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Non-canonical extension of θ-functions and modular integrability of ϑ-constants

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We present new results in the theory of the classical theta functions of Jacobi: series expansions and defining ordinary differential equations (ODEs). The proposed dynamical systems turn out to be Hamiltonian and define fundamental differential properties of theta functions; they also yield an exponential quadratic extension of the canonical θ-series. An integrability condition of these ODEs explains the appearance of the modular ϑ-constants and differential properties thereof. General solutions to all the ODEs are given. For completeness, we also solve the Weierstrassian elliptic modular inversion problem and consider its consequences.

1. Introduction

The theta functions of Jacobi and the Weierstrassian basis of functions $\{\sigma, \zeta, \wp, \wp'\}$ arise in numerous theories and applications. Since their discovery in the late 1820s, this field has become the subject of intensive study. The majority of results and the current form of the theory were obtained in the very works of Jacobi and Weierstrass, and in those of their contemporaries (Hermite [48], Enneper [31], Kiepert, Neumann [66], Halphen [42–44], Hurwitz [50], Frobenius [40], Fricke *et al*. [37, 38, 55, 56]). The current monographic literature on this topic may run into tens of items. Thorough treatises (not only in German [76]) appeared during Jacobi's lifetime. See, for example, [41] (what is now called an elliptic function was named a 'modular-function' by Gudermann in [41]), [87] and, continuing the list further, [15,19,28,35,45,57,59,73,82,84,88], the references in these works and especially the encyclopedic paper by Fricke [36]. Fricke had also planned a third volume to the series [37, 38], but it was compiled (posthumously) only recently [39]. See also the excellent survey by Koenigsberger [58]. We do not claim that the bibliography is a complete list, but the references therein comprehensively cover the known properties of elliptic, modular and θ-functions. In addition to the monographs listed above, we refer the reader to the later presentation of the theory [4–6,16,34,51,61,64,65,71,72] and the handbooks [3, 33, 89]. A great number of specific examples can be found in the classical *'A course of modern analysis'* by Whittaker and Watson [93], and very detailed exposition of the theory is included in Weber's *'Lehrbuch der Algebra'* [88] and in the two-volume Halphen treatise with the posthumous issue of the third volume [42–44]. As a source for the formulae, in most cases Schwarz's collection of the classical results of Weierstrass and Jacobi [89] is by no means lacking, and the four volume set by Tannery and Molk [78–81] contains exhaustive information

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along these lines. The works of Weierstrass [90, 91] and Jacobi [52, 53], being very detailed in presentation, should be referred to herein since they still remain a source of important observations.

1.1. The structure of the paper and comments on the results

In this work we describe some new properties of Jacobi's θ-series that, to the best of our knowledge, have not appeared in the extensive literature on this topic. Among these are series expansions, differential equations and their consequences. A characteristic property of the Jacobi–Weierstrass theory is explicit analytic $\theta, \wp$-formulae for solutions of applied problems. In connection with this, we demonstrate some examples: the modular inversion problem, differential computations of Weierstrassian functions and applications to the famous sixth Painlevé transcendent (see § 10). Other applications were recently presented in [12–14].

1.1.1. *The power series*

The series expansions of elliptic, modular and θ-functions have been widely used in many problems. This is because the coefficients of the series have nice analytic and combinatoric properties. It suffices to mention applications of the function series for various θ-quotients [78–81, 88], number-theoretic q-series, Lambert series [5], the famous McKay–Thompson series and their corollaries, such as the 'Moonshine Conjecture' and its modern extensions. Trigonometric series are used to define Jacobi's θ-functions (see § 2.1) and the power series for Weierstrass's σ-function is very well known [3, 89]. It is frequently reproduced in the literature and has multidimensional generalizations [29]. Considering this, it is somewhat surprising that the power series expansions for θ-functions, i.e. θ-analogs of the Weierstrassian σ-series, are heretofore absent. It is of interest to remark that even Jacobi attempted (in a letter of 1828 to Crelle [52, pp. 259–260], i.e. before the appearance of the *Fundamenta Nova*) to obtain that series and observed that their coefficients resulted in interesting dynamical systems. In § 3 we construct the canonical power expansions for θ-functions in a (fast computable) form that has the simplest structure and is maximally effective from the analytic point of view.

1.1.2. *Differential equations*

In §§ 4–7 and 9 we expound the main material of the work. Namely, dynamical systems satisfied by θ-series, their integrability condition, a non-canonical extension of the canonical θ-series and the modular integrability of ϑ-constants. We see that not only do elliptic functions satisfy ordinary differential equations (ODEs), but θ-functions themselves also satisfy certain ODEs. These ODEs are of interest in their own right, if only because it is more logical to consider *ODEs proper for the θ-functions as basic equations* (see § 4) rather than ODEs for elliptic functions. Moreover, such a viewpoint is natural in a more general pattern since elliptic functions are a subclass of abelian elliptic integrals, and the latter are expressible in terms of θ-functions. In § 9 we also see that solutions of introduced ODEs have a remarkable consequence, namely, an exponential quadratic extension with additional parameters. Under certain parameters this non-canonical extension coincides with the case of canonical θ-series. There is yet another point worth noting. Differential relations between θ-functions are sometimes present in old literature [8, 78–81, 88], but they are regarded there just as differential identities. However, the principal point is a *differential closedness of the finitely-many basic θ-objects* and, together with θ-functions, the theta derivative θ_1' should play an independent part in the theory. Upon introducing this object, analytic and differential manipulations by theta functions are not, in essence, distinct from those by elliptic ones or even elementary functions like sine and cosine.

1.1.3. *Algebraic integrability and Hamiltonicity*

One further remarkable property of the systems mentioned above is the fact that the known basic polynomial theta identities between canonical θ-series are nothing but the specific values of algebraic integrals of the systems (see § 7.1). Clearly, we may take these integrals as the Hamilton functions for these ODEs and the equations, being regarded as dynamical systems, thus possess the remarkable property of being Hamiltonian. In § 9.2 we exemplify a particular case along these lines. Such treatments have a large number of applications and an intimate connection with the theory of integrable systems [9]. A complete Hamiltonian description, along with quantization, will be, however, the subject matter of a separate work, but the theta constant case is discussed at greater length in [14]. In this work we underline (with some counter-examples) the need to distinguish differential identities and defining ODEs. Algebraic integrability of the 'θ-ODEs' is also used in [13] for a new treatment of the finite-gap spectral problems.

1.1.4. *Modularity and integrability conditions*

Yet another consequence of the 'differential viewpoint' is the automatic appearance of the 'modular objects' (see § 6). The differential properties of the 'modular part' of Jacobi's functions are more transcendental and closely related to the classical theory of linear ODEs with infinite groups of Fuchsian monodromies. Moreover, in the early 1990s Ablowitz and Chakravarty with co-authors [1, 21, 22, 77] observed the deep link between this theory and complete integrable equations. In the last two decades, in works of Harnad [46], McKay [47], Ablowitz *et al*. [26, pp. 573–589], [2], Ohyama [67], Hitchin [49], this field was substantially advanced and found nice applications known as monopoles [7], Chazy–Picard–Fuch's equations [77], cosmological metrics of Tod [85], Hitchin [49] etc. In § 7.1 we give further explanations and see that 'modular integrability' of the classical ϑ-constants has the following characterization. It constitutes a compatibility condition of the linear heat equation $4\pi\mathrm{i}\theta_\tau = \theta_{zz}$ and, on the other hand, the quadrature integrable nonlinear 'z-equations' for the functions $\theta(z|\tau)$ and $\theta'(z|\tau)$.

It may also be mentioned here that the proposed 'differential technique' may be applied equally well to the cases of multidimensional Θ-functions when the latter split into a decomposition of the one-dimensional, i.e. Jacobi's θ-functions. It is known that such cases correspond to Jacobians of algebraic curves covering the elliptic tori [9]. The problems, wherein such curves arise, are non-trivial in the one part, and are completely solvable as the pure elliptic case in the other part.

1.1.5. *Modular inversion problem*

The complete form of the 'differential theory' requires a closed-form solution to the Weierstrassian elliptic modular inversion problem. Strange though it may seem, its realization does not appear in the literature and, in § 8, we give that solution and show some consequences and applications thereof.

The content of the next section is a matter of common knowledge and we present it here to fix notation and terminology. In the subsequent sections we do not touch upon the known results and simply use them (with references) where they are required.

2. Definitions and notation

2.1. The Jacobi functions

The four theta functions $\theta_{1,2,3,4}$ and their $\theta_{\alpha\beta}$-equivalents are defined by the canonical series

$$
\begin{aligned}
-\theta_{11}(z|\tau) \equiv \theta_1(z|\tau) &= -\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\tau/4}\sum_{-\infty}^{+\infty}{}_k(-1)^k\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\mathrm{e}^{(2k+1)\pi\mathrm{i}z},\\
\theta_{10}(z|\tau) \equiv \theta_2(z|\tau) &= \mathrm{e}^{\pi\mathrm{i}\tau/4}\sum_{-\infty}^{+\infty}{}_k\,\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\mathrm{e}^{(2k+1)\pi\mathrm{i}z},\\
\theta_{00}(z|\tau) \equiv \theta_3(z|\tau) &= \sum_{-\infty}^{+\infty}{}_k\,\mathrm{e}^{k^2\pi\mathrm{i}\tau}\mathrm{e}^{2k\pi\mathrm{i}z},\\
\theta_{01}(z|\tau) \equiv \theta_4(z|\tau) &= \sum_{-\infty}^{+\infty}{}_k(-1)^k\mathrm{e}^{k^2\pi\mathrm{i}\tau}\mathrm{e}^{2k\pi\mathrm{i}z},
\end{aligned}
$$

where, to avoid further confusion with Weierstrass's branch points e, we use the notation e^z for the exponent of z. We also use the shorthand form $\theta_k := \theta_k(z|\tau)$. The values of θ-functions under $z = 0$ (Thetanullwerthe [52, 53]) are called the theta constants. Because each of the variables z and τ has an 'independent theory', we introduce a separate notation for the nullwerthe: $\vartheta_k := \vartheta_k(\tau) = \theta_k(0|\tau)$. Their series representations follow from the θ-series above:

$$
\left.
\begin{aligned}
\vartheta_2(\tau) &:= \sum_{-\infty}^{\infty}{}_k\,\mathrm{e}^{(k+1/2)^2\pi\mathrm{i}\tau},\\
\vartheta_3(\tau) &:= \sum_{-\infty}^{\infty}{}_k\,\mathrm{e}^{k^2\pi\mathrm{i}\tau},\\
\vartheta_4(\tau) &:= \sum_{-\infty}^{\infty}{}_k(-1)^k\mathrm{e}^{k^2\pi\mathrm{i}\tau}.
\end{aligned}
\right\}
\tag{2.1}
$$

Convergence of the series implies that τ belongs to the upper half-plane: $\tau \in \mathbb{H}^+$, that is, $\Im(\tau) > 0$. For typographical convenience we adopt two pieces of notation for θ-functions with characteristics $[{\alpha \atop \beta}]$ (Hermite 1858):

$$\theta[{\alpha \atop \beta}](z|\tau) \equiv \theta_{\alpha\beta}(z|\tau) = \sum_{-\infty}^{+\infty}{}_k \exp\left(\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)^2\tau + 2\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)\left(z+\frac{\beta}{2}\right)\right). \tag{2.2}$$

Let n and m be arbitrary integers, $n, m = 0, \pm 1, \pm 2, \dots$. We consider only integral characteristics and, hence, by virtue of

$$\theta[{\alpha+2m \atop \beta+2n}] = (-1)^{\alpha n}\theta[{\alpha \atop \beta}], \tag{2.3}$$

functions $\theta_{\alpha\beta}$ always reduce to $\pm\theta_{1,2,3,4}$. When adding a half-period, θ-characteristics undergo a shift:

$$\theta[{\alpha \atop \beta}]\left(z+\frac{n}{2}+\frac{m}{2}\tau\,\middle|\,\tau\right) = (-\mathrm{i})^{(\beta+n)m}\theta[{\alpha+m \atop \beta+n}](z|\tau)\mathrm{e}^{-\pi\mathrm{i}m(4z+m\tau)/4}. \tag{2.4}$$

Two-fold shifts by half-periods yield the law of transformation of θ-function into itself:

$$\theta_{\alpha\beta}(z+n+m\tau|\tau) = (-1)^{n\alpha-m\beta}\theta_{\alpha\beta}(z|\tau)\mathrm{e}^{-\pi\mathrm{i}m(2z+m\tau)}.$$

The value of any θ-function at any half-period is a certain ϑ-constant multiplied by the exponential factor:

$$\theta[{\alpha \atop \beta}]\left(\frac{n}{2}+\frac{m}{2}\tau\,\middle|\,\tau\right) = (-\mathrm{i})^{(\beta+n)m}\vartheta[{\alpha+m \atop \beta+n}](\tau)\mathrm{e}^{\pi\mathrm{i}m^2\tau/4}.$$

In this work we use the 'τ-representation' for ϑ, θ-functions. The transition to the frequently used 'q-representation' ($q = \mathrm{e}^{\pi\mathrm{i}\tau}$) is performed using $\partial_q = \pi\mathrm{i}q\partial_\tau$.

We supplement the set of functions $\{\theta_k\}$ with the derivative $\partial_z\theta_1(z|\tau)$ and consider it as a fifth independent object:

$$\partial_z\theta_1(z|\tau) =: \theta_1'(z|\tau) := \pi\mathrm{e}^{\pi\mathrm{i}\tau/4}\sum_{-\infty}^{+\infty}{}_k(-1)^k(2k+1)\mathrm{e}^{(k^2+k)\pi\mathrm{i}\tau}\mathrm{e}^{(2k+1)\pi\mathrm{i}z}. \tag{2.5}$$

It is interesting to note that the very object and its z-dependence have an intriguing correlation with experimental data in purely physical considerations [75].

2.2. The Weierstrass functions

We use the conventional Weierstrassian notation [89]

$$\sigma(z|\omega,\omega') = \sigma(z; g_2, g_3), \qquad \zeta(z|\omega,\omega') = \zeta(z; g_2, g_3),$$
$$\wp(z|\omega,\omega') = \wp(z; g_2, g_3), \qquad \wp'(z|\omega,\omega') = \wp'(z; g_2, g_3).$$

The invariants (g_2, g_3) are functions of the periods $(2\omega, 2\omega')$ (and vice versa) and the modulus $\tau = \omega'/\omega$. They are defined by the well-known Weierstrass–Eisenstein series [30, 90–92], which are, however, entirely unsuited for numeric computations.

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Hurwitz, in his dissertation [50, p. 547], found a nice transition to the Lambert series [5]

$$\left.\begin{aligned} g_2(\tau) &= 20\pi^4\left\{\frac{1}{240} + \sum_1^\infty k\frac{k^3\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}, \\ g_3(\tau) &= \frac{7}{3}\pi^6\left\{\frac{1}{504} - \sum_1^\infty k\frac{k^5\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}, \end{aligned}\right\} \tag{2.6}$$

which are used in theories, have applications and are most effective in computations.

Determination of the periods $(2\omega, 2\omega')$ by the coefficients (a, b) of an elliptic curve of the Weierstrassian form $w^2 = 4z^3 - az - b$ is know as the elliptic modular inversion problem. Its solution involves the transcendental equation $J(\tau) = A$, where $J(\tau)$ is the classical modular function of Klein [4, 51, 55, 56]. Modular inversion is then realized by the scheme

$$(a, b) \dashrightarrow J(\tau) = \frac{a^3}{a^3 - 27b^2} \dashrightarrow \omega = \pm\sqrt{\frac{a}{b}\frac{g_3(\tau)}{g_2(\tau)}} \dashrightarrow \omega' = \tau\omega. \tag{2.7}$$

The degenerated cases (lemniscatic ($b = 0$) and equianharmonic ($a = 0$)) require separate formulae. In both of these cases there exist exact solutions. The lemniscatic solution ω_{L} was found by Gauss. In our notation it can be written as

$$\omega_{\mathrm{L}} = \sqrt[-4]{8a}\,\pi\vartheta_4^2(2\mathrm{i}), \qquad \omega' = \mathrm{i}\omega_{\mathrm{L}}.$$

See works by Todd [86] and Levin [62] for exhaustive information and a voluminous bibliography on the lemniscate. The exact solution ω_{E} to the equianharmonic case we display here seems to be new:

$$\omega_{\mathrm{E}} = \sqrt[-12]{-27b^2}\,\pi\boldsymbol{\eta}^2(\varrho), \qquad \omega' = \varrho\omega_{\mathrm{E}}, \quad \varrho := -\tfrac{1}{2}(1 - \mathrm{i}\sqrt{3}),$$

where $\boldsymbol{\eta}$ is the Dedekind function (see § 2.3 for a definition). The arbitrary branches of the $\sqrt{\ }$-root are allowed in the previous formulae.

By virtue of homogeneity relations, say $\alpha^2\wp(\alpha z|\alpha\omega, \alpha\omega') = \wp(z|\omega, \omega')$, the couple of half-periods (ω, ω') or invariants (g_2, g_3) can be replaced by one quantity, i.e. the modulus $\tau = \omega'/\omega$. We define the corresponding functions as

$$\begin{aligned} \sigma(z|\tau) &:= \sigma(z|1, \tau), & \zeta(z|\tau) &:= \zeta(z|1, \tau), \\ \wp(z|\tau) &:= \wp(z|1, \tau), & \wp'(z|\tau) &:= \wp'(z|1, \tau). \end{aligned}$$

The Weierstrassian η-function is defined by the formula

$$\eta(\tau) := \zeta(1|1, \tau)$$

and its series representation can be written as

$$\eta(\tau) = 2\pi^2\left\{\frac{1}{24} - \sum_1^\infty k\frac{\mathrm{e}^{2k\pi\mathrm{i}\tau}}{(1-\mathrm{e}^{2k\pi\mathrm{i}\tau})^2}\right\}. \tag{2.8}$$

Modular transformations in the elliptic/modular theory are not only of theoretical interest, since the value of the modulus τ strongly affects the convergence of the series. Moving τ into the fundamental domain of the modular group,

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) =: \boldsymbol{\Gamma}(1)$$

(this process is easily algorithmizable), one obtains values of τ having the minimal imaginary part $\Im(\tau) = \frac{1}{2}\sqrt{3}$. The 'worst' feature of such a scenario is that all the series converge very fast. For example, the modular property of the η-series is

$$\eta\biggl(\frac{a\tau + b}{c\tau + d}\biggr) = (c\tau + d)^2 \eta(\tau) - \frac{\mathrm{i}}{2}\pi c(c\tau + d),$$

where (a, b, c, d) are integers and, as usual, $ad - bc = 1$.

