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A variational inequality arising from optimal exercise perpetual executive stock options†

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We investigate a degenerate parabolic variational inequality arising from optimal continuous exercise perpetual executive stock options. It is also shown in Qin *et al.* (Continuous-Exercise Model for American Call Options with Hedging Constraints, working paper, available at SSRN: http://dx.doi.org/10.2139/ssrn.2757541) that to make this problem non-trivial the stock's growth rate must be no smaller than the discount rate. Well-posedness of the problem is established in Lai *et al.* (2015, Mathematical analysis of a variational inequality modeling perpetual executive stock options, Euro. J. Appl. Math., 26 (2015), 193–213), Qin *et al.* (2015, Regularity free boundary arising from optimal continuous exercise perpetual executive stock options, Interfaces and Free Boundaries, 17 (2015), 69–92), Song & Yu (2011, A parabolic variational inequality related to the perpetual American executive stock options, Nonlinear Analysis, 74 (2011), 6583-6600) for the case when the underlying stock's expected return rate is smaller than the discount rate. In this paper, we consider the remaining case: the discount rate is bigger than the growth rate but no bigger than the return rate. The existence of a unique classical solution as well as a continuous and strictly decreasing free boundary is proved.

Key words: Variational inequality, free boundary, optimal exercise, perpetual executive stock option

1 Introduction

In the absence of frictions, if assets are priced by arbitrage, the value per-unit is invariant to the amount of the asset. In particular, the optimal exercise time is independent of the amount of American style claims. However, it is not necessarily true in the presence of portfolio constraints or other frictions. This fact motivates Rogers and Scheinkman [15] to propose a *continuous exercise* model (claims can be exercised bit by bit) for American

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style claims with a direct application to executive stock options, which are American call options granted by a firm to an executive as a form of benefit to his or her salary usually with long maturity, ranging from 5 to 15 years, and hedge constraint (e.g., no short sell of the underlying stocks). There is an extensive literature on executive stock options, see, for example, Lambert, Larcker and Verechia [11], Carpenter [1], Hall and Murphy [7], Jain and Subramanian [9], Grasselli and Henderson [6], Leung and Sircar [12], and references therein. More precisely, suppose an executive owns A amount of executive stock options initially, he or she can exercise any fractions of options with strike price K at any time before or on the maturity T. We denote an admissible exercise strategy, which describes the amount of remaining options, by an adapted, non-negative, non-increasing, left-limit, and right-continuous process $\{m_t\}_{0\leqslant t\leqslant T}$ with conditions that $m_{0^-} = A$ and $m_T = 0$. And the set of all these admissible exercise strategies is denoted by $\mathcal{M}_{0,T}(A)$. Then the executive's problem is to choose an optimal exercise strategy to maximize his or her utility of the present wealth:

$$V(x,s,A,T) = \sup_{m_t\in\mathcal{M}_{0,T}(A)} \mathbb{E}\Big[U\Big(x - \int_0^T e^{-rt}[S_t - K]^+ dm_t\Big)\,\Big|\, S_0 = s\Big], \tag{1.1}$$

where $r > 0$ is a constant discount rate, U is the executive's utility function which is increasing and concave. $x \in \mathbb{R}$ and $s \geqslant 0$ are the initial wealth of the executive $\{X_t\}_{t\geqslant 0}$ and initial price of the underlying stock price $\{S_t\}_{t\geqslant 0}$, whose dynamics are given as follows respectively:

$$dX_t = -e^{-rt}(S_t - K)^+ dm_t,\ X_0 = x,$$
$$dS_t = \alpha S_t dt + \sigma S_t dW_t, \quad S_0 = s,$$

with constant expected return rate $\alpha \in \mathbb{R}$ and constant volatility $\sigma > 0$. $\{W_t\}_{t\geqslant 0}$ is a standard Brownian motion. As is shown in [14], it is convenient to consider the discounted number of the remaining options $\{A_t\}_{t\geqslant 0}$ (i.e., $A_t = e^{-rt}m_t$), which satisfies

$$dA_t = -rA_t dt + e^{-rt} dm_t,\ \ A_{0^-} = A.$$

In [14], we prove that the value function V defined by (1.1) is a unique viscosity solution to the following variational inequality:

$$\min\left\{\frac{\partial V}{\partial T} + rA\frac{\partial V}{\partial A} - \frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2} - \alpha s\frac{\partial V}{\partial s},\ \frac{\partial V}{\partial A} - [s-K]^+\frac{\partial V}{\partial x}\right\}$$
$$= 0 \quad \text{for } (x,s,A,T) \in \mathbb{R}\times(0,\infty)^3,$$

with appropriate growth conditions, and the following initial and boundary conditions:

$$V|_{sA=0} = U(x), \quad V|_{T=0} = U(x + [s-K]^+A).$$

Note that besides T, the variable A also plays a role of time. Hence, this variational inequality is a parabolic type with double time-like variables, which is seldom seen in the literature.

On the other hand, the perpetual problem (i.e., the maturity is infinite) is not only easy to study but also admits some practical meanings. For example, as mentioned before, the maturity of executive stock options is very long, usually over 10 years, so perpetual one is not a bad approximation. Besides, this continuous exercise model can also be applied to real investment decisions where problems are solved usually with an infinite horizon (see Dixit and Pindyck [4]). In this context, the perpetual problem is naturally defined as the limit of the finite horizon problem, i.e.,

$$v(x,s,A) := \lim_{T\to\infty} V(x,s,A,T).$$

However, when $\alpha - \sigma^2/2 \geqslant r$, the probability that the discounted underlying stock price goes to infinite is one. Thus, the executive will never exercise his or her options until the stock price goes to infinity. So, for the above definition being well-posed, in [14], we establish some limit properties of the value function V as maturity goes to infinite. In particular, under exponential utility, i.e., $U(z) = -e^{-\gamma z}$ with $\gamma > 0$, the condition

$$\alpha - \frac{\sigma^2}{2} < r \tag{1.2}$$

is sufficient and necessary to make the perpetual problem be well-posed. Using dimensionless quantities

$$z = \log\frac{s}{K}, \quad a = \gamma K A, \quad R = \frac{2r}{\sigma^2}, \quad \nu = \frac{2}{\sigma^2}\Big[\alpha - \frac{\sigma^2}{2}\Big],$$

the condition (1.2) is equivalent to $\nu < R$. One can further show that $v(x,s,A) = e^{-\gamma x}u(z,a)$ where u solves

$$\min\Big\{Rau_a - u_{zz} - \nu u_z,\ u_a + g^+u\Big\} = 0 \text{ in } \mathbb{R}\times(0,\infty), \quad u(\cdot,0) = -1. \tag{1.3}$$

The *dimensionless certainty equivalent* is a function $\varphi = \varphi(z,a)$ such that

$$v = U\big(x + \gamma^{-1}\varphi\big).$$

Then $\varphi = -\ln(-u)$ solves the following equation:

$$\min\Big\{\mathcal{A}[\varphi],\ \mathcal{B}\varphi\Big\} = 0 \ \text{ in } \mathbb{R}\times(0,\infty), \quad \varphi(\cdot,0) = 0, \tag{1.4}$$

where

$$\mathcal{A}[\varphi] := Ra\varphi_a - \varphi_{zz} - \nu\varphi_z + \varphi_z^2, \qquad \mathcal{B}\varphi := \varphi_a - g^+,$$

subscripts represent partial derivatives, $g^+ := \max\{g,0\}$, and $g = e^z - 1$. Setting $\phi(z,t) := \varphi(z,e^t)$, the problem can be written as

$$R\phi_t = \max\Big\{\phi_{zz} + \nu\phi_z - \phi_z^2,\ Re^t g^+(z)\Big\} \ \text{ in } \ \mathbb{R}^2, \qquad \phi(\cdot,-\infty) = 0.$$

This is a fully non-linear degenerate parabolic equation with initial value given at

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$t = -\infty$. When $\nu < R - 1$, Song and Yu [16] show the existence of a strong solution by a line method. In [10], we use modified penalty method to prove the existence of a unique classical solution and the existence of a continuous and strictly decreasing function $s(\cdot)$ defined on $[0, \infty)$ such that

$$\begin{aligned} &\mathcal{A}[\varphi] = 0 < \mathcal{B}\varphi \ \text{ in } \mathbf{N} := \{(z,a) \mid a > 0, z < s(a)\}, \\ &\mathcal{A}[\varphi] > 0 = \mathcal{B}\varphi \ \text{ in } \mathbf{T} := \{(z,a) \mid a > 0, z > s(a)\}, \\ &\mathcal{A}[\varphi] = 0 = \mathcal{B}\varphi \ \text{on } \Gamma := \{(z,a) \mid a > 0, z = s(a)\}. \end{aligned} \tag{1.5}$$

We call Γ the *free boundary*. In its original context, **N** is called *no-exercising region* and **T** the *exercising region* and the optimal strategy depends only on the free boundary, which is explicitly constructed in [14] with the following form:

$$A^{\text{optimal}}(t) = \min\Big\{\mathrm{A}, \min_{0 \leqslant \rho \leqslant t} \frac{e^{r\rho}\, a_*\Big(\ln \frac{S(\rho)}{K}\Big)}{\gamma K}\Big\}, \tag{1.6}$$

where $a_*(\cdot)$ is the inverse of $s(\cdot)$, with natural extension $a_*(z) = \infty$ for $z \leqslant 0 = s(\infty)$ and $a_* = 0$ for $z > s(0^+)$.

Moreover, in [10], we discover that the function $\psi := \varphi_a$ satisfies the following variational inequality:

$$\min\Big\{\mathcal{F}[\psi],\ \psi - g\Big\} = 0 \quad \text{in } \mathbb{R} \times [0, \infty), \tag{1.7}$$

where $\mathcal{F}[\psi] := Ra\psi_a + R\psi - \psi_{zz} + \Big(2\int_0^a \psi_z(z,t)dt - \nu\Big)\psi_z$. It is a great simplification that g^+ in (1.4) is replaced by g (see also [3]), since options are not exercised when stock price is below strike price. Also, (1.7) at $a = 0$ provides the equation for the initial value $\psi_0 := \psi(\cdot, 0) = \varphi_a(\cdot, 0)$:

$$\min\Big\{R\psi_0 - \psi_0'' - \nu\psi_0', \quad \psi_0 - g\Big\} = 0 \ \text{ in } \mathbb{R}. \tag{1.8}$$

By establishing a strong solution of (1.7), we obtain a classical solution of (1.4), when $\alpha < r$. Furthermore, in [13], we convert (1.4) to a Stefan type free boundary problem for (φ, ψ, w, s) where $w = \psi_a$. Under the condition $\alpha < r$, we prove that $s \in C^{3/2}([0, \infty)) \cap C^\infty((0, \infty))$ and $\varphi \in C^2(\mathbb{R} \times [0, \infty))$. The analysis in [10, 13] relies on $\varphi_{aa} \leqslant 0$, due to V concave in (x, A); see [2] for a derivation.

