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ON ADDITIVE REPRESENTATION FUNCTIONS

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Abstract

For any finite abelian group G with $|G| = m$, $A \subseteq G$ and $g \in G$, let $R_A(g)$ be the number of solutions of the equation $g = a + b$, $a, b \in A$. Recently, Sándor and Yang ['A lower bound of Ruzsa's number related to the Erdős–Turán conjecture', Preprint, 2016, arXiv:1612.08722v1] proved that, if $m \geq 36$ and $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, then there exists $n \in \mathbb{Z}_m$ such that $R_A(n) \geq 6$. In this paper, for any finite abelian group G with $|G| = m$ and $A \subseteq G$, we prove that (a) if the number of $g \in G$ with $R_A(g) = 0$ does not exceed $\frac{7}{32}m - \frac{1}{2}\sqrt{10m} - 1$, then there exists $g \in G$ such that $R_A(g) \geq 6$; (b) if $1 \leq R_A(g) \leq 6$ for all $g \in G$, then the number of $g \in G$ with $R_A(g) = 6$ is more than $\frac{7}{32}m - \frac{1}{2}\sqrt{10m} - 1$.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For any set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_A(n)$ be the number of solutions of the equation $n = a + b$, $a, b \in A$. A set A is called a basis of $\mathbb{N}$ if $R_A(n) \geq 1$ for all sufficiently large integers n. The classical Erdős–Turán conjecture [7] says that if A is a basis of $\mathbb{N}$, then $R_A(n)$ cannot be bounded. In 2003, Grekos *et al.* [8] proved that, if A is a basis of $\mathbb{N}$, then $R_A(n) \geq 6$ for infinitely many integers n. In 2006, Borwein *et al.* [1] improved the lower bound 6 to 8. In 2013, Konstantoulas [9] proved that, if the upper density of the set of numbers not represented as sums of two elements of A is less than $1/10$, then $R_A(n) \geq 6$ for infinitely many integers n. Nathanson [11] proved that the Erdős–Turán conjecture is wrong in the set $\mathbb{Z}$ of all integers. In 2012, Chen [3] proved that there exists a basis A of $\mathbb{N}$ such that the set of n with $R_A(n) = 2$ has density one. For related results, one may refer to [5, 6] and [14].

For any abelian group G, $A, B \subseteq G$ and $g \in G$, let

$$R_{A,B}(g) = |\{(a, b) : n = a + b, a \in A, b \in B\}|$$

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and let $R_A(g) = R_{A,A}(g)$. A subset A of G is called an additive basis of G if $R_A(g) \geq 1$ for all $g \in G$. Ruzsa [12] proved that, for any positive integer m, there exist an additive basis A of $\mathbb{Z}_m$ and an absolute constant c such that $R_A(n) \leq c$ for all $n \in \mathbb{Z}_m$. Let R_m be the least positive integer r for which there exists an additive basis A of $\mathbb{Z}_m$ with $R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. The numbers R_m are called Ruzsa's numbers. Chen [2] proved that $R_m \leq 288$ for all positive integers m. Tang and Chen [15] obtained better upper bounds for some types of m.

Recently, Sándor and Yang [13] proved that $R_m \geq 6$ for all $m \geq 36$. That is, if $m \geq 36$ and $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, then there exists $n \in \mathbb{Z}_m$ such that $R_A(n) \geq 6$.

In this paper, the following results are proved.

Theorem 1.1. *Let G be a finite abelian group with $|G| = m$, $A \subseteq G$ and let c be an integer with $c \geq 6$. Suppose that $0 \leq R_A(g) \leq c$ for all $g \in G$. Then*

$$|S_0| + \sum_{u\geq 6} |S_u| > \frac{1}{4(c-3)^2 - 4}(7m - 16\sqrt{10m} - 27)$$

and

$$3|S_0| + |S_2| + \sum_{u\geq 8} |S_u| > \frac{1}{4\max\{(c-5)^2 - 1, 8\}}(15m - 48\sqrt{18m} - 112),$$

where

$$S_u = \{g : g \in G, R_A(g) = u\}.$$

In view of Theorem 1.1 with $c = 6$, we have the following corollaries immediately.