The three Weierstrassian σ-functions are defined through Jacobian functions [78–81, 89] by

$$\sigma_\lambda(z|\omega, \omega') = \frac{\theta_{\lambda+1}(z/2\omega|\omega'/\omega)}{\vartheta_{\lambda+1}(\omega'/\omega)} \mathrm{e}^{\eta(\omega,\omega')z^2/2\omega}, \quad \lambda = 1, 2, 3.$$

The Weierstrassian σ-function, as a function of (z, g_2, g_3), satisfies the linear differential equations

$$\begin{aligned} z\frac{\partial\sigma}{\partial z} - 4g_2\frac{\partial\sigma}{\partial g_2} - 6g_3\frac{\partial\sigma}{\partial g_3} - \sigma = 0, \\ \frac{\partial^2\sigma}{\partial z^2} - 12g_3\frac{\partial\sigma}{\partial g_2} - \frac{2}{3}g_2^2\frac{\partial\sigma}{\partial g_3} + \frac{1}{12}g_2 z^2\sigma = 0 \end{aligned}$$

obtained by Weierstrass himself. It immediately follows that there exists a recursive relation for the coefficients C_k of the power series

$$\sigma(z; g_2, g_3) = C_0 z + C_1\frac{z^3}{3!} + \cdots = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 + \cdots, \tag{2.9}$$

where the standard normalization $\sigma(0) = 0$, $\sigma'(0) = 1$, $\sigma''(0) = 0$ has been adopted. The two classical recurrences are known. The first, due to Halphen [42, p. 300], is that

$$C_k = \hat{\mathfrak{D}}C_{k-1} - \tfrac{1}{6}(k-1)(2k-1)g_2 C_{k-2}, \tag{2.10}$$

where

$$\hat{\mathfrak{D}} := 12g_3\frac{\partial}{\partial g_2} + \frac{2}{3}g_2^2\frac{\partial}{\partial g_3},$$

but it was recorded in different notation by Weierstrass [91, p. 49] and even by Jacobi (see § 5). The second recurrence was obtained by Weierstrass as

$$\sigma(z; g_2, g_3) = \sum_{m,n=0}^{\infty} A_{m,n}\biggl(\frac{g_2}{2}\biggr)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}, \tag{2.11}$$

$$\begin{aligned} A_{m,n} = {}& \tfrac{16}{3}(n+1)A_{m-2,n+1} + 3(m+1)A_{m+1,n-1} \\ & - \tfrac{1}{3}(2m+3n-1)(4m+6n-1)A_{m-1,n}, \end{aligned} \tag{2.12}$$

where $A_{0,0} = 1$ and $A_{m,n} = 0$ under $n, m < 0$. Other recurrences are also known [29]. Among all the recurrences, the Weierstrassian one is least expendable because

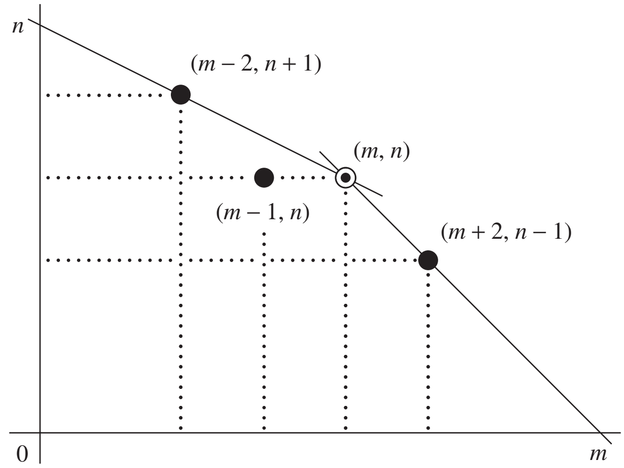

Figure 1. The Weierstrass recurrence (2.12).

it contains only multiplication of integers and the coefficients C_k have already been collected in parameters. It is interesting to note that Weierstrass proves separately the fact that the $A_{m,n}$ have integral values [91, p. 50]. When deriving the power series for Jacobi's θ-functions in § 3 we are guided by the same motivation.

2.3. Dedekind's function

Since the standard notation for the Weierstrassian η-function and for that of Dedekind coincides, we use for the latter the symbol $\boldsymbol{\eta}(\tau)$, defining

$$\begin{aligned}\boldsymbol{\eta}(\tau) &= \mathrm{e}^{\pi\mathrm{i}\tau/12}\prod_{k}^{\infty}{}_{1}(1-\mathrm{e}^{2k\pi\mathrm{i}\tau})\\ &= \mathrm{e}^{\pi\mathrm{i}\tau/12}\sum_{-\infty}^{+\infty}{}_{k}(-1)^k\mathrm{e}^{(3k^2+k)\pi\mathrm{i}\tau}. \qquad \text{(Euler 1748)}\end{aligned}$$

Dedekind's function is connected to the Jacobi–Weierstrass ones through the differential and algebraic relations [71, 88]

$$\frac{1}{\boldsymbol{\eta}}\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\tau} = \frac{\mathrm{i}}{\pi}\eta, \qquad 2\boldsymbol{\eta}^3 = \vartheta_2\vartheta_3\vartheta_4. \tag{2.13}$$

3. Canonical power $\boldsymbol{\theta}$-series

Before proceeding to the θ-series we need some preparatory material on the series for the Weierstrassian σs and a graphical illustration of the recurrence $A_{m,n}$; see figure 1.

Figure 1, (2.11) and (2.12) mean that computation of the point (m, n) involves computation of all the interior points of the delineated quadrangle.

Throughout the paper we use the symbol $[n]$ for the integer part of the number n. Then, one can easily show that the Weierstrassian series (2.11) simplifies to the following expression explicitly collected in variable z:

$$\sigma(z;g_2,g_3)=\sum_{0}^{\infty}{}_k\left\{\sum_{[k/3]}^{k/2}{}_\nu\, 2^{2k-5\nu}A_{3\nu-k,k-2\nu}g_2^{3\nu-k}g_3^{k-2\nu}\right\}\frac{z^{2k+1}}{(2k+1)!}. \tag{3.1}$$

Analogous series exist for all the σ-functions, but their form depends on which parameters $g_{2,3}$ or e_λ are chosen as the basic ones (see Remark 3.3).

3.1. Halphen's operator and series for σ_λ

We define $e_\lambda := \wp(\omega_\lambda|\omega,\omega')$.

Lemma 3.1. *The power series for the functions σ_λ are given by the expression*

$$\sigma_\lambda(z;e_\lambda,g_2)=\sum_{0}^{\infty}{}_k\left\{\sum_{0}^{k/2}{}_\nu\, 2^{-\nu}\mathfrak{B}_{k-2\nu,\nu}e_\lambda^{k-2\nu}g_2^{\nu}\right\}\frac{z^{2k}}{(2k)!},$$

with the integral recurrence

$$\begin{aligned}\mathfrak{B}_{m,n}&=24(n+1)\mathfrak{B}_{m-3,n+1}+(4m-12n-5)\mathfrak{B}_{m-1,n}\\&\quad-\tfrac{4}{3}(m+1)\mathfrak{B}_{m+1,n-1}-\tfrac{1}{3}(m+2n-1)(2m+4n-3)\mathfrak{B}_{m,n-1},\\ \mathfrak{B}_{0,0}&=1\end{aligned}$$

and

$$\mathfrak{B}_{m,n}=0\quad \text{if } m,n<0.$$

Proof. Calculations are based on Halphen's equations satisfied by σ_λ-functions [42–44],

$$\left.\begin{aligned}z\frac{\partial\sigma_\lambda}{\partial z}-2e_\lambda\frac{\partial\sigma_\lambda}{\partial e_\lambda}-4g_2\frac{\partial\sigma_\lambda}{\partial g_2}=0,\\ \frac{\partial^2\sigma_\lambda}{\partial z^2}-(4e_\lambda^2-\tfrac{2}{3}g_2)\frac{\partial\sigma_\lambda}{\partial e_\lambda}-12(4e_\lambda^3-g_2e_\lambda)\frac{\partial\sigma_\lambda}{\partial g_2}+(e_\lambda+\tfrac{1}{12}g_2z^2)\sigma_\lambda=0,\end{aligned}\right\} \tag{3.2}$$

and the proof of integrality of the recurrence $\mathfrak{B}_{m,n}$ is analogous to the Weierstrassian argument in [91, p. 50]. □

We can also include the σ-function in this recurrence. In this case, the analogue of Halphen's equation (3.2) takes the form

$$\left.\begin{aligned}z\frac{\partial\Xi}{\partial z}-2e_\lambda\frac{\partial\Xi}{\partial e_\lambda}-4g_2\frac{\partial\Xi}{\partial g_2}-(1-\varepsilon)\Xi=0,\\ \frac{\partial^2\Xi}{\partial z^2}-(4e_\lambda^2-\tfrac{2}{3}g_2)\frac{\partial\Xi}{\partial e_\lambda}-12(4e_\lambda^3-g_2e_\lambda)\frac{\partial\Xi}{\partial g_2}+(\varepsilon e_\lambda+\tfrac{1}{12}g_2z^2)\Xi=0,\end{aligned}\right\} \tag{3.3}$$

where the case $\Xi=\sigma_\lambda$ corresponds to $\varepsilon=1$ and $\Xi=\sigma$ corresponds to $\varepsilon=0$ with arbitrary e_λ. This is because the σ-function does not depend on permutation of the branch-points e_λ.

Corollary 3.2. *The power series for the Weierstrassian σ-functions are determined by*

$$\Xi(z; e_\lambda, g_2) = \sum_{0}^{\infty}{}_{k} \left\{ \sum_{0}^{k/2}{}_{\nu} 2^{-\nu} \mathfrak{B}^{(\varepsilon)}_{k-2\nu,\nu} e_\lambda^{k-2\nu} g_2^{\nu} \right\} \frac{z^{2k+1-\varepsilon}}{(2k+1-\varepsilon)!} \tag{3.4}$$

under the universal integral recurrence

$$\begin{aligned}\mathfrak{B}^{(\varepsilon)}_{m,n} &= 24(n+1)\mathfrak{B}^{(\varepsilon)}_{m-3,n+1} + (4m - 12n - 4 - \varepsilon)\mathfrak{B}^{(\varepsilon)}_{m-1,n} \\ &\quad - \tfrac{4}{3}(m+1)\mathfrak{B}^{(\varepsilon)}_{m+1,n-1} - \tfrac{1}{3}(m+2n-1)(2m+4n-1-2\varepsilon)\mathfrak{B}^{(\varepsilon)}_{m,n-1}.\end{aligned}$$

Remark 3.3. Weierstrass himself wrote out recurrences not for his σ-functions but for the functions $S_\lambda = \exp\{\frac{1}{2}e_\lambda z^2\}\sigma_\lambda(z)$, with parameters $(e_\lambda, \varepsilon_\lambda = 3e_\lambda^2 - \frac{1}{4}g_2)$ [90, pp. 253–254]. A possible explanation of this choice is that the functions S_λ yield a four-term recurrence like $A_{m,n}$, whereas our recurrences are five-term ones. However, one can show that recurrences for the functions σ_λ through the parameters (g_2, g_3) do not exist. Nevertheless, transition between the pairs (e_λ, g_2) and (e_λ, e_μ) is one-to-one and, therefore, this universal recurrence may be written in any of these 'representations'. It will be completely symmetric in the (e_λ, e_μ)-representation.

Another kind of formula for coefficients of the series is the ϑ-constant one. It is a natural choice for the power θ-series. Indeed, the ϑ-constant expressions for the branch points e_λ are well known. In turn, ϑ-constants are related by the Jacobi identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$. This allows us to pass between the representations, choosing arbitrary pairs. If formulae contain the constants $(\vartheta_{\alpha 0}, \vartheta_{0\beta})$, where $(\alpha, \beta) \neq (0, 0)$, we use the term (α, β)-*representation*.

The preceding Halphen equations and their other representations can be written as one equation if we make use of operator (2.10). We can derive, for example, that

$$\hat{\mathfrak{D}} = (4e_\lambda^2 - \tfrac{2}{3}g_2)\frac{\partial}{\partial e_\lambda} + 12(4e_\lambda^3 - g_2 e_\lambda)\frac{\partial}{\partial g_2}.$$

The ϑ-constant form for the $\hat{\mathfrak{D}}$-operator is also easily calculated, since all the parameters of the theory, i.e. g_2, g_3, e_λ and $\vartheta_{\alpha\beta}$, are related to each other by polynomial relations [78–81]. For example, the $(\vartheta_2, \vartheta_4)$-representation for $\hat{\mathfrak{D}}$ reads as

$$\hat{\mathfrak{D}} = \frac{\pi^2}{3}(\vartheta_4^8 + 2\vartheta_2^4\vartheta_4^4)\frac{\partial}{\partial(\vartheta_4^4)} - \frac{\pi^2}{3}(\vartheta_2^8 + 2\vartheta_2^4\vartheta_4^4)\frac{\partial}{\partial(\vartheta_2^4)}.$$

We define $\langle\alpha\rangle := (-1)^\alpha$. The general (α, β)-representation to the Halphen operator is then given by

$$\hat{\mathfrak{D}} = \frac{\pi^2}{3}(\langle\beta\rangle\vartheta_{\alpha 0}^8 + 2\langle\alpha\rangle\vartheta_{\alpha 0}^4\vartheta_{0\beta}^4)\frac{\partial}{\partial(\vartheta_{\alpha 0}^4)} - \frac{\pi^2}{3}(\langle\alpha\rangle\vartheta_{0\beta}^8 + 2\langle\beta\rangle\vartheta_{\alpha 0}^4\vartheta_{0\beta}^4)\frac{\partial}{\partial(\vartheta_{0\beta}^4)}.$$

Also define the symbol ε depending on the parity of the function $\theta_{\alpha\beta}$ as

$$\varepsilon := \frac{\langle\alpha\beta\rangle + 1}{2} \quad \Rightarrow \quad \begin{cases} \varepsilon = 0 & \text{if } \theta_{\alpha\beta} = \pm\theta_1, \\ \varepsilon = 1 & \text{if } \theta_{\alpha\beta} = \pm\theta_{2,3,4}. \end{cases}$$

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LEMMA 3.4. *Halphen's equation (3.3) for the functions* $\varXi = \{\sigma, \sigma_\lambda\}$ *has the form*

$$\frac{\partial^2 \varXi}{\partial z^2} - \hat{\mathfrak{D}}\varXi + \left\{\varepsilon e_\lambda(\vartheta) + \frac{\pi^4}{12^2}(\vartheta_2^8 + \vartheta_2^4\vartheta_4^4 + \vartheta_4^8)z^2\right\}\varXi = 0,$$

and its general (α, β)*-representation is*

$$\frac{\partial^2 \varXi}{\partial z^2} - \hat{\mathfrak{D}}\varXi + \left\{e_{\gamma\delta}(\vartheta) + \frac{\pi^4}{12^2}[\vartheta_{\alpha 0}^8 + \langle\alpha{+}\beta\rangle\vartheta_{\alpha 0}^4\vartheta_{0\beta}^4 + \vartheta_{0\beta}^8]z^2\right\}\varXi = 0. \tag{3.5}$$

Here, the quantities

$$e_{\gamma\delta}(\vartheta) = \frac{\pi^2}{12}(\langle\gamma\rangle\vartheta_{0\delta}^4 - \langle\delta\rangle\vartheta_{\gamma 0}^4) \tag{3.6}$$

do not dependent on the representation (α, β) *and correspond to the functions* σ*,* σ_λ *through the rules*

$$\sigma \leftrightarrow e_{00} = 0, \qquad \sigma_1 \leftrightarrow e_{01} = e_1, \qquad \sigma_2 \leftrightarrow e_{11} = e_2, \qquad \sigma_3 \leftrightarrow e_{10} = e_3.$$

We say that a representation is a *proper* one (or symmetric) if $(\gamma, \delta) = (\alpha, \beta)$.

REMARK 3.5 (historical). Nice recurrences on the plane (analogs of $A_{m,n}$) were already obtained by Weierstrass in the 1840s. At the time, he was using his old notation Al instead of Jacobi's $\varTheta$ and H [90]. At about the same time, Jacobi considered the power series for his theta functions and introduced the important multiplier e^{Az^2}, much as Weierstrass did for his σ-function; see remark 5.2 in § 5 or lectures by Koenigsberger [57, pp. 79–81].

3.2. Power θ-series

The functions θ are fundamental objects in numerous theories. For this reason we give a representation for their power series in a maximally simplified (canonical) form. Because of this, we collect all the parameters in a similar way as for the Weierstrassian recurrence $A_{m,n}$, so only multiplications of integers remain. It is not difficult to see that the series under question must be series with coefficients being polynomials in variables $\eta(\tau)$, $\vartheta(\tau)$. This follows from the obvious formulae

$$\theta_1(z|\tau) = \pi\boldsymbol{\eta}^3(\tau)\mathrm{e}^{-2\eta(\tau)z^2}\sigma(2z|\tau), \qquad \theta_\lambda(z|\tau) = \vartheta_\lambda(\tau)\mathrm{e}^{-2\eta(\tau)z^2}\sigma_{\lambda-1}(2z|\tau). \tag{3.7}$$

This entails that all the formulae that follow are derivable using various ϑ-representations of the operator $\hat{\mathfrak{D}}$ in Halphen's equations (3.2), (3.3) and (3.5) and then multiplying the result into the series for an exponent. The computations are somewhat lengthy, but routine, and we therefore omit them entirely.

THEOREM 3.6. *The power series for the function*

$$\begin{aligned}\theta_1(z|\tau) &= \sum_{0}^{\infty}{}_k\, C_k(\tau)z^{2k+1} \\ &= 2\pi\boldsymbol{\eta}^3\left\{z - 2\eta z^3 + \left(2\eta^2 - \frac{\pi^4}{180}(\vartheta_2^8 + \vartheta_2^4\vartheta_4^4 + \vartheta_4^8)\right)z^5 + \cdots\right\}\end{aligned} \tag{3.8}$$

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is determined by the analytic expression

$$\theta_1(z|\tau) = 2\pi \sum_{0}^{\infty}{}_{k} \frac{(4\pi \mathrm{i})^k}{(2k+1)!} \frac{\mathrm{d}^k \boldsymbol{\eta}^3}{\mathrm{d}\tau^k} z^{2k+1}$$

$$= 2\pi \boldsymbol{\eta}^3 \sum_{0}^{\infty}{}_{k} (-2)^k \bigg\{ \sum_{0}^{k}{}_{\nu} \bigg(-\frac{\pi^2}{6}\bigg)^{\nu} \frac{\eta^{k-\nu} \boldsymbol{\mathcal{N}}_\nu(\vartheta)}{(k-\nu)!(2\nu+1)!} \bigg\} z^{2k+1}, \tag{3.9}$$

where the ϑ-polynomial

$$\boldsymbol{\mathcal{N}}_\nu(\vartheta) = \sum_{0}^{\nu}{}_{s} \begin{Bmatrix} \mathfrak{G}_{\nu-s,s} \vartheta_4^{4s} \vartheta_2^{4(\nu-s)} \\ (-1)^s \mathfrak{G}_{\nu-s,s} \vartheta_3^{4s} \vartheta_4^{4(\nu-s)} \\ (-1)^s \mathfrak{G}_{s,\nu-s} \vartheta_3^{4s} \vartheta_2^{4(\nu-s)} \end{Bmatrix}$$

is chosen (in braces) according to which ϑ-constant $(4,2)$-, $(3,4)$- or $(3,2)$-representation is taken. Here, the integral recurrence $\mathfrak{G}_{m,n}$ is defined as

$$\mathfrak{G}_{m,n} = 4(n-2m-1)\mathfrak{G}_{m,n-1} - 4(m-2n-1)\mathfrak{G}_{m-1,n}$$
$$- 2(m+n-1)(2m+2n-1)(\mathfrak{G}_{m-2,n} + \mathfrak{G}_{m-1,n-1} + \mathfrak{G}_{m,n-2}),$$
$$\mathfrak{G}_{0,0} = 1$$

and

$$\mathfrak{G}_{m,n} = 0 \quad \text{if } m,n < 0$$

and has the symmetry property that $\mathfrak{G}_{m,n} = (-1)^{m+n} \mathfrak{G}_{n,m}$.

Remark 3.7. We might, of course, derive a representation of the type

$$\theta_1 = \sum C_{mnp} g_2^m g_3^n \eta^p z^k,$$

like the Weierstrassian recurrence, but $\mathfrak{G}_{m,n}$ is more effective than $A_{m,n}$ since all the polynomials have already been collected in ϑ-constants.

It is interesting to observe that odd derivatives $\theta_1^{(2k+1)}(0|\tau)$, i.e. the coefficients in front of z^{2k+1} in (3.8) and (3.9), generate polynomial expressions in the variables (η, ϑ) that are exactly integrable k times in τ.

