J. Inst. Math. Jussieu (2018) **17**(1), 137–170
doi:10.1017/S1474748015000419

ON THE MOTIVIC SPECTRAL SEQUENCE

GRIGORY GARKUSHA[1] AND IVAN PANIN[2,3]

[1] *Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP, UK* (g.garkusha@swansea.ac.uk)
URL: http://math.swansea.ac.uk/staff/gg/

[2] *St. Petersburg Branch of V. A. Steklov Mathematical Institute, Fontanka 27, 191023, St. Petersburg, Russia*

[3] *St. Petersburg State University, Department of Mathematics and Mechanics, Universitetsky prospekt, 28, 198504, Peterhof, St. Petersburg, Russia* (paniniv@gmail.com)

(Received 27 November 2014; revised 17 October 2015; accepted 17 October 2015; first published online 25 November 2015)

Abstract It is shown that the Grayson tower for K-theory of smooth algebraic varieties is isomorphic to the slice tower of S^1-spectra. We also extend the Grayson tower to bispectra, and show that the Grayson motivic spectral sequence is isomorphic to the motivic spectral sequence produced by the Voevodsky slice tower for the motivic K-theory spectrum KGL. This solves Suslin's problem about these two spectral sequences in the affirmative.

Keywords: Motivic spectral sequence; algebraic K-theory; motivic cohomology

2010 *Mathematics subject classification*: 19E08; 55T99

1. Introduction

One of the more significant developments in algebraic K-theory in the 1990s/early 2000s was the construction of an algebraic analog for the Atiyah–Hirzebruch spectral sequence. It is a strongly convergent spectral sequence

$$E_2^{pq} = H_{\mathscr{M}}^{p-q,-q}(X, \mathbb{Z}) \Longrightarrow K_{-p-q}(X)$$

that relates the motivic cohomology groups of a smooth variety to its algebraic K-groups. The existence of this spectral sequence was first conjectured by Beilinson [1]. It is also called the *motivic spectral sequence*. Its construction is given in various forms:

(MSS1) the Bloch–Lichtenbaum motivic spectral sequence [2] for the spectrum of a field together with the Friedlander–Suslin and Levine extensions [4, 14] to the global case for a smooth variety over a field;

This paper was written during a visit of the second author to Swansea University, supported by EPSRC grant EP/J013064/1. He would like to thank the University for its kind hospitality.

(MSS2) the Grayson motivic spectral sequence [5, 9, 26, 34];

(MSS3) the Voevodsky motivic spectral sequence [3, p. 171] produced by the slice filtration of the motivic K-theory spectrum KGL [30, 31].

A problem of Suslin says that the three types of motivic spectral sequence agree with each other. In [15], Levine solved the Voevodsky problem for the slices of the spectrum KGL [30, 31] (over a perfect field). As a consequence he shows that (MSS1) agrees with (MSS3) over perfect fields.

In this paper we show that over perfect fields the Grayson tower for K-theory of smooth algebraic varieties agrees with the slice tower of S^1-spectra (see Theorem 7.7). The Grayson tower is then extended to bispectra. Thanks to this it is proved that (MSS2) agrees with (MSS3) (over perfect fields), answering the Suslin problem in the affirmative for these two spectral sequences (see Theorem 7.12).

To conclude the introduction, we make the following remark, recommended by the referee. In [21], Podkopaev claims that (MSS1) agrees with (MSS2) by comparing Friedlander–Suslin's and Grayson's towers. He shows in six steps that the entries of both towers agree, but does not show the agreement of the tower maps, on which the differentials in both spectral sequences depend. It may be possible to compare the maps in the future.

Throughout the paper we denote by Sm/k the category of smooth separated schemes of finite type over the base field k.

2. Preliminaries

In this section we collect basic facts about the K-theory associated with cubes of additive categories. We mostly follow Grayson [9].

2.1. Bivariant additive categories

Let *AddCats* denote the category of small additive categories and additive functors. Let $AffSm/k$ be the full subcategory of Sm/k whose objects are the affine smooth k-schemes. By a *bivariant additive category* we mean a functor

$$\mathscr{A} : (Sm/k)^{\mathrm{op}} \times AffSm/k \to AddCats.$$

So to any $X \in Sm/k$ and $Y \in AffSm/k$ we associate an additive category $\mathscr{A}(X, Y)$ which is contravariant in X and covariant in Y.

We also require that there is an action of $AffSm/k$ on $\mathscr{A}$ in the following sense. Given $U \in AffSm/k$, there is an additive functor

$$\alpha_U : \mathscr{A}(X, Y) \to \mathscr{A}(X \times U, Y \times U),$$

functorial in X and Y, such that for any morphism $f : U \to V$ in $AffSm/k$ the following square of additive functors is strictly commutative:

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

$$\begin{array}{ccc}
\mathscr{A}(X\times V, Y\times V) & \xrightarrow{\mathscr{A}(1_X\times f,1_{Y\times V})} & \mathscr{A}(X\times U, Y\times V) \\
\uparrow{\scriptstyle \alpha_V} & & \uparrow{\scriptstyle \mathscr{A}(1_{X\times U},1_Y\times f)} \\
\mathscr{A}(X,Y) & \xrightarrow{\alpha_U} & \mathscr{A}(X\times U, Y\times U).
\end{array}$$

By the functoriality of α_U in X we mean that the following square of additive functors is strictly commutative for any $Y \in \mathit{AffSm}/k$ and any morphism $f : X' \to X$ in Sm/k:

$$\begin{array}{ccc}
\mathscr{A}(X\times U, Y\times U) & \xrightarrow{\mathscr{A}(f\times 1_U,1_{Y\times U})} & \mathscr{A}(X'\times U, Y\times U) \\
\uparrow{\scriptstyle \alpha_U} & & \uparrow{\scriptstyle \alpha_U} \\
\mathscr{A}(X,Y) & \xrightarrow{\mathscr{A}(f,1_Y)} & \mathscr{A}(X', Y).
\end{array}$$

By the functoriality of α_U in Y we mean that the following square of additive functors is strictly commutative for any $X \in \mathit{Sm}/k$ and any morphism $g : Y \to Y'$ in AffSm/k:

$$\begin{array}{ccc}
\mathscr{A}(X\times U, Y\times U) & \xrightarrow{\mathscr{A}(1_{X\times U},g\times 1_U)} & \mathscr{A}(X\times U, Y'\times U) \\
\uparrow{\scriptstyle \alpha_U} & & \uparrow{\scriptstyle \alpha_U} \\
\mathscr{A}(X,Y) & \xrightarrow{\mathscr{A}(1_X,g)} & \mathscr{A}(X, Y').
\end{array}$$

Below, we shall associate an explicitly constructed bispectrum to any bivariant additive category. For this we need to collect some facts about the algebraic K-theory of additive categories.

2.2. K-theory for cubes of additive categories

We let *Ord* denote the category of finite nonempty ordered sets. For $A \in \mathit{Ord}$ we define a category $\mathit{Sub}(A)$ whose objects are the pairs (i, j) with $i \leqslant j \in A$, and where there is a (unique) arrow $(i', j') \to (i, j)$ exactly when $i' \leqslant i \leqslant j \leqslant j'$. Given an additive category $\mathscr{M}$, we say that a functor $M : \mathit{Sub}(A) \to \mathscr{M}$ is *additive* if $M(i, i) = 0$ for all $i \in A$, and for all $i \leqslant j \leqslant k \in A$ the map $M(i, k) \to M(i, j) \oplus M(j, k)$ is an isomorphism. Here 0 denotes a previously chosen zero object of $\mathscr{M}$. The set of such additive functors is denoted by $\mathit{Add}(\mathit{Sub}(A), \mathscr{M})$. Given ordered sets $A_1, \dots, A_n$, we let $\mathit{Add}(\mathit{Sub}(A_1) \times \cdots \times \mathit{Sub}(A_n), \mathscr{M})$ denote the set of multiadditive functors, i.e., functors that are additive in each variable.

The Grayson simplicial set $S^{\oplus}\mathscr{M}$ [8, 9] is defined as

$$(S^{\oplus}\mathscr{M})(A) = \mathit{Add}(\mathit{Sub}(A), \mathscr{M}).$$

An n-simplex $M \in S^{\oplus}_n\mathscr{M}$ may be thought of as a compatible collection of direct sum diagrams $M(i, j) \cong M(i, i+1) \oplus \cdots \oplus M(j-1, j)$. There is a natural map $S^{\oplus}\mathscr{M} \to S\mathscr{M}$ (see [9, p. 147]) which converts each direct sum diagram $M(i, k) \cong M(i, j) \oplus M(j, k)$ into the short exact sequence $0 \to M(i, j) \to M(i, k) \to M(j, k) \to 0$. Here $S\mathscr{M}$ stands for the Waldhausen S-construction [33].

We follow the same constructions as in [24, § 8.7] to define Grayson's symmetric spectrum $K^{Gr}(\mathscr{M})$. Given a positive integer n, one can define the n-fold multisimplicial

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

additive category $S^{\oplus,n}\mathcal{M} := S^{\oplus} \overset{n}{\cdots} S^{\oplus}\mathcal{M}$. The nth space of Grayson's K-theory spectrum is given by

$$K^{Gr}(\mathcal{M})_n = |\mathrm{Ob}(S^{\oplus,n}\mathcal{M})|,$$

where the right-hand side is the diagonal of the n-fold multisimplicial set $\mathrm{Ob}(S^{\oplus,n}\mathcal{M})$. The nth symmetric group Σ_n acts on $K^{Gr}(\mathcal{M})_n$ by permuting the order of the $S^{\oplus}$-constructions. Each structure map σ is the composite

$$|\mathrm{Ob}(S^{\oplus,n}\mathcal{M})| \wedge S^1 \cong |\mathrm{Ob}(S^{\oplus,n}S^{\oplus}\mathcal{M})|^{(1)} \subset |\mathrm{Ob}(S^{\oplus,n}S^{\oplus}\mathcal{M})| \cong |\mathrm{Ob}(S^{\oplus,n+1}\mathcal{M})|,$$

where the superscript $^{(1)}$ stands for the 1-skeleton with respect to the last simplicial direction. The k-fold iterated structure map σ^k is then defined as the composite

$$\begin{aligned}|\mathrm{Ob}(S^{\oplus,n}\mathcal{M})| \wedge S^k &\cong |\mathrm{Ob}(S^{\oplus,n}S^{\oplus} \overset{k}{\cdots} S^{\oplus}\mathcal{M})|^{(1,\ldots,1)} \subset |\mathrm{Ob}(S^{\oplus,n}S^{\oplus} \overset{k}{\cdots} S^{\oplus}\mathcal{M})|\\ &\cong |\mathrm{Ob}(S^{\oplus,n+k}\mathcal{M})|,\end{aligned}$$

where the superscript $^{(1,\ldots,1)}$ indicates the multi-1-skeleton with respect to the k last simplicial directions. This map is plainly $(\Sigma_n \times \Sigma_k)$-equivariant. With these definitions $K^{Gr}(\mathcal{M})$ becomes a (semistable) symmetric spectrum. If $\mathcal{M}$ happens to be a multisimplicial additive category, then we define its Grayson K-theory symmetric spectrum $K^{Gr}(\mathcal{M})$ by taking diagonals $K^{Gr}(\mathcal{M})_n := |\mathrm{Ob}(S^{\oplus,n}\mathcal{M})|$ of the multisimplicial sets $\mathrm{Ob}(S^{\oplus,n}\mathcal{M})$, $n \geqslant 0$.

In [8, § 4] there is presented a construction called C which can be applied to a cube of additive categories to convert it into a multisimplicial additive category, the K-theory of which serves as the iterated cofiber space/spectrum of the corresponding cube of K-theory spaces/spectra. We start with preparations.

We let [1] denote the ordered set $\{0 < 1\}$ regarded as a category, and we use ε as notation for an object of [1]. By an n-dimensional cube in a category $\mathscr{C}$ we shall mean a functor from $[1]^n$ to $\mathscr{C}$. An object C in $\mathscr{C}$ gives a 0-dimensional cube denoted by $[C]$, and an arrow $C \to C'$ in $\mathscr{C}$ gives a 1-dimensional cube denoted by $[C \to C']$. If the category $\mathscr{C}$ has products, we may define an external product of cubes as follows. Given an n-dimensional cube X and an n'-dimensional cube Y in $\mathscr{C}$, we let $X \boxtimes Y$ denote the $n+n'$-dimensional cube defined by $(X \boxtimes Y)(\varepsilon_1, \ldots, \varepsilon_{n+n'}) = X(\varepsilon_1, \ldots, \varepsilon_n) \times Y(\varepsilon_{n+1}, \ldots, \varepsilon_{n+n'})$. Let $\mathbb{G}_m^{\wedge n}$ denote the external product of n copies of $[1 \to \mathbb{G}_m]$ in Sm/k. For example, $\mathbb{G}_m^{\wedge 2}$ is the square of schemes

$$\begin{array}{ccc}\operatorname{Spec} k & \longrightarrow & \mathbb{G}_m\\ \downarrow & & \downarrow\\ \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m.\end{array}$$

Let L be a symbol, and consider $\{L\}$ to be an ordered set. Given an ordered set A, by $\{L\}A$ we mean the concatenation ordered set with L declared to be less than every element of A. Given an n-dimensional cube of additive categories $\mathcal{M}$, we define an n-fold multisimplicial additive category $C^{\oplus}\mathcal{M}$ as a functor from $(Ord^n)^{\mathrm{op}}$ to the category of additive categories by letting $C^{\oplus}\mathcal{M}(A_1, \ldots, A_n)$ be the additive category whose objects are the multiadditive natural transformations (we follow here the terminology of [9])

$$Add([Sub(A_1) \to Sub(\{L\}A_1)] \boxtimes \cdots \boxtimes [Sub(A_n) \to Sub(\{L\}A_n)], \mathcal{M}). \tag{1}$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

More precisely, every object in (1) maps each vertex of the cube of the domain to the corresponding vertex of the cube $\mathscr{M}$ by means of a multiadditive functor. If we regard each edge of the cube of the domain as a functor between categories, then one has a commutative diagram of functors, in which one pair of parallel arrows is this edge and the corresponding edge (which is an additive functor) of the cube $\mathscr{M}$. When $n = 0$, we may identify $C^{\oplus}\mathscr{M}$ with $\mathscr{M}$. We define $S^{\oplus}\mathscr{M}$ to be $S^{\oplus}C^{\oplus}\mathscr{M}$, the result of applying the $S^{\oplus}$-construction of Grayson degreewise. It is an $n+1$-fold multisimplicial set (see [9] for details).

If we extend the following lemma to Grayson's K-theory spectra in the obvious way, then we get that Grayson's K-theory $K^{Gr}(C^{\oplus}\mathscr{M})$ of $C^{\oplus}\mathscr{M}$ serves as the iterated cofiber space/spectrum of the corresponding cube of Grayson's K-theory spaces/spectra.

Lemma 2.1 [9, 4.3]. *Suppose that we are given an additive map $\mathscr{M}' \to \mathscr{M}$ of n-dimensional cubes of additive categories. Let $[\mathscr{M}' \to \mathscr{M}]$ denote the corresponding $n+1$-dimensional cube of additive categories:*

(a) *there is a fibration sequence*

$$S^{\oplus}[0 \to \mathscr{M}] \to S^{\oplus}[\mathscr{M}' \to \mathscr{M}] \to S^{\oplus}[\mathscr{M}' \to 0];$$

(b) *the space $S^{\oplus}[\mathscr{M} \xrightarrow{1} \mathscr{M}]$ is contractible;*

(c) *$S^{\oplus}[0 \to \mathscr{M}]$ is homotopy equivalent to $S^{\oplus}\mathscr{M}$;*

(d) *$S^{\oplus}[\mathscr{M} \to 0] = S^{\oplus}S^{\oplus}\mathscr{M}$ is a delooping of $S^{\oplus}\mathscr{M}$;*

(e) *there is a fibration sequence $S^{\oplus}\mathscr{M}' \to S^{\oplus}\mathscr{M} \to S^{\oplus}[\mathscr{M}' \to \mathscr{M}]$.*

Let $\mathscr{A} : (Sm/k)^{\text{op}} \times AffSm/k \to AddCats$ be a bivariant additive category. Given $X \in Sm/k$, $Y \in AffSm/k$, and $n > 0$, the cube of schemes $\mathbb{G}_m^{\wedge n}$ gives rise to a cube of additive categories $\mathscr{A}(X, Y \times \mathbb{G}_m^{\wedge n})$. Its vertexes are $\mathscr{A}(X, Y \times \mathbb{G}_m^{\times \ell})$, $0 \leqslant \ell \leqslant n$. The edges of the cube are given by the natural additive functors $i_s : \mathscr{A}(X, Y \times \mathbb{G}_m^{\times(\ell-1)}) \to \mathscr{A}(X, Y \times \mathbb{G}_m^{\times \ell})$ induced by the embeddings $i_s : \mathbb{G}_m^{\times(\ell-1)} \to \mathbb{G}_m^{\times \ell}$ of the form

$$(x_1, \ldots, x_{\ell-1}) \longmapsto (x_1, \ldots, 1, \ldots, x_{\ell-1}),$$

where 1 is the sth coordinate.