In this paper, we study the remaining case $r \leqslant \alpha < r + \sigma^2/2$ or equivalently $R - 1 \leqslant \nu < R$. Our main result is the following:

Theorem 1 *Assume that* $R > 0$ *and* $0 < \mu := R - \nu \leqslant 1$. *Then* (1.4) *admits a unique solution having the following properties:* $\varphi, \varphi_z, \varphi_{zz}, a\varphi_a \in C(\mathbb{R} \times [0, \infty))$, $\varphi_{aa} \leqslant 0$,

$$\begin{cases} 0 \leqslant \mu\varphi \ \leqslant \max\{(ae^z)^\mu, ae^z\}, & 0 \leqslant \ \varphi_z \ \leqslant \varphi + a, \\ 0 \leqslant a\varphi_a \leqslant \varphi, & 0 \leqslant a\varphi_{az} \leqslant \varphi + a, \\ 0 \leqslant \varphi_{zz} \leqslant R\varphi - \nu\varphi_z + \varphi_z^2, & a\varphi_a \in C^{\mu, \mu/2}(\mathbb{R} \times [0, \infty)), \end{cases} \tag{1.9}$$

$$\varphi_a(z,0) := \lim_{a\searrow 0}\frac{\varphi(z,a)}{a} = \begin{cases} e^z & \text{if } \mu = 1 \\ \infty & \text{if } \mu\in(0,1)\end{cases} \qquad \forall\, z\in\mathbb{R}. \tag{1.10}$$

Also, there exists a strictly decreasing and continuous function s such that (1.5) *holds and*

$$\pi + \ln\Big(1+\frac{1+\nu^2}{2Ra}\Big) \geqslant s(a) \geqslant \begin{cases} \ln[(\mu-\mu^2)/4]^{1/\mu} - \ln a & \textit{if } \mu\in(0,1), \\ \ln(\sqrt{R}/2) - \ln\sqrt{a} & \textit{if } \mu = 1.\end{cases} \tag{1.11}$$

Remark 1.1 It is very delicate to prove the uniqueness of the solution, due to the singular behaviour:

$$\lim_{a\searrow 0}\lim_{z\to\infty}\varphi(z,a) = \infty, \qquad \lim_{z\to\infty}\lim_{a\searrow 0}\varphi(z,a) = 0, \qquad \lim_{a\searrow 0} s(a) = \infty.$$

We prove the uniqueness as follows. It is shown in [14] that the optimal strategy is given by (1.6) if the solution of (1.5) has the following property: there exist positive constants M and δ such that

$$a\,e^{s(a)} \leqslant M, \qquad \varphi(z,a) = M[ae^z]^\delta \quad \forall\, z\leqslant s(a), a\in(0,1].$$

Clearly, our upper bounds of s and φ validate the above assumptions, so by Theorem **3** in [14],

$$\varphi(z,a) = -\ln\Big(-v\Big(0, Ke^z, \frac{a}{\gamma K}\Big)\Big).$$

Since the value function V is well-defined, we see that φ is unique and (1.6) is the optimal strategy.

Remark 1.2 When $\mu\in(0,1)$, the estimate (1.11) implies that

$$\lim_{a\searrow 0}\frac{s(a)}{\ln a} = -1.$$

In a subsequent paper, we shall show that the above limit is $-\frac{1}{2}$ when $\mu = 1$.

The semi-discretization scheme of Song and Yu [16] is based on the function u for (1.3). In terms of (1.4), it can be described as

$$\min\{A_n[\varphi_n],\ B_n[\varphi_n]\} = \mathbf{0}, \tag{1.12}$$

where A_n and B_n are implicit time-discretization of $\mathcal{A}$ and $\mathcal{B}$, respectively. Different from Song and Yu [16], here we use a semi-discretization of (1.7):

$$a_n = a_0 + nh, \qquad \min\{F_n[\psi_n],\ \psi_n - g\} = \mathbf{0}, \qquad \varphi_n = \varphi_{n-1} + h\psi_n, \tag{1.13}$$

where F_n is an implicit time discretization of $\mathcal{F}$ at time $a = a_n$. For a given discretization time step h, we choose carefully an initial time $a_0 = O(\sqrt{h})$ and initial data (φ_0, ψ_0) at

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time $a = a_0$ such that the solution of (1.13) is actually the solution of (1.12)! Therefore, the resulting approximation possesses properties of solutions of both discretizations.

The rest of the paper is organized as follows. In Section 2, we construct approximate solutions of (1.13) and show that they are also solutions of (1.12). In Section 3, we consider the well-studied case $R > \nu+1$, for the purpose of illustrating the idea of the convergence routine, as $h \searrow 0$. In Section 4, we briefly mention the transition case $R = \nu + 1$ and in Section 5, we consider the case $\nu < R \leqslant \nu + 1$.

In the sequel, we use the convention that all functions are left continuous, i.e., $f(z) = \lim_{x\nearrow z} f(x)$. We use $C^{m+1}(\mathbb{R})$ to denote functions with locally Lipschitz continuous mth order derivatives and C to denote a generic constant depending only on R and ν. We always assume that $R > 0$.

2 Approximation

Let $h > 0$ be a given step size for time discretization. For some initial time $a_0 \geqslant 0$ and initial data (φ_0, ψ_0) to be determined later, we define $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ iteratively as the solution of (1.13), where F_n is an implicit time discretization of $\mathcal{F}$ in (1.7):

$$F_n[\zeta] := Rh^{-1}[a_n\zeta - a_{n-1}\psi_{n-1}] - \zeta'' - \nu\zeta' + 2\varphi'_{n-1}\zeta' + h\zeta'^2. \tag{2.1}$$

We shall construct initial data (a_0, φ_0, ψ_0) having the following properties:

(A1) $a_0 \geqslant 0, \varphi_0, \psi_0 \in C^{1+1}(\mathbb{R}), \min\{R\psi_0 - \psi_0'' - \nu\psi_0' + 2\varphi_0'\psi_0' + h\psi_0'^2,\ \psi_0 - g^+\} \geqslant 0.$

(A2) $\min\{Ra_0\psi_0 - \varphi_0'' - \nu\varphi_0' + \varphi_0'^2,\ \psi_0 - g\} = \mathbf{0}.$

(A3) $\varphi_0 \geqslant 0,\ \varphi_0' \geqslant 0,\ \varphi_0'' \geqslant 0.$

(A4) $0 \leqslant \psi_0' \leqslant \psi_0 + 1.$

(A5) There exists a positive b such that $\psi_0 = g$ in $[b, \infty)$ and $\psi_0 > g$ in $(-\infty, b)$.

2.1 Existence

Lemma 2.1 *Given mesh size $h > 0$ and initial data (a_0, φ_0, ψ_0) satisfying* **(A1)**, *there exists a sequence $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ in $[C^{1+1}(\mathbb{R})]^2$ that solves* (1.13), *where F_n is as in* (2.1), *and satisfies*

$$\psi_0 \geqslant \psi_1 \geqslant \psi_2 \geqslant \cdots \geqslant g^+.$$

Proof We use a mathematical induction. Set $\mathcal{F}_n[\zeta] = \min\{F_n[\zeta], \zeta - g\}$.

Hypothesis (H_{n-1}): $\quad \varphi_{n-1}, \psi_{n-1} \in C^{1+1}(\mathbb{R}), \quad \psi_0 \geqslant \psi_{n-1} \geqslant \mathbf{0}, \quad \mathcal{F}_n[\psi_{n-1}] \geqslant \mathbf{0}.$

When $n = 1$, (φ_0, ψ_0) is given and satisfies (H_0), by the condition **(A1)** and the fact that $a_1 = a_0 + h$ and $\psi_0 - g \geqslant \psi_0 - g^+ \geqslant \mathbf{0}$. Thus, the hypothesis is true when $n = 1$.

Next, suppose (H_{n-1}) is true for some $n \geqslant 1$. We construct ψ_n by solving $\mathcal{F}_n[\psi_n] = \mathbf{0}$.

(i) Note that $\mathcal{F}_n[\mathbf{0}] = \min\{-Rh^{-1}a_{n-1}\psi_{n-1},\ -g\} \leqslant \mathbf{0}$. Hence, $\mathbf{0}$ is a subsolution.

(ii) By the induction hypothesis, $\mathcal{F}_n[\psi_{n-1}] \geqslant \mathbf{0}$. Hence, ψ_{n-1} is a supersolution.

(iii) As the supersolution ψ_{n-1} is no smaller than the subsolution $\mathbf{0}$, by a standard pde argument (c.f. [5] or Step **1** of the proof of Lemma 2.4 below), there exists a $C^{1+1}(\mathbb{R})$ solution ψ_n of $\mathcal{F}_n[\psi_n] = \mathbf{0}$, satisfying $\mathbf{0} \leqslant \psi_n \leqslant \psi_{n-1}$. We set $\varphi_n = \varphi_{n-1} + h\psi_n$. Then $\varphi_n \in C^{1+1}(\mathbb{R})$.

(iv) Now we verify hypothesis (H_n). We already know $0 \leqslant \psi_n \leqslant \psi_{n-1} \leqslant \psi_0$. Also,

$$\mathcal{F}_{n+1}[\psi_n] = \min\{F_{n+1}[\psi_n], \psi_n - g\} = \min\{F_n[\psi_n] + \delta_n, \psi_n - g\} \geqslant \mathcal{F}_n[\psi_n] = 0,$$

since $\delta_n := Rh^{-1}a_{n-1}[\psi_{n-1} - \psi_n] + 2h\psi_n'^{\,2} \geqslant 0$. Thus, (φ_n, ψ_n) satisfies (H_n). This completes the mathematical induction and also the proof of the lemma. □

Remark 2.1 We are working on unbounded domain $\mathbb{R}$ with unbounded solutions, so the uniqueness of the solution ψ_n of $\mathcal{F}_n[\psi_n] = 0$ does not follow from any classical pde theory. We can show the uniqueness, but the proof is too technical so we decide to omit it here. Nevertheless, under the full assumptions **(A1)–(A5)**, we can work on the half interval $(-\infty, b]$ with boundary conditions $\psi_n'(b) = g'(b)$ and $\psi_n(-\infty) = 0$ to establish the existence and uniqueness (since $R > 0$) of solutions. The property $\psi_0 \geqslant \psi_n \geqslant g$ still holds so $\psi_n(b) = g(b)$ and we can extend ψ_n to $\mathbb{R}$ by setting $\psi_n \equiv g$ on $[b, \infty)$. The condition **(A1)** implies that $F_n[\psi_n] = F_n[g] \geqslant F_1[g] \geqslant 0$ on $[b, \infty)$ (since $\varphi_n = \varphi_0 + [a_n - a_0]g$ on $[b, \infty)$), so the extended function satisfies $\mathcal{F}_n[\psi_n] = 0$ on $\mathbb{R}$.

In the sequel, $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ is the solution of (1.13) given in Lemma 2.1.