Theorem 3.8. *The power series for the functions $\theta[{\alpha \atop \beta}] = \pm\theta_{2,3,4}$,*

$$\theta[{\alpha \atop \beta}](z|\tau) = \sum_{0}^{\infty}{}_{k} C_k^{(\alpha,\beta)}(\tau) z^{2k}$$

$$= \vartheta[{\alpha \atop \beta}] - \vartheta[{\alpha \atop \beta}]\{2\eta + \tfrac{1}{6}\pi^2(\langle\beta\rangle\vartheta[{\alpha-1 \atop 0}]^4 - \langle\alpha\rangle\vartheta[{0 \atop \beta-1}]^4)\}z^2 + \cdots, \tag{3.10}$$

are determined by the analytic expressions

$$\theta[{\alpha \atop \beta}](z|\tau) = \sum_{0}^{\infty}{}_{k} \frac{(4\pi\mathrm{i})^k}{(2k)!} \frac{\mathrm{d}^k \vartheta[{\alpha \atop \beta}]}{\mathrm{d}\tau^k} z^{2k}. \tag{3.11}$$

The proper representation of the series (3.11) has the form

$$\theta[{}^{\alpha}_{\beta}](z|\tau) = \vartheta[{}^{\alpha}_{\beta}] \sum_{k}^{\infty}{}_{0} (-2)^k \biggl\{ \sum_{\nu}^{k}{}_{0} \biggl(-\frac{\pi^2}{6}\biggr)^{\nu} \frac{\eta^{k-\nu}\boldsymbol{\mathcal{N}}_{\nu}^{(\alpha,\beta)}(\vartheta)}{(k-\nu)!(2\nu)!} \biggr\} z^{2k}, \tag{3.12}$$

with the universal integral recurrence

$$\boldsymbol{\mathcal{N}}_{\nu}^{(\alpha,\beta)}(\vartheta) = \sum_{s}^{\nu}{}_{0} \mathfrak{G}_{s,\nu-s}^{(\alpha,\beta)} \vartheta[{}^{\alpha-1}_{0}]^{4s} \vartheta[{}^{0}_{\beta-1}]^{4(\nu-s)},$$

$$\begin{aligned}\mathfrak{G}_{m,n}^{(\alpha,\beta)} = \langle\alpha\rangle(4n-8m-3)\mathfrak{G}_{m,n-1}^{(\alpha,\beta)} - \langle\beta\rangle(4m-8n-3)\mathfrak{G}_{m-1,n}^{(\alpha,\beta)} \\ - 2(m+n-1)(2m+2n-3)(\mathfrak{G}_{m-2,n}^{(\alpha,\beta)} + \langle\alpha{+}\beta\rangle\mathfrak{G}_{m-1,n-1}^{(\alpha,\beta)} + \mathfrak{G}_{m,n-2}^{(\alpha,\beta)}),\end{aligned}$$

where $\mathfrak{G}_{0,0}^{(\alpha,\beta)} = 1$ *and* $\mathfrak{G}_{m,n}^{(\alpha,\beta)} = 0$ *if* $m, n < 0$.

Some remarks are in order. These recurrences are quite effective but there are additional symmetry properties which reduce computations in half. It is evident from the recurrence $\mathfrak{G}_{m,n}^{(\alpha,\beta)}$ itself that it has a symmetry with respect to permutations of indices:

$$\mathfrak{G}_{n,m}^{(\alpha,\beta)} = (-1)^{(m+n)(\alpha+\beta+1)}\mathfrak{G}_{m,n}^{(\alpha,\beta)}, \qquad \mathfrak{G}_{m,n}^{(\beta,\alpha)} = (-1)^{(m+n)(\alpha+\beta)}\mathfrak{G}_{m,n}^{(\alpha,\beta)}. \tag{3.13}$$

This means that we have in effect only two recurrences $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 0)$, i.e. β is always equal to zero. Redefining $\mathfrak{G}_{m,n}^{(\alpha,\beta)} = \mathfrak{G}_{m,n}^{(\alpha,0)} =: \mathfrak{G}_{m,n}^{(\alpha)}$, we have that

$$\begin{aligned}\mathfrak{G}_{m,n}^{(\alpha)} = \langle\alpha\rangle(4n-8m-3)\mathfrak{G}_{m,n-1}^{(\alpha)} - (4m-8n-3)\mathfrak{G}_{m-1,n}^{(\alpha)} \\ - 2(m+n-1)(2m+2n-3)(\mathfrak{G}_{m-2,n}^{(\alpha)} + \langle\alpha\rangle\mathfrak{G}_{m-1,n-1}^{(\alpha)} + \mathfrak{G}_{m,n-2}^{(\alpha)})\end{aligned}$$

and permutations (3.13) therefore reduce to the simple formulae

$$\mathfrak{G}_{n,m}^{(0)} = (-1)^{(m+n)}\mathfrak{G}_{m,n}^{(0)}, \qquad \mathfrak{G}_{n,m}^{(1)} = \mathfrak{G}_{m,n}^{(1)}.$$

We see that recurrences (3.9) and (3.12) differ only in multipliers. Hence, they can be unified into one recurrence much as we did with (3.4) by introducing the parity ε, but the quantity $\langle\alpha\rangle$ still remains. Computer tests show that (m, n)-entries of the matrices $\mathfrak{G}^{(\beta,\alpha)}$ differ from each other only in sign, but we failed to find this rule.

Corollary 3.9. *All the coefficients* $C_k(\tau)$ *and* $C_k^{(\alpha,\beta)}(\tau)$ *are the k-fold exactly* τ*-integrable* (η, ϑ)*-polynomials.*

In § 7 we show that this integrability is a consequence of one dynamical system. Using the formulae above one can construct series in neighbourhoods of the points $z = \{\pm\frac{1}{2}, \pm\frac{1}{2}\tau\}$. By virtue of (2.4), the resulting series are transformed into each other, with some obvious modifications.

4. Dynamical systems satisfied by θ-series

In this section and the next we describe a new and important property of Jacobi's ϑ-, θ- and θ'-series. These, along with elliptic, elementary or rational functions, are differentially closed and thereby define the calculus in its own right.

Theorem 4.1. *The five functions $\theta_1(z|\tau)$, $\theta_2(z|\tau)$, $\theta_3(z|\tau)$, $\theta_4(z|\tau)$ and $\theta_1'(z|\tau)$ satisfy the following closed autonomous ordinary differential equations over the field of coefficients ϑ_2, ϑ_3, ϑ_4 and η:*

$$\left.\begin{aligned}
\frac{\partial\theta_1}{\partial z} &= \theta_1',\\
\frac{\partial\theta_2}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_2 - \pi\vartheta_2^2\frac{\theta_3\theta_4}{\theta_1},\\
\frac{\partial\theta_4}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_4 - \pi\vartheta_4^2\frac{\theta_2\theta_3}{\theta_1},\\
\frac{\partial\theta_3}{\partial z} &= \frac{\theta_1'}{\theta_1}\theta_3 - \pi\vartheta_3^2\frac{\theta_2\theta_4}{\theta_1},\\
\frac{\partial\theta_1'}{\partial z} &= \frac{\theta_1'^2}{\theta_1} - \pi^2\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1} - 4\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\theta_1.
\end{aligned}\right\} \tag{4.1}$$

Proof. The proof is based on the theta function differential identities, which occur infrequently in the literature [88, p. 82], [79, p. 173]. (These important relations are implicitly present in Jacobi's *Werke* [52] but not in the thorough handbook for elliptic functions [89] compiled by Schwarz from Weierstrass's lectures. Even the lectures themselves [90, 91] do not contain these identities in θ-form. They are present in [89, p. 29], [90, 91] in the form of their (ζ, σ_λ)-equivalents. Differential relations for quotients of θ-functions are of course well known. These are differential equations for elliptic functions [93, § 21.6], [8, 57, 59, 88].) The relationships under question are nothing but those between the Weierstrassian functions $(\sigma, \zeta)(z|\tau)$ taken at different half-periods [89]. We can present them in the compact form

$$\theta_\mu\theta_\nu' - \theta_\nu\theta_\mu' = \operatorname{sgn}(\nu-\mu)\pi\vartheta_k^2\theta_1\theta_k, \tag{4.2}$$

where $k = 2, 3, 4$ and

$$\left.\begin{aligned}
\nu &= \frac{8k-28}{3k-10},\\
\mu &= \frac{10k-28}{3k-8};
\end{aligned}\right\} \tag{4.3}$$

the triple (k, ν, μ) runs over the set $\{(2, 3, 4), (3, 4, 2), (4, 2, 3)\}$. In order to turn (4.2) into differential equations we find their differential closure. Taking the property $\vartheta_1 \equiv 0$ into account, we can solve (4.2) with respect to the θ-derivatives and rewrite the result as the first four equations in (4.1),

$$\frac{\partial\theta_k}{\partial z} = \frac{\theta_1'}{\theta_1}\theta_k - \pi\vartheta_k^2\frac{\theta_\nu\theta_\mu}{\theta_1}, \quad k = 1, 2, 3, 4.$$

It only remains to compute the derivative of the object θ_1'. Consider the Weierstrassian identity

$$(\sigma\zeta)' = \sigma\zeta^2 - \sigma\wp$$

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and convert it into the θ-functions. Then,

$$\zeta(2z|\tau) = 2\eta(\tau)z + \frac{1}{2}\frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)}, \tag{4.4}$$

$$\wp(2z|\tau) = \frac{\pi^2}{12}\left\{\vartheta_3^4(\tau) + \vartheta_4^4(\tau) + 3\vartheta_3^2(\tau)\vartheta_4^2(\tau)\frac{\theta_2^2(z|\tau)}{\theta_1^2(z|\tau)}\right\} \tag{4.5}$$

and (3.7) yield the fifth equation in (4.1). □

Weierstrass himself derived a (ζ, σ_λ)-equivalent of (4.2) in a reverse order [89, §§ 24–25], i.e. by differentiating the σ-identities and then using the differential equations for ratios of σ-functions. It should be noted here that the closed θ-form of the Weierstrassian identities contains not only branch points e_λ, i.e. ϑ-constants, but also the 'constant' η. In other words, the closed differential θ-apparatus inevitably contains a fifth function (any of θ_k') and the period of a meromorphic elliptic integral, i.e. $\eta(\tau)$; the total number of equations is thus equal to five.

As mentioned in § 1, theorem 4.1 appears explicable on the basis of the theory of abelian integrals. Namely, these integrals are differentially closed and elliptic functions are the particular case of the meromorphic integrals (integrals of exact differentials). The logarithmic integral is a logarithmic θ-ratio and the canonical meromorphic integral (Weierstrassian ζ-function) is proportional to the fifth function θ_1'.

It is notable that the famous Jacobi identity $\vartheta_1' = \pi\vartheta_2\vartheta_3\vartheta_4$ turns out to be an automatic consequence of (4.1) taken at the point $z = 0$, and this property pertains equally to generalizations of this identity presented by (6.2)–(6.4). By this means we get one more (simple) proof of Jacobi's identity, while in [93] it is stated that, of all the known proofs of this identity, 'none are simple' [93, § 21.41].

The next step suggests itself. All the θ, θ'-functions satisfy the heat equation

$$\left.\begin{aligned} 4\pi\mathrm{i}\frac{\partial\theta}{\partial\tau} &= \frac{\partial^2\theta}{\partial z^2}, \\ 4\pi\mathrm{i}\frac{\partial\theta'}{\partial\tau} &= \frac{\partial^2\theta'}{\partial z^2}. \end{aligned}\right\} \tag{4.6}$$

Therefore, invoking theorem 4.1, we establish that differential closedness is also shared by theta functions, as functions of their second argument.

Theorem 4.2. *The five Jacobi functions $\theta_k(z|\tau)$ and $\theta_1'(z|\tau)$ satisfy the closed non-autonomous ordinary differential equations*

$$\frac{\partial\theta_1}{\partial\tau} = \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1} + \frac{\mathrm{i}}{4}\pi\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1} + \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4 + \vartheta_4^4)\right\}\theta_1,$$

$$\frac{\partial\theta_2}{\partial\tau} = \frac{-\mathrm{i}}{4\pi}\left\{\frac{\theta_1'}{\theta_1} - \pi\vartheta_2^2\frac{\theta_3\theta_4}{\theta_1\theta_2}\right\}^2\theta_2 + \frac{\mathrm{i}}{4}\pi\vartheta_3^2\vartheta_4^2\frac{\theta_1^2}{\theta_2} + \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4 + \vartheta_4^4)\right\}\theta_2,$$

$$\frac{\partial\theta_3}{\partial\tau} = \frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_3 + \frac{\mathrm{i}}{2}\vartheta_3^2\theta_1'\frac{\theta_2\theta_4}{\theta_1^2} - \frac{\mathrm{i}}{4}\pi\vartheta_2^2\vartheta_3^2\frac{\theta_4^2}{\theta_1^2}\theta_3 + \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4 + \vartheta_4^4)\right\}\theta_3,$$

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$$\frac{\partial\theta_4}{\partial\tau}=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_4+\frac{\mathrm{i}}{2}\vartheta_4^2\theta_1'\frac{\theta_2\theta_3}{\theta_1^2}-\frac{\mathrm{i}}{4}\pi\vartheta_2^2\vartheta_4^2\frac{\theta_3^2}{\theta_1^2}\theta_4+\frac{\mathrm{i}}{\pi}\bigg\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\bigg\}\theta_4,$$

$$\frac{\partial\theta_1'}{\partial\tau}=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2}+\frac{3\mathrm{i}}{\pi}\bigg\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1^2}+\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\bigg\}\theta_1'-\frac{\mathrm{i}}{2}\pi^2\vartheta_2^2\vartheta_3^2\vartheta_4^2\frac{\theta_2\theta_3\theta_4}{\theta_1^2}.$$

When deriving the second of these equations, the standard quadratic theta identities

$$\left.\begin{aligned}\vartheta_2^2\theta_4^2-\vartheta_4^2\theta_2^2&=\vartheta_3^2\theta_1^2,\\ \vartheta_2^2\theta_3^2-\vartheta_3^2\theta_2^2&=\vartheta_4^2\theta_1^2\end{aligned}\right\}\tag{4.7}$$

were used. This system, combined with (4.1), constitutes a complete set of rules for differential computations with theta series and, invoking the notation in (4.3), the rules can be written in the compact form

$$\left.\begin{aligned}\frac{\partial\theta_k}{\partial z}&=\frac{\theta_1'}{\theta_1}\theta_k-\pi\vartheta_k^2\frac{\theta_\nu\theta_\mu}{\theta_1},\\ \frac{\partial\theta_1'}{\partial z}&=\frac{\theta_1'^2}{\theta_1}-\pi^2\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1}-4\bigg\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\bigg\}\theta_1,\end{aligned}\right\}\tag{4.8}$$

$$\left.\begin{aligned}\frac{\partial\theta_k}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta_k+\frac{\mathrm{i}}{2}\vartheta_k^2\theta_1'\frac{\theta_\nu\theta_\mu}{\theta_1^2}\\ &\quad+\frac{\mathrm{i}}{4}\pi\{\vartheta_3^2\vartheta_4^2\theta_2^2-\vartheta_k^2\vartheta_\mu^2\theta_\nu^2-\vartheta_k^2\vartheta_\nu^2\theta_\mu^2\}\frac{\theta_k}{\theta_1^2}+\frac{\mathrm{i}}{\pi}\bigg\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\bigg\}\theta_k,\\ \frac{\partial\theta_1'}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2}+\frac{3\mathrm{i}}{\pi}\bigg\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1^2}+\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\bigg\}\theta_1'-\frac{\mathrm{i}}{2}\pi^2\vartheta_2^2\vartheta_3^2\vartheta_4^2\frac{\theta_2\theta_3\theta_4}{\theta_1^2},\end{aligned}\right\}\tag{4.9}$$

where $k=1,2,3,4$. These formulae, incidentally, are not completely symmetric and no theta identities were involved when deriving them; this important point is discussed in § 9.3.

5. The ϑ-constant differential calculus

Equations (4.9) contain $\vartheta(\tau)$- and $\eta(\tau)$-constants but their derivatives have not yet been defined. On the other hand, Weierstrass's invariants (2.6) have the ϑ-constant equivalents

$$\left.\begin{aligned}g_2(\tau)&=\frac{\pi^4}{24}\{\vartheta_2^8(\tau)+\vartheta_3^8(\tau)+\vartheta_4^8(\tau)\},\\ g_3(\tau)&=\frac{\pi^6}{432}\{\vartheta_2^4(\tau)+\vartheta_3^4(\tau)\}\{\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\}\{\vartheta_4^4(\tau)-\vartheta_2^4(\tau)\}\end{aligned}\right\}\tag{5.1}$$

and satisfy the famous Halphen dynamical system [42, p. 331, 449–450]

$$\frac{\mathrm{d}g_2}{\mathrm{d}\tau}=\frac{\mathrm{i}}{\pi}(8g_2\eta-12g_3),\qquad \frac{\mathrm{d}g_3}{\mathrm{d}\tau}=\frac{\mathrm{i}}{\pi}(12g_3\eta-\tfrac{2}{3}g_2^2),\qquad \frac{\mathrm{d}\eta}{\mathrm{d}\tau}=\frac{\mathrm{i}}{\pi}(2\eta^2-\tfrac{1}{6}g_2),\tag{5.2}$$

involving the η-function. In an implicit form, this system was also found by Weierstrass [90, p. 249], and Ramanujan obtained its equivalent [70] when studying his well-known number-theoretic P, Q, R-series. It immediately follows that differential closure of the ϑs requires an extension of (5.2) to a four-dimensional version. It is a direct corollary of (5.1) and (5.2).

Theorem 5.1. *Jacobi's ϑ-constants are differentially closed upon adjoining the Weierstrassian η-function:*

$$\left.\begin{aligned}\frac{\mathrm{d}\vartheta_2}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\vartheta_2, & \frac{\mathrm{d}\vartheta_4}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta - \frac{\pi^2}{12}(\vartheta_2^4+\vartheta_3^4)\right\}\vartheta_4,\\ \frac{\mathrm{d}\vartheta_3}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_2^4-\vartheta_4^4)\right\}\vartheta_3, & \frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{2\eta^2 - \frac{\pi^4}{12^2}(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8)\right\}.\end{aligned}\right\} \tag{5.3}$$

Remark 5.2 (historical). It is less well known that Jacobi wrote out a different analogue to this four-dimensional dynamical system, but Halphen does not mention this fact. This result was published by Borchardt in 1857 on the basis of manuscripts kept after Jacobi's death [53, pp. 383–398]. Namely, Jacobi introduced the four variables (A, B, a, b) in terms of Legendre's quantities $(k, k', \mathsf{K}, \mathsf{E})$ and showed that they satisfy the nice monomial dynamical system (we keep completely to Jacobi's notation in [52, p. 386])

$$\left.\begin{aligned}\frac{\partial A}{\partial h} &= 2A^2B, & \frac{\partial a}{\partial h} &= -16bA^2,\\ \frac{\partial B}{\partial h} &= bA^3, & \frac{\partial b}{\partial h} &= abA^2,\end{aligned}\right\} \tag{5.4}$$

where $h = \frac{1}{4}\pi\mathrm{i}\tau$. Interestingly enough, Jacobi considered (5.4) in the context of the power series for θ-functions and noticed [53, II: p. 390] that the series would be simple if one extracted the exponential multiplier $\exp\{-\frac{1}{2}ABz^2\}$. He described the corresponding recurrences for θ_k [53, pp. 394–398] and one can readily see that they are equivalent to the Weierstrass–Halphen differential recurrence (2.10) for σ-functions. Exhaustive comments on the Jacobi system and its relation to (5.3) can be found in [14].

