rho$. Now take $a, b \in O_g(x)$ with $d(a, b) \leq \rho$. It follows that

$$\begin{aligned} d(f^n(h(a)), f^n(h(b))) &= d(h(g^n(a)), h(g^n(b))) \\ &\leq d(h(g^n(a)), g^n(a)) + d(g^n(a), g^n(b)) + d(h(g^n(b)), g^n(b)) \\ &\leq 2\epsilon' + \frac{e}{2} \\ &< e, \quad \forall -N \leq n \leq N. \end{aligned}$$

By the choice of N we conclude that $d(h(a), h(b)) \leq \Delta$, and so h is uniformly continuous. Then we can extend h continuously to the orbit closure $\overline{O_g(x)}$ to obtain a continuous

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map, still denoted by $h : \overline{O_g(x)} \to X$. Since $\epsilon' < \epsilon$, (3.2) implies

$$d_{C^0}(h, i_{\overline{O_g(x)}}) \leq \epsilon \quad \text{and} \quad f \circ h = h \circ g \text{ in } \overline{O_g(x)}.$$

As ϵ is arbitrary, x is topologically stable and the proof follows. □

Walters Stability Theorem [**13**] asserts that every expansive homeomorphism with the POTP of a compact metric space is topologically stable. Combining Lemmas 3.14 and 3.15 we obtain a pointwise version of this result.

Corollary 3.16. *Every shadowable point of an expansive homeomorphism of a compact metric space is topologically stable.*

Lemmas 3.11 and 3.15 imply the following result.

Proposition 3.17. *Every α-persistent point of a homeomorphism of a compact manifold of dim ≥ 2 is shadowable.*

3.3. Proof of Theorems 3.2, 3.6, 3.7 and Corollary 3.5

Proof of Theorem 3.2. Let $f : X \to X$ be an expansive homeomorphism of a compact manifold X. Then, $Sh(f) \subseteq T(f)$ by Corollary 3.16. On the other hand, since there is an expansive homeomorphism on X, one has dim ≥ 2 (see [**1**] or [**8**]). Then $T(f) \subseteq Sh(f)$ by Lemma 3.11, yielding $T(f) = Sh(f)$. □

Proof of Corollary 3.5. Let $f : X \to X$ be an expansive homeomorphism of a compact manifold X. If every point is topologically stable, then f has the POTP by Corollary 3.12. Therefore, f is topologically stable by Walters Stability Theorem [**13**]. □

Proof of Theorem 3.6. Let $f : X \to X$ be a pointwise recurrent homeomorphism of a compact manifold. If dim $(X) = 1$, then X is the circle and f is topologically conjugated to a circle rotation. In such a case, $T(f) = \emptyset$ by Theorem 2.8(2) and Example 2.9. Otherwise, $T(f) \subseteq Sh(f)$ by Lemma 3.11. Since $Sh(f) = \emptyset$ by [**9**, Corollary 1.6], $T(f) = \emptyset$ in this case too, proving the result. □

Proof of Theorem 3.7. Let $f : X \to X$ be a chain transitive homeomorphism with topologically stable points of a compact manifold X. We claim that dim $(X) \geq 2$. Otherwise, since f is chain transitive, X is the circle and f is topologically conjugated to an irrational rotation R. Since f has a topologically stable point, R also has such a point by Theorem 2.8(2), contradicting Example 2.9. Then the claim holds, so $T(f) \subseteq Sh(f)$ by Lemma 3.11, and thus $Sh(f) \neq \emptyset$. Since f is chain transitive, Kawaguchi [**6**, Theorem 1.1] implies that $Sh(f) = X$, and then f has the POTP by [**9**, Theorem 1.3]. This completes the proof. □

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3.4. Final remarks

By using Theorem 3.2 and Lemma 3.15, we obtain that the following properties are equivalent for every expansive homeomorphism of a compact manifold X and every $x \in X$:

1. x is α-persistent;
2. x is topologically stable;
3. x is shadowable.

On the other hand, it is known that $Sh(f)$ is measurable for every homeomorphism $f : X \to X$ of a metric space X (see [**6**]). Then, by Theorem 3.2, $T(f)$ is measurable for every expansive homeomorphism $f : X \to X$ of a compact manifold X. Is $T(f)$ measurable for every homeomorphism of every metric space?

Some of the result in this paper have been extended to flows by Aponte [**2**].

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