Thus we obtain a cube of bivariant additive categories $\mathscr{A}\langle\mathbb{G}_m^{\wedge n}\rangle$. Grayson's K-theory of $\mathscr{A}\langle\mathbb{G}_m^{\wedge n}\rangle$ produces a functor

$$K^{Gr}(C^{\oplus}\mathscr{A}\langle\mathbb{G}_m^{\wedge n}\rangle) : (Sm/k)^{\text{op}} \times AffSm/k \to Sp^{\Sigma}, \quad (X, Y) \mapsto K^{Gr}(C^{\oplus}\mathscr{A}(X, Y \times \mathbb{G}_m^{\wedge n})).$$

Here Sp^{Σ} stands for the category of symmetric spectra in the sense of [11]. It is directly verified that

$$K_0^{Gr}(C^{\oplus}\mathscr{A}(X, Y \times \mathbb{G}_m^{\wedge n})) = K_0^{Gr}(\mathscr{A}(X, Y \times \mathbb{G}_m^{\times n})) \Big/ \sum_{s=1}^{n} (i_s)_*(K_0^{Gr}(\mathscr{A}(X, Y \times \mathbb{G}_m^{\times n-1}))). \tag{2}$$

Indeed, the case when $n = 1$ follows from Lemma 2.1(e), and the general case is checked by induction.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

3. The category of bispectra

In this paper we work with the category $Pre^{\Sigma}(Sm/k)$ of presheaves of symmetric spectra. It has three model category structures, each of which we discuss separately.

Definition 3.1. A morphism f in $Pre^{\Sigma}(Sm/k)$ is a *stable weak equivalence* (respectively, *stable projective fibration*) if $f(X)$ is a stable weak equivalence (respectively, stable projective fibration) in Sp^{Σ} for all $X \in Sm/k$. It is a *stable projective cofibration* if f has the left lifting property with respect to all stable projective acyclic fibrations.

Recall that the Nisnevich topology is generated by elementary distinguished squares, i.e., pullback squares

$$\begin{array}{ccc} U' & \longrightarrow & X' \\ \downarrow & Q & \downarrow{\scriptstyle \varphi} \\ U & \underset{\psi}{\longrightarrow} & X. \end{array} \qquad (3)$$

where φ is etale, ψ is an open embedding, and $\varphi^{-1}(X \setminus U) \to (X \setminus U)$ is an isomorphism of schemes (with the reduced structure).

Definition 3.2. (1) A stably fibrant presheaf $M \in Pre^{\Sigma}(Sm/k)$ is *Nisnevich local* if for each elementary distinguished square Q the square of symmetric spectra $M(Q)$ is a homotopy pullback.

(2) A Nisnevich local presheaf $M \in Pre^{\Sigma}(Sm/k)$ is $\mathbb{A}^1$-*local* if the natural map

$$M(X) \to M(X \times \mathbb{A}^1)$$

is a stable equivalence of symmetric spectra for all $X \in Sm/k$.

(3) A map $f : A \to B$ in $Pre^{\Sigma}(Sm/k)$ is a *local weak equivalence* (respectively, *motivic equivalence*) if the map of spaces

$$f^* : \mathrm{Map}(B, M) \to \mathrm{Map}(A, M)$$

is a weak equivalence for any Nisnevich local (respectively, $\mathbb{A}^1$-local) presheaf M.

(4) The *Nisnevich local model category* (respectively, the *motivic model category*) on presheaves of symmetric spectra, denoted by $Pre^{\Sigma}_{nis}(Sm/k)$ (respectively, $Pre^{\Sigma}_{mot}(Sm/k)$), is determined by stable projective cofibrations and local weak equivalences (respectively, motivic equivalences). The fibrations are defined by the corresponding lifting property. The homotopy category of $Pre^{\Sigma}_{mot}(Sm/k)$ will be denoted by $SH_{S^1}(k)$.

We define the mapping cylinder $\mathrm{cyl}(f)$ of a map $f : A \to B$ between cofibrant objects in a simplicial model category $\mathscr{M}$. Let $A \otimes \Delta^1$ denote the standard cylinder object for A.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

One has a commutative diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{\nabla} & A \\ {\scriptstyle i=i_0\sqcup i_1}\downarrow & \nearrow{\scriptstyle \sigma} & \\ A \otimes \Delta^1 & & \end{array}$$

in which i is a cofibration and σ is a weak equivalence. Each i_ε must be a trivial cofibration.

Form the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_{0*}} \\ A \otimes \Delta^1 & \xrightarrow{f_*} & \mathrm{Cyl}(f). \end{array}$$

Then $(f\sigma) \circ i_0 = f$, and so there is a unique map $q : \mathrm{Cyl}(f) \to B$ such that $qf_* = f\sigma$ and $qi_{0*} = 1_B$. Put $\mathrm{cyl}(f) = f_* i_1$; then $f = q \circ \mathrm{cyl}(f)$.

If A, B are cofibrant in $\mathscr{M}$, then so is $\mathrm{Cyl}(f)$. Observe also that q is a weak equivalence. The map $\mathrm{cyl}(f)$ is a cofibration, since the diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{f \sqcup 1_A} & B \sqcup A \\ {\scriptstyle i_0\sqcup i_1}\downarrow & & \downarrow{\scriptstyle i_{0*}\sqcup \mathrm{cyl}(f)} \\ A \otimes \Delta^1 & \xrightarrow{f_*} & \mathrm{Cyl}(f). \end{array}$$

is a pushout.

Consider the category $Pre(Sm/k)$ of presheaves of pointed simplicial sets. We can define the projective model category structure on it, where the weak equivalences and fibrations are defined schemewise. Let $\iota : pt = \mathrm{Spec}\, k \to \mathbb{G}_m$ be the embedding $\iota(pt) = 1 \in \mathbb{G}_m$. The mapping cylinder yields a factorization of the induced map

$$\mathrm{Spec}\, k_+ \hookrightarrow \mathrm{Cyl}(\iota) \xrightarrow{\simeq} (\mathbb{G}_m)_+$$

into a cofibration and a simplicial homotopy equivalence in $Pre(Sm/k)$. Let $\mathbb{G}$ denote the cofibrant pointed presheaf $\mathrm{Cyl}(\iota)/\,\mathrm{Spec}\, k_+$.

Let $Pre^{\Sigma,\mathbb{G}}(Sm/k)$ denote the category of $\mathbb{G}$-spectra in $Pre^{\Sigma}(Sm/k)$. Its objects are the sequences $(X_0, X_1, \ldots)$ of presheaves of symmetric spectra X_i together with bonding maps $X_i \to \Omega_{\mathbb{G}} X_{i+1}$, where $\Omega_{\mathbb{G}} X_{i+1} = \underline{\mathrm{Hom}}(\mathbb{G}, X_{i+1})$. Morphisms are defined levelwise and must be consistent with bonding maps. This category will also be referred as the *category of* $(S^1, \mathbb{G})$*-bispectra* or just *bispectra*. We define the *stable projective model structure* on $Pre^{\Sigma,\mathbb{G}}(Sm/k)$ (respectively, the *Nisnevich local and motivic model structure*) as the stable model category of $\mathbb{G}$-spectra in the sense of Hovey [10] associated with the model category $Pre^{\Sigma}(Sm/k)$ (respectively, $Pre^{\Sigma}_{nis}(Sm/k)$ and $Pre^{\Sigma}_{mot}(Sm/k)$). Using Hovey's notation [10], it is the model category $Sp^{\mathbb{N}}(Pre^{\Sigma}(Sm/k), \mathbb{G} \otimes -)$ (respectively, $Sp^{\mathbb{N}}(Pre^{\Sigma}_{nis}(Sm/k), \mathbb{G} \otimes -)$ and $Sp^{\mathbb{N}}(Pre^{\Sigma}_{mot}(Sm/k), \mathbb{G} \otimes -)$). In what follows we shall

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

denote the homotopy category of $Sp^{\mathbb{N}}(Pre^{\Sigma}_{mot}(Sm/k), \mathbb{G}\otimes -)$ by $SH(k)$. It is one of equivalent definitions of the Voevodsky stable motivic category of the field k [29].

The main bispectrum we shall work with is produced by a bivariant additive category

$$\mathscr{A} : (Sm/k)^{\mathrm{op}} \times AffSm/k \to AddCats.$$

Namely, let

$$A_Y = (A_Y(0), A_Y(1), A_Y(2), \ldots)$$

be the sequence of presheaves of symmetric spectra

$$A_Y(n) = K^{Gr}(C^{\oplus}\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n})), \quad n \geqslant 0.$$

We want to construct bonding maps

$$a_n : A_Y(n) \to \Omega_{\mathbb{G}} A_Y(n+1).$$

Each a_n is uniquely determined by a map

$$\beta : K^{Gr}(C^{\oplus}\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n})) \to K^{Gr}(C^{\oplus}\mathscr{A}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m^{\wedge n+1})) \tag{4}$$

and a homotopy

$$h : K^{Gr}(C^{\oplus}\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n})) \to K^{Gr}(C^{\oplus}\mathscr{A}(- \times \operatorname{Spec} k, Y \times \mathbb{G}_m^{\wedge n+1}))^I \tag{5}$$

such that $d_0 h = \iota^*\beta$ and $d_1 h$ factors trough the zero object levelwise.

We first construct the maps β and h for $n = 0$. By definition of a bivariant additive category, there is an additive functor

$$\alpha_{\mathbb{G}_m} : \mathscr{A}(X, Y) \to \mathscr{A}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m), \quad X \in Sm/k.$$

The map β is induced by the composition

$$\mathscr{A}(X, Y) \xrightarrow{\alpha_{\mathbb{G}_m}} \mathscr{A}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \xrightarrow{p} C^{\oplus}\mathscr{A}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m^{\wedge 1}),$$

where p is a natural simplicial functor of simplicial categories (we consider $\mathscr{A}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ as a simplicial category in a trivial way).

One has a commutative square of additive functors

$$\begin{array}{ccc}
\mathscr{A}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) & \xrightarrow{\mathscr{A}(1_X \times \iota, 1_{Y \times \mathbb{G}_m})} & \mathscr{A}(X \times \operatorname{Spec} k, Y \times \mathbb{G}_m) \\
\uparrow{\scriptstyle \alpha_{\mathbb{G}_m}} & & \uparrow{\scriptstyle \mathscr{A}(1_{X \times \operatorname{Spec} k}, 1_{\operatorname{Spec} k} \times \iota)} \\
\mathscr{A}(X, Y) & \xrightarrow{\alpha_{\operatorname{Spec} k}} & \mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k).
\end{array}$$

On the other hand, there is a commutative diagram of simplicial additive categories,

$$\begin{array}{ccc}
\mathscr{A}(X \times \operatorname{Spec} k, Y \times \mathbb{G}_m) & \xrightarrow{p} & C^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \mathbb{G}_m^{\wedge 1}) \\
\uparrow{\scriptstyle \iota_*} & & \uparrow{\scriptstyle \iota_*} \\
\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k) & \xrightarrow{p'} & C^{\oplus}[\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k) \xrightarrow{\mathrm{id}} \mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k)].
\end{array}$$

Recall that the *path space* PX of a simplicial object $X : \Delta^{\mathrm{op}} \to \mathscr{D}$ in a category $\mathscr{D}$ is defined as the composition of X with the shift functor $P : \Delta \to \Delta$ that takes $[n]$ to $[n+1]$ (by mapping i to $i+1$). The right lower corner of the diagram can be identified with the simplicial path space $P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k))$. By [33, 1.5.1], there is a canonical contraction of this simplicial set into the set of its zero simplices regarded as a constant simplicial set. Since $P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k))$ has only one zero simplex, it follows that there is a canonical simplicial homotopy

$$H : P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k)) \to P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k))^I$$

such that $d_0 H = 1$ and $d_1 H = \mathrm{const}$.

Now the map h (5) is induced by the composite map

$$\begin{array}{ccccc}
 & & & & (C^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \mathbb{G}_m^{\wedge 1}))^I \\
 & & & & \uparrow{\scriptstyle \iota_*^I} \\
 & & P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k)) & \xrightarrow{H} & P(S^{\oplus}\mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k))^I \\
 & & \uparrow{\scriptstyle p'} & & \\
\mathscr{A}(X, Y) & \xrightarrow{\alpha_{\operatorname{Spec} k}} & \mathscr{A}(X \times \operatorname{Spec} k, Y \times \operatorname{Spec} k). & &
\end{array}$$

Since $d_1 \circ \iota_*^I \circ H = \iota_* \circ d_1 \circ H = \mathrm{const}$, $d_1 h$ factors through the zero object. On the other hand,

$$\begin{aligned}
d_0 \circ \iota_*^I \circ H \circ p' \circ \alpha_{\operatorname{Spec} k} &= \iota_* \circ d_0 \circ H \circ p' \circ \alpha_{\operatorname{Spec} k} = \iota_* \circ p' \circ \alpha_{\operatorname{Spec} k} \\
&= p \circ \iota_* \circ \alpha_{\operatorname{Spec} k} = p \circ \iota^* \circ \alpha_{\mathbb{G}_m}.
\end{aligned}$$

Moreover, $p \circ \iota^* \circ \alpha_{\mathbb{G}_m} = \iota^* \circ p \circ \alpha_{\mathbb{G}_m}$, and therefore $d_0 h = \iota^* \beta$. The bonding map $a_0 : A_Y(0) \to \Omega_{\mathbb{G}} A_Y(1)$ is now constructed. The definition of each $a_n : A_Y(n) \to \Omega_{\mathbb{G}} A_Y(n+1)$ is similar to that of a_0. The only difference is that we replace the bivariant additive category $\mathscr{A}(X, Y)$ by the multisimplicial bivariant additive category $C^{\oplus}\mathscr{A}(X, Y \times \mathbb{G}_m^{\wedge n})$.

Given an abelian monoid $(A, +)$, denote by $EM(A)$ its Eilenberg–Mac Lane symmetric spectrum in the sense of [5, Appendix A]. It is defined in terms of the σ-construction and is similar to the definition of the Waldhausen K-theory spectrum that uses the S-construction [33]. By definition, σA is a simplicial set whose n-simplices are the functions

$$a : \operatorname{Ob} \operatorname{Ar}[n] \to A, \quad (i, j) \mapsto a(i, j) = a_{i,j},$$

having the property that, for every j, $a_{j,j} = 0$ and $a_{i,k} = a_{i,j} + a_{j,k}$ whenever $i \leqslant j \leqslant k$.

Given an n-dimensional cube of abelian monoids M, we define an n-fold multisimplicial abelian monoid $C^{\oplus}M$ similar to formula (1). The only difference is that we consider functions from objects of the corresponding posets ignoring poset arrows. It is worth mentioning that Lemma 2.1 is valid for the σ-construction of cubes of abelian monoids. For this one uses the additivity theorem for the σ-construction, just as in [33, § 1.5]. Applying the σ-construction to the multisimplicial abelian monoid $C^{\oplus}M$, one gets a symmetric spectrum $EM(C^{\oplus}M)$. It serves as the iterated cofiber spectrum of the cube of Eilenberg–Mac Lane's spectra $EM(M)$.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Every n-dimensional cube of additive categories $\mathscr{M}$ gives rise to an n-fold cube of abelian groups $K_0^{Gr}(\mathscr{M})$. There is a natural map $S^{\oplus}\mathscr{M} \to \sigma K_0^{Gr}(\mathscr{M})$ induced by the map sending an object of an additive category to its isomorphism class in the Grothendieck group. This map induces a map of symmetric spectra

$$\tau : K^{Gr}(C^{\oplus}\mathscr{M}) \to EM(C^{\oplus}K_0^{Gr}(\mathscr{M})).$$

For each $n \geqslant 0$ we set

$$A_{0,Y}(n) = EM(C^{\oplus}K_0^{Gr}(\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n}))).$$

Note that $\pi_0(A_{0,Y}(n))$ is computed by formula (2), and the other homotopy groups are zero. We have that the sequence of symmetric spectra

$$A_{0,Y} = (A_{0,Y}(0), A_{0,Y}(1), A_{0,Y}(2), \ldots)$$

forms a bispectrum, in which the bonding maps are defined like those for the bispectrum A_Y. Moreover, there is a canonical map of bispectra

$$\tau : A_Y \to A_{0,Y}.$$

This map consists of the collection of canonical maps of symmetric spectra

$$\tau_n : A_Y(n) \to A_{0,Y}(n), \quad n \geqslant 0,$$

defined as above.