Before (5.4) was derived, Jacobi also obtained its analogs (see [53, p. 176]) and, in particular, his remarkable differential equation of third order for the ϑ-series,

$$C^4(\ln C^3C_{\tau\tau})_\tau^2 = 16C^3C_{\tau\tau} - \pi^2, \quad C = \vartheta^{-2}. \tag{5.5}$$

(C is Jacobi's notation.) In turn, a simple computation shows that logarithmic derivatives of the ϑ-series also satisfy a compact differential equation, which we meet in § 9.3. The equation and its general solution are

$$(X_\tau - 2X^2)X_{\tau\tau\tau} - X_{\tau\tau}^2 + 16X^3X_{\tau\tau} + 4(X_\tau - 6X^2)X_\tau^2 = 0, \tag{5.6}$$

$$X = \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{\vartheta_k((a\tau+b)/(c\tau+d))}{\sqrt{c\tau+d}}.$$

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We should also mention that the well-known differential relations on logarithms of ratios $\vartheta_2 : \vartheta_3 : \vartheta_4$ [78–81, 88],

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \frac{\vartheta_2}{\vartheta_3} &= \frac{\mathrm{i}}{4}\pi\vartheta_4^4, \\ \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \frac{\vartheta_3}{\vartheta_4} &= \frac{\mathrm{i}}{4}\pi\vartheta_2^4, \\ \frac{\mathrm{d}}{\mathrm{d}\tau} \ln \frac{\vartheta_2}{\vartheta_4} &= \frac{\mathrm{i}}{4}\pi\vartheta_3^4 \end{aligned}$$

(see also [8]), and the ordinary differential equation of Chazy [24, 25],

$$\pi\eta_{\tau\tau\tau} = 12\mathrm{i}(2\eta\eta_{\tau\tau} - 3\eta_\tau^2), \tag{5.7}$$

are the direct consequences of (5.3).

Remark 5.3 (exercise). If we view the last equation in (5.2) as a Riccati equation, then we get an interesting example of the solvable linear second-order ODE. The coefficient of this equation is proportional to the everywhere holomorphic in $\mathbb{H}^+$ form $g_2(\tau)$, which is automorphic with respect to the group $\boldsymbol{\Gamma}(1)$. We carry out the calculations and bring the equation into the form

$$\Psi'' + \frac{g_2(\tau)}{3\pi^2}\Psi = 0.$$

Using (2.13) we generalize this equation to

$$\Psi'' + \frac{n+2}{\pi\mathrm{i}}\eta(\tau)\Psi' - \frac{n}{6\pi^2}g_2(\tau)\Psi = 0$$

and prove that

$$\Psi = \frac{1}{\boldsymbol{\eta}^n(\tau)}\left(A + B\int^\tau \boldsymbol{\eta}^{2n}(\tau)\,\mathrm{d}\tau\right)$$

is its general solution.

To conclude this section, we note that the formulae for multiply differentiating the ϑ, η-constants are given explicitly as coefficients of series (3.11) and (3.12). The same coefficients provide the general expressions for the quantities $\theta^{(n)}(0|\tau)$; the relations between the derivatives $\theta'(0|\tau)$, $\theta''(0|\tau), \ldots, \theta^{(n)}(0|\tau)$ under small n are often used in the literature as auxiliary identities [33, 45, 57, 59, 61, 64, 65, 73, 78–81, 88, 93] (Baruch's dissertation [8] contains a lot of such identities). See also [18] where modular functions like η, ϑ, ϑ', etc appear in the theory of hydrodynamical chains and the differential calculus described above significantly simplifies computations in this work. Due to the fact that the group $\boldsymbol{\Gamma}(1)$ has a lot of interesting and non-trivial subgroups, the number of known differential systems related to the base one, (5.3), is far from being exhausted. Even next to $\boldsymbol{\Gamma}(1)$, groups like $\boldsymbol{\Gamma}_0(N)$ inspire a rich theory. See, for example, [63], which contains many nice results along these lines and additional references.

6. Unification: θ, θ'-functions with characteristics

In this section we summarize the previous results and other basic properties of theta functions and their derivatives in a unified notation, i.e. in terms of theta characteristics using (α, β)-representations. This will enable us primarily to trivialize and automate analytic manipulation with theta functions by including fundamental operations: shifts by half-periods, modular transformations and differential computations. Apart from the unification of formulae, this can serve as the basis for further generalization to the theta functions of higher genera.

Any object, symmetrical in ϑ-constants, can be written in (α, β)-representation. For example, branch points (3.6) or the (α, β)-representation for invariants (5.1) can be written as

$$g_2(\tau) = \frac{\pi^4}{12}\{\vartheta_{\alpha 0}^8 + \langle\alpha{+}\beta\rangle\vartheta_{\alpha 0}^4\vartheta_{0\beta}^4 + \vartheta_{0\beta}^8\}, \quad (\alpha, \beta) \neq (0, 0),$$
$$g_3(\tau) = \frac{\pi^6}{432}\{2\langle\beta\rangle\vartheta_{\alpha 0}^{12} - 3\vartheta_{\alpha 0}^4\vartheta_{0\beta}^4(\langle\beta\rangle\vartheta_{0\beta}^4 - \langle\alpha\rangle\vartheta_{\alpha 0}^4) - 2\langle\alpha\rangle\vartheta_{0\beta}^{12}\}.$$

Other examples are the Jacobi identity

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau) \tag{6.1}$$

and $\vartheta_1' = 2\pi\boldsymbol{\eta}^3$; they have the (α, β)-representation

$$\vartheta[{}^{\alpha}_{\beta}]^4 = (\langle\beta\rangle\vartheta[{}^{\alpha-1}_{0}]^4 + \langle\alpha\rangle\vartheta[{}^{0}_{\beta-1}]^4)\frac{\langle\alpha\beta\rangle + 1}{2}, \qquad \vartheta_{\alpha\beta}'(\tau) = \mathrm{i}^{\beta+1}(1 - \langle\alpha\beta\rangle)\pi\boldsymbol{\eta}^3(\tau). \tag{6.2}$$

Henceforth $\vartheta_{\alpha\beta}'(\tau)$ is understood to be equal to $\theta_{\alpha\beta}'(0|\tau)$.

6.1. Shifts by half-periods for θ-derivatives

In connection with the appearance of the object θ_1', we should augment (2.4) by involving the fact that the algebraic and differential closedness of the θs entails a transformation law for their derivatives. Hence, it is naturally to be expected that any function $\theta_{\alpha\beta}'(z|\tau)$ with a z-argument shifted by some half-period is expressible through the function $\theta_1'(z|\tau)$ and other functions $\theta_{1,2,3,4}(z|\tau)$. The ultimate solution is, however, not a simple differential consequence of (2.4) and is far from being obvious. It should be put to better use as an independent property.

Theorem 6.1 (transformation law for θ-derivatives). *Let α, β, m, n be integers. Then,*

$$\begin{aligned}\theta_{\alpha\beta}'&\left(z + \frac{n}{2} + \frac{m}{2}\tau\middle|\tau\right)\\ &= \mathrm{i}^{-m(\beta+n)}\mathrm{e}^{-\pi\mathrm{i}m(4z+m\tau)/4}\\ &\qquad\times\Bigl\{(\theta_1'(z|\tau) - \pi\mathrm{i}m\theta_1(z|\tau))\theta[{}^{\alpha+m}_{\beta+n}](z|\tau)\\ &\qquad\quad - \Bigl\langle(\alpha{+}m)\Bigl[\frac{\beta+n}{2}\Bigr]\Bigr\rangle\pi\vartheta[{}^{\alpha+m}_{\beta+n}]^2\theta[{}^{\alpha+m-1}_{0}](z|\tau)\theta[{}^{0}_{\beta+n-1}](z|\tau)\Bigr\}\frac{1}{\theta_1(z|\tau)},\end{aligned}$$

where, for closedness of the formula, identity (2.3) should be taken into account.

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Proof. The proof follows from a combination of (4.8) and the conversion of any $\theta_{\alpha\beta}$-function into the function θ_1 by the formula

$$\theta_1(z|\tau) = \mathrm{i}^{\alpha}\theta\big[{\alpha-1 \atop \beta-1}\big]\left(z - \frac{\alpha}{2}\tau - \frac{\beta}{2}\middle|\tau\right)\mathrm{e}^{-\pi\mathrm{i}\alpha(z-\alpha\tau/4)}, \tag{6.3}$$

wherein we set (α, β) to be the integers $(-m, -n)$. □

Taking the limit at $z = 0$, which exploits the series expansions described above, we get a generalization of Jacobi's derivative formula, i.e. the second formula in (6.2).

COROLLARY 6.2. *The general $\theta'_{\alpha\beta}$-constant, i.e. the value of the θ'-function at any half-period, is expressed through a ϑ-constant and exponential multiplier*

$$\begin{aligned}\theta'_{\alpha\beta}&\left(\frac{n}{2} + \frac{m}{2}\tau\middle|\tau\right)\\ &= \mathrm{i}^{1-(\beta+n)m}\pi\{\mathrm{i}^{\beta+n}(1 - \langle(\alpha{+}m)(\beta{+}n)\rangle)\boldsymbol{\eta}^3 - m\vartheta\big[{\alpha+m \atop \beta+n}\big]\}\mathrm{e}^{-\pi\mathrm{i}m^2\tau/4}.\end{aligned} \tag{6.4}$$

Since $\langle(\alpha{+}m)(\beta{+}n)\rangle$ is equal to ± 1, only one term remains in the right-hand side of (6.4), i.e. $\vartheta\big[{\alpha+m \atop \beta+n}\big]$ or $\boldsymbol{\eta}^3 = \frac{1}{2}\vartheta_2\vartheta_3\vartheta_4$. For tables of some particular cases see [79, p. 256].

6.2. Modular transformations

Transformations of θ-functions with respect to the modular group

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \boldsymbol{\Gamma}(1)$$

belong among fundamental properties of theta functions and have numerous applications [5, 71]. Suffice it to mention that the corresponding transformation of the series θ_1 [78–81, 88, 92],

$$\theta_1\left(\frac{z}{c\tau + d}\middle|\frac{a\tau + b}{c\tau + d}\right) = \boldsymbol{\aleph}^3\sqrt{c\tau + d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta_1(z|\tau), \tag{6.5}$$

where $\boldsymbol{\aleph}^3$ denotes some eighth root of unity, may turn a hyper-convergent series into the never-computable one. Hermite represented the famous multiplier $\boldsymbol{\aleph}^3$ via the sum of quadratic Gaussian exponents (it is known that these sums are not easily computed) and Jacobi's symbol $(\frac{a}{b})$ [48, pp. 482–486] (see also [60, pp. 183–193], [57, pp. 57–58], [88, pp. 124–132] and [79, II]). For this reason, it is interesting that there exists a simpler formula for the modular transformation wherein the multiplier $\boldsymbol{\aleph}$ is merely an exponent of a rational. Without loss of generality, we may normalize c to be positive: $c > 0$.

THEOREM 6.3 ($\boldsymbol{\Gamma}(1)$-transformation law for the general θ-function). *Let $\theta\big[{\alpha \atop \beta}\big]$ be the theta series with arbitrary integer characteristics (2.2) and let $n \in \mathbb{Z}$. Then,*

$$\theta\big[{\alpha-1 \atop \beta}\big](z|\tau + n) = \mathrm{i}^{n(1-\alpha^2)/2}\theta\big[{\alpha-1 \atop \beta+n\alpha}\big](z|\tau), \tag{6.6}$$

$$\theta\big[{\alpha'-1 \atop \beta'-1}\big]\left(\frac{z}{c\tau + d}\middle|\frac{a\tau + b}{c\tau + d}\right) = \boldsymbol{\mathfrak{E}}_{\alpha\beta}\boldsymbol{\aleph}^3\sqrt{c\tau + d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta\big[{\alpha-1 \atop \beta-1}\big](z|\tau), \tag{6.7}$$

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where the multipliers $\mathfrak{E}_{\alpha\beta}$ *and* $\aleph$ *depend on* (a,b,c,d) *as*

$$\mathfrak{E}_{\alpha\beta} = \exp\frac{\pi}{4}\mathrm{i}\{2\alpha(\beta bc - d + 1) - \beta c(\beta a - 2) - \alpha^2 db\},$$

$$\aleph := \exp\pi\mathrm{i}\left\{\frac{a-d}{12c} - \frac{d}{6}(2c-3) + \frac{c-1}{4}\,\mathrm{sgn}(d) - \frac{1}{4} + \frac{1}{c}\sum_{|[c/d]|+1}^{c-1}{}_{k}\left[\frac{d}{c}k\right]k\right\} \quad (6.8)$$

and the characteristics (α,β), (α',β') *are related through the linear transformation*

$$\left.\begin{aligned} \alpha' &= d\alpha - c\beta, & \alpha &= a\alpha' + c\beta',\\ \beta' &= -b\alpha + a\beta, & \beta &= b\alpha' + d\beta'. \end{aligned}\right\} \quad (6.9)$$

Proof. Formula (6.6) is an elementary consequence of the series (2.2). Proof of (6.7) consists of two steps. The first is to use accurate manipulations/simplifications by Dedekind's sums [5, 69] to determine the multiplier $\aleph$ and enter it into the transformation formula for the $\boldsymbol{\eta}$-function:

$$\boldsymbol{\eta}\left(\frac{a\tau+b}{c\tau+d}\right) = \aleph\sqrt{c\tau+d}\,\boldsymbol{\eta}(\tau).$$

Subsequent use of the fact that the multiplier for θ_1 in (6.5) is a cube of multiplier $\aleph$ for $\boldsymbol{\eta}$ [78–81, 92] yields (6.8). The second step exploits the fact that the function θ_1 transforms into itself and any of the functions $\theta_{\alpha\beta}(z|\tau)$ can be transformed into the function $\theta_1(z|\tau)$ (and vice versa) by a half-period shift of its z-argument, as in (6.3). This gives the linear transformation between characteristics (6.9) and, with it, the multiplier $\mathfrak{E}_{\alpha\beta}$. Characteristics, as appeared in (6.7), have been chosen in order that the formula be most symmetric. □

Remark 6.4. Hermite also gave *nonlinear* formulae for the transformation of characteristics $(\alpha,\beta)\mapsto(\alpha',\beta')$ [48, p. 483], which are reproduced in subsequent works [34, 72, 88] (some linear forms can be found in [60, p. 183]). It is somewhat surprising that no such self-contained formula seems to have hitherto been presented in the literature. A ratio of any θ-functions contains no multiplier $\aleph$ and Hermite used this fact to build the functions $\varphi(\tau)$, $\psi(\tau)$, $\chi(\tau)$ and tables of transformation between them [48, 78–81] when constructing his famous solution to the quintic equation $x^5 - x = a$ in terms of φ, ψ, χ [48, p. 10]. These functions are in fact certain ϑ-constants, so their transformations are consequences of the ϑ-constant ones.

Corollary 6.5. *The* $\boldsymbol{\Gamma}(1)$*-transformations for the general* $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$*-constants are*

$$\vartheta\left[\begin{smallmatrix}\alpha-1\\ \beta\end{smallmatrix}\right](\tau+n) = \mathrm{i}^{n(1-\alpha^2)/2}\vartheta\left[\begin{smallmatrix}\alpha-1\\ \beta+n\alpha\end{smallmatrix}\right](\tau),$$

$$\vartheta\left[\begin{smallmatrix}\alpha'-1\\ \beta'-1\end{smallmatrix}\right]\left(\frac{a\tau+b}{c\tau+d}\right) = \mathrm{i}^{\{2\alpha(\beta bc-d+1)-\beta c(\beta a-2)-\alpha^2 db\}/2}\aleph^3\sqrt{c\tau+d}\,\vartheta\left[\begin{smallmatrix}\alpha-1\\ \beta-1\end{smallmatrix}\right](\tau).$$

The known property that each θ_k-function transforms into itself under the group $\boldsymbol{\Gamma}(2)$ is also a consequence of theorem 6.3 and (6.9).

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Corollary 6.6. *Let (m, n, p, q) be integers. Then, the group*

$$\boldsymbol{\Gamma}(2) \ni \begin{pmatrix} 2n+1 & 2m \\ 2p & 2q+1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is necessary and sufficient for each function θ_k to transform into itself. The transformations are

$$\begin{aligned}
\theta_1\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta_1(z|\tau), \\
\theta_2\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{i}^{2q(p-1)+p}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta_2(z|\tau), \\
\theta_3\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{i}^{2q(p+1)-m(2n+1)+p}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta_3(z|\tau), \\
\theta_4\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{i}^{2n(m-1)-m}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta_4(z|\tau).
\end{aligned}$$

Proof. Using (6.9) and (6.7) we get that

$$\begin{aligned}
\theta_2\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{e}^{\pi\mathrm{i}(2-d)c/4}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta\left[\begin{smallmatrix} c-1 \\ d-1 \end{smallmatrix}\right](z|\tau), \\
\theta_3\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{e}^{\pi\mathrm{i}\{2(a+c-ad)-ab-cd\}/4}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta\left[\begin{smallmatrix} a+c-1 \\ b+d-1 \end{smallmatrix}\right](z|\tau), \\
\theta_4\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) &= \mathrm{e}^{\pi\mathrm{i}(2a-ab-2)/4}\aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\theta\left[\begin{smallmatrix} a-1 \\ b-1 \end{smallmatrix}\right](z|\tau).
\end{aligned}$$

Requiring now that $\theta\left[\begin{smallmatrix} c-1 \\ d-1 \end{smallmatrix}\right] \simeq \theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \theta_2$, $\theta\left[\begin{smallmatrix} a+c-1 \\ b+d-1 \end{smallmatrix}\right] \simeq \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \theta_3$ and $\theta\left[\begin{smallmatrix} a-1 \\ b-1 \end{smallmatrix}\right] \simeq \theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \theta_4$, we obtain *linear* equations for a, b, c and d. Their solution yields the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2n+1 & 2m \\ 2p & 2q+1 \end{pmatrix}.$$

□

The transformation of the fifth function θ_1' follows from a derivative of (6.5):

$$\theta_1'\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) = \aleph^3\sqrt{c\tau+d}\,\mathrm{e}^{\pi\mathrm{i}cz^2/(c\tau+d)}\{(c\tau+d)\theta_1'(z|\tau) + 2\pi\mathrm{i}cz\theta_1(z|\tau)\}.$$

(Exercise: using theorem 6.1 derive the $\boldsymbol{\Gamma}(1)$-transformation law for the general $\theta'_{\alpha\beta}$-function.) It is not difficult to see that the general transformation can always be brought to the form $\theta_k \mapsto \theta_k$ if we involve the inhomogeneous transformations of the argument $z \mapsto (z + s\tau + r)/(c\tau + d)$.

6.3. Differential equations

The next theorem describes completely differential calculus of the classical Jacobi $\vartheta, \theta, \theta'$-series in both the variables z and τ.