2.2 Variation structure of φ

Lemma 2.2 *Assume* **(A1)** *and* **(A2)**. *Then the solution of* (1.13) *satisfies* (1.12) *where*

$$A_n[\zeta] := Ra_n h^{-1}[\zeta - \varphi_{n-1}] - \zeta'' - \nu\zeta' + \zeta'^2, \qquad B_n[\zeta] := h^{-1}[\zeta - \varphi_{n-1}] - g.$$

Proof For each integer $n \geqslant 1$, using $2\varphi_{n-1}'\psi_n' + h\psi_n'^2 = h^{-1}(\varphi_n'^2 - \varphi_{n-1}'^2)$, we obtain

$$hF_n[\psi_n] = [Ra_n\psi_n - \varphi_n'' - \nu\varphi_n' + \varphi_n'^2] - [Ra_{n-1}\psi_{n-1} - \varphi_{n-1}'' - \nu\varphi_{n-1}' + \varphi_{n-1}'^2].$$

Hence, defining $\gamma_0 := Ra_0\psi_0 - \varphi_0'' - \nu\varphi_0' + \varphi_0'^{\,2}$, we have

$$A_n[\varphi_n] = Ra_n\psi_n - \varphi_n'' - \nu\varphi_n' + \varphi_n'^{\,2} = \gamma_0 + \sum_{i=1}^{n} hF_i[\psi_i].$$

(i) The condition **(A2)** implies that $\gamma_0 \geqslant 0$. The equation (1.13) implies that $F_i[\psi_i] \geqslant 0$ for each $i \geqslant 1$. Hence, $A_n[\varphi_n] \geqslant 0$ in $\mathbb{R}$. Also, $B_n[\varphi_n] = \psi_n - g \geqslant 0$ in $\mathbb{R}$.

(ii) Suppose $z \in \mathbb{R}$ is a point at which $B_n[\varphi_n](z) = \psi_n(z) - g(z) > 0$. Then $\psi_0(z) \geqslant \psi_1(z) \geqslant \cdots \geqslant \psi_n(z) > g(z)$. Consequently, from $\min\{F_i[\psi_i], \psi_i - g\} = 0$ and $\min\{\gamma_0, \psi_0 - g\} = 0$, we have $\gamma_0(z) = 0$ and $F_i[\psi_i](z) = 0$ for $i = 1, \cdots, n$. Thus, $A_n[\varphi_n](z) = 0$.

In conclusion, $\min\{A_n[\varphi_n],\ B_n[\varphi_n]\} = \mathbf{0}$. □

Remark 2.2 The scheme of Song and Yu [16], based on (1.3), is almost the same as (1.12). Using (1.13), we can estimate directly $\psi = \varphi_a$, retaining the variation structure (1.12).

2.3 Positivity, monotonicity, and convexity of φ

Lemma 2.3 *Assume* **(A1)–(A3)**. *Then $\varphi_n \geqslant 0, \varphi_n' \geqslant 0$, and $\varphi_n'' \geqslant 0$ in $\mathbb{R}$ for every $n \geqslant 1$.*

Proof We use an induction argument. Assume that $n \geqslant 1$ and $\varphi_{n-1} \geqslant 0, \varphi_{n-1}' \geqslant 0$ and $\varphi_{n-1}'' \geqslant 0$.

For each fixed constant $\varepsilon > 0$, we consider the evolution problem, for $\phi = \phi(z,t)$,

$$\phi_t - \phi_{zz} - \nu\phi_z + \phi_z^2 + Rh^{-1}a_n(\phi - \varphi_{n-1}) + \beta_\varepsilon(\phi - \varphi_{n-1} - hg) = 0 \text{ in } \mathbb{R}\times(0,\infty), \tag{2.2}$$
$$\phi(\cdot,0) = 0 \text{ on } \mathbb{R}\times\{0\}.$$

Here, $\beta_\varepsilon(s) = \beta(s/\varepsilon)$ and β is a smooth function satisfying

$$\beta = 0 \text{ in } [0,\infty), \quad \beta'' < 0 < \beta' \text{ in } (-1,0), \quad \beta'' = 0 \text{ in } (-\infty,-1]. \tag{2.3}$$

(i) One can apply maximum principle to obtain an *a priori* estimate $\phi \geqslant 0$ in $\mathbb{R}\times(0,\infty)$. Also, since $\min\{A_n[\varphi_n], B_n[\varphi_n]\} = 0$, one can use comparison to obtain an *a priori* estimate $\phi \leqslant \varphi_n$ in $\mathbb{R}\times[0,\infty)$. These upper and lower bounds ensure that the non-linear parabolic equation admits a unique solution in the function space $\{\phi \in C^{3+1,2}(\mathbb{R}\times[0,\infty)) \mid 0 \leqslant \phi \leqslant \varphi_n\}$.

(ii) Differentiating (2.2) with respect to z, we obtain

$$\begin{aligned}
&(\phi_z)_t - (\phi_z)_{zz} + [2\phi_z - \nu](\phi_z)_z + [Rh^{-1}a_n + \beta_\varepsilon']\phi_z \\
&\quad = Rh^{-1}a_n\varphi_{n-1}' + [\varphi_{n-1}' + hg']\beta_\varepsilon' \geqslant 0, \\
&(\phi_{zz})_t - (\phi_{zz})_{zz} + [2\phi_z - \nu](\phi_{zz})_z + [Rh^{-1}a_n + \beta_\varepsilon' + 2\phi_{zz}]\phi_{zz} \\
&\quad = Rh^{-1}a_n\varphi_{n-1}'' + [\varphi_{n-1}'' + hg'']\beta_\varepsilon' - [\phi_z - \varphi_{n-1}' - hg']^2\beta_\varepsilon'' \geqslant 0.
\end{aligned}$$

Hence, by maximum principle, $\phi_z \geqslant 0$ and $\phi_{zz} \geqslant 0$ in $\mathbb{R}\times[0,\infty)$.

(iii) Since $\phi \geqslant 0$, we have $\phi_t(\cdot,0) \geqslant 0$. We can differentiate (2.2) with respect to t to obtain a parabolic equation for ϕ_t and show by maximum principle that $\phi_t \geqslant 0$ in $\mathbb{R}\times[0,\infty)$. Hence, $\phi_\varepsilon(x) := \lim_{t\to\infty}\phi(x,t)$ exists; the limit satisfies $\varphi_n \geqslant \phi_\varepsilon \geqslant 0$, $\phi_\varepsilon' \geqslant 0$ and $\phi_\varepsilon'' \geqslant 0$ in $\mathbb{R}$, and

$$-\phi_\varepsilon'' - \nu\phi_\varepsilon' + \phi_\varepsilon'^2 + Rh^{-1}a_n(\phi_\varepsilon - \varphi_{n-1}) + \beta_\varepsilon(\phi_\varepsilon - \varphi_{n-1} - hg) = 0 \text{ in } \mathbb{R}. \tag{2.4}$$

(iv) Finally, sending $\varepsilon \searrow 0$, we obtain $\varphi_n = \lim_{\varepsilon\searrow 0}\phi_\varepsilon$, being the unique solution of the variational inequality (1.12). The properties $\phi_\varepsilon \geqslant 0, \phi_\varepsilon' \geqslant 0$, and $\phi_\varepsilon'' \geqslant 0$ carry over to the limit so $\varphi_n \geqslant 0, \varphi_n' \geqslant 0$, and $\varphi_n'' \geqslant 0$ in $\mathbb{R}$.

This completes the mathematical induction argument and also the proof of the lemma. □

Remark 2.3

(1) In step (iv), we need the uniqueness of solutions of (1.12); see Remark 2.1.

(2) The classical approach (e.g. [5]) for the obstacle problem (1.12) uses (2.4) for approximation, where there is a technical difficulty in applying the maximum principle for ϕ''_ε, e.g., *a priori* it is not known if $Rh^{-1}a_n + \beta'_\varepsilon + 2\phi''_\varepsilon$ is positive. We overcome this difficulty by the parabolic approximation for which positivity of $Rh^{-1}a_n + \beta'_\varepsilon + 2\phi_{zz}$ is not needed for maximum principle.

2.4 Free boundary approximation

At $a = a_n$, the approximate free boundary is defined as the boundary of the set

$$\{z \mid \psi_n(z) > g(z)\} = \{z \mid [\psi_n(z) + 1]e^{-z} > 1\}.$$

If $[\psi_n(z) + 1]e^{-z}$ is decreasing, i.e., $\psi'_n \leqslant \psi_n + 1$, the free boundary is at most a singleton.

Lemma 2.4 *Assume* **(A1)–(A4)**. *Then for each* $n \geqslant 1$, $0 \leqslant \psi'_n \leqslant \psi_n + 1$ *in* $\mathbb{R}$.

Proof We use mathematical induction. Assume that $n \geqslant 1$ and $0 \leqslant \psi'_{n-1} \leqslant \psi_{n-1} + 1$.

(1) For each fixed $\varepsilon > 0$, consider the problem, for $\psi_\varepsilon(z)$, that approximates ψ_n:

$$Rh^{-1}(a_n\psi_\varepsilon - a_{n-1}\psi_{n-1}) - \psi''_\varepsilon + [2\varphi'_{n-1} + h\psi'_\varepsilon - \nu]\psi'_\varepsilon + \beta_\varepsilon(\psi_\varepsilon - g) = 0 \ \text{ in } \mathbb{R}, \quad (2.5)$$

where $\beta_\varepsilon(s) = \beta(s/\varepsilon)$ and β is as in (2.3). One can show that $\mathbf{0}$ is a subsolution and ψ_{n-1} is a supersolution (since $\mathcal{F}_n[\psi_{n-1}] \geqslant 0$). Hence, there exists a solution satisfying $0 \leqslant \psi_\varepsilon \leqslant \psi_{n-1}$. Sending $\varepsilon \searrow 0$, we obtain a limit $\psi_n \in C^{1+1}(\mathbb{R})$ which is the solution of $\mathcal{F}_n[\psi_n] = 0$ satisfying $0 \leqslant \psi_n \leqslant \psi_{n-1}$; see, for example, [5] for the standard $\varepsilon \searrow 0$ process.

(2) Differentiating (2.5) with respect to z, we obtain

$$-\psi'''_\varepsilon + (2\varphi'_{n-1} + 2h\psi'_\varepsilon - \nu)\psi''_\varepsilon + (Rh^{-1}a_n + 2\varphi''_{n-1} + \beta'_\varepsilon)\psi'_\varepsilon = Rh^{-1}a_{n-1}\psi'_{n-1} + \beta'_\varepsilon g' \geqslant 0. \quad (2.6)$$

It then follows from the maximum principle that $\psi'_\varepsilon \geqslant 0$ in $\mathbb{R}$. Sending $\varepsilon \searrow 0$, we obtain $\psi'_n \geqslant 0$.