4. The Grayson tower of bispectra

In this section we work in the framework of simplicial additive categories $\mathscr{M}$ over contractible simplicial rings R. Given such a pair $(\mathscr{M}, R)$, Grayson [9] constructs a tower of spaces which is also referred to as the *Grayson tower*. Each space of the Grayson tower is defined as a K-theory space of some 'multisimplicial additive category with commuting automorphisms' associated with $(\mathscr{M}, R)$. In practice, the Grayson tower gives rise to a motivic spectral sequence (see [5, 9, 26, 34]). The Grayson tower for $(\mathscr{M}, R)$ can be extended to symmetric spectra [5]. We shall mostly adhere to [5] in this section.

In our setting, the contractible ring R is

$$k[\Delta] : d \mapsto k[\Delta^d] = k[t_0, t_1, \ldots, t_d]/(t_0 + t_1 + \cdots + t_d - 1).$$

In what follows, we require

$$d \mapsto \mathscr{A}(X \times \Delta^d, Y)$$

to be a $k[\Delta]$-linear additive category, where $\mathscr{A} : (Sm/k)^{\mathrm{op}} \times AffSm/k \to AddCats$ is a bivariant additive category.

In order to make Grayson's machinery applicable to our setting, throughout this section we work with a bivariant additive category

$$\mathscr{A} : (Sm/k)^{\mathrm{op}} \times AffSm/k \to AddCats$$

satisfying the following property.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

($\mathfrak{Aut}$) For every $X \in Sm/k$, $Y \in AffSm/k$ and $n > 0$, the additive category $\mathscr{A}(X, Y \times \mathbb{G}_m^{\times n})$ can be identified with the additive category $\mathscr{A}(X, Y)(\mathbb{G}_m^{\times n})$ whose objects are the tuples $(P, \theta_1, \dots, \theta_n)$, where $P \in \mathscr{A}(X, Y)$ and $(\theta_1, \dots, \theta_n)$ are commuting automorphisms of P. More precisely, there is an isomorphism of additive categories (not only an equivalence of categories)

$$\rho_{X,Y,n} : \mathscr{A}(X, Y \times \mathbb{G}_m^{\times n}) \to \mathscr{A}(X, Y)(\mathbb{G}_m^{\times n})$$

such that the diagram of functors

$$\begin{array}{ccc} \mathscr{A}(X, Y \times \mathbb{G}_m^{\times n-1}) & \xrightarrow{\rho_{X,Y,n-1}} & \mathscr{A}(X, Y)(\mathbb{G}_m^{\times n-1}) \\ \downarrow{\scriptstyle i_s} & & \downarrow{\scriptstyle j_s} \\ \mathscr{A}(X, Y \times \mathbb{G}_m^{\times n}) & \xrightarrow{\rho_{X,Y,n}} & \mathscr{A}(X, Y)(\mathbb{G}_m^{\times n}) \end{array}$$

is commutative. Here i_s (respectively, j_s) stands for the functor induced by the map $(x_1, \dots, x_{n-1}) \in \mathbb{G}_m^{\times n-1} \mapsto (x_1, \dots, 1, \dots, x_{n-1}) \in \mathbb{G}_m^{\times n}$ with 1 the sth coordinate (respectively, $(\theta_1, \dots, \theta_{n-1}) \mapsto (\theta_1, \dots, 1, \dots, \theta_{n-1})$). We also require each identification $\rho_{X,Y,n}$ to be functorial in both arguments.

We can form a cube of additive categories $\mathscr{A}(X, Y)(\mathbb{G}_m^{\wedge n})$ whose vertexes are $\mathscr{A}(X, Y)(\mathbb{G}_m^{\times \ell})$, $\ell \leqslant n$, and whose edges are given by the functors j_s. The ($\mathfrak{Aut}$) property implies that the cubes $\mathscr{A}(X, Y)(\mathbb{G}_m^{\wedge n})$ and $\mathscr{A}(X, Y \times \mathbb{G}_m^{\wedge n})$ are isomorphic.

Consider the map of bispectra

$$\tau : A_Y \to A_{0,Y}.$$

For every $n \geqslant 0$ there is a triangle in the homotopy category $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$ (see [5, § 7] for details)

$$|d \mapsto A_Y(n+1)(- \times \Delta^d)| \xrightarrow{\gamma_n} \Omega|d \mapsto A_Y(n)(- \times \Delta^d)| \xrightarrow{\Omega\tau_n} \Omega|d \mapsto A_{0,Y}(n)(- \times \Delta^d)|.$$

The map γ_1 is induced by a zigzag map of spectra

$$K^{Gr}(C^{\oplus}\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge 1})) \xrightarrow{v} \Omega K^{Gr}(S^{-1}S\mathscr{A}(-, Y)) \xleftarrow{\Omega s} \Omega K^{Gr}(\mathscr{A}(-, Y)).$$

Here $S^{-1}S$ stands for Quillen's construction (see § 6 for more details), $s : K^{Gr}(\mathscr{A}(-, Y)) \to K^{Gr}(S^{-1}S\mathscr{A}(-, Y))$ is a stable equivalence induced by the map sending an object $M \in \mathscr{A}(-, Y)$ to $(M, 0) \in S^{-1}S\mathscr{A}(-, Y)$ (see [9, 9.3] and § 6), and v is a natural map that exists by [9, 9.4] and [34, p. 16]. The map γ_n is defined as γ_1 by replacing $\mathscr{A}(-, Y)$ with $\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n})$. Note that each map in the zigzag agrees with the bonding map in $\mathbb{G}$-direction.

Since the category $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$ is triangulated with $\Sigma_s = - \wedge S^1$ a shift functor, the latter triangle gives a triangle

$$\Sigma_s|d \mapsto A_Y(n+1)(- \times \Delta^d)| \to |d \mapsto A_Y(n)(- \times \Delta^d)| \to |d \mapsto A_{0,Y}(n)(- \times \Delta^d)|.$$

We shall also call it the *Grayson triangle.*

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

We obtain a tower in $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$,

$$\cdots \to \Sigma_s^{n+1}|d \mapsto A_Y(n+1)(-\times\Delta^d)| \to \Sigma_s^n|d \mapsto A_Y(n)(-\times\Delta^d)| \to \cdots \\ \to |d \mapsto A_Y(0)(-\times\Delta^d)|, \qquad (6)$$

in which successive cones are of the form

$$\Sigma_s^n|d \mapsto A_{0,Y}(n)(-\times\Delta^d)|.$$

We shall also refer to it as the *Grayson tower* for $\mathscr{A}$.

Given $F, G \in Pre^{\Sigma}_{nis}(Sm/k)$ we shall use the following notation:

$$[F, G] := \mathrm{Hom}_{\mathrm{Ho}\, Pre^{\Sigma}_{nis}(Sm/k)}(F, G).$$

Given $X \in Sm/k$, $Y \in Affsm/k$ and $p, q \in \mathbb{Z}$, we also set

$$H^{p,q}_{\mathscr{A}}(X, Y) := [X_+, \Sigma_s^{p-q}|d \mapsto A_{0,Y}(q)(-\times\Delta^d)|]$$

and

$$K_i^{\mathscr{A}}(X, Y) := [X_+, \Sigma_s^{-i}|d \mapsto A_Y(0)(-\times\Delta^d)|], \quad i \in \mathbb{Z}.$$

Remark 4.1. Let $D(NSh)$ be the derived category of Nisnevich sheaves of abelian groups on Sm/k. The Dold–Kan correspondence yields a complex of presheaves of abelian groups

$$C^*(A_{0,Y}(q))$$

which uniquely corresponds to the simplicial presheaf

$$d \mapsto K_0^{Gr}(C^{\oplus}\mathscr{A}(-\times\Delta^d, Y\times\mathbb{G}_m^{\wedge q})).$$

After sheafifying $C^*(A_{0,Y}(q))$ degreewise in the Nisnevich topology, we get a bounded above complex $C^*(A_{0,Y}(q))_{\mathrm{nis}} \in D(NSh)$ (the indexing is cohomological). It is then proved similarly to [5, 7.8] that

$$H^{p,q}_{\mathscr{A}}(X, Y) = H^p_{\mathrm{nis}}(X, C^*(A_{0,Y}(q))_{\mathrm{nis}}[-q]),$$

where the right-hand side stands for Nisnevich hypercohomology groups of X with coefficients in $C^*(A_{0,Y}(q))_{\mathrm{nis}}[-q]$ (the shift is cohomological).

Theorem 4.2 (Grayson). *The Grayson tower (6) produces a strongly convergent spectral sequence*

$$E_2^{pq} = H^{p-q,-q}_{\mathscr{A}}(X, Y) \Longrightarrow K^{\mathscr{A}}_{-p-q}(X, Y), \quad X \in Sm/k, Y \in Affsm/k, \qquad (7)$$

which will also be referred to as the Grayson spectral sequence *for* $\mathscr{A}$.

Proof. This is proved similarly to [5, 7.9]. □

Corollary 4.3. *If the groups $H^{p,q}_{\mathscr{A}}(X, Y)$ are homotopy invariant in the first argument, then so are the groups $K_i^{\mathscr{A}}(X, Y)$. In particular, every fibrant replacement of $|d \mapsto A_Y(0)(-\times\Delta^d)|$ in the Nisnevich local model category $Pre^{\Sigma}_{nis}(Sm/k)$ is fibrant in the motivic model category $Pre^{\Sigma}_{mot}(Sm/k)$.*

Below, we shall study conditions when the Grayson spectral sequence (7) is expressed in terms of bispectra.

Given a bispectrum $X = (X_0, X_1, \ldots)$, the *shift in $\mathbb{G}$-direction* $\Sigma_{\mathbb{G}} X$ is the bispectrum $(X_1, X_2, \ldots)$. Similarly, the nth shift $\Sigma^n_{\mathbb{G}} X$ is the bispectrum $(X_n, X_{n+1}, \ldots)$. For each $n \geqslant 0$, we have a triangle in the homotopy category of bispectra $\mathrm{Ho}(Pre^{\Sigma,\mathbb{G}}(Sm/k))$

$$\Sigma^{n+1}_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \xrightarrow{\gamma} \Omega \Sigma^n_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \xrightarrow{\Omega\tau} \Omega \Sigma^n_{\mathbb{G}} |d \mapsto A_{0,Y}(- \times \Delta^d)|.$$

Since the category $\mathrm{Ho}(Pre^{\Sigma,\mathbb{G}}(Sm/k))$ is triangulated with $\Sigma_s = - \wedge S^1$ a shift functor, the latter triangle gives a triangle

$$\Sigma_s \Sigma^{n+1}_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \to \Sigma^n_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \xrightarrow{\tau} \Sigma^n_{\mathbb{G}} |d \mapsto A_{0,Y}(- \times \Delta^d)|.$$

We shall also call it the *Grayson triangle of bispectra.*

We obtain a tower of bispectra in $\mathrm{Ho}(Pre^{\Sigma,\mathbb{G}}(Sm/k))$

$$\cdots \to \Sigma^{n+1}_s \Sigma^{n+1}_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \to \Sigma^n_s \Sigma^n_{\mathbb{G}} |d \mapsto A_Y(- \times \Delta^d)| \to \cdots \to |d \mapsto A_Y(- \times \Delta^d)|, \tag{8}$$

in which successive cones are of the form

$$\Sigma^n_s \Sigma^n_{\mathbb{G}} |d \mapsto A_{0,Y}(- \times \Delta^d)|.$$

We shall also refer to it as the *Grayson tower of bispectra* for $\mathscr{A}$.

Given a presheaf $\mathscr{F}$ of abelian groups and a scheme $X \in Sm/k$, one sets

$$\mathscr{F}(X \wedge \mathbb{G}_m) := \mathrm{Ker}(\iota^* : \mathscr{F}(X \times \mathbb{G}_m) \to \mathscr{F}(X \times \mathrm{Spec}\, k)).$$

There is a map of complexes

$$\beta : C^*(A_{0,Y}(q)) \to C^*(A_{0,Y}(q+1))(- \times \mathbb{G}_m),$$

where the left arrow is induced by map (4). Homotopy (5) implies that β uniquely factors through $C^*(A_{0,Y}(q+1))(- \wedge \mathbb{G}_m)$. Therefore one gets maps

$$\beta^{p,q} : H^{p,q}_{\mathscr{A}}(X, Y) \to H^{p+1,q+1}_{\mathscr{A}}(X \wedge \mathbb{G}_m, Y).$$

Definition 4.4. We say that the bigraded presheaves $H^{*,*}_{\mathscr{A}}(-, Y)$ satisfy the *cancelation property* if all maps $\beta^{p,q}$ are isomorphisms.

Let $X = (X_0, X_1, \ldots)$ and $Y = (Y_0, Y_1, \ldots)$ be two bispectra. Recall that a map of bispectra $f : X \to Y$ is a level Nisnevich local equivalence if so is each $f_n : X_n \to Y_n$ in $Pre^{\Sigma}_{nis}(Sm/k)$. By common facts for model categories (see, e.g., [10]), there is a level Nisnevich local equivalence of bispectra

$$u : X \to \widetilde{X}$$

such that each map $u_n : X_n \to \widetilde{X}_n$ is a cofibration and each $\widetilde{X}_n$ is fibrant in $Pre^{\Sigma}_{nis}(Sm/k)$. Moreover, the map is functorial in X.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Consider the bispectrum $A_{0,Y} = (A_{0,Y}(0), A_{0,Y}(1), \ldots)$. Denote by $A^{\Delta}_{0,Y}$ and $\widetilde{A}^{\Delta}_{0,Y}$ the bispectra

$$(|d \mapsto A_{0,Y}(0)(- \times \Delta^d)|, |d \mapsto A_{0,Y}(1)(- \times \Delta^d)|, \ldots)$$

and

$$(\widetilde{(A^{\Delta}_{0,Y})_0}, \widetilde{(A^{\Delta}_{0,Y})_1}, \widetilde{(A^{\Delta}_{0,Y})_2}, \ldots),$$

respectively. Then there is a map of bispectra (see above)

$$u : A^{\Delta}_{0,Y} \to \widetilde{A}^{\Delta}_{0,Y}.$$

Observe that there is an isomorphism

$$H^{p,q}_{\mathscr{A}}(X, Y) \cong \pi(X_+, \Sigma^{p-q}_s(\widetilde{A}^{\Delta}_{0,Y})_q),$$

where the right-hand side stands for the usual homotopy equivalence classes of maps.

Lemma 4.5. *The bigraded presheaves $H^{*,*}_{\mathscr{A}}(-, Y)$ satisfy the cancelation property if and only if each structure map of the bispectrum $\widetilde{A}^{\Delta}_{0,Y}$*

$$(\widetilde{A}^{\Delta}_{0,Y})_n \to \Omega_{\mathbb{G}}(\widetilde{A}^{\Delta}_{0,Y})_{n+1}$$

is a stable weak equivalence in $Pre^{\Sigma}(Sm/k)$.