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THEOREM 6.7. *Jacobi's $\theta_{\alpha\beta}$, $\theta'(z|\tau)$-series (2.2) and (2.5), with arbitrary integer characteristics (α,β), as functions of the variables z and τ, are differentially closed over the field of $\eta(\tau)$- and $\vartheta^2(\tau)$-constants. Corresponding rules for differentiating are defined by the (α,β)-representation of z-equations (4.8) as*

$$\left.\begin{aligned}\frac{\partial\theta[{}^{\alpha}_{\beta}]}{\partial z}&=\frac{\theta_1'}{\theta_1}\theta[{}^{\alpha}_{\beta}]-(-1)^{[\beta/2]\alpha}\pi\vartheta[{}^{\alpha}_{\beta}]^2\frac{\theta[{}^{\alpha-1}_{0}]\theta[{}^{0}_{\beta-1}]}{\theta_1},\\ \frac{\partial\theta_1'}{\partial z}&=\frac{\theta_1'^2}{\theta_1}-\pi^2\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1}-4\left\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\theta_1,\end{aligned}\right\}\tag{6.10}$$

and by the (α,β)-equivalent of τ-equations (4.9) as

$$\left.\begin{aligned}\frac{\partial\theta[{}^{\alpha}_{\beta}]}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^2}{\theta_1^2}\theta[{}^{\alpha}_{\beta}]+\frac{\mathrm{i}}{2}(-1)^{[\beta/2]\alpha}\vartheta[{}^{\alpha}_{\beta}]^2\theta_1'\frac{\theta[{}^{\alpha-1}_{0}]\theta[{}^{0}_{\beta-1}]}{\theta_1^2}\\ &\quad+\frac{\mathrm{i}}{4}\pi\{\vartheta_3^2\vartheta_4^2\theta_2^2-(\vartheta[{}^{0}_{\beta-1}]^2\theta[{}^{\alpha-1}_{0}]^2+\vartheta[{}^{\alpha-1}_{0}]^2\theta[{}^{0}_{\beta-1}]^2)\vartheta[{}^{\alpha}_{\beta}]^2\}\frac{\theta[{}^{\alpha}_{\beta}]}{\theta_1^2}\\ &\quad+\frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\theta[{}^{\alpha}_{\beta}],\\ \frac{\partial\theta_1'}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\frac{\theta_1'^3}{\theta_1^2}+\frac{3\mathrm{i}}{\pi}\left\{\frac{\pi^2}{4}\vartheta_3^2\vartheta_4^2\frac{\theta_2^2}{\theta_1^2}+\eta+\frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\theta_1'-\frac{\mathrm{i}}{2}\pi^2\vartheta_2^2\vartheta_3^2\vartheta_4^2\frac{\theta_2\theta_3\theta_4}{\theta_1^2}.\end{aligned}\right\}\tag{6.11}$$

The constants $\eta(\tau)$ and $\vartheta^2(\tau)$ form a differential ring $\mathbb{C}_\partial[\eta,\vartheta^2]$ that is defined by the system of polynomial ODEs

$$\left.\begin{aligned}\frac{\mathrm{d}\vartheta[{}^{\alpha}_{\beta}]}{\mathrm{d}\tau}&=\frac{\mathrm{i}}{\pi}\left\{\eta+\frac{\pi^2}{12}((-1)^{\beta}\vartheta[{}^{\alpha-1}_{0}]^4-(-1)^{\alpha}\vartheta[{}^{0}_{\beta-1}]^4)\right\}\vartheta[{}^{\alpha}_{\beta}],\\ \frac{\mathrm{d}\eta}{\mathrm{d}\tau}&=\frac{\mathrm{i}}{\pi}\left\{2\eta^2-\frac{\pi^4}{72}(\vartheta[{}^{\alpha}_{0}]^8+(-1)^{\alpha+\beta}\vartheta[{}^{\alpha}_{0}]^4\vartheta[{}^{0}_{\beta}]^4+\vartheta[{}^{0}_{\beta}]^8)\right\},\end{aligned}\right\}\tag{6.12}$$

where $(\alpha,\beta)\neq(0,0)$ for the second of these equations.

We now comment on some connections of these dynamical systems to the classical properties of the theta series. There are two fundamental algebraic relations between θ-series, i.e. (4.7), and they are, of course, compatible with (6.10) and (6.11) (the proof is a calculation). However, these relations are satisfied not only by the θ-series themselves but by solutions of the equations as well. As before, a simple calculation shows that the corresponding solutions contain three constants A, B and C and have the form

$$\left.\begin{aligned}\theta[{}^{\alpha}_{\beta}]&=C\mathrm{e}^{\pi\mathrm{i}A(2z+A\tau)}\theta[{}^{\alpha}_{\beta}](z+A\tau+B|\tau),\\ \theta_1'&=C\mathrm{e}^{\pi\mathrm{i}A(2z+A\tau)}\{\theta_1'(z+A\tau+B|\tau)-2\pi\mathrm{i}A\theta[{}^{1}_{1}](z+A\tau+B|\tau)\}.\end{aligned}\right\}\tag{6.13}$$

Another point that should be noted is that the heat equation (4.6) must be treated as a *corollary* of the above equations, rather than the reverse, because (6.10)–(6.11) are the *ordinary* differential equations, while (4.6) is an equation in *partial* derivatives. The heat equation has a lot of solutions having nothing to do with theta functions. In order to extract some special, say θ-, solutions to this equation we must impose additional (periodic, differential, modular, etc) conditions on them. Once this has been done for θ-functions, we arrive at the ODEs above, so the initial consideration with the heat equation may be dropped out or 'forgotten'. In other words, (6.10) and (6.11) would be better considered as fundamental differential properties of theta functions.

The third point we would like to mention here is the fact that the modular transformation considered in § 6.2 may be derived as a consequence of these equations rather than as an 'internal property' of the theta series themselves. This can be briefly outlined as follows. Equations (6.10) and (6.11) admit automorphisms that can be found through some linear fractional ansatz. That this ansatz is a linear fractional one can be determined using Lie's symmetries of these equations. This does not even use the fact that the solutions to (6.10) and (6.11) are the θ-series. Moreover, the availability of a discrete automorphism follows from the property that each function θ_k satisfies a *common ODE* (see § 9.1). An analogous property holds for the ϑ-constant equation (5.5). This means that one can find two basic transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ generating the group $\boldsymbol{\Gamma}(1)$. For lack of space we omit the proofs of these statements, but they can be partially derived from the procedure of integration of the equations, which is detailed in § 9.4. Good examples of the application of Lie's symmetries to Chazy's equation (5.7) and many other of Jacobi's 'modular/elliptic' equations are presented in [25, 74]. As with (4.7) and (6.13) we can write solutions to (6.12) that respect Jacobi's identity (6.1). These are separate solutions to (5.5) and (5.7) and they are, of course, known:

$$\left.\begin{aligned} \vartheta\big[{}^{\alpha}_{\beta}\big] &= \frac{1}{\sqrt{c\tau+d}}\vartheta\big[{}^{\alpha}_{\beta}\big]\left(\frac{a\tau+b}{c\tau+d}\right) \quad \text{(Jacobi [53, pp. 186–187])},\\ \eta &= \frac{1}{(c\tau+d)^2}\eta\left(\frac{a\tau+b}{c\tau+d}\right) + \frac{1}{2}\frac{\pi i c}{c\tau+d} \quad \text{(Chazy [24])}, \end{aligned}\right\} \tag{6.14}$$

where (a, b, c, d) are the integration constants, $ad - bc = 1$ and the right-hand sides of these formulae are the ϑ, η-series.

7. Compatible integrability of (4.8)–(4.9)

An important corollary of the preceding section is that ODEs satisfied by Jacobi's functions have a wider class of solutions that are not bound to be the canonical θ-series (2.2) or (6.13). The latter contain only three free constants, while equations (4.8) are of fifth order. In applications, variations of (4.8) may occur in their own right and, hence, the quantities ϑ, being parameters in (4.8), are not bound to be the values of θ-series at zero. For the same reason the quantity η must not necessarily be given by any known expression related to the ϑ, θ-series (see, for example, [78–81, 89]),

$$\eta(\tau) = -\frac{1}{12}\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} = -\frac{1}{4}\frac{\theta_3''(0|\tau)}{\vartheta_3(\tau)} - \frac{\pi^2}{12}\{\vartheta_2^4(\tau) - \vartheta_4^4(\tau)\} = \cdots .$$

Thus, (4.8) and (4.9) may serve as an independent origin of the ϑ, θ-functions, since the equations are no less fundamental objects than their solutions.

7.1. The θ-identities as algebraic integrals

We assume that η and ϑ are the undetermined quantities in (6.10) and (6.11) or in (4.8) and (4.9).

Theorem 7.1. *Nonlinear equations (4.8) and (4.9) are compatible if and only if their coefficients η, ϑ_k satisfy the dynamical system*

$$\left.\begin{aligned}
\frac{\mathrm{d}\vartheta_2}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4)\right\}\vartheta_2,\\
\frac{\mathrm{d}\vartheta_3}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4-3\boldsymbol{B}^4\vartheta_4^4)\right\}\vartheta_3,\\
\frac{\mathrm{d}\vartheta_4}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4-3\boldsymbol{A}^4\vartheta_3^4)\right\}\vartheta_4,\\
\frac{\mathrm{d}\eta}{\mathrm{d}\tau} &= \frac{\mathrm{i}}{\pi}2\eta^2 - \frac{\pi^3}{72}\mathrm{i}\{\vartheta_3^8 + (9\boldsymbol{A}^4\boldsymbol{B}^4 - 6\boldsymbol{A}^4 - 6\boldsymbol{B}^4+2)\vartheta_3^4\vartheta_4^4 + \vartheta_4^8\},
\end{aligned}\right\} \tag{7.1}$$

where constants $\boldsymbol{A}^4$ and $\boldsymbol{B}^4$ are the algebraic (rational) integrals of the systems (4.8) and (4.9), defined as

$$\boldsymbol{A}^4\vartheta_3^2\theta_1^2 = \vartheta_2^2\theta_4^2 - \vartheta_4^2\theta_2^2, \qquad \boldsymbol{B}^4\vartheta_4^2\theta_1^2 = \vartheta_2^2\theta_3^2 - \vartheta_3^2\theta_2^2. \tag{7.2}$$

The three functions $(\vartheta_3, \vartheta_4, \eta)$ are differentially closed.

Proof. Considering the compatibility condition $\theta_{z\tau} = \theta_{\tau z}$ of systems (4.8) and (4.9), we obtain not only certain restrictions on the coefficients η, ϑ but also algebraic relations between the θs. A straightforward check of (7.2) shows that $\boldsymbol{A}, \boldsymbol{B}(\theta;\vartheta)$ are the arbitrary constants, indeed, that is

$$\frac{\partial \boldsymbol{A}}{\partial z} = \frac{\partial \boldsymbol{B}}{\partial z} \equiv 0, \qquad \frac{\partial \boldsymbol{A}}{\partial \tau} = \frac{\partial \boldsymbol{B}}{\partial \tau} \equiv 0,$$

so relations (7.2) do determine two independent algebraic integrals. □

Relations (7.2) generalize the well-known quadratic identities between canonical θ-series

$$\operatorname{sgn}(\nu-\mu)\vartheta_k^2\theta_1^2 = \vartheta_\mu^2\theta_\nu^2 - \vartheta_\nu^2\theta_\mu^2 \quad (k=2,3,4), \tag{7.3}$$

only two of which, say (4.7), are independent [64, 65, 88] since ϑ-constants satisfy the Jacobi identity (6.1). This suggests that there exists one more integral; this is indeed the case.

Theorem 7.2. *The system (7.1) has the algebraic (rational) integral $\mathfrak{A}^4(\vartheta)$:*

$$\mathfrak{A}^4\vartheta_2^4 = \boldsymbol{A}^4\vartheta_3^4 - \boldsymbol{B}^4\vartheta_4^4 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathfrak{A} \equiv 0. \tag{7.4}$$

Integrals (7.2) and (7.4), as generalizations of the famous Jacobi relations (7.3) and (6.1), mean that the various polynomial θ-identities, e.g. (7.3), are the additional constraints on the basic equations (4.8) and (4.9). We do not dwell on degenerations of system (7.1) into elementary functions and consider only a generic situation describing a non-canonical version of θ-functions. The quantities $\boldsymbol{A}$, $\boldsymbol{B}$ define initial conditions for (4.8) and (4.9) and are parameters for system (7.1).

7.2. Canonical θ-series and elliptic functions

We explain how reduction to the canonical case of Jacobi and Weierstrass is performed. This procedure discloses some interesting facts.

Set $\boldsymbol{A} = \boldsymbol{B} = 1$. Then, as follows from (7.1), function η satisfies the Chazy equation (5.7) and the functions $\vartheta_{3,4}$ satisfy the Jacobi equation (5.5). Integral (7.4) is still free. Further setting $\mathfrak{A} = 1$, we can rewrite equations/identities (7.1) in the symmetrical form (5.3). This procedure, however, changes not only the structure of the equations but their algebraic integral as well.

Proposition 7.3. *The algebraic integral* $\mathfrak{A}(\vartheta)$ *of symmetrical equations (5.3) has the form*

$$(\mathfrak{A}^4 - 1)\vartheta_2^4\vartheta_3^4\vartheta_4^4 = (\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4)^3. \tag{7.5}$$

Remark 7.4. This identity should be treated as a correct form of the 'complete Jacobi identity' if determining ODEs for the quantities ϑ, η have been used in the symmetrized form (5.3). It is particularly remarkable that if we consider algebraic integrals (7.4) and (7.5) as algebraic curves in projective coordinates $\vartheta_2 : \vartheta_3 : \vartheta_4$, then we find that curve (7.4) has genus three, while (7.5) is a curve of genus nineteen! In addition to this complication under $\mathfrak{A} \neq 1$, none of the functions η, $\vartheta_{2,3,4}$, or logarithmic derivatives $\ln_\tau \vartheta$ satisfies any equation of third order, as for (5.5) and (5.6). These assertions can be proved using polynomial Gröbner bases techniques over the variables η, ϑ, $\dot{\eta}$, $\dot{\vartheta}$, $\ldots$ [27], but we omit complete proofs for reasons of space. Broadly speaking, we lose a differential closedness of the three functions ϑ_3, ϑ_4 and η.

From the aforementioned it appears that we must do away with the rules of differential computation in the symmetrical form (5.3), (6.12) and redefine them according to (7.1) under $\boldsymbol{A} = \boldsymbol{B} = 1$. Some heuristic arguments lead to the following formulae.

Proposition 7.5. *Integrable rules of differentiating the* ϑ, η*-constants are as follows. The* η*-derivative reads as*

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \frac{\mathrm{i}}{\pi}\left\{2\eta^2 - \frac{\pi^4}{72}(\vartheta_3^8 - \vartheta_3^4\vartheta_4^4 + \vartheta_4^8)\right\}$$

and ϑ*-constants are differentiated as*

$$\frac{\mathrm{d}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}{\mathrm{d}\tau} = \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{24}[(2 - 3\langle\alpha\rangle - 3\langle\beta\rangle)\vartheta_4^4 - (1 - 3\langle\beta\rangle)\vartheta_3^4]\right\}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$$

or, equivalently,

$$\frac{\mathrm{d}\vartheta_k}{\mathrm{d}\tau} = \frac{\mathrm{i}}{\pi}\left\{\eta + \frac{\pi^2}{24}[2(3k^2 - 18k + 25)\vartheta_4^4 - (3k^2 - 15k + 16)\vartheta_3^4]\right\}\vartheta_k$$

under $k = 1, 2, 3, 4$ *(recall that* $\vartheta_1 \equiv 0$*) and, as before, the arbitrary integral* (α, β)*.*

We define $\mathbf{P} := \theta_2^2/\theta_1^2$. We then derive that the identity

$$\mathbf{P}_z^2 = 4\pi^2(\vartheta_4^2\mathbf{P} + \boldsymbol{A}^4\vartheta_3^2)(\vartheta_3^2\mathbf{P} + \boldsymbol{B}^4\vartheta_4^2)\mathbf{P} \tag{7.6}$$

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holds. Therefore, $\mathbf{P}$ is expressible in terms of Weierstrass's $\wp$-function, which is proportional to the ratio of Jacobi's θ-series by formula (4.5):

$$\wp(2z|\boldsymbol{\tau}) = \frac{\pi^2}{12}\left\{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau}) + 3\vartheta_3^2(\boldsymbol{\tau})\vartheta_4^2(\boldsymbol{\tau})\frac{\theta_2^2(z|\boldsymbol{\tau})}{\theta_1^2(z|\boldsymbol{\tau})}\right\}.$$

From this point on we use the notation $\vartheta_{2,3,4}(\boldsymbol{\tau})$, $\theta_k(z|\boldsymbol{\tau})$ for explicit pointing out the canonical ϑ, θ-series, their modulus $\boldsymbol{\tau}$ and the argument z. The same symbols without arguments will denote dynamical variables entering into our ODEs. Bringing (7.6) into the form of Weierstrass's cubic and applying the standard technique [45, 93], we obtain that the elliptic modulus $\boldsymbol{\tau}$ for (7.6) is determined as a root of the transcendental equation (we recall (2.7))

$$J(\boldsymbol{\tau}) = \frac{1}{54}\frac{(\mathfrak{A}^8\vartheta_2^8 + \boldsymbol{A}^8\vartheta_3^8 + \boldsymbol{B}^8\vartheta_4^8)^3}{\mathfrak{A}^8\boldsymbol{A}^8\boldsymbol{B}^8\vartheta_2^8\vartheta_3^8\vartheta_4^8}. \tag{7.7}$$

Here, for symmetry, we have used integral (7.4). Assuming for the moment that $\boldsymbol{\tau}$ has been determined, one derives that the following formula for the ratio $\mathbf{P}$ must exist:

$$\begin{aligned}\frac{\theta_2^2}{\theta_1^2} &= \frac{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})}{3\vartheta_3^2\vartheta_4^2} - \frac{\boldsymbol{A}^4\vartheta_3^4 + \boldsymbol{B}^4\vartheta_4^4}{3\vartheta_3^2\vartheta_4^2} + \frac{\vartheta_3^2(\boldsymbol{\tau})\vartheta_4^2(\boldsymbol{\tau})}{\vartheta_3^2\vartheta_4^2}\frac{\theta_2^2(z+z_0|\boldsymbol{\tau})}{\theta_1^2(z+z_0|\boldsymbol{\tau})}\\ &= \frac{4}{\pi^2}\frac{\wp(2(z+z_0)|\boldsymbol{\tau})}{\vartheta_3^2\vartheta_4^2} - \frac{\boldsymbol{A}^4}{3}\frac{\vartheta_3^2}{\vartheta_4^2} - \frac{\boldsymbol{B}^4}{3}\frac{\vartheta_4^2}{\vartheta_3^2}.\end{aligned}$$

Analogous formulae can be obtained for the quotients θ_k/θ_j without squares. It is not difficult to see that such a variation would lead to the Jacobi elliptic functions $\mathrm{sn} \sim \theta_1/\theta_4$, etc:

$$\left(\frac{\theta_1}{\theta_4}\right)_z^2 = \pi^2\left\{\boldsymbol{A}^4\vartheta_3^2\left(\frac{\theta_1}{\theta_4}\right)^2 - \vartheta_2^2\right\}\left\{\mathfrak{A}^4\vartheta_2^2\left(\frac{\theta_1}{\theta_4}\right)^2 - \vartheta_3^2\right\}. \tag{7.8}$$

We thus infer that a ratio of any two θ-solutions to (4.8) and (4.9) is proportional to a ratio of canonical θ-series with a new modulus $\boldsymbol{\tau}$ determined from (7.7). Before proceeding to further integration, we need a closed formula solution to the problem of finding said modulus.

8. Modular inversion problem and related topics

As mentioned in § 1, no explicit formula realization of the scheme (2.7) is hitherto available if the elliptic curve has been given in the Weierstrassian form

$$y^2 = 4x^3 - ax - b \tag{8.1}$$

or in a more general form

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4. \tag{8.2}$$

An analytic solution to the problem is only known for the canonical Legendre form

$$y^2 = (1 - x^2)(1 - k^2x^2). \tag{8.3}$$

In this case it is given by the famous formula of Jacobi,

$$\tau = \mathrm{i}\frac{\mathsf{K}'(k)}{\mathsf{K}(k)}, \tag{8.4}$$

where K and K' are complete elliptic integrals [33, 52]. By virtue of the classical formula

$$k^2 = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}$$

this solution implies the identity

$$\tau \equiv \mathrm{i}\frac{\mathsf{K}'(\vartheta_2^2(\tau)/\vartheta_3^2(\tau))}{\mathsf{K}(\vartheta_2^2(\tau)/\vartheta_3^2(\tau))} \quad \mathrm{mod}\ \boldsymbol{\Gamma}(2) \quad \forall\tau \in \mathbb{H}^+. \tag{8.5}$$

Elliptic curves are, however, parametrized by the group $\boldsymbol{\Gamma}(1)$, not by $\boldsymbol{\Gamma}(2)$. Moreover, transitions between (8.1), (8.2) and (8.3) require knowledge of the roots of the x-polynomials (8.2) or (8.1) (see [33, 13.5]) and the modulus τ is computed via ratios of certain hypergeometric series. Owing to the fact that the ${}_2F_1(J)$-series converges only inside the unity circle, the ${}_2F_1$-solutions will differ in structure depending on whether $|J| > 1$ or $|J| < 1$. For instance, in the Weierstrassian representation (8.1) the resulting formulae constitute rather cumbersome expressions and, in addition to that, they involve the series in logarithmic derivative of Euler's Γ-function. See, for example, [54, pp. 27–28], the collection of formulae in [33, § 14.6.2, (22)–(27)] or enumeration of all the particular cases in [42, pp. 341–348], though a more compact hypergeometric form of the solution was obtained by Bruns [17]. Such forms of solutions are not convenient in applications since they can not be manipulated analytically. Meanwhile the problem has an elegant solution.