(3) Set $\zeta = \psi_\varepsilon + 1 + \varepsilon - \psi'_\varepsilon$. Taking the difference of (2.5) and (2.6), we obtain, using $g' = g + 1$,

$$\begin{aligned}
&-\zeta'' + [2\varphi'_{n-1} + 2h\psi'_\varepsilon - \nu]\zeta' + [Rh^{-1}a_n + \beta'_\varepsilon]\zeta \\
&= R + Rh^{-1}a_n\varepsilon + h\psi'^{2}_\varepsilon + 2\varphi''_{n-1}\psi'_\varepsilon + Rh^{-1}a_{n-1}[\psi_{n-1} + 1 - \psi'_{n-1}] + (\psi_\varepsilon + \varepsilon - g)\beta'_\varepsilon - \beta_\varepsilon \\
&\geqslant (s+1)\beta'(s) - \beta(s)\Big|_{s=\frac{\psi_\varepsilon - g}{\varepsilon}} \geqslant 0;
\end{aligned}$$

here we have used the following fact: $[(s + 1)\beta'(s) - \beta(s)]' = (s + 1)\beta''(s) \leqslant 0$, so

$$\min_{s\in\mathbb{R}} \Big\{(s+1)\beta'(s) - \beta(s)\Big\} = (s+1)\beta'(s) - \beta(s)\Big|_{s=0} = 0.$$

Hence, applying the maximum principle, we obtain $\zeta \geqslant 0$ in $\mathbb{R}$. This implies that $\psi_\varepsilon + 1 + \varepsilon - \psi'_\varepsilon \geqslant 0$. Sending $\varepsilon \searrow 0$, we obtain $\psi_n + 1 - \psi'_n \geqslant 0$.

Hence, $0 \leqslant \psi'_n \leqslant \psi_n + 1$. This completes the mathematical induction and also the proof. □

Lemma 2.5 *Assume* **(A1)–(A5)**. *Then for each* $n \geqslant 1$, *there exists* $z_n \in \mathbb{R}$ *such that*

$$\psi_n > g \ \ in \ \ (-\infty, z_n), \qquad \psi_n = g \ \ in \ \ [z_n, \infty).$$

In addition, $b = z_0 > z_1 > z_2 > \cdots > 0$.

Proof Suppose that for some $n \geqslant 1$, there exists $z_{n-1} \in (0, b]$ such that $\psi_{n-1} > g$ in $(-\infty, z_{n-1})$ and $\psi_{n-1} = g$ on $[z_{n-1}, \infty)$.

(i) Since $\psi_{n-1} \geqslant \psi_n \geqslant g$ in $\mathbb{R}$, $\psi_n \equiv g$ on $[z_{n-1}, \infty)$.

(ii) Consider the function $\zeta_n(z) = [\psi_n(z) + 1]e^{-z}$. On $[z_{n-1}, \infty)$, $\zeta_n \equiv 1$. On $(-\infty, z_{n-1}]$, $\zeta'_n = e^{-z}[\psi'_n - (\psi_n + 1)] \leqslant 0$. Also, since $\psi_n \geqslant 0$, $\zeta_n(-\infty) = \infty$. Hence, there exists a unique $z_n \in (-\infty, z_{n-1}]$ such that $\zeta_n > 1$ in $(-\infty, z_n)$ and $\zeta_n \equiv 1$ on $[z_n, \infty)$; namely, $\psi_n > g$ in $(-\infty, z_n)$ and $\psi_n \equiv g$ on $[z_n, \infty)$.

(iii) By strong maximum principle, $\psi_n > 0$ on $\mathbb{R}$. This implies that $\psi(z_n) = g(z_n) > 0$, so $z_n > 0$.

(iv) Since $\psi_n \in C^1(\mathbb{R})$, that $\psi_n - g$ attains a global minimum at z_n implies that $\psi'_n(z_n) = g'(z_n)$.

(v) On $(-\infty, z_n]$, $F_n[\psi_n] = 0 \leqslant F_n[\psi_{n-1}]$. If $z_{n-1} = z_n$, we would have $\psi_n(z_n) = \psi_{n-1}(z_n)$, from which we can apply Hopf's Lemma to derive that $\psi'_n(z_n) - \psi'_{n-1}(z_n) > 0$, contradicting $\psi(z_n)' = g'(z_n)$ and $\psi'_{n-1}(z_{n-1}) = g'(z_{n-1})$. Thus, $z_n < z_{n-1}$. This completes the mathematical induction and also the proof of the lemma. □

2.5 The latent heat

We call $\eta := (\mathcal{A}[\varphi])_a = \mathcal{F}[\psi]$ the latent heat; see [13].

Lemma 2.6 *Assume* **(A1)–(A5)**. *For* $n \geqslant 1$, *define*

$$\eta_n := F_n[\psi_n], \quad l_n := e^z(\varphi'_n + \varphi'_{n-1} + R - 1 - \nu) - R. \tag{2.7}$$

For each $n \geqslant 1$, *the following holds:*

(1) $\eta_n = 0$ *in* $(-\infty, z_n]$, $0 < \eta_n < l_n$ *in* $[z_n^+, z_{n-1})$, $\eta_n = l_n$ *on* $[z_{n-1}, \infty)$;

(2) $\eta_n(z_n^+) - \eta_n(z_n^-) = \psi''_n(z_n) - g''(z_n) > 0$;

(3) $\eta_{n+1} \geqslant \eta_n \geqslant 0$ *and* $\eta'_n \geqslant 0$ *on* $\mathbb{R}$.

Proof

(i) The definition of ψ_n reads $\min\{\eta_n, \psi_n - g\} = 0$. Hence, $\eta_n \geqslant 0$ in $\mathbb{R}$. Also, by the definition of z_n, we have $\eta_n = 0$ in $(-\infty, z_n]$ and $\psi_n = g$ in $[z_n, \infty)$. Since $\psi_{n-1} > g = \psi_n$

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on $[z_n, z_{n-1})$, the first assertion follows by the definition $\eta_n = Rh^{-1}a_{n-1}[\psi_n - \psi_{n-1}] + R\psi_n - \psi_n'' + [\varphi_n' + \varphi_{n-1}' - \nu]\psi_n'$.

(ii) Since $\psi_n \in C^{1+1}$, one sees that $\eta_n(z_n^+) - \eta_n(z_n^-) = \psi_n''(z_n^-) - \psi_n''(z_n^+) = \psi_n''(z_n) - g''(z_n) = \zeta_n'(z_n) > 0$, by the Hopf's Lemma, since $\zeta_n := \psi_n' - (\psi_n + 1)$ is negative in $(-\infty, z_n)$, equals to zero at z_n, and in $(-\infty, z_n)$,

$$Rh^{-1}a_n\zeta_n - \zeta_n'' + (\varphi_n' + \varphi_{n-1}' - \nu)\zeta_n' = -R + Rh^{-1}a_{n-1}\zeta_{n-1} - (\varphi_n'' + \varphi_{n-1}'')\psi_n' < -R.$$

(iii) On $[z_n^+, \infty)$, $\psi_n = g$ so $\eta_n = l_n + Rh^{-1}a_{n-1}(\psi_n - \psi_{n-1})$; using $\psi_n' = g' = \psi_n + 1$, we obtain

$$\eta_n' = R + \eta_n + e^z[\varphi_n'' + \varphi_{n-1}''] + Rh^{-1}a_{n-1}[\psi_{n-1} + 1 - \psi_{n-1}'] > R + \eta_n > 0.$$

(iv) Assume $n \geqslant 2$. On (z_{n-1}, ∞), $\psi_n = g = \psi_{n-1}$, so $\eta_n = l_n$. As $\eta_{n-1} \leqslant l_{n-1}$ on $[z_{n-1}, \infty)$, we have $\eta_n - \eta_{n-1} \geqslant l_n - l_{n-1} = e^z[h\psi_n' + h\psi_{n-1}'] = 2he^{2z} > 0$. On $(z_n, z_{n-1}]$, $\eta_n > 0 = \eta_{n-1}$. On $(-\infty, z_n]$, $\eta_n = 0 = \eta_{n-1}$. Thus, $\eta_n - \eta_{n-1} \geqslant 0$ on $\mathbb{R}$. The assertion of the lemma thus follows.

□

2.6 Hölder continuity of $a\psi$ in a

We write the equation $F_n[\psi_n] = \eta_n$ as

$$Rh^{-1}[a_n\psi_n - a_{n-1}\psi_{n-1}] - \psi_n'' = f_n := \eta_n + [\nu - \varphi_n' - \varphi_{n-1}']\psi_n'.$$

Since $\eta_n = 0$ in $(-\infty, z_n]$ and $0 \leqslant \eta_n \leqslant l_n = [\varphi_n' + \varphi_{n-1}' + R - \nu - 1]\psi_n' - R$ on $[z_n^+, \infty)$, we have

$$|f_n| \leqslant \max\Big\{(R-1)^+, |\nu| + 2\varphi_n'\Big\}\psi_n' \quad \text{ on } \mathbb{R}.$$

For integer $0 \leqslant m < n$ and constant $\delta > 0$ to be chosen,

$$\begin{aligned} a_n\psi_n(z) - a_m\psi_m(z) &= a_n\Big\{\psi_n(z) - \fint_{z-\delta}^{z} \psi_n(y)dy\Big\} - a_m\Big\{\psi_m(z) - \fint_{z-\delta}^{z} \psi_m(y)dy\Big\} \\ &\quad + \fint_{z-\delta}^{z} \sum_{i=m+1}^{n} [a_i\psi_i(y) - a_{i-1}\psi_{i-1}(y)]dy = I - II + III. \end{aligned}$$

Since $\psi_i' > 0$, both I and II are positive so $|I - II| \leqslant \max_{m\leqslant i\leqslant n} \|a_i\psi_i'\|_{L^\infty((-\infty,z])}\delta$. Next, using equation $a_i\psi_i - a_{i-1}\psi_{i-1} = R^{-1}h[\psi_i'' + f_i]$, we have

$$\begin{aligned} |III| &= \frac{h}{R}\Big|\fint_{z-\delta}^{z} \sum_{i=m+1}^{n} [\psi_i''(y) + f_i(y)]dy\Big| \\ &\leqslant \frac{h}{R\delta} \sum_{i=m+1}^{n} |\psi_i'(z) - \psi_i'(z-\delta)| + \frac{h}{R} \fint_{z-\delta}^{z} \sum_{i=m+1}^{n} |f_i(y)|dy \\ &\leqslant \frac{a_n - a_m}{R\delta} \max_{m\leqslant i\leqslant n} \|\psi_i'\|_{L^\infty((-\infty,z])} + \frac{a_n - a_m}{R} \max_{m+1\leqslant i\leqslant n} \|f_i\|_{L^\infty((-\infty,z])}. \end{aligned}$$

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Hence, taking $\delta = \sqrt{a_n - a_m}/\sqrt{Ra_n}$ and using $0 \leqslant \psi_i' \leqslant \psi_i + 1 \leqslant \psi_m + 1$ for $i \geqslant m$, we obtain the following.