We can also define bispectra A^{Δ}_Y and $\widetilde{A}^{\Delta}_Y$ similar to the bispectra $A^{\Delta}_{0,Y}$ and $\widetilde{A}^{\Delta}_{0,Y}$. There is a map of towers in $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \Sigma^{n+1}_s(A^{\Delta}_Y)_{n+1} & \longrightarrow & \Sigma^n_s(A^{\Delta}_Y)_n & \longrightarrow \cdots \longrightarrow & (A^{\Delta}_Y)_0 \\
 & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & \Sigma^{n+1}_s(\widetilde{A}^{\Delta}_Y)_{n+1} & \longrightarrow & \Sigma^n_s(\widetilde{A}^{\Delta}_Y)_n & \longrightarrow \cdots \longrightarrow & (\widetilde{A}^{\Delta}_Y)_0,
\end{array}$$

where the upper tower is the Grayson tower and the vertical morphisms are Nisnevich local equivalences in $Pre^{\Sigma}_{nis}(Sm/k)$. Moreover, we have maps of successive cones of both towers

$$\Sigma^n_s u_n : \Sigma^n_s(A^{\Delta}_{0,Y})_n \to \Sigma^n_s(\widetilde{A}^{\Delta}_{0,Y})_n.$$

For every $q \geqslant 0$ and $p \in \mathbb{Z}$, one sets

$$K^{\mathscr{A}}_{-p}(X, Y \wedge \mathbb{G}^q_m) := [X_+, \Sigma^{p-q}_s(A^{\Delta}_Y)_q].$$

Observe that there is an isomorphism

$$K^{\mathscr{A}}_{-p}(X, Y \wedge \mathbb{G}^q_m) \cong \pi(X_+, \Sigma^{p-q}_s(\widetilde{A}^{\Delta}_Y)_q).$$

Theorem 4.6 (Cancelation for K-theory). *Suppose that the bigraded presheaves $H^{*,*}_{\mathscr{A}}(-, Y)$ satisfy the cancelation property. Then each structure map of the bispectrum $\widetilde{A}^{\Delta}_Y$*

$$(\widetilde{A}^{\Delta}_Y)_q \to \Omega_{\mathbb{G}}(\widetilde{A}^{\Delta}_Y)_{q+1}, \quad q \geqslant 0, \tag{9}$$

is a stable weak equivalence of symmetric spectra. In particular, the natural map

$$K^{\mathscr{A}}_{-p}(X, Y \wedge \mathbb{G}^q_m) \to K^{\mathscr{A}}_{-p-1}(X \wedge \mathbb{G}_m, Y \wedge \mathbb{G}^{q+1}_m),$$

induced by (9), is an isomorphism for all $p \in \mathbb{Z}$.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Proof. We have a map of towers in $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Sigma_s^{n+1}(\widetilde{A}_Y^{\Delta})_{n+1} & \longrightarrow & \Sigma_s^{n}(\widetilde{A}_Y^{\Delta})_{n} & \longrightarrow & \cdots & \longrightarrow & (\widetilde{A}_Y^{\Delta})_0 \\ & \downarrow & & \downarrow & & & & \downarrow \\ \cdots \longrightarrow & \Sigma_s^{n+1}\Omega_{\mathbb{G}}(\widetilde{A}_Y^{\Delta})_{n+2} & \longrightarrow & \Sigma_s^{n}\Omega_{\mathbb{G}}(\widetilde{A}_Y^{\Delta})_{n+1} & \longrightarrow & \cdots & \longrightarrow & \Omega_{\mathbb{G}}(\widetilde{A}_Y^{\Delta})_1, \end{array}$$

where the upper tower is the Grayson tower and the vertical morphisms are structure morphisms of the bispectrum $\widetilde{A}_Y^{\Delta}$. By Lemma 4.5, all maps of successive cones

$$\Sigma_s^n(\widetilde{A}_{0,Y}^{\Delta})_n \to \Sigma_s^n\Omega_{\mathbb{G}}(\widetilde{A}_{0,Y}^{\Delta})_{n+1}$$

are stable weak equivalences in $Pre^{\Sigma}(Sm/k)$.

For every $X \in Sm/k$, Theorem 4.2 implies that the upper tower produces a strongly convergent spectral sequence

$$E_{pq}^2 = \pi_{p+q}(\Sigma_s^q(\widetilde{A}_{0,Y}^{\Delta})_q(X)) \Longrightarrow \pi_{p+q}((\widetilde{A}_Y^{\Delta})_0(X))$$

and the lower tower produces a strongly convergent spectral sequence

$$E_{pq}^2 = \pi_{p+q}(\Sigma_s^q\Omega_{\mathbb{G}}(\widetilde{A}_{0,Y}^{\Delta})_{q+1}(X)) \Longrightarrow \pi_{p+q}(\Omega_{\mathbb{G}}(\widetilde{A}_Y^{\Delta})_1(X)).$$

Since both spectral sequences are isomorphic, we conclude that the map

$$(A_Y^{\Delta})_0 \to \Omega_{\mathbb{G}}(A_Y^{\Delta})_1$$

is a stable weak equivalence in $Pre^{\Sigma}(Sm/k)$. It is proved in a similar fashion that all other vertical maps in the diagram above are stable weak equivalences in $Pre^{\Sigma}(Sm/k)$, and hence so are their desuspensions. □

We are now in a position to prove the main result of the section.

Theorem 4.7. *Suppose that the groups $H_{\mathscr{A}}^{p,q}(X,Y)$ are homotopy invariant in the first argument and that they satisfy the cancelation property. Then the bispectra $\widetilde{A}_Y^{\Delta}$ and $\widetilde{A}_{0,Y}^{\Delta}$ are motivically fibrant in $Pre_{mot}^{\Sigma,\mathbb{G}}(Sm/k)$, and the maps*

$$A_Y \to \widetilde{A}_Y^{\Delta}, \quad A_{0,Y} \to \widetilde{A}_{0,Y}^{\Delta}$$

are motivic weak equivalences of bispectra. As a result, one has a tower in $SH(k)$,

$$\cdots \to \Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_Y \to \Sigma_s^q\Sigma_{\mathbb{G}}^q A_Y \to \cdots \to A_Y, \tag{10}$$

in which successive cones are of the form $\Sigma_s^q\Sigma_{\mathbb{G}}^q A_{0,Y}$. This tower produces a strongly convergent spectral sequence

$$E_{pq}^2 = SH(k)(X_+, \Sigma_s^{-p}\Sigma_{\mathbb{G}}^q A_{0,Y}) \Longrightarrow SH(k)(X_+, \Sigma_s^{-p-q}A_Y),$$

which is isomorphic to the Gayson spectral sequence for $\mathscr{A}$.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Proof. By Corollary 4.3, $(\widetilde{A}^{\Delta}_Y)_0$ is fibrant in $Pre^{\Sigma}_{mot}(Sm/k)$. It is proved in a similar way that each $(\widetilde{A}^{\Delta}_Y)_q$, $q > 0$, is fibrant in $Pre^{\Sigma}_{mot}(Sm/k)$. So the bispectrum $\widetilde{A}^{\Delta}_Y$ is level motivically fibrant. Theorem 4.6 implies that it is fibrant in $Pre^{\Sigma,\mathbb{G}}_{mot}(Sm/k)$. The fact that the bispectrum $\widetilde{A}^{\Delta}_{0,Y}$ is fibrant in $Pre^{\Sigma,\mathbb{G}}_{mot}(Sm/k)$ follows from Lemma 4.5 and the homotopy invariance of the groups $H^{p,q}_{\mathscr{A}}(X, Y)$ in the first argument.

Let us show that $A_Y \to \widetilde{A}^{\Delta}_Y$ is a motivic weak equivalence of bispectra. This map is the composition

$$A_Y \to A^{\Delta}_Y \to \widetilde{A}^{\Delta}_Y,$$

where the right arrow is a Nisnevich local equivalence, and hence a motivic weak equivalence. The left arrow is a motivic weak equivalence by [18, 3.8]. Similarly, $A_{0,Y} \to \widetilde{A}^{\Delta}_{0,Y}$ is a motivic weak equivalence of bispectra.

The Grayson tower of bispectra (8) in $\mathrm{Ho}(Pre^{\Sigma,\mathbb{G}})(Sm/k)$ yields a tower of bispectra in $SH(k)$

$$\cdots \to \Sigma^{q+1}_s \Sigma^{q+1}_{\mathbb{G}} A_Y \to \Sigma^q_s \Sigma^q_{\mathbb{G}} A_Y \to \cdots \to A_Y, \tag{11}$$

in which successive cones are of the form $\Sigma^q_s \Sigma^q_{\mathbb{G}} A_{0,Y}$. This tower produces a spectral sequence

$$E^2_{pq} = SH(k)(X_+, \Sigma^{-p}_s \Sigma^q_{\mathbb{G}} A_{0,Y}) \Longrightarrow SH(k)(X_+, \Sigma^{-p-q}_s A_Y),$$

which is the same as the spectral sequence

$$E^2_{pq} = SH(k)(X_+, \Sigma^{-p}_s \Sigma^q_{\mathbb{G}} \widetilde{A}^{\Delta}_{0,Y}) \Longrightarrow SH(k)(X_+, \Sigma^{-p-q}_s \widetilde{A}^{\Delta}_Y).$$

Since all bispectra involved in the latter spectral sequence are motivically fibrant, it follows that it is isomorphic to the spectral sequence

$$E^2_{pq} = \mathrm{Hom}_{\mathrm{Ho}(Pre^{\Sigma}(Sm/k))}(X_+, \Sigma^{-p}_s (\widetilde{A}^{\Delta}_{0,Y})_q) \Longrightarrow \mathrm{Hom}_{\mathrm{Ho}(Pre^{\Sigma}(Sm/k))}(X_+, \Sigma^{-p-q}_s (\widetilde{A}^{\Delta}_Y)_0).$$

It is plainly isomorphic to the Grayson spectral sequence (7). □

5. Postnikov towers in $SH_{S^1}(k)$ and $SH(k)$

Voevodsky [31] has defined a canonical tower on the motivic stable homotopy category $SH_{S^1}(k)$, which is called the *motivic Postnikov tower*.

Let $\Sigma^n_{\mathbb{G}} SH_{S^1}(k)$ be the localizing subcategory of $SH_{S^1}(k)$ generated by objects of the form $\Sigma^n_{\mathbb{G}} E$, $E \in SH_{S^1}(k)$. This gives us the tower of localizing subcategories,

$$\cdots \subset \Sigma^{n+1}_{\mathbb{G}} SH_{S^1}(k) \subset \Sigma^n_{\mathbb{G}} SH_{S^1}(k) \subset \cdots \subset SH_{S^1}(k).$$

Take $E \in SH_{S^1}(k)$, and consider the cohomological functor

$$\mathrm{Hom}_{\Sigma^n_{\mathbb{G}} SH_{S^1}(k)}(-, E) : \Sigma^n_{\mathbb{G}} SH_{S^1}(k) \to Ab.$$

By [19], this functor is represented by an object $r_n E$ of $\Sigma^n_{\mathbb{G}} SH_{S^1}(k)$. Sending E to $r_n E$ defines a right adjoint $r_n : SH_{S^1}(k) \to \Sigma^n_{\mathbb{G}} SH_{S^1}(k)$ to the inclusion $i_n : \Sigma^n_{\mathbb{G}} SH_{S^1}(k) \to SH_{S^1}(k)$. Let $f_n := i_n \circ r_n$ with counit $f_n \to \mathrm{id}$. Thus, for each $E \in SH_{S^1}(k)$, there is a canonical tower in $SH_{S^1}(k)$

$$\cdots \to f_{n+1} E \to f_n E \to \cdots \to f_0 E = E,$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

the *motivic Postnikov tower for S^1-spectra*. We write $f_{n/n+r}E$ for the cofiber of $f_{n+r} \to f_n E$; we use the notation $s_n := f_{n/n+1}$ to denote the nth slice in the Postnikov tower.

Let $EM : Ab \to Sp^{\Sigma}$ be the Eilenberg–Mac Lane functor in the sense of [5, Appendix A], and let $\mathbb{Z}(n)$, $n \geqslant 0$, be the Suslin–Voevodsky complexes [27].

Proposition 5.1. *Suppose that k is a perfect field. Then for every $n \geqslant 0$ we have $EM(\mathbb{Z}(n)) = s_n(EM(\mathbb{Z}(n)))$.*

Proof. By [13, 1.4.3], $EM(\mathbb{Z}(n)) \in \Sigma^n_{\mathbb{G}} SH_{S^1}(k)$. The cancellation theorem of Voevodsky [32] and [31, 4.2] imply that

$$\Omega^{n+1}_{\mathbb{G}}(EM(\mathbb{Z}(n))_f) \cong \Omega_{\mathbb{G}}(EM(\mathbb{Z}(0))_f) = 0,$$

where $EM(\mathbb{Z}(n))_f$ is a local fibrant replacement of $EM(\mathbb{Z}(n))$. It follows that $EM(\mathbb{Z}(n))$ is right orthogonal to $\Sigma^{n+1}_{\mathbb{G}} SH_{S^1}(k)$, and hence $EM(\mathbb{Z}(n)) = s_n(EM(\mathbb{Z}(n)))$. □

Given an integer ℓ, call $E \in SH_{S^1}(k)$ *ℓ-connected* if, for each $n \leqslant \ell$, the Nisnevich sheaf $\pi_n(\widetilde{E})$ is zero, where $\widetilde{E}$ is a fibrant model for E in $Pre^{\Sigma}_{mot}(Sm/k)$. We shall also refer to (-1)-connected spectra as *connected*.

Lemma 5.2. *Suppose that k is a perfect field and that $\mathscr{A}$ is a bivariant additive category with the $(\mathfrak{Aut})$ property. Suppose further that each $A_{0,Y}(n) \in \Sigma^n_{\mathbb{G}} SH_{S^1}(k)$. If the presheaves $H^{p,q}_{\mathscr{A}}(-, Y)$ are homotopy invariant, then $A_Y(n) \in \Sigma^n_{\mathbb{G}} SH_{S^1}(k)$ for every $n \geqslant 0$.*

Proof. By Corollary 4.3, $(\widetilde{A}^{\Delta}_Y)_0$ is fibrant in $Pre^{\Sigma}_{mot}(Sm/k)$. It is proved in a similar way that each $(\widetilde{A}^{\Delta}_Y)_n$, $n > 0$, is fibrant in $Pre^{\Sigma}_{mot}(Sm/k)$. The map $A_Y(n) \to (\widetilde{A}^{\Delta}_Y)_n$ is a motivic weak equivalence. It is the composition

$$A_Y(n) \to (A^{\Delta}_Y)_n \to (\widetilde{A}^{\Delta}_Y)_n,$$

where the right arrow is a Nisnevich local equivalence, and hence a motivic weak equivalence. The left arrow is a motivic weak equivalence by [18, 3.8].

We have $A_{0,Y}(m) \cong f_n(A_{0,Y}(m)) \in \Sigma^n_{\mathbb{G}} SH_{S^1}(k)$ for all $m \geqslant n$. Applying the functor f_n to the Grayson tower (6), we get a map of towers of motivically fibrant presheaves of spectra,

$$\begin{array}{ccccc}
\cdots \longrightarrow & \Sigma^{n+1}_s f_n((\widetilde{A}^{\Delta}_Y)_{n+1}) & \longrightarrow & \Sigma^n_s f_n((\widetilde{A}^{\Delta}_Y)_n) \\
& \downarrow & & \downarrow \\
\cdots \longrightarrow & \Sigma^{n+1}_s (\widetilde{A}^{\Delta}_Y)_{n+1} & \longrightarrow & \Sigma^n_s (\widetilde{A}^{\Delta}_Y)_n,
\end{array}$$

in which successive cones are of the form $\Sigma^{n+q}_s (\widetilde{A}^{\Delta}_{0,Y})_{n+q}$.

Every spectrum $\Sigma^{n+q}_s (\widetilde{A}^{\Delta}_Y)_{n+q}$ is $(n+q-1)$-connected. If $X \in Sm/k$ is of dimension d, then $\Sigma^{n+q}_s (\widetilde{A}^{\Delta}_Y)_{n+q}(X)$ is $(n+q-d-1)$-connected in Sp^{Σ}, by [17, 4.3.1]. By [4, 6.1.1], the lower tower produces a strongly convergent spectral sequence

$$E^2_{pq} = \pi_{p+q}(\Sigma^{n+q}_s (\widetilde{A}^{\Delta}_{0,Y})_{n+q}(X)) \Longrightarrow \pi_{p+q}((\widetilde{A}^{\Delta}_Y)_n(X)).$$

To show that the spectral sequence produced by the upper tower

$$E^2_{pq} = \pi_{p+q}(\Sigma^{n+q}_s (\widetilde{A}^{\Delta}_{0,Y})_{n+q}(X)) \Longrightarrow \pi_{p+q}(f_n((\widetilde{A}^{\Delta}_Y)_n)(X))$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

is strongly convergent, we need to know that each $f_n((\widetilde{A}_Y^{\Delta})_{n+q})$ has the same connectivity properties as $(\widetilde{A}_Y^{\Delta})_{n+q}$. Since $\Sigma_s^{n+q}(\widetilde{A}_Y^{\Delta})_{n+q}$ is $(n+q-1)$-connected, it follows from [16, 3.2] and Proposition A.2 that so is $\Sigma_s^{n+q} f_n((\widetilde{A}_Y^{\Delta})_{n+q}) = f_n(\Sigma_s^{n+q}(\widetilde{A}_Y^{\Delta})_{n+q})$. If $X \in Sm/k$ is of dimension d, then $\Sigma_s^{n+q} f_n((\widetilde{A}_Y^{\Delta})_{n+q})(X)$ is $(n+q-d-1)$-connected in Sp^{Σ}, by [17, 4.3.1].

It follows from [4, 6.1.1] that the second spectral sequence is strongly convergent. We conclude that the map in $\mathrm{Ho}(Sp^{\Sigma})$

$$f_n((\widetilde{A}_Y^{\Delta})_n)(X) \to (\widetilde{A}_Y^{\Delta})_n(X)$$

is an isomorphism. We see that $A_Y(n)$ is isomorphic in $SH_{S^1}(k)$ to $f_n((\widetilde{A}_Y^{\Delta})_n) \in \Sigma_{\mathbb{G}}^n SH_{S^1}(k)$. □

The following theorem says that Grayson's tower of S^1-spectra (6) is isomorphic in $SH_{S^1}(k)$ to the motivic Postnikov tower of the K-theory S^1-spectrum $K^{Gr}(\mathscr{A}(-,Y))$. In Theorem 7.10 we shall extend this result to bispectra.