8.1. Analytic formula solution

The modulus τ depends only on the value of the absolute $\mathbf{J}$-invariant, which, in turn, is computed via the coefficients a through the two invariants [4, 45, 88]

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \tag{8.6}$$

according to Klein's definition

$$\mathbf{J} = \frac{g_2^3}{g_2^3 - 27g_3^2}. \tag{8.7}$$

It is well known that the function $J(\tau)$ is related to a hypergeometric equation of the form [23]

$$J(J-1)\psi'' + \tfrac{1}{6}(7J-4)\psi' + \tfrac{1}{144}\psi = 0 \tag{8.8}$$

and generic solutions of such equations are usually designated as ${}_2F_1(\alpha, \beta;\ \gamma|z)$. However, under some restrictions on the parameters (α, β, γ) the solutions are representable in terms of known special functions; this occurs when the ${}_2F_1$-series admits a quadratic transformation [32]. In this case the ${}_2F_1$-equation reduces to an

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equation with two parameters; for example, to Legendre's equation [3, 32, 93]. This is just the case of the ψ-equation (8.8). Its solution is a linear combination

$$\psi = \sqrt[6]{J}\{AP_\nu^\mu(\sqrt{1-J}) + BQ_\nu^\mu(\sqrt{1-J})\}$$

of Legendrian functions with parameters $(\nu,\mu) = (-\frac{1}{2}, \frac{1}{3})$. Recall that the functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ are independent solutions of the linear equation

$$(1-z^2)\psi'' - 2z\psi' + \left\{\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right\}\psi = 0. \tag{8.9}$$

See [32, ch. 3] for definitions and exhaustive properties of these functions. From the aforementioned it appears that

$$\tau = \frac{\boldsymbol{a}P_\nu^\mu(\sqrt{1-J}) + \boldsymbol{b}Q_\nu^\mu(\sqrt{1-J})}{\boldsymbol{c}P_\nu^\mu(\sqrt{1-J}) + \boldsymbol{d}Q_\nu^\mu(\sqrt{1-J})}$$

must hold, where parameters $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ have definite numeric values. In order to find them, we only need any three values of τ under which the quantity $J(\tau)$ has known exact values. There exist many such points and all of them correspond to tori with complex multiplication [88]. For instance,

$$J(\mathrm{i}) = 1, \qquad J(\tfrac{1}{2} + \mathrm{i}\tfrac{1}{2}\sqrt{3}) = 0, \qquad J(\sqrt{2}\mathrm{i}) = \frac{5^3}{3^3}, \qquad \text{etc.}$$

The asymptotic property that $P(z)/Q(z) \to \infty$ as $z \to \infty$ implies that $\boldsymbol{c} = 0$, since $J(\mathrm{i}\infty) = \infty$. It therefore suffices to consider only the two simplest points $J = \{0, 1\}$, and the corresponding values of the functions $P, Q(\sqrt{1-J})$ are easily computed. We obtain these values using (9)–(10) and table cases (22) and (40) in [32, § 3.2].

THEOREM 8.1 (Weierstrassian analogue of (8.4)). *For an elliptic curve in the Weierstrassian form (8.1), its modulus $\tau = \omega'/\omega$, i.e. the solution of the transcendental equation*

$$J(\tau) = \frac{a^3}{a^3 - 27b^2},$$

is given by the expression

$$\tau = \mathrm{i}\frac{P_{-1/6}^0(-\sqrt{\mathfrak{g}})}{P_{-1/6}^0(\sqrt{\mathfrak{g}})}, \quad \mathfrak{g} := 27\frac{b^2}{a^3}. \tag{8.10}$$

If a curve has the generic form (8.2), then $\mathfrak{g} = 1 - \mathbf{J}^{-1}$ is computed according to (8.6) and (8.7).

Proof. We use the relations between Legendrian equations (8.9) with different indices. More precisely, we carry out the above-mentioned quadratic transformation

$$z \mapsto J = \frac{az^2 + b}{cz^2 + d}$$

and demand that the normal form of (8.8) is preserved, that is,

$$\tilde{\psi}'' = -\frac{1}{12^2}\frac{36J^2 - 41J + 32}{(J-1)^2J^2}\tilde{\psi}.$$

The only possibilities we then have are

$$z^2 = 1 - J, \qquad (\nu, \mu) = (-\tfrac{1}{2}, \pm\tfrac{1}{3})$$

and

$$z^2 = 1 - \frac{1}{J}, \qquad (\nu, \mu) = (-\tfrac{1}{6}, 0) \quad \text{or} \quad (\nu, \mu) = (-\tfrac{5}{6}, 0).$$

The coefficients $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ are derived as above. For the first case we obtain that

$$\tau = \left\{ \pi \mathrm{i} \frac{P_\nu^\mu}{Q_\nu^\mu}(\sqrt{1-J}) - 1 \right\} \mathrm{e}^{\pi \mathrm{i}/3}, \tag{8.11}$$

where $(\nu, \mu) = (-\frac{1}{2}, \frac{1}{3})$. A simpler version comes from the second case; for example, from the case $(\nu, \mu) = (-\frac{1}{6}, 0)$. The functions (P, Q) therewith are interchanged. As before, their proportionality coefficient is derived using the tables in [32, § 3.2]. The last step is to use [32, 3.2(10)] to express a certain linear combination of $P(z)$ and $Q(z)$ through a single $P(-z)$; see [3, (8.2.3)]. Under our parameters this relation reads as

$$\pi P^0_{-1/6}(-z) = \pi \mathrm{e}^{\pi \boldsymbol{s} \mathrm{i}/6} P^0_{-1/6}(z) + Q^0_{-1/6}(z), \qquad \boldsymbol{s} := \mathrm{sgn}(\Im z).$$

The multi-valued functions P and Q are defined such that this identity holds for $\Im z \lessgtr 0$ (see also [32, 3.3.1(10)]), but this restriction disappears when passing to the ratio P/Q and the solution simplifies into the ultimate formula (8.10) under arbitrary a and b. □

Result (8.11) was recently announced in [12] and used there in connection with a non-trivial application to the soliton theory when considering a linear spectral problem of the form $\Psi''' + u(x)\Psi' - \frac{1}{2}u'(x)\Psi = \lambda\Psi$.

The function $\sqrt{1 - J(\tau)}$ is single valued and, therefore, (8.11) entails an interesting analogue of (8.5).

Corollary 8.2. *For all $\tau \in \mathbb{H}^+$ the following $\boldsymbol{\Gamma}(1)$-analogue of (8.5) holds:*

$$\tau \equiv \left\{ \pi \mathrm{i} \frac{P_\nu^\mu}{Q_\nu^\mu}\left(\mathrm{i} \frac{\sqrt{27}}{\pi^6} \frac{g_3(\tau)}{\boldsymbol{\eta}^{12}(\tau)} \right) - 1 \right\} \mathrm{e}^{\pi \mathrm{i}/3} \quad \mathrm{mod}\ \boldsymbol{\Gamma}(1)$$

under $(\nu, \mu) = (-\frac{1}{2}, \frac{1}{3})$.

8.2. Consequences

An interrelation between (8.10) and Jacobi's formula (8.4) needs to be understood if the elliptic curve (8.2) has already been given in the canonical form (8.3) as

$$y^2 = (1 - x^2)(1 - \varkappa^2 x^2).$$

Its absolute J-invariant is determined not by the classical expression

$$J = \frac{4}{27} \frac{(k^4 - k^2 + 1)^3}{k^4(k^2 - 1)^2}, \tag{8.12}$$

wherein $k^2 = \varkappa^2$, but by an expression of the form

$$J = \frac{1}{108}\frac{(\varkappa^4 + 14\varkappa^2 + 1)^3}{\varkappa^2(\varkappa^2 - 1)^4}. \tag{8.13}$$

In the standard Legendre–Jacobi theory of (8.3) [42–44, 78–81, 88, 89, 93], the function x is proportional to Jacobi's sn function, whereas Weierstrass's $\wp$ is proportional to the square of sn; hence, $\wp \rightleftarrows \mathrm{sn}$ is not a *birational* transformation. Function x also solves (7.8) under $\boldsymbol{A} = \mathfrak{A} = 1$ and, hence, has periods $2\mathrm{i}\mathsf{K}'$ and $4\mathsf{K}$, so their ratio is equal to $\frac{1}{2}\tau$ rather than τ. To be more precise, one can carry out some standard calculations and derive birational transformations between Weierstrass's $\{\wp, \wp'\}$ and the θ-ratios, i.e. Jacobi's basis

$$\mathrm{sn}(\pi\vartheta_3^2(\tau)z; k) = \frac{\vartheta_3(\tau)}{\vartheta_2(\tau)}\frac{\theta_1(z|\tau)}{\theta_4(z|\tau)},$$
$$\mathrm{cn}(\pi\vartheta_3^2(\tau)z; k) = \frac{\vartheta_4(\tau)}{\vartheta_2(\tau)}\frac{\theta_2(z|\tau)}{\theta_4(z|\tau)},$$
$$\mathrm{dn}(\pi\vartheta_3^2(\tau)z; k) = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}\frac{\theta_3(z|\tau)}{\theta_4(z|\tau)}.$$

Proposition 8.3 (inversion of (4.5)). *All homogeneous $\theta(z|\tau)$-ratios and, consequently, Jacobi's functions $\{\mathrm{sn}, \mathrm{cn}, \mathrm{dn}\}$ are rationally represented via the Weierstrassian $\{\wp, \wp'\}$-functions. The three basic θ-ratios read as*

$$\begin{aligned}
\frac{\theta_2(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_2(\tau)}{\vartheta_1'(\tau)}\frac{\wp'(2z|2\tau)}{\wp(2z|2\tau) - e'(2\tau)} \\
&= 2\frac{\vartheta_2(\tau)}{\vartheta_1'(\tau)}\{\zeta(2z|2\tau) - \zeta(2z - 2\tau|2\tau) - \eta'(2\tau)\}, \\
\frac{\theta_3(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_3(\tau)}{2\vartheta_1'(\tau)}\frac{\wp'(z|(\tau+1)/2)}{\wp(z|(\tau+1)/2) - e((\tau+1)/2)} \\
&= \frac{\vartheta_3(\tau)}{\vartheta_1'(\tau)}\left\{\zeta\left(z\,\middle|\,\frac{\tau+1}{2}\right) - \zeta\left(z-1\,\middle|\,\frac{\tau+1}{2}\right) - \eta\left(\frac{\tau+1}{2}\right)\right\}, \\
\frac{\theta_4(z|\tau)}{\theta_1(z|\tau)} &= -\frac{\vartheta_4(\tau)}{2\vartheta_1'(\tau)}\frac{\wp'(z|\tau/2)}{\wp(z|\tau/2) - e(\tau/2)} \\
&= \frac{\vartheta_4(\tau)}{\vartheta_1'(\tau)}\left\{\zeta\left(z\,\middle|\,\frac{\tau}{2}\right) - \zeta\left(z-1\,\middle|\,\frac{\tau}{2}\right) - \eta\left(\frac{\tau}{2}\right)\right\},
\end{aligned}$$

and nine others are obtained by the three half-period shifts $z \mapsto z + \frac{1}{2}\{1, \tau, \tau+1\}$.

The half-moduli on the right-hand sides of these equations explain the 'distinction' between Weierstrass's modulus for $(\zeta, \wp, \wp')$ and Jacobi's for sn. By this we mean that the transition $\varkappa^2 \rightleftarrows k^2$, i.e. the transformation (8.12)$\rightleftarrows$(8.13), is realized through a duplication of the modulus

$$\varkappa^2 = k^2(2\tau).$$

To put it differently, the map $\tau \mapsto 2\tau$ is a one-to-one transformation (it is just a normalization of $\tau \in \mathbb{H}^+$), and every elliptic curve is uniquely determined by the value of τ (or 2τ). We might, of course, work with the modulus 2τ instead of τ; however, in this case, the classical integral representation of the fundamental group $\boldsymbol{\Gamma}(1)$ must be changed to the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\Gamma}(1) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

As far as we know, this precise correlation between the modular inversions of Weierstrass and Jacobi has not been mentioned in the literature [4, 6, 31, 33, 42–45, 61, 78–81, 88–91, 93].

One further comment is in order. Transformation of the curve (8.2) from the general form into the Weierstrassian one (and vice versa) is performed through the linear fractional change of the variable x [23, 33, 45, 88]. However, this requires knowledge of the roots of x-polynomial (8.2), i.e. the solution of a quartic equation. This is not easy to find if the coefficients of the polynomial do not have definite numerical values. For this reason it would be useful to have a transformation over the field of coefficients $\mathbb{C}(a_0, \ldots, a_4)$, i.e. *without resorting to the solution of any equations*. Such a birational change does indeed exist, and a version of it is shown below. To simplify the formulae we make a trivial transformation, bringing (8.2) to a shortened form with $(a_0, a_1) = (1, 0)$.

Proposition 8.4. *The elliptic curve*

$$y^2 = x^4 - 6\alpha x^2 + 4\beta x + \gamma \tag{8.14}$$

is equivalent to the canonical Weierstrassian form through a birational change over $\mathbb{C}(\alpha, \beta)$ *(no* γ *here). The corresponding Weierstrassian cubic has the form*

$$\boldsymbol{w}^2 = 4\boldsymbol{z}^3 - (3\alpha^2 + \gamma)\boldsymbol{z} - (\alpha^3 - \gamma\alpha - \beta^2),$$

and the transformation between these curves reads as

$$\begin{aligned} \boldsymbol{w} &= x^3 - yx - 3\alpha x + \beta, \\ \boldsymbol{z} &= \tfrac{1}{2}(x^2 - y - \alpha), \\ x &= \frac{1}{2}\frac{\boldsymbol{w} - \beta}{\boldsymbol{z} - \alpha}, \\ y &= \frac{1}{4}\frac{(\boldsymbol{w} - \beta)^2}{(\boldsymbol{z} - \alpha)^2} - 2\boldsymbol{z} - \alpha \\ &= \frac{\beta}{2}\frac{\beta - \boldsymbol{w}}{(\boldsymbol{z} - \alpha)^2} + \frac{1}{4}\frac{9\alpha^2 - \gamma}{\boldsymbol{z} - \alpha} - \boldsymbol{z} + \alpha. \end{aligned}$$

The Legendrian form (8.3) corresponds, in these formulae, to the substitution

$$y \dashrightarrow \frac{y}{k}, \qquad (\alpha, \beta, \gamma) = \left(\frac{k^2 + 1}{6k^2}, 0, \frac{1}{k^2}\right).$$

Proof. The variable x, as a function on the curve (8.14), has two simple poles. If $\mathfrak{u}$ denotes a uniformizer for (8.14), then we may place these poles at the points

$\mathfrak{u} = \{0, v\}$ and, hence, write that

$$\left.\begin{aligned} x &= \zeta(\mathfrak{u}; g_2, g_3) - \zeta(\mathfrak{u} - v; g_2, g_3) - C, \\ y &= \wp(\mathfrak{u} - v; g_2, g_3) - \wp(\mathfrak{u}; g_2, g_3), \end{aligned}\right\} \tag{8.15}$$

that is, $y = \mathrm{d}x/\mathrm{d}\mathfrak{u}$. We manage the parameters g_2, g_3, v and C to turn (8.14), (8.15) into an identity in $\mathfrak{u}$. A calculation using the $\zeta, \wp$-addition theorems yields the constant C and the mutual computability of the parameters $(\alpha, \beta, \gamma) \leftrightarrows (v, g_2, g_3)$. We obtain that $C = \zeta(v; g_2, g_3)$ and

$$\alpha = \wp(v; g_2, g_3), \qquad \beta = \wp'(v; g_2, g_3), \qquad \gamma = g_2 - 3\wp^2(v; g_2, g_3),$$
$$v = \wp^{-1}(\alpha; g_2, g_3), \qquad g_2 = 3\alpha^2 + \gamma, \qquad g_3 = \alpha^3 - \gamma\alpha - \beta^2.$$

These expressions follow from the fact that the pair (α, β) lies on the curve $\beta^2 = 4\alpha^3 - g_2\alpha - g_3$. This also implies the transcendental version

$$y^2 = x^4 - 6\wp(v)x^2 + 4\wp'(v)x + \{g_2 - 3\wp^2(v)\}$$

of algebraic equation (8.14), and that the birational isomorphism stated in the proposition is just a $(\wp, \wp')$-version of (8.15) under the notation $\boldsymbol{z} = \wp(\mathfrak{u})$ and $\boldsymbol{w} = \wp'(\mathfrak{u})$. □

In connection to this, we mention Weierstrass's $\wp, \wp'$-formulae in [42, pp. 118–120], [89, V], [4], [81, pp. 66–67] and in the inaugural dissertation by Biermann [10, § 1] (a student to whom Weierstrass left the problem as an exercise; it became an introductory clause of [10]), wherein the problem of transition between Weierstrass's cubic and quartic equation (8.14) was posed for the first time. The complete form of Biermann's birational transformations was given in [93, 20.6, ex. 2–3]. The formulae presented in [10, p. 6] and [93, 20.6] are, however, rather complicated, so their simplest form (supplemented by a computation of parameters) is given by proposition 8.4, (8.15), and theorem 8.1. It is known that transformations between two forms of one elliptic curve may contain a free parameter, and we can readily introduce it into proposition 8.4. To do this it is sufficient to change $\mathfrak{u} \mapsto \mathfrak{u} - \mathfrak{u}_0$ and make use of the addition theorems in the (re)definitions $\boldsymbol{z} = \wp(\mathfrak{u} - \mathfrak{u}_0)$ and $\boldsymbol{w} = \wp'(\mathfrak{u} - \mathfrak{u}_0)$. Clearly, $\mathfrak{u}_0$ is a more convenient quantity than the 'algebraic parameters' entering into the Biermann–Whittaker–Watson formulae.

The immediate (and known [4, 42–44]) consequence of the technique above is an application to the cubic and quartic equations. Using theorem 8.1 we can present their roots in a completely closed and analytic form. Solutions to the cubic $4x^3 - ax - b = 0$ are obvious; these are the Weierstrassian points $x_\kappa = \{e(\tau), e'(\tau), e''(\tau)\}$ under $\tau = \tau(a, b)$, as above.

Corollary 8.5. *The closed and radical-free formula for the roots x_κ of the quartic equation*

$$x^4 - 6\alpha x^2 + 4\beta x + \gamma = 0$$

reads as

$$x_\kappa = 2\zeta(\tfrac{1}{2}v + \omega_\kappa) - \zeta(v + 2\omega_\kappa), \quad \omega_\kappa = \omega\{0, 1, \tau, \tau + 1\}.$$

Here,

$$v = \int_{\infty}^{\alpha} \frac{\mathrm{d}z}{\sqrt{4z^3 - (3\alpha^2+\gamma)z - \alpha^3 + \gamma\alpha + \beta^2}}, \qquad \omega^2 = \frac{3\alpha^2+\gamma}{\alpha^3 - \gamma\alpha - \beta^2}\frac{g_3(\tau)}{g_2(\tau)},$$

$$\tau = \mathrm{i}\frac{P^0_{-1/6}(-\sqrt{\mathfrak{g}})}{P^0_{-1/6}(\sqrt{\mathfrak{g}})}, \qquad \mathfrak{g} = 27\frac{(\alpha^3-\gamma\alpha-\beta^2)^2}{(3\alpha^2+\gamma)^3}$$

and the arbitrary value of the Weierstrassian elliptic integral is taken.

These formulae provide analytic single-valued expressions for the roots x_κ as functions of the coefficients and, thereby, the problem of multi-valuedness and jumps between roots does not appear here, in contrast to the standard (cumbersome) radical-type formulae.

To summarize briefly, we may conclude that both the transformation between elliptic curves and the modular inversion do not require any auxiliary constructions; in each case the ultimate answer is given by an explicit analytic formula independent of the Weierstrassian or Legendrian representations or the general representation (8.2). Hence, the solution to (7.7) may be thought of as complete and we return to integration of the basic equations (4.8), (4.9) and (7.1).