Lemma 2.7 *Assume* **(A1)–(A5)**. *Then for every integer $n > m \geqslant 0$,*

$$|a_n\psi_n - a_m\psi_m| \leqslant \sqrt{a_n - a_m}\frac{\sqrt{a_n}}{R}\Big(2\sqrt{R} + \max\big\{(R-1)^+, |\nu| + 2\varphi_n'\big\}\Big)[1 + \psi_m]. \tag{2.8}$$

3 The case $\mu > 1$

In this section, we assume that $\mu := R - \nu > 1$, which has been well-studied in [10,13,16]. The purpose that we revisit this studied problem is to illustrate the routine convergence technique that will be used in subsequent Sections for the case $0 < \mu \leqslant 1$.

3.1 Construction of the initial data

We set $a_0 = 0$, $\varphi_0 = \mathbf{0}$ and

$$\lambda := \frac{\sqrt{\nu^2 + 4R} - \nu}{2}, \quad b := \ln\frac{\lambda}{\lambda - 1}, \quad \psi_0(x) := \begin{cases} g(b)e^{\lambda(x-b)} & \text{if} \quad x \leqslant b, \\ g(x) & \text{if} \quad x > b. \end{cases} \tag{3.1}$$

Here, λ is the positive root of $\lambda^2 + \nu\lambda = R$. Note that $\lambda > 1$ since $R > 1 + \nu$.

Lemma 3.1 *Assume that $R > \max\{0, 1+\nu\}$. Let $a_0 = 0$, $\varphi_0 \equiv 0$, and ψ_0 be defined in* (3.1). *Then for each $h > 0$, (a_0, φ_0, ψ_0) satisfies* **(A1)–(A5)**. *In addition, $\gamma_0 := a_0\psi_0 - \varphi_0'' - \nu\varphi_0' + \varphi_0'^2 \equiv 0$.*

Proof

(i) First, we show that ψ_0 is the solution of (1.8).

In the set $(-\infty, b]$, $R\psi_0 - \psi_0'' - \nu\psi_0' = 0$ since $\lambda^2 + \nu\lambda = R$. Also, $\psi_0(b) = g(b)$ and $\psi_0'(b) = \lambda g(b) = g'(b)$. Moreover, $\{\psi_0(x) - g(x)\}' = g'(b)[e^{\lambda(x-b)} - e^{x-b}] < 0$ for $x < b$ (since $\lambda > 1$). Hence, $\psi_0 - g > 0$ in $(-\infty, b)$ and $\psi_0 \in C^{1+1}(\mathbb{R})$.

In the set (b, ∞), $\psi_0 = g$, and $R\psi_0 - \psi_0'' - \nu\psi_0' = (R-1-\nu)e^z - R > (R-1-\nu)e^b - R = \lambda$. Hence, ψ_0 satisfies (1.8).

(ii) Since $a_0 = 0$ and $\varphi_0 \equiv 0$, we see that conditions **(A1)–(A3)** and **(A5)** are satisfied, and $\gamma_0 \equiv 0$. In addition, when $x \geqslant b$, $\psi_0 = g$ so $\psi_0' = \psi_0 + 1$; when $x \leqslant b$, $\psi_0' - \psi_0 - 1 = [g'(b) - g(b)]e^{\lambda(x-b)} - 1 = e^{\lambda(x-b)} - 1 < 0$. Hence, (a_0, φ_0, ψ_0) satisfies **(A1)–(A5)**.

□

3.2 Convergence routine

For simplicity, we suppress the dependence on h, denoting, for each fixed $h > 0$, the solution of (1.13) by $\{(\varphi_n, \psi_n)\}_{n=0}^{\infty}$ and the sequence in Lemma 2.5 by $\{z_n\}_{n=0}^{\infty}$. Also, $\eta_n =$

$F_n[\psi_n]$. We denote by $\mathbf{1}_A$ the characteristic function of the set A: $\mathbf{1}_A(a) = 1$ if $a \in A$ and $\mathbf{1}_A(a) = 0$ if $a \notin A$.

We define, for $a \geqslant 0$ and $z \in \mathbb{R}$,

$$
\begin{aligned}
s^h(a) &:= \sum_{i=1}^{\infty} z_i \mathbf{1}_{[a_{i-1},a_i)}(a),\\
\varphi^h(z,a) &:= \sum_{i=1}^{\infty} \Big(\frac{a_i - a}{h}\varphi_{i-1} + \frac{a - a_{i-1}}{h}\varphi_i \Big) \mathbf{1}_{[a_{i-1},a_i)}(a),\\
\eta^h(z,a) &:= \sum_{i=1}^{\infty} \eta_i(z) \mathbf{1}_{(a_{i-1},a_i]}(a),\\
\xi^h(z,a) &:= \gamma_0(z) + \int_0^a \eta^h(z,t)dt.
\end{aligned}
$$

Then $s^h(\cdot)$ is a decreasing function, valued in $(0,b]$. Also, we have the following L^∞ bounds:

$$
\begin{aligned}
0 \leqslant \varphi^h &= \int_0^a \sum_{i=1}^{\infty} \psi_i \mathbf{1}_{(a_{i-1},a_i]}(t)dt \leqslant \psi_0 \int_0^a \sum_{i=1}^{\infty} \mathbf{1}_{(a_{i-1},a_i]}(t)dt = a\psi_0,\\
0 \leqslant \varphi^h_a &= \sum_{i=1}^{\infty} \psi_i \mathbf{1}_{(a_{i-1},a_i]}(a) \leqslant \psi_0,\\
0 \leqslant \varphi^h_z &= \int_0^a \sum_{i=1}^{\infty} \psi_i' \mathbf{1}_{(a_{i-1},a_i]}(t)dt \leqslant a[\psi_0 + 1],\\
0 \leqslant \varphi^h_{za} &= \sum_{i=1}^{\infty} \psi_i' \mathbf{1}_{(a_{i-1},a_i]}(a) \leqslant \psi_0 + 1,\\
0 \leqslant \varphi^h_{zz} &\leqslant Ra\psi_0 + |\nu| a[\psi_0 + 1] + (a+h)^2[\psi_0 + 1]^2,
\end{aligned}
$$

where in the last estimate, we have used $A_n[\varphi_n] \geqslant 0$ so $\varphi_n'' \leqslant Ra_n\psi_n - \nu\varphi_n' + \varphi_n'^2$.

Also, we have, in distribution,

$$
\eta^h \geqslant 0, \quad \eta^h_z \geqslant 0, \quad \eta^h_a \geqslant 0, \quad \xi^h \geqslant 0, \quad \xi^h_z \geqslant 0, \quad \xi^h_a = \eta^h \geqslant 0.
$$

Hence, there exists a decreasing function $s : [0,\infty) \to [0,b]$, a locally Lipschitz continuous function φ, and monotonic non-negative functions ξ and η such that along a sequence $h \searrow 0$,

$$
\begin{aligned}
s^h(a) &\longrightarrow s(a) && \forall\, a \geqslant 0,\\
(\varphi^h, \varphi^h_z) &\longrightarrow (\varphi, \varphi_z) && \text{in } C^\alpha(\mathbb{R}\times[0,T]) \quad \forall\, \alpha \in [0,1), T > 0,\\
(\eta^h, \xi^h, \varphi^h_a, \varphi^h_{zz}) &\rightharpoonup (\eta, \xi, \varphi_a, \varphi_{zz}) && \text{in distribution.}
\end{aligned}
$$

We remark that the following:

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(1) on $[b,\infty)\times[0,\infty)$, $\varphi^h \equiv ag$;

(2) the uniform convergence of (φ^h,φ^h_z) in the unbounded region $(-\infty,0]\times[0,T]$ is due to the uniform exponential decay of the solution, as $z\to-\infty$;

(3) by Lemma 2.7, $a\varphi_a \in C^{0+1,1/2}(\mathbb{R}\times[0,\infty))$ and

$$a\varphi^h_a \longrightarrow a\varphi_a \quad \text{uniformly in any compact subset of } \mathbb{R}\times[0,\infty);$$

(4) the convergence of $\xi^h \to \xi$ is indeed pointwise, by the following argument: (i) since $\xi^h_z \geqslant 0$, along a sequence of $h \searrow 0$, $\xi^h(z,a)\to\xi(z,a)$ for every $z\in\mathbb{R}$ and every rational $a\geqslant 0$, (ii) since $\xi^h_a = \eta^h$ is locally bounded, we have $\xi^h(z,a)\to\xi(z,a)$ at every $(z,a)\in\mathbb{R}\times[0,\infty)$;

(5) the convergence of $\eta^h\to\eta$ can be made in L^2_{loc} and a.e., since $\{\eta^h\}$ is a bounded family of bounded variation in any compact subset of $\mathbb{R}\times[0,\infty)$.

The limit (ξ,η) has the following properties:

$$\eta = 0 \ \text{ in } (-\infty, s(a^+)), \qquad 0\leqslant\eta\leqslant l \ \text{ in } \ [s(a^+),s(a^-)], \qquad \eta = l \ \text{ in } (s(a^-),\infty),$$
$$\xi(z,a) = \int_{\min\{a,a_*(z)\}}^{a} l(z,t)dt \quad \text{a.e.},$$

where $a_*(\cdot)$ is the inverse function of $s(\cdot)$, with natural extension $a_*(z)=\infty$ for $z\leqslant s(\infty)$ and $a_*(z)=0$ for $z\geqslant b$, and

$$l(z,a) := e^z\{2\varphi_z(z,a)+R-1-\nu\}-R \quad \in C(\mathbb{R}\times[0,\infty)).$$

3.3 The limit equation

Set $\mathbf{N}=\{(z,a)\mid a>0, z<s(a^+)\}$, $\Gamma=\{(z,a)\mid a>0, z\in[s(a^+),s(a^-)]\}$, and $\mathbf{T}=\{(z,a)\mid a>0, z>s(a^-)\}$. Then $\varphi_a = g$ on $\Gamma\cup\mathbf{T}$ since $\lim_{h\to0}\varphi^h_a=\varphi_a$ uniformly in any compact subset of $\mathbb{R}\times(0,\infty)$. Also, using $A_n[\varphi_n]=\gamma_0+\sum_{i=1}^n h\eta_i = \xi^h(\cdot,a_n)$, we have

$$\mathcal{A}[\varphi] = Ra\varphi_a - \varphi_{zz} - \nu\varphi_z + \varphi_z^2 = \xi(z,a) \ \text{ a.e. and in } \ L^2_{\text{loc}}(\mathbb{R}\times(0,\infty)).$$

Applying the Hopf's Lemma to $\psi+1-\psi'$ on the rectangular domain $(-\infty,s(a^-)]\times(a-\varepsilon,a^-]$, we can show as in [10] that the free boundary has neither horizontal part nor vertical part; that is, $s(\cdot)$ is continuous and strictly decreasing on $(0,\infty)$. This implies that the inverse $a_*(\cdot)$ of $s(\cdot)$ is continuous so $\xi(z,a)=\int_{\min\{a,a_*(z)\}}^a l(z,t)dt$ is a continuous function on $\mathbb{R}\times[0,\infty)$. As $a\varphi_a\in C(\mathbb{R}\times[0,\infty))$, we obtain from $\mathcal{A}[\varphi]=\xi$ in L^2_{loc} that $\varphi_{zz}=\xi+Ra\varphi_a+\varphi_z^2-\nu\varphi_z\in C(\mathbb{R}\times[0,\infty))$ and $\mathcal{A}[\varphi]=\xi$ in $C(\mathbb{R}\times[0,\infty))$.