Theorem 5.3. *Suppose that k is a perfect field and that $\mathscr{A}$ is a bivariant additive category with the $(\mathfrak{Aut})$ property. Suppose further that each $A_{0,Y}(q) = s_q(A_{0,Y}(q))$. If the presheaves $H_{\mathscr{A}}^{p,q}(-,Y)$ are homotopy invariant, then the Grayson tower (6) is isomorphic in $SH_{S^1}(k)$ to the motivic Postnikov tower*

$$\cdots \to f_{q+1}(K^{Gr}(\mathscr{A}(-,Y))) \to f_q(K^{Gr}(\mathscr{A}(-,Y))) \to \cdots$$
$$\to f_0(K^{Gr}(\mathscr{A}(-,Y))) = K^{Gr}(\mathscr{A}(-,Y)).$$

Proof. We have $f_0(K^{Gr}(\mathscr{A}(-,Y))) = K^{Gr}(\mathscr{A}(-,Y))$. Suppose that an isomorphism $\theta_q : \Sigma_s^q A_Y^{\Delta}(q) \cong f_q(K^{Gr}(\mathscr{A}(-,Y)))$, $q \geqslant 0$, is constructed. Since $\Sigma_s^{q+1} A_Y(q+1) \in \Sigma_{\mathbb{G}}^{q+1} SH_{S^1}(k)$ by the preceding lemma and $s_q(K^{Gr}(\mathscr{A}(-,Y)))$ is right orthogonal to $\Sigma_{\mathbb{G}}^{q+1} SH_{S^1}(k)$, it follows that there is a unique morphism

$$\theta_{q+1} : \Sigma_s^{q+1} A_Y(q+1) \to f_{q+1}(K^{Gr}(\mathscr{A}(-,Y)))$$

making the diagram

$$\begin{array}{ccc} \Sigma_s^{q+1} A_Y(q+1) & \longrightarrow & \Sigma_s^q A_Y(q) \\ \downarrow{\scriptstyle \theta_{q+1}} & & \downarrow{\scriptstyle \theta_q} \\ f_{q+1}(K^{Gr}(\mathscr{A}(-,Y))) & \longrightarrow & f_q(K^{Gr}(\mathscr{A}(-,Y))) \end{array}$$

commutative. We claim that θ_{q+1} is an isomorphism in $SH_{S^1}(k)$.

By assumption, $f_{q+1}(\Sigma_s^q A_{0,Y}(q)) = f_{q+1}(\Sigma_s^q s_q(A_{0,Y}(q))) = \Sigma_s^q f_{q+1} s_q(A_{0,Y}(q)) = 0$. We also have that $f_{q+1} s_q(K^{Gr}(\mathscr{A}(-,Y))) = 0$, and hence the horizontal arrows of the commutative diagram

$$\begin{array}{ccc} f_{q+1}(\Sigma_s^{q+1} A_Y(q+1)) & \longrightarrow & f_{q+1}(\Sigma_s^q A_Y(q)) \\ \downarrow{\scriptstyle f_{q+1}(\theta_{q+1})} & & \downarrow{\scriptstyle f_{q+1}(\theta_q)} \\ f_{q+1}(f_{q+1}(K^{Gr}(\mathscr{A}(-,Y)))) & \longrightarrow & f_{q+1}(f_q(K^{Gr}(\mathscr{A}(-,Y)))) \end{array}$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

are isomorphisms. But $f_{q+1}(\theta_q)$ is an isomorphism, and hence so is $f_{q+1}(\theta_{q+1})$. By the previous lemma, $\Sigma_s^{q+1} A_Y(q+1)$ is in $\Sigma_{\mathbb{G}}^{q+1} SH_{S^1}(k)$. Since $f_{q+1}(K^{Gr}(\mathscr{A}(-,Y)))$ belongs to $\Sigma_{\mathbb{G}}^{q+1} SH_{S^1}(k)$ as well and $f_{q+1}(\theta_{q+1})$ is an isomorphism, we conclude that θ_{q+1} is an isomorphism. □

The next result computes the slices of the K-theory S^1-spectrum $K^{Gr}(\mathscr{A}(-,Y))$. It will be extended to bispectra in Theorem 7.11.

Theorem 5.4. *Under the assumptions of Theorem 5.3 there are isomorphisms in $SH_{S^1}(k)$,*

$$s_q(K^{Gr}(\mathscr{A}(-,Y))) \cong \Sigma_s^q A_{0,Y}(q), \quad q \geqslant 0.$$

Proof. The proof of the previous theorem shows that there is a commutative diagram in $SH_{S^1}(k)$,

$$\begin{array}{ccccccc}
\Sigma_s^{q+1} A_Y(q+1) & \longrightarrow & \Sigma_s^q A_Y(q) & \longrightarrow & \Sigma_s^q A_{0,Y}(q) & \longrightarrow & \Sigma_s^{q+2} A_{0,Y}(q+1) \\
\Big\downarrow{\scriptstyle \theta_{q+1}} & & \Big\downarrow{\scriptstyle \theta_q} & & & & \Big\downarrow{\scriptstyle \Sigma_s \theta_{q+1}} \\
f_{q+1}(K^{Gr}(\mathscr{A}(-,Y))) & \longrightarrow & f_q(K^{Gr}(\mathscr{A}(-,Y))) & \longrightarrow & s_q(K^{Gr}(\mathscr{A}(-,Y))) & \longrightarrow & \Sigma_s f_{q+1}(K^{Gr}(\mathscr{A}(-,Y))),
\end{array}$$

where the vertical arrows are isomorphisms. Since $SH_{S^1}(k)$ is triangulated, then there exists an isomorphism

$$\Sigma_s^q A_{0,Y}(q) \cong s_q(K^{Gr}(\mathscr{A}(-,Y))),$$

as required. □

Voevodsky [30] defines the slice filtration in $SH(k)$ just as it is defined in $SH_{S^1}(k)$. Let $SH^{eff}(k)$ be the smallest localizing subcategory of $SH(k)$ containing all suspension spectra $\Sigma_{\mathbb{G}}^\infty \Sigma_s^\infty X_+$ with $X \in Sm/k$; this is the same as the smallest localizing subcategory containing all the $\mathbb{G}$-suspension spectra $\Sigma_{\mathbb{G}}^\infty E$ for $E \in SH_{S^1}(k)$. For each integer p, let $\Sigma_{\mathbb{G}}^p SH^{eff}(k)$ denote the smallest localizing subcategory of $SH(k)$ containing the $\mathbb{G}$-spectra $\Sigma_{\mathbb{G}}^p \mathscr{E}$ for $\mathscr{E} \in SH^{eff}(k)$. The inclusion $i_p : \Sigma_{\mathbb{G}}^p SH^{eff}(k) \to SH(k)$ admits the right adjoint $r_p : SH(k) \to \Sigma_{\mathbb{G}}^p SH^{eff}(k)$; setting $f_p := i_p \circ r_p$, one has for each $\mathscr{E} \in SH(k)$ the functorial *slice tower*

$$\cdots \to f_{d+1}\mathscr{E} \to f_d\mathscr{E} \to \cdots \to f_0\mathscr{E} \to f_{-1}\mathscr{E} \to \cdots \to \mathscr{E}.$$

As for the slice tower in $SH_{S^1}(k)$, the existence of the adjoint follows from [19], and the map $f_p\mathscr{E} \to \mathscr{E}$ is universal for maps $\mathscr{F} \to \mathscr{E}$, $\mathscr{F} \in \Sigma_{\mathbb{G}}^p SH^{eff}(k)$. The cofiber of $f_{d+1}\mathscr{E} \to f_d\mathscr{E}$ is denoted by $s_d\mathscr{E}$.

Lemma 5.5. *Under the assumptions of Lemma 5.2, the bispectra A_Y and $A_{0,Y}$ belong to $SH^{eff}(k)$.*

Proof. We prove the assertion for A_Y, because the same arguments will hold for $A_{0,Y}$. It is shown similarly to [20, A.33] that every bispectrum $\mathscr{E}$ is the colimit of a natural sequence

$$Tr_0\mathscr{E} \to Tr_1\mathscr{E} \to \cdots,$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

where $Tr_i\mathscr{E}$ stands for the ith truncation. Moreover, $Tr_i\mathscr{E}$ is naturally stably equivalent to $\Omega^i_{\mathbb{G}}((\Sigma^\infty_{\mathbb{G}} E_i)^f)$ ('f' for fibrant resolution).

Lemma 5.2 implies that $(\Sigma^\infty_{\mathbb{G}} A_Y(i))^f \in \Sigma^i_{\mathbb{G}} SH^{eff}(k)$, and hence $\Omega^i_{\mathbb{G}}((\Sigma^\infty_{\mathbb{G}} A_Y(i))^f) \in SH^{eff}(k)$. We see that A_Y is the colimit of the sequence

$$Tr_0 A_Y \to Tr_1 A_Y \to \cdots,$$

where each $Tr_i A_Y$ is in $SH^{eff}(k)$. It follows from [20, A.34] that A_Y is isomorphic in $SH(k)$ to the homotopy colimit of $Tr_i A_Y$, which is in $SH^{eff}(k)$. We conclude that $A_Y \in SH^{eff}(k)$. □

6. The bispectrum $KGL_{\mathscr{A},Y}$

Given an additive category $\mathscr{M}$, Quillen defines a new category $S^{-1}S\mathscr{M}$ whose objects are pairs (A, B) of objects of $\mathscr{M}$. A morphism $(A, B) \to (C, D)$ in $S^{-1}S\mathscr{M}$ is given by a pair of split monomorphisms

$$f : A \overset{\twoheadleftarrow}{\rightarrowtail} C, \quad g : B \overset{\twoheadleftarrow}{\rightarrowtail} D$$

together with an isomorphism $h : \operatorname{Coker} f \to \operatorname{Coker} g$. By a split monomorphism we mean a monomorphism together with a chosen splitting. The nerve of the category $S^{-1}S\mathscr{M}$ which is also denoted by $S^{-1}S\mathscr{M}$ is homotopy equivalent to Quillen's K-theory space of $\mathscr{M}$ by [7].

The set $S^{-1}S_k\mathscr{M}$ of k-simplices of the category $S^{-1}S\mathscr{M}$ can be regarded as the set of objects of an additive category in the usual way, and we use exactly the same notation to denote that category. In this way $S^{-1}S\mathscr{M}$ becomes a simplicial additive category. Its symmetric Grayson K-theory spectrum will be denoted by $K^{Gr}(S^{-1}S\mathscr{M})$. It follows from the proof of [9, 9.3] that the map $\operatorname{Ob}\mathscr{M} \to S^{-1}S\mathscr{M}$ sending M to $(M, 0)$ induces a homotopy equivalence

$$S^{\oplus}\mathscr{M} \to S^{\oplus}(S^{-1}S)\mathscr{M}.$$

Therefore the induced map of symmetric spectra

$$K^{Gr}(\mathscr{M}) \to K^{Gr}(S^{-1}S\mathscr{M})$$

is a stable weak equivalence, which is a level weak equivalence in positive degrees.

Let $\mathscr{A}$ be a bivariant additive category. Then $S^{-1}S\mathscr{A}$ is a simplicial bivariant additive category. For any $Y \in AffSm/k$ we can form a bispectrum

$$S^{-1}SA_Y = (S^{-1}SA_Y(0), S^{-1}SA_Y(1), \ldots),$$

where each $S^{-1}SA_Y(n) = K^{Gr}(C^{\oplus}S^{-1}S\mathscr{A}(-, Y \times \mathbb{G}_m^{\wedge n}))$, and define a natural map of bispectra

$$s : A_Y \to S^{-1}SA_Y.$$

This map is a level stable weak equivalence.

In order to construct K-theory spectra with entries being sectionwise fibrant spaces, we use the category of topological symmetric spectra $TopSp^{\Sigma}$ (see [25, § I.1]). We can apply adjoint functors 'geometric realization', denoted by $|-|$, and 'singular complex', denoted by $\mathscr{S}$, levelwise to go back and forth between simplicial and topological symmetric spectra

$$|-|:Sp^{\Sigma}\rightleftarrows TopSp^{\Sigma}:\mathscr{S}. \tag{12}$$

Remark 6.1. By the standard abuse of notation, $|-|$ denotes both the functor from Sp^{Σ} to $TopSp^{\Sigma}$ and the realization functor from simplicial spectra to spectra. It will always be clear from the context which of the meanings is used.

Given a (multisimplicial) additive category $\mathscr{M}$, denote by $\hat{K}^{Gr}(\mathscr{M})$ the symmetric spectrum $\mathscr{S}|K^{Gr}(\mathscr{M})|$. The unit of the adjunction induces a map of symmetric spectra

$$K^{Gr}(\mathscr{M})\to\hat{K}^{Gr}(\mathscr{M}),$$

functorial in $\mathscr{M}$. Observe that $|K^{Gr}(S^{-1}S\mathscr{M})|$ is an Ω-spectrum in $TopSp^{\Sigma}$, and hence so is $\hat{K}^{Gr}(S^{-1}S\mathscr{M})$ in Sp^{Σ}.

Suppose that $\mathscr{A}$ is a bivariant additive category satisfying property $(\mathfrak{Aut})$. For any $Y\in AffSm/k$ we can form a bispectrum

$$S^{-1}S\hat{A}_Y=(S^{-1}S\hat{A}_Y(0),S^{-1}S\hat{A}_Y(1),\ldots),$$

where each $S^{-1}S\hat{A}_Y(n)=\hat{K}^{Gr}(C^{\oplus}S^{-1}S\mathscr{A}(-,Y\times\mathbb{G}_m^{\wedge n}))$, and define a natural map of bispectra

$$t:S^{-1}SA_Y\to S^{-1}S\hat{A}_Y.$$

This map is a level stable weak equivalence.

By [9, 9.4] and [34, p. 16], there is a natural map of symmetric topological spectra,

$$|K^{Gr}(C^{\oplus}S^{-1}S\mathscr{A}(X,Y\times\mathbb{G}_m^{\wedge 1}))|\to\Omega|K^{Gr}((S^{-1}S)(S^{-1}S)\mathscr{A}(X,Y))|.$$

It gives a natural map in $Pre^{\Sigma}(Sm/k)$,

$$v_1:S^{-1}S\hat{A}_Y(1)\to\Omega\hat{K}^{Gr}((S^{-1}S)(S^{-1}S)\mathscr{A}(-,Y)).$$

We can get more generally a map (see [34, p. 16] as well)

$$\begin{aligned}v_n:S^{-1}S\hat{A}_Y(n)&\to\Omega\hat{K}^{Gr}(C^{\oplus}(S^{-1}S)(S^{-1}S)\mathscr{A}(-,Y\times\mathbb{G}_m^{\wedge n-1}))\to\cdots\\&\to\Omega^n\hat{K}^{Gr}((S^{-1}S)^{n+1}\mathscr{A}(-,Y)).\end{aligned}$$

One sets

$$\varkappa_0:=\Omega_{\mathbb{G}}(v_1)\circ\hat{a}_0:\hat{K}^{Gr}(S^{-1}S\mathscr{A}(-,Y))\to\Omega_{\mathbb{G}}\Omega\hat{K}^{Gr}((S^{-1}S)^2\mathscr{A}(-,Y)),$$

where $\hat{a}_0:\hat{A}_Y(0)\to\Omega_{\mathbb{G}}\hat{A}_Y(1)$ is the structure map. Applying the above construction to the multisimplicial bivariant additive category $(S^{-1}S)^n\mathscr{A}$, we get a map

$$\varkappa_n:\Omega^n\hat{K}^{Gr}((S^{-1}S)^{n+1}\mathscr{A}(-,Y))\to\Omega_{\mathbb{G}}\Omega^{n+1}\hat{K}^{Gr}((S^{-1}S)^{n+2}\mathscr{A}(-,Y)).$$

Definition 6.2. Let $\mathscr{A}$ be a bivariant additive category with the $(\mathfrak{Aut})$ property, and let $Y\in AffSm/k$. Then the bispectrum $KGL_{\mathscr{A},Y}$ is defined by the sequence in $Pre^{\Sigma}(Sm/k)$

$$(\hat{K}^{Gr}((S^{-1}S)\mathscr{A}(-,Y)),\Omega\hat{K}^{Gr}((S^{-1}S)^2\mathscr{A}(-,Y)),\Omega^2\hat{K}^{Gr}((S^{-1}S)^3\mathscr{A}(-,Y)),\ldots).$$

Its structure maps are given by the maps $\varkappa_n$.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

The maps v_n determine a map of bispectra

$$v : S^{-1}S\hat{A}_Y \to KGL_{\mathscr{A},Y}.$$

So we have a map of bispectra

$$\chi := v \circ t \circ s : A_Y \to KGL_{\mathscr{A},Y}. \tag{13}$$

In the next section we shall work with the bispectrum $KGL_{\mathscr{A},\operatorname{Spec} k}$ for a certain bivariant additive category $\mathscr{A}$. It will be shown that it represents Quillen's K-theory of algebraic varieties.