9. Non-canonical $\boldsymbol{\theta}$-functions

9.1. Exponential quadratic extension of $\boldsymbol{\theta}$-functions

The system of equations (4.8) ($\Leftrightarrow$(6.10)) has fifth order, whereas the Weierstrassian base of functions $(\sigma, \zeta, \wp)$ is of order three. In the canonical case $\boldsymbol{A} = \boldsymbol{B} = 1$, one can easily derive the differential equation satisfied by any of Jacobi's canonical θ-functions,

$$F_z^2 = -4\{F + \tfrac{1}{3}\pi^2(\vartheta_3^4 + \vartheta_4^4)\}\{F + \tfrac{1}{3}\pi^2(\vartheta_2^4 - \vartheta_4^4)\}\{F - \tfrac{1}{3}\pi^2(\vartheta_2^4 + \vartheta_3^4)\}, \tag{9.1}$$

where $F := \ln_{zz}\theta_k(z|\tau) + 4\eta$ and $\vartheta = \vartheta(\tau)$, $\eta = \eta(\tau)$.

In the non-canonical case $\boldsymbol{A} \neq 1 \neq \boldsymbol{B}$, this equation must have an analogue in the form of some differential equation of fifth order. It is not difficult to obtain from (6.10) that each θ-solution of (6.10) satisfies

$$\left.\begin{aligned} &F^2F_{zzz} - 2FF_zF_{zz} + F_z^3 + (F^4)_z = 0,\\ &F := (\ln\theta)_{zz} + 4\{\eta + \tfrac{1}{12}\pi^2(\vartheta_3^4 + \vartheta_4^4)\}. \end{aligned}\right\} \tag{9.2}$$

This important equation, as an equation of third order for the function F, is the generalization of a second-order differential consequence of the canonical Weierstrassian equation $F_z^2 = 4F^3 - g_2F - g_3$, that is, $F_{zzz} = 12FF_z$. The latter *is not a reduction* of (9.2), although (9.2) is also solved by the $\wp$-function

$$F = \wp(\omega|\omega, \omega') - \wp(z + c|\omega, \omega'),$$

with free constants (ω, ω', c). The distinction between them lies in the fact that, as long as we do not require the differential closure of θs, it is sufficient to use one Weierstrassian equation (9.1). In both of these cases the periods $(2\omega, 2\omega')$ and the modulus τ appear as integration constants. Equation (9.2) is easily integrated if we rewrite it in the form

$$\left(\frac{1}{F_z}\left(\frac{F_z^2}{F}\right)_z\right)_z + 8F_z = 0. \tag{9.3}$$

We thus derive the complete integral of (6.10), whatever the values of parameters ϑ, η. Let

$$M := \varkappa^2\{\eta(\boldsymbol{\tau}) + \tfrac{1}{12}\pi^2[\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})]\} - \{\eta + \tfrac{1}{12}\pi^2[\vartheta_3^4 + \vartheta_4^4]\}.$$

Then, some routine computations yield the following result.

THEOREM 9.1. *Differential equations (4.8) and (6.10) have the general solution*

$$\left.\begin{aligned}
\pm\theta_1 &= \frac{\vartheta_2\vartheta_3\vartheta_4}{2\boldsymbol{\eta}^3(\boldsymbol{\tau})}C\theta_1(\varkappa z + B|\boldsymbol{\tau})\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_2 &= \frac{\varkappa\vartheta_2}{\vartheta_2(\boldsymbol{\tau})}C\theta_2(\varkappa z + B|\boldsymbol{\tau})\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_3 &= \frac{\varkappa\vartheta_3}{\vartheta_3(\boldsymbol{\tau})}C\theta_3(\varkappa z + B|\boldsymbol{\tau})\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_4 &= \frac{\varkappa\vartheta_4}{\vartheta_4(\boldsymbol{\tau})}C\theta_4(\varkappa z + B|\boldsymbol{\tau})\mathrm{e}^{2M(z+A)^2},\\
\pm\theta_1' &= \frac{\vartheta_2\vartheta_3\vartheta_4}{2\boldsymbol{\eta}^3(\boldsymbol{\tau})}C\{\varkappa\theta_1'(\varkappa z + B|\boldsymbol{\tau}) + 4M(z+A)\theta_1(\varkappa z + B|\boldsymbol{\tau})\}\mathrm{e}^{2M(z+A)^2},
\end{aligned}\right\} \tag{9.4}$$

where $\{A, B, C, \varkappa, \boldsymbol{\tau}\}$ is a complete set of integration constants and the signs $\pm$ may be freely changed for the arbitrary pair (θ_j, θ_k).

Solution (9.4) shows that its dependence on $\varkappa$ and $\boldsymbol{\tau}$ is rather non-trivial in comparison with its dependence on the constants B, C and the linear exponent e^{Az}; these are easily 'guessable' in (6.13). An additional point to emphasize is that the dependence of both (9.2) and its solution (9.4) on the parameters (ϑ, η) is represented, omitting the trivial multiplicative constant C, through the one essential parameter

$$\tfrac{1}{4}\Lambda = \eta + \tfrac{1}{12}\pi^2(\vartheta_3^4 + \vartheta_4^4). \tag{9.5}$$

Using (9.4), we can, after some algebra, rewrite integrals (7.2) in the '$(\varkappa, \boldsymbol{\tau})$-representation'.

COROLLARY 9.2 (generalization of Jacobi's identities). *The identities*

$$\vartheta_2^2\theta_4^2 - \vartheta_4^2\theta_2^2 = \varkappa^2\frac{\vartheta_3^4(\boldsymbol{\tau})}{\vartheta_3^4}\vartheta_3^2\theta_1^2, \qquad \vartheta_2^2\theta_3^2 - \vartheta_3^2\theta_2^2 = \varkappa^2\frac{\vartheta_4^4(\boldsymbol{\tau})}{\vartheta_4^4}\vartheta_4^2\theta_1^2 \tag{9.6}$$

are satisfied by the non-canonical ϑ, θ-functions.

The canonical Weierstrass–Jacobi case is defined by the restriction $\boldsymbol{A} = \boldsymbol{B} = 1$ and, therefore, is equivalent to the conditions on constants $(\boldsymbol{\tau}, \varkappa)$: $\vartheta(\boldsymbol{\tau}) = \vartheta$, $\eta(\boldsymbol{\tau}) = \eta$, and $\varkappa = \pm 1$.

The following should be particularly emphasized. Notwithstanding the fact that the non-canonical case is realized through the elementary function (the quadratic

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exponent e^{2Mz^2}), it depends non-trivially on the constants $(\varkappa, \boldsymbol{\tau})$ and generates a *transcendental extension* since the canonical σ- and θ-functions are defined up to a linear exponent by (6.13). The dependence of the extension on the parameters ϑ and η is also non-trivial. The quasi-periodicity properties of the θ_1', θ-extensions, i.e. analogs of (2.4), are readily established from (9.4) and theorem 6.1; we do not display them here.

Remark 9.3. A brief mention of the quadratic exponential multiplier in front of the Jacobian function $\Theta(u)$ can be found in [20, pp. 156, 189]. Such a function was also considered by Jacobi himself; [52, pp. 307–318] are devoted to study of the object $\chi(u) = e^{ruu}\Omega(u)$, where $\int_0^u E(u)\,du = \log\Omega(u)$ under the standard Jacobi notation for $E(u)$, and r has a special (not generic) value. The first appearance of the quadratic exponent can be found in *Fundamenta Nova* [52, p. 226]. In connection with certain differential identities and the heat equation for θ-functions, this exponent also appears in [67].

Before going further we pause to comment on the nature of the integrability of (4.8).

9.2. Algebraic integrability of ODEs for θ-functions

Equation (9.3) discloses an interesting feature of the canonical and non-canonical θ-series. We define the term algebraic integrability as a property of differential equations that have solutions in terms of finitely many integrals of algebraic functions.

Theorem 9.4. *Differential equations (4.8) are algebraically integrable upon adjoining an inversion of integrals.*

Proof. By virtue of (9.3) we may write that

$$\int^F \frac{dx}{\sqrt{x(x-\boldsymbol{a})(x-\boldsymbol{b})}} = 2iz + \boldsymbol{c} \quad \Rightarrow \quad F = \Xi(z; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), \tag{9.7}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are some integration constants. Integration is thus completed if we introduce the inversion operation Ξ:

$$\theta = \exp \int^z \left\{ \int^x \Xi(y; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\,dy \right\} dx e^{-\Lambda z^2/2 + \boldsymbol{d}z + \boldsymbol{e}}, \tag{9.8}$$

where $\boldsymbol{d}$, $\boldsymbol{e}$ are new constants. The inversion function Ξ here is, of course, not understood to be a ratio of the θ-series. Integration to the θ_1' is obvious. □

The two-fold integration of the inversion operation in (9.8) can now be reduced to a one-fold integration of the algebraic function and this leads to a meromorphic integral rather than the holomorphic one as in (9.7). By this means we obtain the following non-standard way of introducing the theta function.

Corollary 9.5. *The canonical θ-series, along with its non-canonical extension, can be defined through a meromorphic elliptic integral.*

To prove this it will suffice to make the following change to (9.8):

$$\int^z \Xi(y;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\,\mathrm{d}y = \int^{\Xi(z;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})} \frac{z\,\mathrm{d}z}{\sqrt{z(z-\boldsymbol{a})(z-\boldsymbol{b})}}.$$

To all appearances, Tikhomandritskiĭ [83] was the first to point out a way of defining θ (different from that described above) through a meromorphic integral, but his old note [83] went unnoticed in the literature.

Theorem 9.6. *The dynamical systems defining the non-canonical extensions of the θ-functions (4.8) and (4.9) and ϑ-constants (7.1) are Hamiltonian. They admit the gradient flow forms* $\dot{X} = \Omega\nabla H(X)$ *with Poisson brackets* $\Omega = \Omega(X)$, *which may not be the constant ones.*

This theorem enhances the results on algebraic integrability (theorem 9.4) but its proof and consequences will be given elsewhere because the systems under question require the even-dimensional extensions; these are not obvious in advance. Here, we just give an example that, on the one hand, generalizes the Hamiltonicity of the canonical version of (7.1) described in [14, theorem 13] and, on the other hand, is a rational subcase of a pencil of the brackets found in the same work. Let U, V and W be defined as three vector field components for system (7.1) as

$$\begin{aligned}
U(\vartheta,\eta) &:= \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4-3\boldsymbol{B}^4\vartheta_4^4)\Big\}\vartheta_3,\\
V(\vartheta,\eta) &:= \frac{\mathrm{i}}{\pi}\Big\{\eta + \frac{\pi^2}{12}(\vartheta_3^4+\vartheta_4^4-3\boldsymbol{A}^4\vartheta_3^4)\Big\}\vartheta_4,\\
W(\vartheta,\eta) &:= \frac{\mathrm{i}}{\pi}2\eta^2 - \frac{\pi^3}{72}\mathrm{i}\{\vartheta_3^8 + (9\boldsymbol{A}^4\boldsymbol{B}^4 - 6\boldsymbol{A}^4 - 6\boldsymbol{B}^4+2)\vartheta_3^4\vartheta_4^4 + \vartheta_4^8\},
\end{aligned}$$

and let the $\mathfrak{A}$-integral (7.4) be taken as a Hamilton function. Then, one can show and verify by a straightforward computation the following result.

Theorem 9.7. *System (7.1) admits the gradient flow form*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\vartheta_2\\ \vartheta_3\\ \vartheta_4\\ \eta\end{pmatrix} = \frac{\vartheta_2}{4H}\begin{pmatrix}0 & U & V & W\\ -U & 0 & 0 & 0\\ -V & 0 & 0 & 0\\ -W & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}H_{\vartheta_2}\\ H_{\vartheta_3}\\ H_{\vartheta_4}\\ H_\eta\end{pmatrix}$$

under the Hamilton function

$$H(\vartheta_2,\vartheta_3,\vartheta_4,\eta) = \boldsymbol{A}^4\frac{\vartheta_3^4}{\vartheta_2^4} - \boldsymbol{B}^4\frac{\vartheta_4^4}{\vartheta_2^4}.$$

The corresponding Poisson bracket is degenerated ($\det\Omega(\vartheta,\eta)\equiv 0$) *but single valued.*

In [13] we also showed that the algebraic integrability of θ-functions may be treated as a Liouvillian extension of certain differential fields. It, therefore, has an intimate connection to the Picard–Vessiot solvability of spectral problems defined by linear ODEs.

9.3. Renormalization of θ-functions

Solution (9.4) suggests that we make the renormalization $\theta \mapsto \boldsymbol{\theta}$, where

$$\boldsymbol{\theta}_1 = \theta_1, \qquad \boldsymbol{\theta}_2 = \pi\vartheta_3\vartheta_4\theta_2, \qquad \boldsymbol{\theta}_3 = \pi\vartheta_2\vartheta_4\theta_3, \qquad \boldsymbol{\theta}_4 = \pi\vartheta_2\vartheta_3\theta_4.$$

Then, (4.8) and (9.2) contain the single parameter (9.5). We get that

$$\left.\begin{aligned} &\frac{\partial\boldsymbol{\theta}_1}{\partial z} = \boldsymbol{\theta}_1', \qquad \frac{\partial\boldsymbol{\theta}_1'}{\partial z} = \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1} - \frac{\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1} - \Lambda\boldsymbol{\theta}_1, \\ &\frac{\partial\boldsymbol{\theta}_2}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_2 - \frac{\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1}, \qquad \frac{\partial\boldsymbol{\theta}_3}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_3 - \frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1}, \qquad \frac{\partial\boldsymbol{\theta}_4}{\partial z} = \frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1}\boldsymbol{\theta}_4 - \frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1}. \end{aligned}\right\} \tag{9.9}$$

It immediately follows that the τ-dependence, that is, τ-differentiating the θ-functions (4.9), is also simplified. Setting, for simplicity, that $\tau = 4\pi \mathrm{i}t$, we obtain the system

$$\left.\begin{aligned} \frac{\partial\boldsymbol{\theta}_1}{\partial t} &= \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1} - \frac{\boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1} - \Lambda\boldsymbol{\theta}_1, \qquad \frac{\partial\boldsymbol{\theta}_1'}{\partial t} = \frac{\boldsymbol{\theta}_1'^3}{\boldsymbol{\theta}_1^2} - 3(\boldsymbol{\theta}_2^2 + \Lambda\boldsymbol{\theta}_1^2)\frac{\boldsymbol{\theta}_1'}{\boldsymbol{\theta}_1^2} + 2\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2}, \\ \frac{\partial\boldsymbol{\theta}_2}{\partial t} &= \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_2 - 2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_3\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2} - (\boldsymbol{\theta}_2^2 - \boldsymbol{\theta}_3^2 - \boldsymbol{\theta}_4^2)\frac{\boldsymbol{\theta}_2}{\boldsymbol{\theta}_1^2} - \{\Lambda - \ln_t(\vartheta_3\vartheta_4)\}\boldsymbol{\theta}_2, \\ \frac{\partial\boldsymbol{\theta}_3}{\partial t} &= \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_3 - 2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2} + \boldsymbol{\theta}_4^2\frac{\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1^2} - \{\Lambda - \ln_t(\vartheta_2\vartheta_4)\}\boldsymbol{\theta}_3, \\ \frac{\partial\boldsymbol{\theta}_4}{\partial t} &= \frac{\boldsymbol{\theta}_1'^2}{\boldsymbol{\theta}_1^2}\boldsymbol{\theta}_4 - 2\boldsymbol{\theta}_1'\frac{\boldsymbol{\theta}_2\boldsymbol{\theta}_3}{\boldsymbol{\theta}_1^2} + \boldsymbol{\theta}_3^2\frac{\boldsymbol{\theta}_4}{\boldsymbol{\theta}_1^2} - \{\Lambda - \ln_t(\vartheta_2\vartheta_3)\}\boldsymbol{\theta}_4. \end{aligned}\right\} \tag{9.10}$$

Using (9.9) and (9.10) in this form, the mechanism of integrability for the θ-functions (and analysis at all) becomes very simple. Another point that should be emphasized here is an asymmetry of the equations. It manifests in the fact that we may not use Jacobi's polynomial θ-identities (7.3) in advance since those identities are just particular integrals of (9.9) and (9.10); the latter are constructed from the heat equation $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{zz}$, by definition. Furthermore, (9.10) contain not the variables ϑ but their logarithmic derivatives. Because of this, the compatibility conditions $\boldsymbol{\theta}_{tz} = \boldsymbol{\theta}_{zt}$ for these equations will be

(1) *algebraic* relations between functions $\boldsymbol{\theta}$ over the field of coefficients Λ, $\ln_t \vartheta$ and

(2) the only differential relation containing $\Lambda_t =: \dot{\Lambda}$.

A computation yields that

$$\left.\begin{aligned} &\frac{\dot{\vartheta}_2}{\vartheta_2} + \Lambda = 0, \qquad \frac{\dot{\vartheta}_3}{\vartheta_3} + \Lambda = \frac{\boldsymbol{\theta}_3^2 - \boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1^2}, \qquad \frac{\dot{\vartheta}_4}{\vartheta_4} + \Lambda = \frac{\boldsymbol{\theta}_4^2 - \boldsymbol{\theta}_2^2}{\boldsymbol{\theta}_1^2}, \\ &\dot{\Lambda} - 2\left(\frac{\dot{\vartheta}_3}{\vartheta_3} + \frac{\dot{\vartheta}_4}{\vartheta_4}\right)\Lambda - 2\frac{\dot{\vartheta}_3}{\vartheta_3}\frac{\dot{\vartheta}_4}{\vartheta_4} = 0 \end{aligned}\right\} \tag{9.11}$$

and we see that the parameter $\ln_t \vartheta_2$ is not an independent one but enters into the theory fictitiously, through Λ. The first of (9.11) is in effect the first equation in (7.1). The right-hand sides of the second and third equations in (9.11) are functions of z, but their left-hand sides are functions of t. This leads again to the algebraic integrals (7.2),

$$\boldsymbol{\theta}_3^2 - \boldsymbol{\theta}_2^2 = \boldsymbol{B}^4 \vartheta_4^4 \boldsymbol{\theta}_1^2,$$
$$\boldsymbol{\theta}_4^2 - \boldsymbol{\theta}_2^2 = \boldsymbol{A}^4 \vartheta_3^4 \boldsymbol{\theta}_1^2,$$

and to the second and third equations in (7.1). The fourth equation is obvious. An important point here is the fact that the primary object of the theory (the compatibility condition) manifests itself not as the symmetrical equations (5.3) but as the non-symmetrical ones in (7.1). Furthermore, because systems (9.9) and (9.10) are nonlinear ones, we use an additional differentiation in (9.11) to eliminate $\boldsymbol{\theta}$ completely. For symmetry we take the three quantities

$$2\left(\frac{\dot{\vartheta}_2}{\vartheta_2}, \frac{\dot{\vartheta}_3}{\vartheta_3}, \frac{\dot{\vartheta}_4}{\vartheta_4}\right) =: (X, Y, Z)$$

and obtain at once the equations

$$\left.\begin{aligned} \dot{X} &= (Y+Z)X - YZ, \\ \dot{Y} &= (X+Z)Y - XZ, \\ \dot{Z} &= (X+Y)Z - XY. \end{aligned}\right\} \tag{9.12}$$

This is the famous Darboux–Halphen system [42, p. 331] and one of its consequences is equivalent to (5.6). The scale change $X \mapsto \frac{1}{2}X$ turns it into

$$(\dot{X} - X^2)\dddot{X} = \ddot{X}(\ddot{X} - 4X^3) - 2\dot{X}^2(\dot{X} - 3X^2). \tag{9.13}$$

The applications of system (9.12) are very well known. See, for example, works by Ablowitz et al. [1], [26, p. 577], Takhtajan [77] and Conte [26, pp. 143, 147]. As a vacuum cosmological model these equations come from a particular case of the Bianchi-IX model [21].