Note that $\xi(z,a)=0$ for $z<s(a)$; $l(s(a),a)\geqslant 0$; and $l_z(z,a)>R+l(z,a)>0$ for $z>s(a)$ (since $\varphi_{zz}\geqslant 0$). Hence, $\xi(z,a)>0$ for $z>s(a)$. This implies that $\mathcal{A}[\varphi]=0$ when $z\leqslant s(a)$ and $\mathcal{A}[\varphi]>0$ when $z>s(a)$.

Using comparison, one can show that $\psi_z<1+\psi$ in $\mathbf{N}$. This implies that $\mathcal{B}\varphi=\psi-g>0$ in $\mathbf{N}$. Hence, we have the following:

Theorem 2 *Assume that $R > \max\{0, 1 + \nu\}$. Let (b, ψ_0) be defined as in* (3.1). *Then* (1.4) *admits a classical solution having the following properties: $\varphi, \varphi_z, \varphi_{zz}, a\varphi_a \in C(\mathbb{R} \times [0, \infty))$; on $[b, \infty) \times [0, \infty)$, $\varphi(z, a) = ag(z)$; on $(-\infty, b] \times [0, \infty)$, $\varphi_{aa} \leqslant 0$ and*

$$\begin{array}{lll} 0 \leqslant \varphi \leqslant a\psi_0, & 0 \leqslant \varphi_z \leqslant a[\psi_0 + 1], & 0 \leqslant \varphi_{zz} \leqslant Ra\psi_0 - \nu\varphi_z + \varphi_z^2, \\ 0 \leqslant \varphi_a \leqslant \psi_0, & 0 \leqslant \varphi_{az} \leqslant \psi_0 + 1, & a\varphi_a \in C^{0+1,1/2}(\mathbb{R} \times [0, \infty)). \end{array}$$

There exists a strictly decreasing and continuous function $s : [0, \infty) \to (0, b]$ such that (1.5) *holds.*

Remark 3.1 Set $w_n = h^{-1}[\psi_n - \psi_{n-1}]$. Then $w_n \leqslant 0$. Also, since $F_n[\psi_n] = \eta_n$, for $n \geqslant 2$,

$$0 \leqslant h^{-1}[\eta_n - \eta_{n-1}] = Rh^{-1}[a_n w_n - a_{n-2} w_{n-1}] - w_n'' + [\varphi_n' + \varphi_{n-1}' - \nu] w_n + [\psi_n' + \psi_{n-1}'] \psi_{n-1}'.$$

Note that $w_n \equiv 0$ when $z \geqslant b$. Constructing a subsolution, we can show that $0 \geqslant w_n \geqslant -C[1 + a_n]$. From which, we conclude that $\psi_a = w$ is locally bounded. Hence, $\psi = \varphi_a$ is locally Lipschitz continuous; in particular, $\varphi_a(z, 0) = \psi(z, 0) = g(b)e^{\lambda(z-b)}$ for $z \leqslant b$.

4 The case $\mu \in (0, 1]$

In the sequel, we assume that $R > 0$ and $0 < \mu := R - \nu \leqslant 1$.

4.1 Construction of initial data

Let $h > 0$ be fixed. We construct $a_0 > 0$ and (φ_0, ψ_0) by the following steps:

(1) We fix the following relation:

$$\varphi_0(z) := a_0 \int_{-\infty}^{z} \psi_0(x)dx \quad \forall z \in \mathbb{R}.$$

Then $\varphi_0' = a_0\psi_0$ and the condition **(A2)** is equivalent to

$$\min\{-\psi_0' + \mu\psi_0 + a_0\psi_0^2,\ \psi_0 - g\} = 0 \ \text{ in } \mathbb{R}. \tag{4.1}$$

This leads to the following definition:

$$\psi_0(z) := \begin{cases} \dfrac{1}{a_0} \dfrac{\mu\varrho e^{\mu(z-z_0)}}{\mu + \varrho - \varrho e^{\mu(z-z_0)}} & \text{if } z \leqslant z_0, \\ g(z) & \text{if } z > z_0, \end{cases} \tag{4.2}$$

$$\varphi_0(z) := \begin{cases} \ln(\mu + \varrho) - \ln(\mu + \varrho - \varrho e^{\mu(z-z_0)}) & \text{if } z \leqslant z_0, \\ \ln(1 + \varrho/\mu) + a_0 \int_{z_0}^{z} g(x)dx & \text{if } z > z_0, \end{cases} \tag{4.3}$$

where

$$z_0 := \ln \frac{a_0 + \varrho}{a_0}, \qquad \varrho := \frac{1 - \mu + \sqrt{(1 - \mu)^2 + 4a_0}}{2}. \tag{4.4}$$

(2) Now we verify that ψ_0 defined in (4.2) satisfies (4.1).

Direct differentiation shows that ψ_0 satisfies the following:

$$\psi_0' = [\mu + a_0\psi_0]\psi_0 \ \text{ in } \ (-\infty, z_0], \qquad \psi_0(-\infty) = 0, \quad \psi_0(z_0) = \frac{\varrho}{a_0}.$$

Note that $g(z_0) = e^{z_0} - 1 = \varrho/a_0$ and ϱ is the roots of $[\mu + \varrho]\varrho = a_0 + \varrho$. Hence,

$$\psi_0(z_0) = g(z_0), \qquad \psi_0'(z_0) = [\mu + a_0 g(z_0)]g(z_0) = 1 + g(z_0) = g'(z_0).$$

Therefore, $\psi_0 \in C^{1+1}(\mathbb{R})$ and $\varphi_0 \in C^{2+1}(\mathbb{R})$. In addition, **(A3)** holds.

In $(-\infty, z_0]$, we have $\psi_0' - [\psi_0 + 1] = [\mu + a_0\psi_0]\psi_0 - [\psi_0 + 1] = [\psi_0 - g(z_0)][a_0\psi_0 + 1/g(z_0)] < 0$. Using integrating factor e^{-z}, this also implies that $\psi_0 > g$ in $(-\infty, z_0)$.

In (z_0, ∞), $\psi_0 = g$, and $-\psi_0' + \mu\psi_0 + a_0\psi_0^2 = a_0 g^2 + (\mu - 1)g - 1 = [g - g(z_0)][a_0 g + 1/g(z_0)] > 0$. Thus, ψ_0 satisfies (4.1).

Therefore, **(A2)–(A5)** are all satisfied.

(3) It remains to verify the condition **(A1)**. In $(-\infty, z_0]$, differentiating $\psi_0' = [\mu + a_0\psi_0]\psi_0$, one finds that $\psi_0'' = [\mu + 2a_0\psi_0]\psi_0' = [R - \nu + 2\varphi_0']\psi_0'$. Hence,

$$\begin{aligned}
\eta_0 &:= R\psi_0 - \psi_0'' - \nu\psi_0' + 2\varphi_0'\psi_0' + h{\psi_0'}^2 \\
&= R\psi_0 - R\psi_0' + h{\psi_0'}^2 \\
&= \psi_0\{R(1 - \mu - a_0\psi_0) + h(\mu + a_0\psi_0)^2\psi_0\}.
\end{aligned}$$

One finds that if $0 < Ra_0 \leqslant h\mu^2$, then $\eta_0 > 0$ in $(-\infty, z_0]$.

For later application, we use a different a_0 for the case $\mu \in (0, 1)$. We consider two scenarios:

(1) If $\mu + a_0\psi_0(z) < 1$, we have $\eta_0(z) > 0$.

(2) If $\mu + a_0\psi_0(z) \geqslant 1$, we have

$$\begin{aligned}
1 - \mu - a_0\psi_0 &\geqslant 1 - \mu - a_0 g(z_0) = 1 - \mu - \varrho \\
&= -\frac{2a_0}{\sqrt{(1-\mu)^2 + 4a_0} + (1-\mu)} \geqslant -\frac{a_0}{1-\mu}, \\
h(\mu + a\psi_0)^2\psi_0 &\geqslant h\psi_0 \geqslant \frac{h(1-\mu)}{a_0}.
\end{aligned}$$

Thus, $\eta_0(z) \geqslant 0$ if $Ra_0^2 \leqslant h(1-\mu)^2$.

In conclusion, $\eta_0 \geqslant 0$ in $(-\infty, z_0]$ if we take

$$a_0 := \begin{cases} R^{-1}h & \text{if } \ \mu = 1, \\ (1-\mu)\sqrt{h/R} & \text{if } \ \mu \in (0, 1). \end{cases} \tag{4.5}$$

In $[z_0, \infty)$, $\psi_0 = g$, and $\varphi_0' = a_0\psi_0 = a_0 g$. Using $g' = g'' = g + 1$, we obtain

$$\begin{aligned} \eta_0 &= R\psi_0 - \psi_0'' - \nu\psi_0' + 2\varphi_0'\psi_0' + h\psi_0'^2 \\ &= 2a_0 g^2 + (2a_0 + \mu - 1)g - 1 - \nu + hg'^2. \end{aligned}$$

Then $\eta_0' = [4a_0 g + 2a_0 + \mu - 1 + 2hg'']g' > 0$ since $a_0 g(z) \geqslant a_0 g(z_0) = \varrho > 1 - \mu$. Also, $\eta_0(z_0^+) - \eta_0(z_0) = \psi_0''(z_0) - g''(z_0) \geqslant 0$, since $\psi_0 \geqslant g$ in $(-\infty, z_0]$, $\psi_0(z_0) = g(z_0)$, and $\psi_0'(z_0) = g'(z_0)$. Thus, $\eta_0(z) \geqslant \eta_0(z_0^+) \geqslant \eta_0(z_0) \geqslant 0$.

Hence, $\eta_0 \geqslant 0$ in $\mathbb{R}$, so **(A1)** holds.

In conclusion, we have the following:

Lemma 4.1 *Assume that $R > 0$ and $0 < \mu := R - \nu \leqslant 1$. For each fixed $h > 0$, define a_0 as in* (4.5) *and (φ_0, ψ_0) as in* (4.2)–(4.4). *Then $(a_0, h, \varphi_0, \psi_0)$ satisfies* **(A1)–(A5)**.