Corollary 1.2. *Let G be a finite abelian group with $|G| = m$ and let A be a subset of G. If $1 \leq R_A(g) \leq 6$ for all $g \in G$, then*

$$|\{g : g \in G, R_A(g) = 6\}| > \tfrac{7}{32}m - \tfrac{1}{2}\sqrt{10m} - 1$$

and

$$|\{g : g \in G, R_A(g) = 2\}| > \tfrac{15}{32}m - \tfrac{3}{2}\sqrt{18m} - \tfrac{7}{2}.$$

Corollary 1.3. *Let G be a finite abelian group with $|G| = m$ and $A \subseteq G$. If*

$$|\{g : g \in G, R_A(g) = 0\}| \leq \tfrac{7}{32}m - \tfrac{1}{2}\sqrt{10m} - 1,$$

then there exists $g \in G$ such that $R_A(g) \geq 6$.

Remark 1.4. It is known that there are infinitely many positive integers m for which there exists $A \subseteq \mathbb{Z}_m$ with $1 \leq R_A(n) \leq 6$ for all $n \in \mathbb{Z}_m \setminus \{0\}$ (see [4, Remark 3]).

2. Proofs

Firstly we prove the following lemmas.

LEMMA 2.1. *Let G be a finite abelian group with $|G| = m > 1$ and let k be a real number. Then, for any $A \subseteq G$,*

$$\sum_{g\in G}(R_A(g) - k)^2 \geq \frac{1}{m-1}(|A|^2 - |A|)^2 - (2k-1)|A|^2 + k^2 m.$$

PROOF. We follow the proof of [10, Theorem 1]. Noting that

$$\sum_{g\in G} R_A(g) = |A|^2, \quad \sum_{g\in G} R_A(g)^2 = \sum_{g\in G} R_{A,-A}(g)^2,$$

it follows from Cauchy's inequality that

$$\begin{aligned}
\sum_{g\in G}(R_A(g) - k)^2 &= \sum_{g\in G} R_A(g)^2 - 2k\sum_{g\in G} R_A(g) + k^2 m \\
&= \sum_{g\in G} R_{A,-A}(g)^2 - 2k|A|^2 + k^2 m \\
&= \sum_{g\in G\setminus\{0\}} R_{A,-A}(g)^2 - (2k-1)|A|^2 + k^2 m \\
&\geq \frac{1}{m-1}\Big(\sum_{g\in G\setminus\{0\}} R_{A,-A}(g)\Big)^2 - (2k-1)|A|^2 + k^2 m \\
&= \frac{1}{m-1}(|A|^2 - |A|)^2 - (2k-1)|A|^2 + k^2 m.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *Let a and b be two positive real numbers and let f be the function defined by $f(x) = x^4 - 2x^3 - ax^2 - bx$ for $x \geq 0$. Then*

$$\begin{aligned}
f(x) \geq -\frac{1}{4}a^2 - \frac{\sqrt{2}}{2}a\sqrt{a} - \frac{\sqrt{2}}{2}b\sqrt{a} - \frac{15}{8}a - \frac{7}{4}b - \frac{3}{8}\frac{b^2}{a} \\
- \frac{9\sqrt{2}}{16}\sqrt{a} - \frac{3\sqrt{2}}{16}\frac{b}{\sqrt{a}} - \frac{9}{16}\frac{b}{a} - \frac{27}{16}.
\end{aligned}$$

PROOF. Note that

$$f'(x) = 4x^3 - 6x^2 - 2ax - b \quad \text{and} \quad f''(x) = 12x^2 - 12x - 2a.$$

The polynomial $f''(x)$ has two roots:

$$x_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{2a}{3}}, \quad x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2a}{3}}.$$