7. Comparing Grayson's and slice towers for KGL

In this section we prove the main results of the paper. For technical reasons we have dealt with general bivariant additive categories so far. Below, a concrete example of such a category $\mathscr{A}$ is given. It will lead to solutions for the problems mentioned in the introduction in § 1. Its definition is extracted from [6]. We start with preparations.

Let $U, X \in Sm/k$. Define $\operatorname{Supp}(U \times X/X)$ as the set of all closed subsets in $U \times X$ of the form $A = \cup A_i$, where each A_i is a closed irreducible subset in $U \times X$ which is finite and surjective over U. The empty subset in $U \times X$ is also regarded as an element of $\operatorname{Supp}(U \times X/X)$.

Given $U, X \in Sm/k$ and $A \in \operatorname{Supp}(U \times X/X)$, let $I_A \subset \mathscr{O}_{U\times X}$ be the ideal sheaf of the closed set $A \subset U \times X$. Denote by A_m the closed subscheme in $U \times X$ of the form $(A, \mathscr{O}_{U\times X}/I_A^m)$. If $m = 0$, then A_m is the empty subscheme. Define $\operatorname{SubSch}(U \times X/X)$ as the set of all closed subschemes in $U \times X$ of the form A_m.

For any $Z \in \operatorname{SubSch}(U \times X/X)$ we write $p_U^Z : Z \to U$ to denote $p \circ i$, where $i : Z \hookrightarrow U \times X$ is the closed embedding and $p : U \times X \to U$ is the projection. If there is no likelihood of confusion we shall write p_U instead of p_U^Z, dropping Z from the notation.

Clearly, for any $Z \in \operatorname{SubSch}(U \times X/X)$ the reduced scheme Z^{red}, regarded as a closed subset of $U \times X$, belongs to $\operatorname{Supp}(U \times X/X)$.

For any $U, X \in Sm/k$ we define objects of $\mathscr{A}(U, X)$ as equivalence classes for the triples

$$(n, Z, \varphi : p_{U,*}(\mathscr{O}_Z) \to M_n(\mathscr{O}_U)),$$

where n is a non-negative integer, $Z \in \operatorname{SubSch}(U \times X/X)$, and φ is a non-unital homomorphism of sheaves of $\mathscr{O}_U$-algebras. Let $p(\varphi)$ be the idempotent $\varphi(1) \in M_n(\Gamma(U, \mathscr{O}_U))$; then $P(\varphi) = \operatorname{Im}(p(\varphi))$ can be regarded as a $p_{U,*}(\mathscr{O}_Z)$-module by means of φ.

By definition, two triples (n, Z, φ), (n', Z', φ') are equivalent if $n = n'$ and there is a triple (n'', Z'', φ'') such that $n = n' = n''$, $Z, Z' \subset Z''$ are closed subschemes in Z'', and the diagrams

$$\begin{array}{ccc} p_{U,*}(\mathscr{O}_Z) & \xrightarrow{\varphi} & M_n(\mathscr{O}_U) \\ & {\scriptstyle can}\nwarrow \quad \nearrow{\scriptstyle \varphi''} & \\ & p_{U,*}(\mathscr{O}_{Z''}) & \end{array} \qquad \begin{array}{ccc} p'_{U,*}(\mathscr{O}_{Z'}) & \xrightarrow{\varphi'} & M_n(\mathscr{O}_U) \\ & {\scriptstyle can}\nwarrow \quad \nearrow{\scriptstyle \varphi''} & \\ & p_{U,*}(\mathscr{O}_{Z''}) & \end{array}$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

are commutative. We shall often denote an equivalence class for the triples by Φ. Though Z is not uniquely defined by Φ, nevertheless we shall also refer to $Z \subset U \times X$ as the *support* of Φ.

Given $\Phi, \Phi' \in \mathscr{A}(U, X)$, we first equalize supports Z, Z' of the objects Φ, Φ' and then set

$$\mathrm{Hom}_{\mathscr{A}(U,X)}(\Phi, \Phi') = \mathrm{Hom}_{p_{U,*}(\mathscr{O}_Z)}(P(\varphi), P(\varphi')).$$

Given any three objects $\Phi, \Phi', \Phi'' \in \mathscr{A}(U, X)$, a composition law

$$\mathrm{Hom}_{\mathscr{A}(U,X)}(\Phi, \Phi') \circ \mathrm{Hom}_{\mathscr{A}(U,X)}(\Phi', \Phi'') \to \mathrm{Hom}_{\mathscr{A}(U,X)}(\Phi, \Phi'')$$

is defined in the obvious way. This therefore makes $\mathscr{A}(U, X)$ an additive category. The zero object is the equivalence class of the triple $(0, \emptyset, 0)$. By definition,

$$\begin{aligned}\Phi_1 \oplus \Phi_2 \;=\; &(n_1 + n_2, Z_1 \cup Z_2, p_{U,*}(\mathscr{O}_{Z_1 \cup Z_2}) \to p_{U,*}(\mathscr{O}_{Z_1}) \times p_{U,*}(\mathscr{O}_{Z_2}) \\ &\to M_{n_1}(\mathscr{O}_U)) \times M_{n_2}(\mathscr{O}_U)) \hookrightarrow M_{n_1+n_2}(\mathscr{O}_U)).\end{aligned}$$

Clearly, $P(\varphi_1 \oplus \varphi_2) \cong P(\varphi_1) \oplus P(\varphi_2)$.

If $f : X' \to X$ and $g : Y \to Y'$ are in Sm/k, then, following [6, 4.13] and [6, 4.14], set

$$\mathscr{A}(f, g) = f^* \circ g_* = g_* \circ f^* : \mathscr{A}(X, Y) \to \mathscr{A}(X', Y').$$

By [6, 4.14; 4.12], the assignments $(X, Y) \mapsto \mathscr{A}(X, Y)$ and $(f, g) \mapsto \mathscr{A}(f, g)$ determine a functor

$$(U, X) \in (Sm/k)^{\mathrm{op}} \times Sm/k \mapsto \mathscr{A}(U, X) \in AddCats.$$

Throughout this section, by $\mathscr{A}$ we shall mean this bivariant additive category.

Next we shall introduce an action of *AffSm*$/k$ on $\mathscr{A}$ in the sense of § 2.1. Following the notation introduced just below Theorem 4.15 of [6], we set

$$\alpha_U := (1_U)^\star : \mathscr{A}(X, Y) \to \mathscr{A}(X \times U, Y \times U).$$

To check that the assignment $U \mapsto \alpha_U$ defines an action of *AffSm*$/k$ on $\mathscr{A}$, we need to check the commutativity of the three squares in § 2.1. Commutativity of the second and the third squares follows from [6, 4.17; 4.18]. Commutativity of the first square, that is the equality $\mathscr{A}(1_{X\times U}, 1_Y \times f) \circ \alpha_U = \mathscr{A}(1_{X\times U}, g \times 1_U) \circ \alpha_V$, is exactly commutativity of the diagram from [6, 4.24]. Thus the assignment $U \mapsto \alpha_U$ defines an action of *AffSm*$/k$ on $\mathscr{A}$. Below we shall consider the bivariant category $\mathscr{A}$ equipped with this specific action of *AffSm*$/k$.

For any $U, X \in Sm/k$ the bivariant category $\mathscr{A}$ produces a simplicial additive category

$$d \mapsto \mathscr{A}(U \times \Delta^d, X).$$

It is straightforward to check that this simplicial additive category is a $k[\Delta]$-linear additive category in the sense of [9, p. 158].

By [6, 4.27; 4.28], the bivariant category $\mathscr{A}$ also satisfies the property $(\mathfrak{Aut})$ from § 4.

Now we have a spectral category $\mathbb{K}$ whose objects are those of Sm/k, i.e., a category enriched over Sp^Σ. Its morphism symmetric spectra are of the form

$$\mathbb{K}(U, X) = K^{Gr}(\mathscr{A}(U, X)).$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

We refer the reader to [6] for details. One can associate a ringoid $\mathbb{K}_0$ to it. By definition, the objects of $\mathbb{K}_0$ are those of Sm/k, and

$$\mathbb{K}_0(U, X) = \pi_0(\mathbb{K}(U, X)), \quad U, X \in Sm/k.$$

In what follows we shall write $H^{p,q}_{\mathbb{K}}(U, \mathbb{Z})$ to denote $H^{p,q}_{\mathscr{A}}(U, \operatorname{Spec} k)$. In Remark 4.1 we defined complexes of presheaves $C^*(A_{0,Y}(q))$ (indexing is cohomological). We set

$$\mathbb{Z}^{\mathbb{K}}(q) := C^*(A_{0,\operatorname{Spec} k}(q)[-q])_{\mathrm{nis}}.$$

It follows from Remark 4.1 that

$$H^{p,q}_{\mathbb{K}}(U, \mathbb{Z}) = H^p_{\mathrm{nis}}(U, \mathbb{Z}^{\mathbb{K}}(q)).$$

We shall also denote by KGL the bispectrum $KGL_{\mathscr{A},\operatorname{Spec} k}$.

There is another ringoid which is important in our analysis. Let $\tilde{\mathscr{P}}(U, X)$, $U, X \in Sm/k$, be the additive category of big coherent $\mathscr{O}_{U\times X}$-modules P such that $\operatorname{Supp} P$ is finite over U and the coherent $\mathscr{O}_U$-module $(p_U)_*(P)$ is locally free (see [4, 6, 9]). We shall write $\tilde{\mathscr{P}}(U)$ to denote $\tilde{\mathscr{P}}(U, \operatorname{Spec} k)$. Define a ringoid $K_0^{\oplus}$ as

$$K_0^{\oplus}(U, X) = K_0(\tilde{\mathscr{P}}(U, X)), \quad U, X \in Sm/k.$$

Here the right-hand side stands for the Grothendieck group of the additive category $\tilde{\mathscr{P}}(U, X)$.

If X is affine, then there is a natural additive functor (see [6] for details)

$$F_{U,X} : \mathscr{A}(U, X) \to \tilde{\mathscr{P}}(U, X)$$

which is an equivalence of categories whenever U is affine. By [6], $F_{U,X}$ is functorial in U. These functors can naturally be extended to a map of ringoids

$$F : \mathbb{K}_0 \to K_0^{\oplus}.$$

Given $n \geqslant 0$, we denote by $\mathbb{Z}^{Gr}(n)$ (respectively, $\mathbb{Z}(n)$) the Grayson complex [26, 34] corresponding to the ringoid $K_0^{\oplus}$ (respectively, the Suslin–Voevodsky [27] complex corresponding to the ringoid Cor). The complexes are defined in the same fashion as $\mathbb{Z}^{\mathbb{K}}(n)$. Recall that motivic cohomology is defined as

$$H^{p,q}_{\mathscr{M}}(X, \mathbb{Z}) := H^p_{\mathrm{nis}}(X, \mathbb{Z}(q)).$$

Theorem 7.1 (Suslin [26]). *For any $n \geqslant 0$, the canonical homomorphism of complexes of Nisnevich sheaves $\mathbb{Z}^{Gr}(n) \to \mathbb{Z}(n)$ is a quasi-isomorphism.*

Corollary 7.2. *For any $n \geqslant 0$, the canonical homomorphism of complexes of Nisnevich sheaves $\mathbb{Z}^{\mathbb{K}}(n) \to \mathbb{Z}^{Gr}(n)$, induced by the map of ringoids $F : \mathbb{K}_0 \to K_0^{\oplus}$, is an isomorphism. Hence, for any smooth scheme $X \in Sm/k$, cohomology groups $H^{p,q}_{\mathbb{K}}(X, \mathbb{Z})$ coincide with motivic cohomology groups $H^{p,q}_{\mathscr{M}}(X, \mathbb{Z})$.*

Proof. As we have mentioned above, the additive functor $F_{U,X} : \mathscr{A}(U, X) \to \mathscr{P}(U, X)$ is an equivalence whenever U and X are affine. It follows that the map of simplicial abelian groups

$$(d \mapsto \mathbb{K}_0(U \times \Delta^d, \mathbb{G}_m^{\times n})) \to (d \mapsto K_0^{\oplus}(U \times \Delta^d, \mathbb{G}_m^{\times n})), \quad n \geqslant 0,$$

is an isomorphism for every smooth affine scheme U. Hence the map of simplicial abelian sheaves

$$(d \mapsto \mathbb{K}_0(- \times \Delta^d, \mathbb{G}_m^{\times n})_{\mathrm{nis}}) \to (d \mapsto K_0^{\oplus}(- \times \Delta^d, \mathbb{G}_m^{\times n})_{\mathrm{nis}})$$

is an isomorphism of motivic spaces. Our assertion now follows. □

Corollary 7.3. *The cohomology groups $H^{*,*}_{\mathbb{K}}(X, \mathbb{Z})$ are homotopy invariant and satisfy the cancelation property.*

Proof. The proof follows from [26, 3.1; 4.13] and Corollary 7.2. □

Corollary 7.4. *Let k be a perfect field. Then $A_{0,\operatorname{Spec} k}(n) = s_n(A_{0,\operatorname{Spec} k}(n))$ for each $n \geqslant 0$.*

Proof. The proof follows from Proposition 5.1, Theorem 7.1, and Corollary 7.2. □

By definition, by the K-theory of X we shall mean the Waldhausen algebraic K-theory symmetric spectrum of big vector bundles (regarded as an exact category)

$$K(X) = K(\tilde{\mathscr{P}}(X)).$$

We set $G_X := F_{X,\operatorname{Spec} k}$. Observe that G_X is functorial in X. So we get a map in $Pre^{\Sigma}(Sm/k)$,

$$G : K^{Gr}(\mathscr{A}(-, \operatorname{Spec} k)) \to K(-),$$

where the left-hand side spectrum is defined on p. 140.

Proposition 7.5. *G is a Nisnevich local weak equivalence and it induces canonical isomorphisms*

$$K_p^{\mathscr{A}}(X, \operatorname{Spec} k) \cong K_p(X),$$

for any smooth scheme X and any integer p, where the left-hand side group is defined on p. 148.

Proof. The fact that G is a Nisnevich local weak equivalence follows from the fact that G_X is an equivalence of categories whenever X is affine. So we also have that

$$K^{Gr}(\mathscr{A}(- \times \Delta^d, \operatorname{Spec} k)) \to K(- \times \Delta^d), \quad d \geqslant 0,$$

is a Nisnevich local weak equivalence in $Pre^{\Sigma}(Sm/k)$.

Consider a commutative diagram in $Pre^{\Sigma}(Sm/k)$,

$$\begin{array}{ccccc} K^{Gr}(\mathscr{A}(-, \operatorname{Spec} k)) & \longrightarrow & |K^{Gr}(\mathscr{A}(- \times \Delta^{\cdot}, \operatorname{Spec} k))| & \longrightarrow & |K^{Gr}(\mathscr{A}(- \times \Delta^{\cdot}, \operatorname{Spec} k))|_f \\ \downarrow{\scriptstyle G} & & \downarrow & & \downarrow{\scriptstyle \gamma} \\ K(-) & \xrightarrow{\alpha} & |K(- \times \Delta^{\cdot})| & \xrightarrow{\beta} & |K(- \times \Delta^{\cdot})|_f. \end{array} \tag{14}$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Here the lower f-symbol refers to a fibrant replacement functor in $Pre^{\Sigma}_{nis}(Sm/k)$. The vertical arrows are Nisnevich local weak equivalences. The left horizontal arrows are motivic weak equivalences, by [18, 3.8].

Since $K(-)$ is homotopy invariant, α is a stable weak equivalence. By [28], $K(-)$ is Nisnevich excisive, and hence β, γ are stable weak equivalences. It remains to observe that

$$K_p^{\mathscr{A}}(X, \operatorname{Spec} k) = \pi_p(|K^{Gr}(\mathscr{A}(- \times \Delta^{\cdot}, \operatorname{Spec} k))|_f(X))$$

for any $X \in Sm/k$. □

Corollary 7.6. *Let $K(-) \to \tilde{K}(-)$ be any fibrant replacement of $K(-)$ in the stable projective model structure of $Pre^{\Sigma}(Sm/k)$. Then the composite map*

$$K^{Gr}(\mathscr{A}(-, \operatorname{Spec} k)) \xrightarrow{G} K(-) \to \tilde{K}(-)$$

is a motivic fibrant replacement of $K^{Gr}(\mathscr{A}(-, \operatorname{Spec} k))$ in $Pre^{\Sigma}_{mot}(Sm/k)$.