To end this subsection, we give some remarks on the special case $\mu = 1$ as follows. For each $h > 0$, we take $a_0 = h/R$ and

$$\psi_0(z) = \begin{cases} \dfrac{e^z}{[1+\sqrt{a_0}]^2 - a_0 e^z}, & \text{if } z \leqslant z_0 := \ln[1 + \frac{1}{\sqrt{a_0}}], \\ g(z) & \text{if } z > z_0, \end{cases}$$

$$\varphi_0(z) = \begin{cases} -\ln\Big(1 - \dfrac{a_0 e^z}{[1+\sqrt{a_0}]^2}\Big), & \text{if } z \leqslant z_0, \\ \ln(1+\sqrt{a_0}) + a_0 \int_{z_0}^{z} g(x)dx & \text{if } z > z_0. \end{cases}$$

Then $\psi_0 \in C^{1+1}(\mathbb{R})$, $\varphi_0' = a_0\psi_0$, $\varphi_0 \in C^{2+1}(\mathbb{R})$, and

$$\frac{e^z}{[1+\sqrt{a_0}]^2} < \psi_0(z) \leqslant e^z, \qquad \frac{a_0 e^z}{[1+\sqrt{a_0}]^2} < \varphi_0(z) < a_0 e^z \quad \forall z \in \mathbb{R}.$$

As shown in this subsection, such a choice satisfies **(A1)–(A5)**. Since ψ_0 is bounded (independent of h and a_0), we can proceed the same proof as that in the case $\mu > 1$ to show the existence of a solution. We omit the details.

4.2 A few properties of φ_0

Lemma 4.2 *Let $a_0 \in (0, \mu/3]$ and φ_0 be defined as in* (4.3) *and* (4.4). *Then for each $z \in \mathbb{R}$ and $a > 0$,*

$$-a_0 \leqslant \varphi_0'(z) - \varphi_0''(z) \leqslant \tfrac{1}{4}(1-\mu)^2, \tag{4.6}$$

$$\frac{\varrho \max\{(ae^z)^\mu, ae^z\}}{(a_0+\varrho)^\mu(\mu+\varrho)} \leqslant \varphi_0\Big(z + \ln\tfrac{a}{a_0}\Big) \leqslant \mu^{-1} \max\{(ae^z)^\mu,\ ae^z\}, \tag{4.7}$$

$$\Phi(z,a) := \lim_{a_0 \searrow 0} \varphi_0\Big(z + \ln\frac{a}{a_0}\Big) = \begin{cases} -\ln(1 - [1-\mu]^{1-\mu}[ae^z]^\mu) & \text{if } ae^z < 1-\mu, \\ ae^z - \ln\mu - (1-\mu) & \text{if } ae^z \geqslant 1-\mu. \end{cases} \tag{4.8}$$

In addition, when $z \leqslant z_0$, $\varphi_0'(z) - \varphi_0(z) - a_0 \leqslant 0$.

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Proof

(i) When $z > z_0$, we have $\varphi_0'' = a_0\psi_0' = a_0 g' = a_0 g + a_0 = \varphi_0' + a_0$, so $\varphi_0' - \varphi_0'' = -a_0$.

When $z \leqslant z_0$, we have $\varphi_0'' = a_0\psi_0' = a_0[\mu + a\psi_0]\psi_0 = [\mu + \varphi_0']\varphi_0'$ so $\varphi_0' - \varphi_0'' = [1 - \mu - \varphi_0']\varphi_0'$. Since $\varphi_0'' > 0$ and $\varphi_0'(z_0) = a_0 g(z_0) = \varrho > 1 - \mu$, we find that

$$\max_{z\in(-\infty,z_0]}\{\varphi_0'(z) - \varphi_0''(z)\} = \max_{s\in[0,\varrho]}[1-\mu-s]s = \frac{(1-\mu)^2}{4},$$
$$\min_{z\in(-\infty,z_0]}\{\varphi_0'(z) - \varphi_0''(z)\} = \min_{s\in[0,\varrho]}[1-\mu-s]s = [1-\mu-\varrho]\varrho = -a_0,$$

by the definition of ϱ. Hence, (4.6) holds.

(ii) When $0 < a_0 \leqslant \mu$, one sees from the definition of ϱ that $\sqrt{a_0} \leqslant \varrho \leqslant 1$.

When $z \leqslant z_0$, by (4.2),

$$\varphi_0'(z) = a_0\psi_0(z) \leqslant \varrho e^{\mu(z-z_0)} = \varrho^{1-\mu} e^{\mu(z+\ln a_0)+\mu[\ln\varrho - \ln(a_0+\varrho)]} \leqslant e^{\mu[z+\ln a_0]}.$$

When $z > z_0$, we have $\varphi_0'(z) = a_0\psi_0(z) = a_0 g(z) \leqslant a_0 e^z = e^{z+\ln a_0}$. Hence, for every $x \in \mathbb{R}$,

$$\varphi_0(x - \ln a_0) \leqslant \int_{-\infty}^{x} \max\{e^{\mu y}, e^y\}dy \leqslant \mu^{-1}\max\{e^{\mu x}, e^x\}.$$

Set $x = z + \ln a$, we obtain upper bound in (4.7).

(iii) When $a_0 \in (0, \mu/3]$, one can check that $a_0 + \varrho < 1$.

When $z < z_0$, by (4.2),

$$\varphi_0'(z) = a\psi_0(z) \geqslant \frac{\mu\varrho}{\mu+\varrho}e^{\mu(z-z_0)}, \qquad \varphi_0(z) = \int_{-\infty}^{z}\varphi_0'(y)dy > \frac{\varrho e^{\mu(z-z_0)}}{\mu+\varrho}.$$

When $z > z_0$, $\varphi_0'(z) = a_0(e^z - 1) = [a_0 + \varrho]e^{z-z_0} - a_0 \geqslant \varrho e^{z-z_0}$. Since $\mu + \varrho > 1$,

$$\varphi_0(z) = \varphi_0(z_0) + \int_{z_0}^{z}\varphi_0'(y)dy > \frac{\varrho}{\mu+\varrho} + \varrho(e^{z-z_0} - 1) \geqslant \frac{\varrho}{\mu+\varrho}e^{z-z_0}.$$

Since $\max\{e^{\mu(z-z_0)}, e^{z-z_0}\} = e^{\mu(z-z_0)}$ for $z < z_0$ and $\max\{e^{\mu(z-z_0)}, e^{z-z_0}\} = e^{z-z_0}$ for $z > z_0$ and $a_0 + \varrho \leqslant (a_0+\varrho)^\mu$, we have

$$\varphi_0(z) \geqslant \frac{\varrho}{\mu+\varrho}\max\{e^{\mu(z-z_0)}, e^{z-z_0}\} = \frac{\varrho}{\mu+\varrho}\max\left\{\frac{e^{\mu(z+\ln a_0)}}{(a_0+\varrho)^\mu}, \frac{e^{z+\ln a_0}}{a_0+\varrho}\right\}$$
$$\geqslant \frac{\varrho\max\{e^{\mu(z+\ln a_0)}, e^{z+\ln a_0}\}}{(a_0+\varrho)^\mu(\mu+\varrho)}.$$

Replacing z by $z + \ln a - \ln a_0$, we obtain the lower bound in (4.7).

(iv) The limit (4.8) follows by direct computation.

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(v) When $z \leqslant z_0$, using the upper bound of φ_0' in (ii) and the lower bound of φ_0 in (iii), we obtain

$$\varphi_0'(z) - \varphi_0(z) \leqslant \varrho e^{\mu(z-z_0)} - \frac{\varrho}{\mu+\varrho} e^{\mu(z-z_0)} \leqslant \frac{\varrho[\varrho+\mu-1]}{\mu+\varrho} = \frac{a_0}{\mu+\varrho} < a_0,$$

since $\varrho[\varrho+\mu-1] = a_0$ and $\mu + \varrho > 1$. This completes the proof of the lemma.

□

4.3 A comparison principle

When $0 < \mu < 1$, the constructed initial data ψ_0 approaches ∞ as $a_0 \searrow 0$; as a consequence, the *a priori* estimates used in Section 2 cannot be used in the $h \searrow 0$ limit process. We need new *a priori* estimates. We shall first establish a comparison principle, and then construct sub/supersolutions to find upper and lower bounds.

Lemma 4.3 *Let* $\{\varphi_i\}_{i=0}^{\infty}$ *be the solution of* (1.12). *Assume that* $\{\phi_i\}_{i=1}^{\infty}$ *is a sequence such that* $\phi_0 \geqslant (\leqslant)\, \varphi_0$ *and for every* $n \geqslant 1$,

$$\hat{\mathcal{A}}_n[\phi_n] := \min\{Ra_n h^{-1}[\phi_n - \phi_{n-1}] - \phi_n'' - \nu\phi_n' + \phi_n'^2,\ h^{-1}[\phi_n - \phi_{n-1}] - g\} \geqslant\ (\leqslant)\ 0.$$

Then $\phi_n \geqslant (\leqslant)\, \varphi_n$ *for every* n.

Proof Consider supersolution case. Suppose $n \geqslant 1$ and $\varphi_{n-1} \leqslant \phi_{n-1}$. Then

$$\begin{aligned}\min\{A_n[\varphi_n], B_n[\varphi_n]\} &= 0 \leqslant \hat{\mathcal{A}}_n[\phi_n] \\ &= \min\{Ra_n h^{-1}[\phi_n - \phi_{n-1}] - \phi_n'' - \nu\phi_n' + \phi_n'^2,\ h^{-1}[\phi_n - \phi_{n-1}] - g\} \\ &\leqslant \min\{Ra_n h^{-1}[\phi_n - \varphi_{n-1}] - \phi_n'' - \nu\phi_n' + \phi_n'^2,\ h^{-1}[\phi_n - \varphi_{n-1}] - g\} \\ &= \min\{A_n[\phi_n], B_n[\phi_n]\}.\end{aligned}$$

It then follows from comparison for variational inequality that $\varphi_n \leqslant \phi_n$. Hence, by mathematical induction, $\varphi_n \leqslant \phi_n$ for all $n \geqslant 1$. The proof for subsolution is similar. □

Now we consider the sequence $\{\phi_n\}_{n=0}^{\infty}$ defined by

$$\phi_n(z) = \varphi_0(z + \ln a_n - \ln a_0) + h[1 - a_0 a_n^{-1}].$$

Set $x = z - \ln a_0$. We can calculate, for $n \geqslant 1$,

$$\begin{aligned}\frac{\phi_n(z) - \phi_{n-1}(z)}{h} &= \int_{a_{n-1}}^{a_n} \varphi_0'(x + \ln a)\frac{d(a - a_{n-1})}{ha} + \int_{a_{n-1}}^{a_n} \frac{a_0}{a^2} da \\ &= \frac{\varphi_0'(x + \ln a_n)}{a_n} + \int_{a_{n-1}}^{a_n} \Big(\frac{a - a_{n-1}}{ha^2}[\varphi_0' - \varphi_0''] + \frac{a_0}{a^2}\Big) da \\ &> \frac{\varphi_0'(z + \ln a_n - \ln a_0)}{a_n},\end{aligned}$$

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since (4.6) $\varphi_0' - \varphi_0'' \geqslant -a_0$. Hence, setting $y = z + \ln a_n - \ln a_0$,

$$\begin{aligned}
&Ra_n h^{-1}[\phi_n - \phi_{n-1}] - \phi_n'' - \nu\phi_n' + {\phi_n'}^2 \\
&\qquad \geqslant R\varphi_0'(y) - \varphi_0''(y) - \nu\varphi_0'(y) + {\varphi_0'}^2(y) = a_0\{-\psi_0' + \mu\psi_0 + a_0\psi_0^2\} \geqslant 0, \\
&h^{-1}[\phi_n - \phi_{n-1}] - g(z) \geqslant \frac{\varphi_0'(y)}{a_n} - e^z + 1 = \frac{a_0}{a_n}\Big(\psi_0(y) - e^y + \frac{a_n}{a_0}\Big) \geqslant 0.
\end{aligned}$$

Thus, $\hat{\mathcal{A}}_n[\phi_n] \geqslant 0$ for all $n \geqslant 1$. It is easy to see that $\varphi_0 = \phi_0$. Hence, $\varphi_n \leqslant \phi_n$.