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Since $a > 0$, we have $x_1 < 0 < 1 < x_2$. It follows that $f''(x) < 0$ for $0 \le x < x_2$ and $f''(x) > 0$ for $x > x_2$. So, $f'(x)$ is decreasing for $0 \le x \le x_2$ and increasing for $x \ge x_2$. Since $f'(0) = -b < 0$, it follows that $f'(x) < 0$ for $0 \le x \le x_2$. In particular, $f'(x_2) < 0$. Noting that $f'(x) \to +\infty$ as $x \to +\infty$, there exists a unique real number x_3 with $x_2 < x_3 < +\infty$ such that $f'(x_3) = 0$. Now we have $f'(x) < 0$ for $0 \le x < x_3$ and $f'(x) > 0$ for $x > x_3$. Hence, $f(x) \ge f(x_3)$ for all $x \ge 0$. From $f'(x_3) = 0$,

$$x_3^3 = \frac{3}{2}x_3^2 + \frac{a}{2}x_3 + \frac{b}{4}. \tag{2.1}$$

It follows that

$$x_3^4 = \frac{3}{2}x_3^3 + \frac{a}{2}x_3^2 + \frac{b}{4}x_3, \quad x_3^2 = \frac{3}{2}x_3 + \frac{a}{2} + \frac{b}{4x_3}, \quad x_3 = \frac{3}{2} + \frac{a}{2x_3} + \frac{b}{4x_3^2}. \tag{2.2}$$

In view of (2.2) and $x_3 > 0$, we have $x_3 > \sqrt{a/2}$. By (2.1) and (2.2),

$$\begin{aligned}
f(x_3) &= x_3^4 - 2x_3^3 - ax_3^2 - bx_3 \\
&= \frac{3}{2}x_3^3 + \frac{a}{2}x_3^2 + \frac{b}{4}x_3 - 2x_3^3 - ax_3^2 - bx_3 \\
&= -\frac{1}{2}\Big(\frac{3}{2}x_3^2 + \frac{a}{2}x_3 + \frac{b}{4}\Big) - \frac{a}{2}x_3^2 - \frac{3}{4}bx_3 \\
&= -\frac{1}{4}(3+2a)x_3^2 - \frac{1}{4}(a+3b)x_3 - \frac{1}{8}b \\
&= -\frac{1}{4}(3+2a)\Big(\frac{3}{2}x_3 + \frac{a}{2} + \frac{b}{4x_3}\Big) - \frac{1}{4}(a+3b)x_3 - \frac{1}{8}b \\
&= -\frac{1}{8}(8a+6b+9)x_3 - \frac{1}{8}(2a^2+3a+b) - \frac{1}{16x_3}(3+2a)b \\
&= -\frac{1}{8}(8a+6b+9)\Big(\frac{3}{2} + \frac{a}{2x_3} + \frac{b}{4x_3^2}\Big) - \frac{1}{8}(2a^2+3a+b) - \frac{1}{16x_3}(3+2a)b \\
&= -\frac{1}{16}(4a^2+30a+20b+27) - \frac{1}{16x_3}(8a^2+8ab+9a+3b) \\
&\quad - \frac{1}{32x_3^2}(8ab+6b^2+9b) \\
&\ge -\frac{1}{16}(4a^2+30a+20b+27) - \frac{\sqrt{2}}{16\sqrt{a}}(8a^2+8ab+9a+3b) \\
&\quad - \frac{1}{16a}(8ab+6b^2+9b) \\
&= -\frac{1}{4}a^2 - \frac{\sqrt{2}}{2}a\sqrt{a} - \frac{\sqrt{2}}{2}b\sqrt{a} - \frac{15}{8}a - \frac{7}{4}b - \frac{3}{8}\frac{b^2}{a} \\
&\quad - \frac{9\sqrt{2}}{16}\sqrt{a} - \frac{3\sqrt{2}}{16}\frac{b}{\sqrt{a}} - \frac{9}{16}\frac{b}{a} - \frac{27}{16}.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

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Proof of Theorem 1.1. We begin by proving the first inequality in Theorem 1.1. For $m \le 59$, Theorem 1.1 is trivial since

$$7m - 16\sqrt{10m} - 27 < 0.$$

So, we assume that $m \ge 60$. If $R_A(g)$ is odd, then $g = 2a$ for some $a \in A$. It follows that