Proof. All maps of diagram (14) are motivic weak equivalences. The proof of the preceding proposition shows that $\tilde{K}(-)$ is fibrant in $Pre^{\Sigma}_{mot}(Sm/k)$. □

We are now in a position to prove the following.

Theorem 7.7. *Let k be a perfect field. Then the Grayson tower (6) of S^1-spectra in $SH_{S^1}(k)$*

$$\cdots \to \Sigma_s^{q+1} A^{\Delta}_{\operatorname{Spec} k}(q+1) \to \Sigma_s^{q} A^{\Delta}_{\operatorname{Spec} k}(q) \to \cdots \to A^{\Delta}_{\operatorname{Spec} k}$$

is isomorphic to the tower

$$\cdots \to f_{q+1}(K(-)) \to f_q(K(-)) \to \cdots \to f_0(K(-)).$$

Moreover, $s_q(K(-)) = EM(\mathbb{Z}(q))$ for every $q \geqslant 0$.

Proof. This is a consequence of Theorems 5.3, 5.4, and 7.1, Corollaries 7.2–7.4, and Proposition 7.5. □

The next theorem says that the bispectrum KGL represents algebraic K-theory.

Theorem 7.8. *For any smooth scheme X, one has canonical isomorphisms*

$$KGL^{p,q}(X_+) = SH(k)(\Sigma^{\infty}_{\mathbb{G}} \Sigma^{\infty}_s X_+, \Sigma_s^{p-q} \Sigma^q_{\mathbb{G}} KGL) \cong K_{2q-p}(X),$$

where $K(X)$ is algebraic K-theory of X.

Proof. Given a bispectrum X, let X^{Δ} be the bispectrum $(|X_0(- \times \Delta^{\cdot})|, |X_1(- \times \Delta^{\cdot})|, \ldots)$. Taking a fibrant replacement of X^{Δ} in the level Nisnevich local model structure of $Pre^{\Sigma,\mathbb{G}}(Sm/k)$, we get a bispectrum X^{Δ}_f. So one has maps of bispectra

$$X \to X^{\Delta} \to X^{\Delta}_f,$$

where the left arrow is a level motivic weak equivalence by [18, 3.8] and the right arrow is a level Nisnevich local weak equivalence.

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

Consider the bispectra $(S^{-1}S\hat{A}^{\Delta}_{\mathrm{Spec}\,k})_f$ and KGL^{Δ}_f. Note that the first bispectrum is equivalent to $\widetilde{A}^{\Delta}_{\mathrm{Spec}\,k}$. We claim that each structure map

$$\rho_n : (KGL^{\Delta}_f)_n = (KGL^{\Delta}_n)_f \to \Omega_{\mathbb{G}}(KGL^{\Delta}_{n+1})_f$$

is a stable weak equivalence in $Pre^{\Sigma}(Sm/k)$. Corollary 7.6 and [9, 9.3] imply that each $(KGL^{\Delta}_n)_f$ has homotopy type of $\Omega^n\tilde{K}(-) \in Pre^{\Sigma}(Sm/k)$.

By construction, the map ρ_0 factors as

$$S^{-1}S\hat{A}_{\mathrm{Spec}\,k}(0)^{\Delta}_f \to \Omega_{\mathbb{G}}S^{-1}S\hat{A}_{\mathrm{Spec}\,k}(1)^{\Delta}_f \to \Omega_{\mathbb{G}}(KGL^{\Delta}_1)_f.$$

Corollary 7.3 and the cancelation theorem for K-theory 4.6 imply that the left arrow is a stable weak equivalence. It follows from [9, 9.6] that a homotopy cofiber of the right arrow is $\Omega_{\mathbb{G}}\Omega(\widetilde{A}^{\Delta}_{0,\mathrm{Spec}\,k})_0$. By Corollary 7.2, we have

$$\pi_{p-1}(\Omega_{\mathbb{G}}\Omega(\widetilde{A}^{\Delta}_{0,\mathrm{Spec}\,k})_0(X)) \cong H^{p,0}_{\mathbb{K}}(X \wedge \mathbb{G}_m, \mathbb{Z}) \cong H^{p,0}_{\mathcal{M}}(X \wedge \mathbb{G}_m, \mathbb{Z}), \quad X \in Sm/k, p \in \mathbb{Z}.$$

The proof of [31, 4.2] implies that $H^{p,0}_{\mathcal{M}}(X \wedge \mathbb{G}_m, \mathbb{Z}) = 0$. So $\Omega_{\mathbb{G}}(\widetilde{A}^{\Delta}_{0,\mathrm{Spec}\,k})_0$ is zero in $\mathrm{Ho}(Pre^{\Sigma}(Sm/k))$, and hence ρ_0 is a stable weak equivalence. The fact that each ρ_n is a stable weak equivalence is proved in a similar fashion. The only difference with ρ_0 is that one iterates the $S^{-1}S$-construction at each step.

We conclude that KGL^{Δ}_f is a motivically fibrant bispectrum. Therefore,

$$\begin{aligned} KGL^{p,q}(X_+) &= SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^{p-q}_s \Sigma^{q}_{\mathbb{G}} KGL^{\Delta}_f) \\ &\cong SH_{S^1}(k)(\Sigma^{\infty}_s X_+, \Sigma^{p-q}_s (KGL^{\Delta}_f)_q) \\ &\cong SH_{S^1}(k)(\Sigma^{\infty}_s X_+, \Sigma^{p-q}_s \Omega^q \tilde{K}(-)) \cong K_{2q-p}(X), \end{aligned}$$

as was to be shown. □

Lemma 7.9. *Let k be a perfect field. Then the bispectrum $A_{\mathrm{Spec}\,k}$ is isomorphic in $SH(k)$ to $f_0(KGL)$.*

Proof. It follows from Corollaries 7.3 and 7.4 and Lemma 5.5 that $A_{\mathrm{Spec}\,k}$ is in $SH^{eff}(k)$. Then map (13) of bispectra $\chi : A_{\mathrm{Spec}\,k} \to KGL$. factors as

$$A_{\mathrm{Spec}\,k} \xrightarrow{\theta} f_0(KGL) \xrightarrow{\zeta} KGL.$$

For any $X \in Sm/k$ and any $p \in \mathbb{Z}$, the induced map

$$\zeta_* : SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s f_0(KGL)) \to SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s KGL)$$

is an isomorphism by construction of $f_0(KGL)$. On the other hand, Theorem 7.8 implies that the induced map

$$\chi_* : SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s A_{\mathrm{Spec}\,k}) \to SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s KGL)$$

is an isomorphism, and hence so is

$$\theta_* : SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s A_{\mathrm{Spec}\,k}) \to SH(k)(\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+, \Sigma^p_s f_0(KGL)).$$

Since $\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty}_s X_+$ generate the compactly generated triangulated category $SH^{eff}(k)$, we conclude that θ is an isomorphism in $SH(k)$. □

The following result gives an explicit model for the non-negative part of the slice tower of the bispectrum KGL.

Theorem 7.10. *Let k be a perfect field. Then the tower (11) of bispectra in $SH(k)$*

$$\cdots \to \Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k} \to \Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{\operatorname{Spec} k} \to \cdots \to A_{\operatorname{Spec} k}$$

is isomorphic to the tower

$$\cdots \to f_{q+1}(KGL) \to f_q(KGL) \to \cdots \to f_0(KGL).$$

Proof. By Lemma 7.9, there is an isomorphism $\theta : A_{\operatorname{Spec} k} \to f_0(KGL)$ in $SH(k)$. Suppose that an isomorphism $\theta_q : \Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{\operatorname{Spec} k} \cong f_q(KGL)$, $q \geqslant 0$, is constructed. Since $\Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k} \in \Sigma_{\mathbb{G}}^{q+1}SH(k)$ and $s_q(KGL)$ is orthogonal to $\Sigma_{\mathbb{G}}^{q+1}SH(k)$, it follows that there is a unique morphism

$$\theta_{q+1} : \Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k} \to f_{q+1}(KGL)$$

making the diagram

$$\begin{array}{ccc} \Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k} & \longrightarrow & \Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{\operatorname{Spec} k} \\ {\scriptstyle\theta_{q+1}}\downarrow & & \downarrow{\scriptstyle\theta_q} \\ f_{q+1}(KGL) & \longrightarrow & f_q(KGL) \end{array}$$

commutative. We claim that θ_{q+1} is an isomorphism in $SH(k)$.

By Theorem 4.7 and Corollary 7.3, a homotopy cofiber of the upper horizontal arrow is $\Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{0,\operatorname{Spec} k}$. Therefore,

$$\begin{aligned} SH(k)(\Sigma_{\mathbb{G}}^{q+1}\Sigma_{\mathbb{G}}^{\infty}\Sigma_s^{\infty}X_+, \Sigma_s^{p}\Sigma_{\mathbb{G}}^{q}A_{0,\operatorname{Spec} k}) &= SH(k)(\Sigma_{\mathbb{G}}\Sigma_{\mathbb{G}}^{\infty}\Sigma_s^{\infty}X_+, \Sigma_s^{p}A_{0,\operatorname{Spec} k}) \\ &\cong H_{\mathbb{K}}^{p,0}(X\wedge\mathbb{G}_m, \mathbb{Z}) \end{aligned}$$

for any $X \in Sm/k$ and integer p. The proof of Theorem 7.8 shows that $H_{\mathbb{K}}^{p,0}(X\wedge\mathbb{G}_m, \mathbb{Z}) = 0$, and hence $f_{q+1}(\Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{0,\operatorname{Spec} k}) = 0$.

Since $f_{q+1}(s_q(KGL)) = 0$, we see that the horizontal arrows of the commutative diagram

$$\begin{array}{ccc} f_{q+1}(\Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k}) & \longrightarrow & f_{q+1}(\Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{\operatorname{Spec} k}) \\ {\scriptstyle f_{q+1}(\theta_{q+1})}\downarrow & & \downarrow{\scriptstyle f_{q+1}(\theta_q)} \\ f_{q+1}(f_{q+1}(KGL)) & \longrightarrow & f_{q+1}(f_q(KGL)) \end{array}$$

are isomorphisms. But $f_{q+1}(\theta_q)$ is an isomorphism, and hence so is $f_{q+1}(\theta_{q+1})$. Lemma 7.9 implies that $\Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec} k}$ is in $\Sigma_{\mathbb{G}}^{q+1}SH(k)$. Since $f_{q+1}(KGL)$ belongs to $\Sigma_{\mathbb{G}}^{q+1}SH(k)$ as well and $f_{q+1}(\theta_{q+1})$ is an isomorphism, we conclude that θ_{q+1} is an isomorphism. □

One of the equivalent models for the motivic Eilenberg–Mac Lane bispectrum $H_{\mathbb{Z}}$ is as follows. Let Cor be the ringoid of finite correspondences over Sm/k (see, e.g., [27]). The

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

cube of sheaves $Cor(-,\mathbb{G}_m^{\wedge n})$ is defined similarly to the cube $K_0^{Gr}(\mathscr{A}(-,Y\times\mathbb{G}_m^{\wedge n}))$. Its vertexes are sheaves $Cor(-,\mathbb{G}_m^{\times k})$, $k\leqslant n$. By definition,

$$H_{\mathbb{Z}}=(EM(Cor(-,\operatorname{Spec}k)),EM(C^{\oplus}Cor(-,\mathbb{G}_m^{\wedge 1})),\ldots),$$

where EM stands for the Eilenberg–Mac Lane functor in the sense of [5, Appendix A] from abelian groups to Sp^{Σ}.

The composite map of ringoids

$$\mathbb{K}_0\xrightarrow{F}K_0^{\oplus}\to Cor$$

(see [26, 34] for the definition of the second arrow) yields a map of bispectra

$$\lambda: A_{0,\operatorname{Spec}k}\to H_{\mathbb{Z}}.$$

The proof of Theorem 4.7 and Corollaries 7.2–7.3 shows that λ is an isomorphism in $SH(k)$.

The next result was first conjectured by Voevodsky [30, 31] and solved by Levine [15] by using the coniveau tower (over perfect fields).

Theorem 7.11. *Let k be a perfect field. Then for every $q\geqslant 0$ we have isomorphisms in $SH(k)$,*

$$s_q(KGL)\cong\Sigma_s^q\Sigma_{\mathbb{G}}^q H_{\mathbb{Z}}.$$

Proof. The proof of Theorem 7.10 shows that there is a commutative diagram in $SH(k)$,

$$\begin{array}{ccccccc}
\Sigma_s^{q+1}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec}k} & \longrightarrow & \Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{\operatorname{Spec}k} & \longrightarrow & \Sigma_s^{q}\Sigma_{\mathbb{G}}^{q}A_{0,\operatorname{Spec}k} & \longrightarrow & \Sigma_s^{q+2}\Sigma_{\mathbb{G}}^{q+1}A_{\operatorname{Spec}k} \\
\downarrow{\scriptstyle\theta_{q+1}} & & \downarrow{\scriptstyle\theta_q} & & & & \downarrow{\scriptstyle\Sigma_s\theta_{q+1}} \\
f_{q+1}(KGL) & \longrightarrow & f_q(KGL) & \longrightarrow & s_q(KGL) & \longrightarrow & \Sigma_s f_{q+1}(KGL),
\end{array}$$

where the vertical arrows are isomorphisms. Since $SH(k)$ is triangulated, there exists an isomorphism

$$\Sigma_s^q\Sigma_{\mathbb{G}}^q A_{0,\operatorname{Spec}k}\cong s_q(KGL).$$

It remains to observe that $\lambda: A_{0,\operatorname{Spec}k}\to H_{\mathbb{Z}}$ induces an isomorphism $\Sigma_s^q\Sigma_{\mathbb{G}}^q A_{0,\operatorname{Spec}k}\cong\Sigma_s^q\Sigma_{\mathbb{G}}^q H_{\mathbb{Z}}$ in $SH(k)$. □

Let $\tilde{\mathscr{P}}(\mathbb{G}_m^{\times q})(X)$ be the additive category whose objects are the tuples $(P,\theta_1,\ldots,\theta_q)$ with $P\in\tilde{\mathscr{P}}(X)$ and $(\theta_1,\ldots,\theta_q)$ commuting automorphisms. The cube of affine schemes $\mathbb{G}_m^{\wedge q}$ gives rise to a cube of additive categories $\tilde{\mathscr{P}}(\mathbb{G}_m^{\wedge q})(X)$ with vertexes being $\tilde{\mathscr{P}}(\mathbb{G}_m^{\times q})(X)$, $0\leqslant k\leqslant q$. The edges of the cube are given by the additive functors $i_s:\tilde{\mathscr{P}}(\mathbb{G}_m^{\times k-1})(X)\to\tilde{\mathscr{P}}(\mathbb{G}_m^{\times k})(X)$,

$$(P,(\theta_1,\ldots,\theta_{k-1}))\longmapsto(P,(\theta_1,\ldots,1,\ldots,\theta_{k-1})),$$

where 1 is the sth coordinate.

Grayson's machinery [9] (see [5, 34] as well) produces a tower in $\operatorname{Ho}(Pre_{nis}^{\Sigma}(Sm/k))$,

$$\cdots\to\Sigma_s^q|K^{Gr}(C^{\oplus}\tilde{\mathscr{P}}(\mathbb{G}_m^{\wedge q})(-\times\Delta^{\cdot}))|\to\cdots\to|K^{Gr}(\tilde{\mathscr{P}}(-\times\Delta^{\cdot}))|. \tag{15}$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

By [9, 10.5], $|K^{Gr}(\tilde{\mathscr{P}}(-\times\Delta^{\cdot}))| = |K(\tilde{\mathscr{P}}(-\times\Delta^{\cdot}))|$. This tower produces the *Grayson motivic spectral sequence* for $\tilde{\mathscr{P}}(X)$ (see [5, 9, 26, 34])

$$E_2^{pq} = H_{\text{nis}}^{p-q}(X, \mathbb{Z}^{Gr}(-q)) \Longrightarrow K_{-p-q}(X), \quad X \in Sm/k. \tag{16}$$

In view of Theorem 7.1, it takes the form

$$E_2^{pq} = H_{\mathscr{M}}^{p-q,-q}(X, \mathbb{Z}) \Longrightarrow K_{-p-q}(X), \quad X \in Sm/k.$$

We are now in a position to prove the following.

Theorem 7.12. *Let k be a perfect field. Then the Grayson motivic spectral sequence (16) is isomorphic to the Voevodsky motivic spectral sequence [3, p. 171]*

$$E_2^{pq} = SH(k)(\Sigma_{\mathbb{G}}^{\infty}\Sigma_s^{\infty}X_+, \Sigma_s^{p-q}\Sigma_{\mathbb{G}}^{q}s_0(KGL)) \Longrightarrow K_{-p-q}(X),$$

produced by the slice tower for the bispectrum KGL.