Similarly, we can show that the following is a subsolution:

$$\phi_n(z) = \varphi_0(z + \ln a_n - \ln a_0) - \frac{(1-\mu)^2}{4}\Big(\frac{h}{a_0} - \frac{h}{a_n}\Big) - (a_n - a_0).$$

Hence, we have the following:

Lemma 4.4 *Assume that $a_0 \in (0, \mu/3]$. Then for every integer $n \geqslant 0$,*

$$h\Big[1 - \frac{a_0}{a_n}\Big] \geqslant \varphi_n(z) - \varphi_0\Big(z + \ln\frac{a_n}{a_0}\Big) \geqslant -\frac{(1-\mu)^2}{4}\Big(\frac{h}{a_0} - \frac{h}{a_n}\Big) - [a_n - a_0]. \tag{4.9}$$

In particular, using Lemma 4.2, we obtain

$$\begin{aligned}
\varphi_n(z) &\leqslant \mu^{-1}\max\{(a_n e^z)^\mu, a_n e^z\} + h, \\
\varphi_n(z) &\geqslant \frac{\varrho\max\{(a_n e^z)^\mu, a_n e^z\}}{(a_0+\varrho)^\mu(\mu+\varrho)} - \frac{(1-\mu)^2}{4}\Big(\frac{h}{a_n} - \frac{h}{a_0}\Big) - [a_n - a_0].
\end{aligned}$$

4.4 Upper and lower bounds of free boundary

Upper Bound. For each $n \geqslant 1$, in $(-\infty, z_n)$,

$$\begin{aligned}
\varphi_n'' &= {\varphi_n'}^2 - \nu\varphi_n + Ra_n\psi_n \geqslant {\varphi_n'}^2 - \frac{{\varphi_n'}^2 + \nu^2}{2} + Ra_n g \\
&= \frac{\varphi_n'^2 + 1}{2} + Ra_n\Big(e^z - 1 - \frac{1+\nu^2}{2Ra_n}\Big).
\end{aligned}$$

If $z_n > Z_n := \ln[1 + \frac{1+\nu^2}{2Ra_n}]$, then in $[Z_n, z_n]$, we have $2\varphi_n'' \geqslant 1 + \varphi_n'^2$, i.e., $1 \leqslant 2\varphi_n''/(1+\varphi_n'^2)$. After integrating over $[Z_n, z_n]$, we find that $z_n - Z_n \leqslant 2[\arctan\varphi_n'(z_n) - \arctan\varphi_n'(Z_n)] \leqslant \pi$.

Lower bound. In $[z_n, \infty)$, $0 \leqslant \eta_n \leqslant l_n$, so

$$0 \leqslant [\varphi_n' + \varphi_{n-1}' - (1-\mu)]e^z - R. \tag{4.10}$$

For $z \in \mathbb{R}$,

$$\varphi_n' = \varphi_0' + \sum_{i=1}^n h\psi_i' \leqslant \varphi_0' + \sum_{i=1}^n h[\psi_i + 1] = \varphi_0' + [\varphi_n - \varphi_0] + [a_n - a_0].$$

When $z \leqslant z_0$, by Lemma 4.2, $\varphi_0' - \varphi_0 - a_0 \leqslant 0$. Hence, when $0 < z \leqslant z_0$ (and $a_n e^z \leqslant 1$ if $\mu \in (0,1)$),

$$\varphi_n' + \varphi_{n-1}' \leqslant \varphi_n + a_n + \varphi_{n-1} + a_{n-1} \leqslant \frac{a_n^\mu e^{\mu z} + a_{n-1}^\mu e^{\mu z}}{\mu} + h + 2a_n \leqslant \frac{4(a_n e^z)^\mu}{\mu};$$

here we have used $a_n \leqslant a_n e^z \leqslant (a_n e^z)^\mu$ (and $\leqslant 1$ if $\mu \in (0,1)$) and $a_{n-1}^\mu + \mu h \leqslant (a_{n-1}+h)^\mu = a_n^\mu$.

(i) If $\mu \in (0,1)$, then (4.10) gives $\varphi_n'(z_n) + \varphi_{n-1}'(z_n) \geqslant 1-\mu$, so $z_n \geqslant \ln[(\mu-\mu^2)/4]^{1/\mu} - \ln a_n$.

(ii) If $\mu = 1$, then (4.10) gives $e^z[\varphi_n'(z_n) + \varphi_{n-1}'(z_n)] \geqslant R$, so $z_n \geqslant \frac{1}{2}\ln\frac{R}{4a_n}$.

Hence, we have the following:

Lemma 4.5 *For every $n \geqslant 1$,*

$$\pi + \ln\Big(1 + \frac{1+\nu^2}{2Ra_n}\Big) \geqslant z_n \geqslant \begin{cases} \ln[(\mu-\mu^2)/4]^{1/\mu} - \ln a_n & \textit{if } \mu \in (0,1), \\ \ln(\sqrt{R}/2) - \ln\sqrt{a_n} & \textit{if } \mu = 1. \end{cases}$$

4.5 L^∞ estimates

We start from the basic estimate:

$$0 \leqslant \varphi_n \leqslant \mu^{-1}\max\{[a_n e^z]^\mu, a_n e^z\} + h.$$

Using $\varphi_n - \varphi_0 = \sum_{i=1}^n h\psi_i \geqslant [a_n - a_0]\psi_n$, we obtain, for each $n \geqslant 1$,

$$\begin{aligned} &0 \leqslant \psi_n \leqslant \frac{\varphi_n - \varphi_0}{a_n - a_0} \leqslant \frac{\varphi_n}{a_n - a_0}, \\ &0 \leqslant \psi_n' \leqslant \psi_n + 1, \\ &0 \leqslant \varphi_n' = \varphi_0' + \sum_{i=1}^n h\psi_n' \leqslant \varphi_0' + \sum_{i=1}^n h[\psi_i + 1] \\ &\qquad = \varphi_0' + \varphi_n - \varphi_0 + a_n - a_0 \leqslant \varphi_n + a_n \quad \text{(if } z \leqslant z_0\text{, by Lemma 4.2)}, \\ &0 \leqslant \varphi_n'' \leqslant Ra_n\psi_n - \nu\varphi_n' + \varphi_n'^2. \end{aligned}$$

Finally, we derive from (2.8) that for $n > m \geqslant 1$ and $\beta = \min\{\frac{1}{2}, \mu\} \geqslant \frac{\mu}{2}$,

$$\frac{|a_n\psi_n - a_m\psi_m|}{(a_n - a_m)^\beta} \leqslant \begin{cases} Ca_n^{1-\beta}[1+\varphi_n']\Big(1 + \frac{\varphi_m}{a_m - a_0}\Big) & \text{if } a_n - a_m \leqslant \frac{a_n}{2}, \\ 2^\beta\Big(\frac{a_n^{1-\beta}\varphi_n}{a_n - a_0} + \frac{a_m^{1-\beta}\varphi_m}{a_m - a_0}\Big) & \text{if } a_n - a_m \geqslant \frac{a_n}{2}. \end{cases}$$

4.6 Proof of Theorem 1

(1) Define $s^h = \sum_{i=1}^\infty z_i \mathbf{1}_{[a_{i-1}, a_i)}(a)$. Then along a sequence $h \searrow 0$,

$$s^h(a) \longrightarrow s(a) \quad \forall\, a > 0.$$

In addition, s is decreasing and satisfies (1.11).

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(2) Define φ^h as in §3.2. Then along a (sub)sequence of $h \searrow 0$, we have

$$(\varphi^h, \varphi^h_z, a\varphi^h_a) \longrightarrow (\varphi, \varphi_z, a\varphi_a) \text{ in } C^{\alpha,\alpha/2}([-T,T]\times[0,T]) \quad \forall\, T>0, \alpha\in(0,\mu).$$

The limit φ satisfies the estimates listed in (1.9).

(3) When $\mu \in (0,1)$, we have (1.10) since the lower bound in Lemma 4.4 gives us

$$\varphi(z,a) \geqslant (1-\mu)^{1-\mu}(ae^z)^\mu - a.$$

When $\mu = 1$, the upper bound $\varphi(a,z) \leqslant ae^z$ gives us $\overline{\lim}_{a\searrow 0} a^{-1}\varphi(a,z) \leqslant e^z$. Since the solution is increasing in ν (which is a scaled growth rate of stock price), comparing the solution with $\tilde{\varphi}$ associated with $\tilde{\nu}$ and using Remark 3.1 or results in [13] for the case $\tilde{\nu} < \nu = R-1$, we have

$$\underline{\lim}_{a\searrow 0} \frac{\varphi(z,a)}{a} \geqslant \lim_{a\searrow 0} \frac{\tilde{\varphi}(z,a)}{a} = \tilde{\varphi}_a(z,0) = g(b)e^{\lambda(z-b)} = e^{\lambda z}[1-e^{-b}]e^{[1-\lambda]b},$$

for $z < b$, where (b,λ) is as in (3.1) with ν replaced by $\tilde{\nu}$. Sending $\tilde{\nu} \nearrow \nu = R-1$, we obtain $b \to \infty$ and $\underline{\lim}_{a\searrow 0} a^{-1}\varphi(z,a) \geqslant e^z$. Hence, we obtain (1.10).

(4) Following the same proof as that of Theorem 2 (Sections 3.2 and 3.3) and/or the techniques in [10] (for continuity and strictly decreasing of s and (1.5)), we complete the proof of Theorem 1.

Remark 4.1

(i) From (4.9), (4.8), and (4.5), we derive that

$$0 \leqslant \Phi(z,a) - \varphi(z,a) \leqslant a \qquad \forall\,(z,a) \in \mathbb{R}\times(0,\infty). \tag{4.11}$$

Here, Φ is the solution of the original problem with $K=0$, i.e., with $g(z)=e^z$; see [14].

(ii) Once we have the tight estimate near $a=0$, uniqueness of solution in the class satisfying (4.11) follows from the standard technique illustrated in [10]. Indeed, if $\tilde{\varphi}$ is another solution, one can compare $\tilde{u} = -e^{-\tilde{\varphi}}$ with $u = -e^{-\varphi} \pm \varepsilon[e^a + ae^{\lambda_+ z} + ae^{\lambda_- z}]$ to show that $\tilde{u} = -e^{-\varphi}$; here $\lambda_\pm$ are the two roots of $\lambda^2 + \nu\lambda = R$.

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