$$\sum_{\substack{g\in G\\ R_A(g) \text{ is odd}}} 1 \le |A|.$$

By Lemma 2.1 for $k = 3$,

$$\sum_{g\in G}(R_A(g)-3)^2 \ge \frac{1}{m-1}(|A|^2-|A|)^2 - 5|A|^2 + 9m.$$

Since

$$\begin{aligned}\sum_{g\in G}(R_A(g)-3)^2 &= \sum_{\substack{g\in G\\ |R_A(g)-3|\ge 3}}(R_A(g)-3)^2 + \sum_{\substack{g\in G\\ R_A(g)=2,4}} 1 + \sum_{\substack{g\in G\\ R_A(g)=1,5}} 4\\ &\le \sum_{\substack{g\in G\\ |R_A(g)-3|\ge 3}}((R_A(g)-3)^2-1) + \sum_{g\in G} 1 + \sum_{\substack{g\in G\\ R_A(g)=1,5}} 3\\ &\le \sum_{\substack{g\in G\\ |R_A(g)-3|\ge 3}}((R_A(g)-3)^2-1) + \sum_{g\in G} 1 + \sum_{\substack{g\in G\\ R_A(g) \text{ is odd}}} 3\\ &\le m + 3|A| + \sum_{\substack{g\in G\\ |R_A(g)-3|\ge 3}}((R_A(g)-3)^2-1),\end{aligned}$$

it follows that

$$\begin{aligned}\sum_{\substack{g\in G\\ |R_A(g)-3|\ge 3}}((R_A(g)-3)^2-1) &\ge \frac{1}{m-1}(|A|^2-|A|)^2 - 5|A|^2 - 3|A| + 8m\\ &= \frac{1}{m-1}(|A|^4 - 2|A|^3 - (5m-6)|A|^2 - (3m-3)|A|) + 8m.\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}&|A|^4 - 2|A|^3 - (5m-6)|A|^2 - (3m-3)|A|\\ &\ge -\frac{1}{4}(5m-6)^2 - \frac{\sqrt{2}}{2}(5m-6)\sqrt{5m-6} - \frac{\sqrt{2}}{2}(3m-3)\sqrt{5m-6}\\ &\quad - \frac{15}{8}(5m-6) - \frac{7}{4}(3m-3) - \frac{3}{8}\frac{(3m-3)^2}{5m-6}\\ &\quad - \frac{9\sqrt{2}}{16}\sqrt{5m-6} - \frac{3\sqrt{2}}{16}\frac{3m-3}{\sqrt{5m-6}} - \frac{9}{16}\frac{3m-3}{5m-6} - \frac{27}{16}.\end{aligned}$$

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Since $m \geq 60$,

$$\begin{aligned}
&\sum_{\substack{g\in G\\ |R_A(g)-3|\geq 3}} ((R_A(g)-3)^2-1)\\
&\geq -\Big(\frac{25}{4}(m-1)-\frac{5}{2}\Big)-\Big(\frac{5\sqrt{10}}{2}\sqrt{m}-\frac{\sqrt{2}}{2}\frac{\sqrt{5m-6}}{m-1}\Big)-\frac{3\sqrt{10}}{2}\sqrt{m}\\
&\quad -\Big(\frac{75}{8}-\frac{15}{8(m-1)}\Big)-\frac{21}{4}-\frac{27}{8}\frac{m-1}{5m-6}-\frac{9\sqrt{2}}{16}\frac{\sqrt{5m-6}}{m-1}\\
&\quad -\frac{9\sqrt{2}}{16}\frac{1}{\sqrt{5m-6}}-\frac{27}{16}\frac{1}{5m-6}-\frac{27}{16(m-1)}+8m\\
&=\frac{7}{4}m-4\sqrt{10m}-\frac{47}{8}-\frac{\sqrt{2}}{16}\frac{\sqrt{5m-6}}{m-1}-\frac{27}{16}\frac{2m-1}{5m-6}+\frac{3}{16(m-1)}-\frac{9\sqrt{2}}{16\sqrt{5m-6}}\\
&>\frac{7}{4}m-4\sqrt{10m}-6.7.
\end{aligned}$$