Proof. Recall that there is an additive functor $G_X : \mathscr{A}(X, \operatorname{Spec} k) \to \tilde{\mathscr{P}}(X)$, functorial in X, which is an equivalence whenever X is affine. It induces a map of multisimplicial additive categories,

$$G_{q,X} : C^{\oplus}\mathscr{A}(X, \operatorname{Spec} k)(\mathbb{G}_m^{\wedge q}) \to C^{\oplus}\tilde{\mathscr{P}}(\mathbb{G}_m^{\wedge q})(X),$$

which is an equivalence whenever X is affine. In view of the $(\mathfrak{Aut})$ property for $\mathscr{A}$, we can identify $\mathscr{A}(X, \operatorname{Spec} k)(\mathbb{G}_m^{\wedge q})$ with $\mathscr{A}(X, \operatorname{Spec} k \times \mathbb{G}_m^{\wedge q})$.

It follows that Grayson's tower (15) for $\tilde{\mathscr{P}}(X)$ is isomorphic in $\operatorname{Ho}(Pre_{nis}^{\Sigma}(Sm/k))$ to Grayson's tower (6) for $\mathscr{A}(X, \operatorname{Spec} k)$. Corollary 7.2 and Proposition 7.5 imply that Grayson's motivic spectral sequence (16) is isomorphic to Grayson's motivic spectral sequence (7) for $\mathscr{A}$,

$$E_2^{pq} = H_{\mathbb{K}}^{p-q,-q}(X, \mathbb{Z}) \Longrightarrow K_{-p-q}^{\mathscr{A}}(X, \operatorname{Spec} k), \quad X \in Sm/k.$$

Theorems 4.7, 7.10, and 7.11 now finish the proof. □

Appendix A. Some facts on spectra

We prove here a couple of useful facts. First we wish to compare the agreement of the bispectrum KGL with the classical K-theory $\mathbb{P}^1$-spectrum BGL (see, e.g., [18, 20, 29]).

The functor $diag : SH_{S^1,\mathbb{G}}(k) \to SH_{S^1\wedge\mathbb{G}}(k)$ sending a bispectrum to its diagonal $S^1 \wedge \mathbb{G}$-spectrum is an equivalence of categories. In particular, $diag(KGL)$ is isomorphic to the following $S^1 \wedge \mathbb{G}$-spectrum:

$$\begin{aligned} KGL_1 = (&\widehat{K}^{Gr}(S^{-1}S\mathscr{A}(-, \operatorname{Spec} k))_f^{(0)}, \Omega\widehat{K}^{Gr}((S^{-1}S)^2\mathscr{A}(-, \operatorname{Spec} k))_f^{(1)}, \\ &\Omega^2\widehat{K}^{Gr}((S^{-1}S)^3\mathscr{A}(-, \operatorname{Spec} k))_f^{(2)}, \ldots), \end{aligned}$$

where f refers to motivic fibrant replacement with respect to the injective model structure of motivic spaces (see [12]) and the superscript $^{(n)}$ refers to the nth space of the S^1-spectrum $\Omega^n\widehat{K}^{Gr}((S^{-1}S)^{n+1}\mathscr{A}(-, \operatorname{Spec} k))$. Let $\mathscr{K}$ be a motivic fibrant replacement of the K-theory presheaf $U \mapsto K(\tilde{\mathscr{P}}(U))$. Then we can choose homotopy equivalences

$$\alpha_n : \Omega^n\widehat{K}^{Gr}((S^{-1}S)^{n+1}\mathscr{A}(-, \operatorname{Spec} k))_f^{(n)} \to \mathscr{K}, \quad n \geqslant 0,$$

https://doi.org/10.1017/S1474748015000419 Published online by Cambridge University Press

and maps $\beta_n : \mathscr{K} \wedge (S^1 \wedge \mathbb{G}) \to \mathscr{K}$ which are defined as

$$\mathscr{K} \wedge (S^1 \wedge \mathbb{G}) \xrightarrow{\gamma_n \wedge 1} \Omega^n \widehat{K}^{Gr}((S^{-1}S)^{n+1}\mathscr{A}(-, \operatorname{Spec} k))_f^{(n)}$$
$$\wedge (S^1 \wedge \mathbb{G}) \to \Omega^{n+1} \widehat{K}^{Gr}((S^{-1}S)^{n+2}\mathscr{A}(-, \operatorname{Spec} k))_f^{(n+1)} \xrightarrow{\alpha_{n+1}} \mathscr{K}$$

with γ_n being some homotopy inverse of α_n. We then get a $S^1 \wedge \mathbb{G}$-spectrum

$$KGL_2 = (\mathscr{K}, \mathscr{K}, \mathscr{K}, \ldots)$$

with structure maps given by β_n.

It follows from [22, 6.3] that KGL_1 and KGL_2 are isomorphic in $SH_{S^1 \wedge \mathbb{G}}(k)$. By the same result and [23, 1.1.2], KGL_2 is isomorphic in $SH_{S^1 \wedge \mathbb{G}}(k)$ to the spectrum

$$KGL_3 = (\mathscr{K}, \mathscr{K}, \mathscr{K}, \ldots)$$

with each structure map given by β_0.

There is a zigzag of motivic equivalences,

$$S^1 \wedge \mathbb{G} \xleftarrow{\sim}_{\mathbb{A}^1} \widetilde{T} \xrightarrow{\sim}_{\mathbb{A}^1} T \xleftarrow{\sim}_{\mathbb{A}^1} \mathbb{P}^1,$$

where $\widetilde{T}$ is the mapping cylinder for the inclusion $(\mathbb{G}_m)_+ \to \mathbb{A}^1_+$. By [12, 2.13], the zigzag induces an equivalence of categories,

$$\theta : SH_{S^1 \wedge \mathbb{G}}(k) \to SH_{\mathbb{P}^1}(k).$$

Consider a $\mathbb{P}^1$-spectrum

$$KGL_4 = (\mathscr{K}, \mathscr{K}, \mathscr{K}, \ldots),$$

where each structure map $\mathscr{K} \to \Omega_{\mathbb{P}^1}\mathscr{K}$ is given by

$$\mathscr{K} \to \Omega_{S^1 \wedge \mathbb{G}}\mathscr{K} \xrightarrow{\sim} \Omega_{\mathbb{P}^1}\mathscr{K}.$$

Here the left arrow is adjoint to β_0 and the right arrow is a chosen homotopy equivalence induced by the zigzag above (recall that $\mathscr{K}$ is a motivically fibrant space).

It follows from [22, 6.3] that $\theta(KGL_3)$ is isomorphic to KGL_4 in $SH_{\mathbb{P}^1}(k)$. It remains to apply [23, 1.1.2] to show that KGL_4 is isomorphic in $SH_{\mathbb{P}^1}(k)$ to the $\mathbb{P}^1$-spectrum BGL defined in [20, 1.2.1].

We document the above arguments as follows.

Theorem A.1. *The image of the bispectrum KGL under the equivalence of triangulated categories $\theta \circ diag : SH_{S^1,\mathbb{G}}(k) \to SH_{\mathbb{P}^1}(k)$ is isomorphic to the K-theory $\mathbb{P}^1$-spectrum BGL in the sense of [20, 1.2.1].*

Although the authors have not found the following result in the literature, they do not have pretensions to originality. It is used in the proof of Lemma 5.2.

Proposition A.2. *If E is a connected motivically fibrant S^1-spectrum, then so is $\Omega_{\mathbb{G}}E$.*

Proof. Clearly, $\Omega_{\mathbb{G}}E$ is motivically fibrant. To prove that it is connected, it suffices to check that for any smooth local Henselian scheme U one has $\pi_{n<0}(E(\mathbb{G}_{m,U}))=0$ (recall that $\mathbb{G}$ is sectionwise equivalent to the pointed motivic space $(\mathbb{G}_m,1)$). Since $\Omega_{\mathbb{G}_m}E$ is motivically fibrant, by [17, 6.1.6], it is enough to verify that for any k-smooth variety X one has $\pi_{n<0}(E(\mathbb{G}_{m,K}))=0$ with $K=k(X)$ its function field.

Sublemma. *If $\mathscr{F}$ is a strictly homotopy invariant Nisnevich sheaf of Abelian groups on Sm/k, then $H^n_{\mathrm{nis}}(\mathbb{G}_{m,k(X)},\mathscr{F})=0$ for all $n>0$ and $X\in Sm/k$.*

Proof. The result is well known for $n>1$. One has

$$H^1_{\mathrm{nis}}(\mathbb{G}_{m,k(X)},\mathscr{F}) \overset{(1)}{=} H^2_{\mathrm{nis}}(S^1\wedge\mathbb{G}_{m,k(X)},\mathscr{F}) \overset{(2)}{=} [S^1\wedge\mathbb{G}_{m,k(X)},K(\mathscr{F},2)]_{H_{\mathbb{A}^1}(k)}$$
$$\overset{(3)}{=} [\mathbb{P}^1_{k(X)},K(\mathscr{F},2)]_{H_{\mathbb{A}^1}(k)} \overset{(4)}{=} H^2_{\mathrm{nis}}(\mathbb{P}^1,\mathscr{F}) \overset{(5)}{=} 0.$$

Here (1) is given by the suspension isomorphism, (2) holds because $K(\mathscr{F},2)$ is an $\mathbb{A}^1$-local motivic space, (3) holds because $S^1\wedge\mathbb{G}_{m,k(X)}\cong\mathbb{P}^1_{k(X)}$ in $H_{\mathbb{A}^1}(k)$, (4) holds because $K(\mathscr{F},2)$ is an $\mathbb{A}^1$-local motivic space, and finally (5) follows from the fact that $\dim\mathbb{P}^1=1<2$. □

Now the spectral sequence

$$H^p(\mathbb{G}_{m,k(X)},\underline{\pi}_q(E)) \Longrightarrow \pi_{q-p}(E(\mathbb{G}_{m,k(X)}))$$

together with the sublemma above shows that $H^0(\mathbb{G}_{m,k(X)},\underline{\pi}_q(E))=\pi_q(E(\mathbb{G}_{m,k(X)}))=0$ for $q<0$, because $\underline{\pi}_{q<0}(E)=0$. □

To conclude the paper, we remark that all presheaves of symmetric spectra forming the main bispectra $A_{\mathrm{Spec}\,k}$, $A_{0,\mathrm{Spec}\,k}$ we work with are $\mathbb{K}$-modules in the sense of [6]. Moreover, their structure maps are $\mathbb{K}$-module morphisms. Also, Grayson's tower (6) for $\mathscr{A}$ is in fact a tower in the homotopy category $\mathrm{Ho}(\mathrm{Mod}\,\mathbb{K})$ of $\mathbb{K}$-modules. It produces a tower of compact objects in the motivic homotopy category of $\mathbb{K}$-modules in the sense of [5]. This point of view of the motivic spectral sequence motivated the authors to develop the 'enriched motivic homotopy theory' of spectral categories and modules over them [5, 6]. As an application, the motivic spectral sequence is realized in associated triangulated categories. Though we tried to avoid the use of this language here, it is this theory that led the authors to the main results of this paper.

References

1. A. Beilinson, Letter to Christophe Soulé, 1982, K-theory Preprint archives 694.
2. S. Bloch and S. Lichtenbaum, A spectral sequence for motivic cohomology, 1995, K-theory Preprint archives 62.

3. B. I. Dundas, M. Levine, P. A. Østvær, O. Röndigs and V. Voevodsky, *Motivic Homotopy Theory*, Universitext (Springer-Verlag, Berlin, 2007).
4. E. M. Friedlander and A. Suslin, The spectral sequence relating algebraic K-theory to motivic cohomology, *Ann. Sci. Éc. Norm. Supér.* (4) **35** (2002), 773–875.
5. G. Garkusha and I. Panin, K-motives of algebraic varieties, *Homology, Homotopy Appl.* (2) **14** (2012), 211–264.
6. G. Garkusha and I. Panin, The triangulated category of K-motives $DK_{-}^{eff}(k)$, *J. K-Theory* (1) **14** (2014), 103–137.
7. D. Grayson, Higher algebraic K-theory II [after Daniel Quillen], in *Algebraic K-theory, Evanston 1976*, Lecture Notes in Mathematics, Volume 551, pp. 217–240 (Springer-Verlag, Berlin, Heidelberg, New York, 1976).
8. D. Grayson, Adams operations on higher K-theory, *K-Theory* **6** (1992), 97–111.
9. D. Grayson, Weight filtrations via commuting automorphisms, *K-Theory* **9** (1995), 139–172.
10. M. Hovey, Spectra and symmetric spectra in general model categories, *J. Pure Appl. Algebra* (1) **165** (2001), 63–127.
11. M. Hovey, B. Shipley and J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* **13** (2000), 149–208.
12. J. F. Jardine, Motivic symmetric spectra, *Doc. Math.* **5** (2000), 445–552.
13. B. Kahn and M. Levine, Motives of Azumaya algebras, *J. Inst. Math. Jussieu* (3) **9** (2010), 481–599.
14. M. Levine, Techniques of localization in the theory of algebraic cycles, *J. Algebraic Geom.* (2) **10** (2001), 299–363.
15. M. Levine, The homotopy coniveau filtration, *J. Topol.* **1** (2008), 217–267.
16. M. Levine, The slice filtration and Grothendieck–Witt groups, *Pure Appl. Math. Q.* (4) **7** (2011), 1543–1584.
17. F. Morel, The stable $\mathbb{A}^1$-connectivity theorems, *K-theory* **35** (2006), 1–68.
18. F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, *Publ. Math. Inst. Hautes Études Sci.* **90** (1999), 45–143.
19. A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, *J. Amer. Math. Soc.* (1) **9** (1996), 205–236.
20. I. Panin, K. Pimenov and O. Röndigs, On Voevodsky's algebraic K-theory spectrum *BGL*, *Abel. Symp. Proc.* **4** (2009), 279–330.
21. O. Podkopaev, The equivalence of Grayson and Friedlander–Suslin spectral sequences, Preprint, 2013, arXiv:1309.7597.
22. J. Riou, Catégorie homotopique stable d'un site suspendu avec intervalle, *Bull. Soc. Math. France* **135** (2007), 495–547.
23. J. Riou, Algebraic K-theory, $\mathbb{A}^1$-homotopy and Riemann–Roch theorems, *J. Topol.* **3** (2010), 229–264.
24. J. Rognes, *Introduction to Higher Algebraic K-theory*, Lecture Notes 2010, available at folk.uio.no/rognes/kurs/mat9570v10/akt.pdf.
25. S. Schwede, An untitled book project about symmetric spectra, available at www.math.uni-bonn.de/~schwede (version v3.0/April 2012).
26. A. Suslin, On the Grayson spectral sequence, *Tr. Mat. Inst. Steklova* **241** (2003), 218–253. (Russian). English transl. in Proc. Steklov Inst. Math. (2) 241 (2003), 202–237.
27. A. Suslin and V. Voevodsky, Bloch–Kato conjecture and motivic cohomology with finite coefficients, in *The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998)*, NATO Science Series C: Mathematics, Physics and Science, Volume 548, pp. 117–189 (Kluwer Academic Publishers, Dordrecht, 2000).

28. R. W. Thomason and T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in *The Grothendieck Festschrift III*, Progress in Mathematics, Volume 88, pp. 247–435 (Birkhäuser, Boston, 1990).
29. V. Voevodsky, $\mathbb{A}^1$-Homotopy theory, *Doc. Math.* **ICM**(1) (1998), 417–442.
30. V. Voevodsky, Open problems in the motivic stable homotopy theory. I, in *Motives, Polylogarithms and Hodge Theory, Part I (Irvine, CA, 1998)*, International Press Lecture Series, Volume 3, pp. 3–34 (International Press, Somerville, MA, 2002).
31. V. Voevodsky, A possible new approach to the motivic spectral sequence for algebraic K-theory, in *Recent Progress in Homotopy Theory (Baltimore, MD, 2000)*, Contemporary Mathematics, Volume 293, pp. 371–379 (American Mathematical Society, Providence, RI, 2002).
32. V. Voevodsky, Cancellation theorem, Doc. Math. Extra Volume in honor of A. Suslin (2010), 671–685.
33. F. Waldhausen, Algebraic K-theory of spaces, in *Algebraic and Geometric Topology, Proceedings Conference, New Brunswick/USA 1983*, Lecture Notes in Mathematics, Volume 1126, pp. 318–419 (Springer-Verlag, Berlin, Heidelberg, New York, 1985).
34. M. E. Walker, Motivic cohomology and the K-theory of automorphisms, PhD thesis, University of Illinois at Urbana-Champaign, 1996.