Since $0 \leq R_A(g) \leq c$ and $c \geq 6$, it follows that

$$\begin{aligned}
\sum_{\substack{g\in G\\ |R_A(g)-3|\geq 3}} ((R_A(g)-3)^2-1) &= \sum_{\substack{g\in G\\ R_A(g)=0}} ((R_A(g)-3)^2-1)+\sum_{\substack{g\in G\\ R_A(g)\geq 6}} ((R_A(g)-3)^2-1)\\
&\leq \sum_{\substack{g\in G\\ R_A(g)=0}} 8+\sum_{\substack{g\in G\\ R_A(g)\geq 6}} ((c-3)^2-1)\\
&\leq ((c-3)^2-1)\Big(\sum_{\substack{g\in G\\ R_A(g)=0}} 1+\sum_{\substack{g\in G\\ R_A(g)\geq 6}} 1\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\{g : g\in G, R_A(g)=0\}|+|\{g : g\in G, R_A(g)\geq 6\}|\\
&> \frac{1}{(c-3)^2-1}\Big(\frac{7}{4}m-4\sqrt{10m}-6.7\Big)\\
&> \frac{1}{4(c-3)^2-4}(7m-16\sqrt{10m}-27).
\end{aligned}$$

Now we prove the second inequality in Theorem 1.1. For $m \leq 197$, Theorem 1.1 is trivial since $15m - 48\sqrt{18m} - 112 < 0$. So, we assume that $m \geq 198$. By Lemma 2.1 for $k = 5$,

$$\sum_{g\in G}(R_A(g)-5)^2 \geq \frac{1}{m-1}(|A|^2-|A|)^2-9|A|^2+25m.$$

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Since

$$\begin{aligned}\sum_{g\in G}(R_A(g)-5)^2 &\le \sum_{\substack{g\in G\\ R_A(g)\ge 8}}(R_A(g)-5)^2+\sum_{\substack{g\in G\\ R_A(g)=0}}25\\ &\quad+\sum_{\substack{g\in G\\ R_A(g)=2}}9+\sum_{\substack{g\in G\\ R_A(g)=4,6}}1+\sum_{\substack{g\in G\\ R_A(g)=1,3,5,7}}16\\ &\le \sum_{\substack{g\in G\\ R_A(g)\ge 8}}((R_A(g)-5)^2-1)+\sum_{\substack{g\in G\\ R_A(g)=0}}24+\sum_{\substack{g\in G\\ R_A(g)=2}}8+\sum_{g\in G}1+\sum_{\substack{g\in G\\ R_A(g)\text{ is odd}}}15\\ &\le m+15|A|+\sum_{\substack{g\in G\\ R_A(g)\ge 8}}((R_A(g)-5)^2-1)+\sum_{\substack{g\in G\\ R_A(g)=0}}24+\sum_{\substack{g\in G\\ R_A(g)=2}}8,\end{aligned}$$

it follows that

$$\begin{aligned}&\max\{(c-5)^2-1,8\}\Big(3\sum_{\substack{g\in G\\ R_A(g)=0}}1+\sum_{\substack{g\in G\\ R_A(g)=2\text{ or }R_A(g)\ge 8}}1\Big)\\ &\ge \sum_{\substack{g\in G\\ R_A(g)\ge 8}}((R_A(g)-5)^2-1)+\sum_{\substack{g\in G\\ R_A(g)=0}}24+\sum_{\substack{g\in G\\ R_A(g)=2}}8\\ &\ge \frac{1}{m-1}(|A|^2-|A|)^2-9|A|^2-15|A|+24m\\ &=\frac{1}{m-1}(|A|^4-2|A|^3-(9m-10)|A|^2-15(m-1)|A|)+24m.\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}&|A|^4-2|A|^3-(9m-10)|A|^2-15(m-1)|A|\\ &\ge -\frac{1}{4}(9m-10)^2-\frac{\sqrt{2}}{2}(9m-10)\sqrt{9m-10}-\frac{\sqrt{2}}{2}(15m-15)\sqrt{9m-10}\\ &\quad-\frac{15}{8}(9m-10)-\frac{7}{4}(15m-15)-\frac{3}{8}\frac{(15m-15)^2}{9m-10}\\ &\quad-\frac{9\sqrt{2}}{16}\sqrt{9m-10}-\frac{3\sqrt{2}}{16}\frac{15m-15}{\sqrt{9m-10}}-\frac{9}{16}\frac{15m-15}{9m-10}-\frac{27}{16}.\end{aligned}$$

Since $m\ge 198$,

$$\begin{aligned}&\max\{(c-5)^2-1,8\}\Big(3\sum_{\substack{g\in G\\ R_A(g)=0}}1+\sum_{\substack{g\in G\\ R_A(g)=2\text{ or }R_A(g)\ge 8}}1\Big)\\ &\ge -\Big(\frac{81}{4}(m-1)-\frac{9}{2}\Big)-\Big(\frac{9\sqrt{18}}{2}\sqrt{m}-\frac{\sqrt{2}}{2}\frac{\sqrt{9m-10}}{m-1}\Big)-\frac{15\sqrt{18}}{2}\sqrt{m}\end{aligned}$$

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$$
\begin{aligned}
&\quad - \Big(\frac{15\times 9}{8} - \frac{15}{8(m-1)}\Big) - \frac{7\times 15}{4} - \frac{3\times 15^2}{8}\frac{m-1}{9m-10} \\
&\quad - \frac{9\sqrt{2}}{16}\frac{\sqrt{9m-10}}{m-1} - \frac{45\sqrt{2}}{16}\frac{1}{\sqrt{9m-10}} - \frac{9\times 15}{16}\frac{1}{9m-10} - \frac{27}{16(m-1)} + 24m \\
&= \frac{15}{4}m - 12\sqrt{18m} - \frac{147}{8} - \frac{\sqrt{2}}{16}\frac{\sqrt{9m-10}}{m-1} \\
&\quad - \frac{9\times 15}{16}\frac{10m-9}{9m-10} + \frac{3}{16(m-1)} - \frac{45\sqrt{2}}{16}\frac{1}{\sqrt{9m-10}} \\
&> \frac{15}{4}m - 12\sqrt{18m} - 28.
\end{aligned}
$$

Therefore,

$$
3\sum_{\substack{g\in G\\ R_A(g)=0}} 1 + \sum_{\substack{g\in G\\ R_A(g)=2 \text{ or } R_A(g)\ge 8}} 1 > \frac{1}{4\max\{(c-5)^2-1, 8\}}(15m - 48\sqrt{18m} - 112).
$$

This completes the proof of Theorem 1.1. □

References

[1] P. Borwein, S. Choi and F. Chu, 'An old conjecture of Erdős–Turán on additive bases', *Math. Comp.* **75** (2006), 475–484.

[2] Y.-G. Chen, 'The analogue of Erdős–Turán conjecture in $\mathbb{Z}_m$', *J. Number Theory* **128** (2008), 2573–2581.

[3] Y.-G. Chen, 'On the Erdős–Turán conjecture', *C. R. Acad. Sci. Paris Sér. I* **350** (2012), 933–935.

[4] Y.-G. Chen and T. Sun, 'The difference basis and bi-basis of $\mathbb{Z}_m$', *J. Number Theory* **130** (2010), 716–726.

[5] Y.-G. Chen and M. Tang, 'On a generalization of a theorem of Sárközy and Sós', *European J. Combin.* **54** (2016), 201–206.

[6] A. Dubickas, 'A basis of finite and infinite sets with small representation function', *Electron. J. Combin.* **19** (2012), #P6, 16 pages.

[7] P. Erdős and P. Turán, 'On a problem of Sidon in additive number theory, and on some related problems', *J. Lond. Math. Soc.* **16** (1941), 212–215.

[8] G. Grekos, L. Haddad, C. Helou and J. Pihko, 'On the Erdős–Turán conjecture', *J. Number Theory* **102** (2003), 339–352.

[9] I. Konstantoulas, 'Lower bounds for a conjecture of Erdős and Turán', *Acta Arith.* **159** (2013), 301–313.

[10] V. F. Lev and A. Sárközy, 'An Erdős–Fuchs type theorem for finite groups', *Integers* **11** (2011), 487–494.

[11] M. B. Nathanson, 'Unique representation bases for the integers', *Acta Arith.* **108** (2003), 1–8.

[12] I. Z. Ruzsa, 'A just basis', *Monatsh. Math.* **109** (1990), 145–151.

[13] C. Sándor and Q.-H. Yang, 'A lower bound of Ruzsa's number related to the Erdős–Turán conjecture', Preprint, 2016, arXiv:1612.08722v1.

[14] M. Tang, 'On the Erdős–Turán conjecture', *J. Number Theory* **150** (2015), 74–80.

[15] M. Tang and Y.-G. Chen, 'The new upper bounds of some Ruzsa numbers R_m', *J. Math. Res. Exposition* **30** (2010), 557–561.

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