Thus, renormalization of the θ-functions trivializes the integration scheme of the defining ODEs and clarifies the interrelations between differential properties of ϑ, θ-functions, the heat equation, the Darboux–Halphen system and their consequences. All the equations are integrated in terms of canonical and non-canonical theta series.

Remark 9.8. We continue the renormalization $\boldsymbol{\theta} \mapsto \tilde{\boldsymbol{\theta}}$ by setting $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}\exp(\frac{1}{2}\Lambda z^2)$. Parameter Λ then disappears in (9.9) as if we have set $\Lambda = 0$ there. The integrability conditions then become the simple algebraic relations

$$Y = X + 2\pi^2\varkappa^2\vartheta_4^4(\boldsymbol{\tau}), \qquad Z = X + 2\pi^2\varkappa^2\vartheta_3^4(\boldsymbol{\tau}),$$

and the only differential equation of Riccati type for function $X(t)$, with variable coefficients $\varkappa = \varkappa(t)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$,

$$\dot{X} = X^2 + 2\pi^2\varkappa^2\{\vartheta_3^4(\boldsymbol{\tau}) + \vartheta_4^4(\boldsymbol{\tau})\}X - 4\pi^4\varkappa^4\vartheta_3^4(\boldsymbol{\tau})\vartheta_4^4(\boldsymbol{\tau}).$$

This equation is also integrable since it is a consequence of (9.11)–(9.13).

9.4. General integrals

Insomuch as (9.12), (5.6) and (9.13) serve both the canonical and non-canonical case, the general integral of (7.1) is a variation of (6.14). We denote the four integration constants for the system (7.1) as (a, b, c, d). We next set $\mathrm{T} := (a\tau + b)/(c\tau + \delta)$, where, as usual, $a\delta - bc = 1$, so δ is not a free parameter.

THEOREM 9.9. *The general solution to the non-canonical dynamical system (7.1) is*

$$\left.\begin{aligned} \vartheta_2 = d\frac{\vartheta_2(\mathrm{T})}{\sqrt{c\tau+\delta}}, \qquad \vartheta_3 = \frac{1}{\boldsymbol{A}}\frac{\vartheta_3(\mathrm{T})}{\sqrt{c\tau+\delta}}, \qquad \vartheta_4 = \frac{1}{\boldsymbol{B}}\frac{\vartheta_4(\mathrm{T})}{\sqrt{c\tau+\delta}}, \\ \eta = \frac{1}{(c\tau+\delta)^2}\left\{\eta(\mathrm{T}) + \frac{\pi^2}{12}[(1-\boldsymbol{A}^{-4})\vartheta_3^4(\mathrm{T}) + (1-\boldsymbol{B}^{-4})\vartheta_4^4(\mathrm{T})]\right\} + \frac{1}{2}\frac{\pi \mathrm{i} c}{c\tau+\delta}, \end{aligned}\right\} \tag{9.14}$$

where $\vartheta_k(\mathrm{T})$ *and* $\eta(\mathrm{T})$ *are understood to be the canonical* ϑ, η*-series (2.1) and (2.8).*

Proof. The verification is straightforward. □

As a corollary we found that the principal parameter of the theory acquires the form

$$\Lambda = \frac{4}{(c\tau+\delta)^2}\left\{\eta(\mathrm{T}) + \frac{\pi^2}{12}[\vartheta_3^4(\mathrm{T}) + \vartheta_4^4(\mathrm{T})]\right\} + \frac{2\pi \mathrm{i} c}{c\tau+\delta};$$

it contains integration constants but does not contain the integrals $\boldsymbol{A}$, $\boldsymbol{B}$. Therefore, it does not depend on whether the canonical or non-canonical case is taken. It also satisfies the third-order equation (5.6), wherein we set $X = \mathrm{i}\Lambda/4\pi$.

In addition to remark 7.4, we note that, in spite of seeming simplicity, the symmetrical system (5.3) is not amenable to integration. Among other things, it is not a compatibility condition for (4.8) and (4.9) and, thus, may not be used as an alternative to the correct and integrable system (7.1). The algebraic integral for the system (5.3), i.e. (7.5), is *nonlinear* in variables ϑ^4.

Regarding remark 5.2, we note that Jacobi's system (5.4) is integrated in its full generality along with the system (7.1). Computations show that Jacobi's $a(h)$ is

$$a = 2I\frac{\vartheta_2^4}{\vartheta_3^4}\left(\frac{\alpha h+\beta}{\gamma h+\delta}\right) + I \quad (\alpha\delta - \beta\gamma = 1),$$

and the remaining functions $b(h)$, $A(h)$ and $B(h)$ are easily computed from (5.4) by differentiation followed by trivial simplification. We thus obtain the complete set of integration constants for (5.4). All this material is discussed at greater length in [14].

We now assume that the quantities η, ϑ in (4.8) are functions of τ according to (7.1), and that the integration constants $\{A, B, C, \varkappa, \boldsymbol{\tau}\}$ are unknown functions of τ. Substituting (9.4) into (4.9), we get a system of ODEs for these functions. The calculations can be reduced in advance since we already have two integrals from (7.2) ($\Leftrightarrow$ (9.6)):

$$\varkappa\frac{\vartheta_3^2(\boldsymbol{\tau})}{\vartheta_3^2} = \boldsymbol{A}^2, \qquad \varkappa\frac{\vartheta_4^2(\boldsymbol{\tau})}{\vartheta_4^2} = \boldsymbol{B}^2. \tag{9.15}$$

These relations describe a point transformation between the algebraic form of the integrals $(\boldsymbol{A}, \boldsymbol{B})$ and their transcendental counterpart, i.e. the pair $(\varkappa, \boldsymbol{\tau})$; the quantities ϑ_3 and ϑ_4 are parameters. Hence, we can obtain the sought-for equations/solutions in a simpler way. Let the dot above a symbol denote the derivative with respect to τ. Then, we derive that

$$\dot{\boldsymbol{\tau}} = \varkappa^2, \qquad \pi\mathrm{i}\frac{\dot{\varkappa}}{\varkappa} = 2M, \qquad \dot{A} = 0, \qquad \dot{B} = A\dot{\varkappa}, \qquad \frac{\dot{C}}{C} = -\frac{\dot{\varkappa}}{\varkappa}.$$

The first two equations have the solution (implicit in $\boldsymbol{\tau}$)

$$\frac{\vartheta_3(\boldsymbol{\tau})}{\vartheta_4(\boldsymbol{\tau})} = \boldsymbol{p}\left\{\frac{\vartheta_3(\mathrm{T})}{\vartheta_4(\mathrm{T})}\right\}^{\boldsymbol{q}}, \qquad \varkappa = \frac{\sqrt{\boldsymbol{q}}}{c\tau+\delta}\frac{\vartheta_2^2(\mathrm{T})}{\vartheta_2^2(\boldsymbol{\tau})}, \tag{9.16}$$

where $\boldsymbol{p}$, $\boldsymbol{q}$ are new integration constants. The remaining equations are easily integrated to give

$$A = \boldsymbol{E}, \qquad B = \frac{\boldsymbol{E}\sqrt{\boldsymbol{q}}}{c\tau+\delta}\frac{\vartheta_2^2(\mathrm{T})}{\vartheta_2^2(\boldsymbol{\tau})} + \boldsymbol{D}, \qquad C = \frac{\boldsymbol{C}}{\sqrt{\boldsymbol{q}}}\frac{\vartheta_2^2(\boldsymbol{\tau})}{\vartheta_2^2(\mathrm{T})}(c\tau+\delta),$$

where $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ are further integration constants. By virtue of (9.15) and (9.14) we set $\boldsymbol{p} = \boldsymbol{q} = 1$. Such a reduction in the number of integration constants is dictated by the fact that integrals (9.15) are integrals of both z- and τ-equations and the equations themselves are nonlinear. The first equation in (9.16) becomes a relation between $\boldsymbol{\tau}$ and T. One can show that this relation is controlled by the standard modular group $\boldsymbol{\Gamma}(4)$ [64, 65]:

$$\frac{\vartheta_3(\boldsymbol{\tau})}{\vartheta_4(\boldsymbol{\tau})} = \frac{\vartheta_3(\mathrm{T})}{\vartheta_4(\mathrm{T})} \quad \Rightarrow \quad \boldsymbol{\tau} = \widehat{\boldsymbol{\Gamma}(4)}(\mathrm{T}).$$

We choose the simplest case $\boldsymbol{\tau}(\tau) = \mathrm{T}$ and, hence, the functions $A(\tau)$, $B(\tau)$, $C(\tau)$ and $\varkappa(\tau)$ are immediately determined. We thus arrive at the ultimate answer.

THEOREM 9.10. *Let the integrability conditions of (4.8) and (4.9), or, equivalently, (6.10) and (6.11), be given by equations (7.1) and their solution (9.14). Then, the general and simultaneous integral of equations (4.8) and (4.9) reads as*

$$\pm\theta_1 = \frac{1}{\boldsymbol{AB}}\frac{d\boldsymbol{C}}{\sqrt{c\tau+\delta}}\theta_1\left(\frac{z+\boldsymbol{E}}{c\tau+\delta} + \boldsymbol{D}\,\middle|\,\frac{a\tau+b}{c\tau+\delta}\right)\exp\left(\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2\right),$$

$$\pm\theta_2 = \frac{d\boldsymbol{C}}{\sqrt{c\tau+\delta}}\theta_2\left(\frac{z+\boldsymbol{E}}{c\tau+\delta} + \boldsymbol{D}\,\middle|\,\frac{a\tau+b}{c\tau+\delta}\right)\exp\left(\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2\right),$$

$$\pm\theta_3 = \frac{1}{\boldsymbol{A}}\frac{\boldsymbol{C}}{\sqrt{c\tau+\delta}}\theta_3\left(\frac{z+\boldsymbol{E}}{c\tau+\delta} + \boldsymbol{D}\,\middle|\,\frac{a\tau+b}{c\tau+\delta}\right)\exp\left(\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2\right),$$

$$\pm\theta_4 = \frac{1}{\boldsymbol{B}}\frac{\boldsymbol{C}}{\sqrt{c\tau+\delta}}\theta_4\left(\frac{z+\boldsymbol{E}}{c\tau+\delta} + \boldsymbol{D}\,\middle|\,\frac{a\tau+b}{c\tau+\delta}\right)\exp\left(\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2\right).$$

Here, $a\delta - bc = 1$ and the formula for θ_1' is a z-derivative of the first of these formulae.

Proof. Straightforward calculation shows that these expressions do indeed solve the systems (4.8) and (4.9) under the arbitrary constants $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}\}$. The coefficients η and ϑ contain the parameters $\{a, b, c, d\}$, which are free. □

Reduction to the canonical case (6.13) is brought about by setting $\boldsymbol{A} = \boldsymbol{B} = d = 1$ and by choosing the transformation

$$\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \boldsymbol{\Gamma}(2),$$

since the group $\boldsymbol{\Gamma}(2)$ does not permute the functions θ_k or ϑ_k. Indeed,

$$\frac{1}{\sqrt{c\tau+\delta}}\theta_k\left(\frac{z+\boldsymbol{E}}{c\tau+\delta} + \boldsymbol{D}\middle|\frac{a\tau+b}{c\tau+\delta}\right)\exp\left(\frac{-\pi\mathrm{i}c}{c\tau+\delta}(z+\boldsymbol{E})^2\right),$$

under the parameters above, becomes

$$\theta_k \sim \frac{1}{\sqrt{2\tau+1}}\theta_k\left(\frac{z+\boldsymbol{E}}{2\tau+1} + \boldsymbol{D}\middle|\frac{1\tau+0}{2\tau+1}\right)\exp\left(\frac{-\pi\mathrm{i}2}{2\tau+1}(z+\boldsymbol{E})^2\right) = \cdots,$$

and, according to Corollary 6.6,

$$\begin{aligned}\cdots &= \frac{\text{const.}}{\sqrt{2\tau{+}1}}\sqrt{2\tau{+}1}\exp\left(\frac{2\pi\mathrm{i}}{2\tau+1}\{z+\boldsymbol{E}+\boldsymbol{D}(2\tau+1)\}^2\right) \\ &\quad\times\theta_k(z+\boldsymbol{E}+\boldsymbol{D}(2\tau{+}1)|\tau)\exp\left(\frac{-2\pi\mathrm{i}}{2\tau+1}(z+\boldsymbol{E})^2\right) \\ &= \text{const.}\theta_k(z+2\boldsymbol{D}\tau+\boldsymbol{E}+\boldsymbol{D}|\tau)\exp(2\pi\mathrm{i}\boldsymbol{D}(2z+2\boldsymbol{D}\tau)),\end{aligned}$$

that is, the 'linearly exponential' form (6.13) under $(A, B) = (2\boldsymbol{D}, \boldsymbol{E}+\boldsymbol{D})$.

10. An application: the sixth Painlevé transcendent

In addition to the applications mentioned in [12–14], in this section we briefly present one more, non-trivial, application to the famous 6th Painlevé equation [26]

$$\begin{aligned} y_{xx} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y_x \\ &\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left\{\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} - \left(\delta - \frac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right\}. \end{aligned} \tag{10.1}$$

A deep connection between this equation and elliptic functions was established by Painlevé himself in [68], wherein he derived a remarkable form of (10.1),

$$-\frac{\pi^2}{4}\frac{\mathrm{d}^2\boldsymbol{z}}{\mathrm{d}\tau^2} = \alpha\wp'(\boldsymbol{z}|\tau) + \beta\wp'(\boldsymbol{z}-1|\tau) + \gamma\wp'(\boldsymbol{z}-\tau|\tau) + \delta\wp'(\boldsymbol{z}-1-\tau|\tau), \tag{10.2}$$

by performing the transcendental change of variables $(y, x) \rightleftarrows (\boldsymbol{z}, \tau)$ such that

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \qquad y = \frac{1}{3} + \frac{1}{3}\frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} - \frac{4}{\pi^2}\frac{\wp(\boldsymbol{z}|\tau)}{\vartheta_3^4(\tau)}. \tag{10.3}$$

In 1995 Hitchin [49] found a beautiful solution to (10.1), which remains the most non-trivial of all those currently known. It corresponds to the parameters $\alpha = \beta = \gamma = \delta = \frac{1}{8}$ and reads parametrically as [49, pp. 74, 78]

$$\wp(\boldsymbol{z}|\tau) = \wp(A\tau + B|\tau) + \frac{1}{2}\frac{\wp'(A\tau + B|\tau)}{\zeta(A\tau + B|\tau) - (A\tau + B)\eta(\tau) + \mathrm{i}\pi A/2}, \tag{10.4}$$

where A, B are free constants. It is not difficult to show that, in the case of Hitchin's parameters $\{\alpha, \beta, \gamma, \delta\}$, equation (10.2) can be written in the θ-function form

$$-\pi^2 \frac{\mathrm{d}^2\boldsymbol{z}}{\mathrm{d}\tau^2} = 4\wp'(2\boldsymbol{z}|\tau) \quad \Leftrightarrow \quad \frac{\mathrm{d}^2\boldsymbol{z}}{\mathrm{d}\tau^2} = 4\pi\boldsymbol{\eta}^9(\tau)\frac{\theta_1(2\boldsymbol{z}|\tau)}{\theta_1^4(\boldsymbol{z}|\tau)}.$$

With complete rules of differential θ-computations in hand, we can obtain their analogue for the Weierstrassian functions and, thereby, automate and simplify manipulation, with all solutions to (10.1) expressible in terms of elliptic or θ-functions.

10.1. Weierstrass's functions and Hitchin's solution

In order to derive rules for the derivatives of the Weierstrassian functions with respect to the half-periods (ω, ω'), one can use the preceding θ-apparatus supplemented with (3.7), (4.4) and (4.5), or, alternatively, transformations between derivatives in (g_2, g_3) and (ω, ω') [35, pp. 263–265]. The (g_2, g_3)-derivatives were considered by Weierstrass [91], [42], [81] and, at about the same time, by Frobenius and Stickelberger [40]. We define a sign of the period ratio as $\mathfrak{s} := \operatorname{sgn}\{\Im(\omega'/\omega)\}$. Applying any of the techniques above and carrying out some simplification, we obtain that the sought-for formulae acquire a very compact and symmetrical form (not presented in the literature).

Theorem 10.1. *The rules for differentiating Weierstrass's $(\sigma, \zeta, \wp, \wp')$-functions are*

$$\mathfrak{s}\frac{\partial\sigma}{\partial\omega} = -\frac{\mathrm{i}}{\pi}\{\omega'(\wp - \zeta^2 - \tfrac{1}{12}g_2 z^2) + 2\eta'(z\zeta - 1)\}\sigma,$$
$$\mathfrak{s}\frac{\partial\sigma}{\partial\omega'} = \frac{\mathrm{i}}{\pi}\{\omega(\wp - \zeta^2 - \tfrac{1}{12}g_2 z^2) + 2\eta(z\zeta - 1)\}\sigma,$$

$$\mathfrak{s}\frac{\partial\zeta}{\partial\omega} = -\frac{\mathrm{i}}{\pi}\{2(\omega'\zeta - z\eta')\wp + \omega'(\wp' - \tfrac{1}{6}g_2 z) + 2\eta'\zeta\},$$
$$\mathfrak{s}\frac{\partial\zeta}{\partial\omega'} = \frac{\mathrm{i}}{\pi}\{2(\omega\zeta - z\eta)\wp + \omega(\wp' - \tfrac{1}{6}g_2 z) + 2\eta\zeta\},$$

$$\mathfrak{s}\frac{\partial\wp}{\partial\omega} = \frac{\mathrm{i}}{\pi}\{2(\omega'\zeta - z\eta')\wp' + 4(\omega'\wp - \eta')\wp - \tfrac{2}{3}\omega' g_2\},$$
$$\mathfrak{s}\frac{\partial\wp}{\partial\omega'} = -\frac{\mathrm{i}}{\pi}\{2(\omega\zeta - z\eta)\wp' + 4(\omega\wp - \eta)\wp - \tfrac{2}{3}\omega g_2\},$$

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$$\mathfrak{s}\frac{\partial\wp'}{\partial\omega} = \frac{\mathrm{i}}{\pi}\{6(\omega'\wp-\eta')\wp' + (\omega'\zeta - z\eta')(12\wp^2 - g_2)\},$$
$$\mathfrak{s}\frac{\partial\wp'}{\partial\omega'} = -\frac{\mathrm{i}}{\pi}\{6(\omega\wp-\eta)\wp' + (\omega\zeta - z\eta)(12\wp^2 - g_2)\}.$$

Setting, in these equations, $\omega = 1$, $\omega' = \tau$ and $\mathfrak{s} = 1$, we arrive at a dynamical system containing the parameter z:

$$\left.\begin{aligned}
\frac{\partial\sigma}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\{\wp - \zeta^2 + 2\eta(z\zeta - 1) - \tfrac{1}{12}g_2z^2\}\sigma,\\
\left.\begin{aligned}
\frac{\partial\zeta}{\partial\tau} &= \frac{\mathrm{i}}{\pi}\{\wp' + 2(\zeta - z\eta)\wp + 2\eta\zeta - \tfrac{1}{6}g_2z\},\\
\frac{\partial\wp}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\{2(\zeta - z\eta)\wp' + 4(\wp-\eta)\wp - \tfrac{2}{3}g_2\},\\
\frac{\partial\wp'}{\partial\tau} &= -\frac{\mathrm{i}}{\pi}\{6(\wp-\eta)\wp' + (\zeta - z\eta)(12\wp^2 - g_2)\},
\end{aligned}\right\}
\end{aligned}\right\} \tag{10.5}$$

where we use the additional right brace to denote the differential closedness of the functions $(\zeta, \wp, \wp')$. It follows that the triple of the functions $\zeta(z|\tau)$, $\wp(z|\tau)$ and $\wp'(z|\tau)$ is differentially closed with respect to both the variables z and τ. We also have to close the derivatives of the coefficients g_2 and η. This is realized by the Halphen system (5.2), but the third variable g_3 is not present in system (10.5) or in the system of z-equations

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = -\wp, \qquad \frac{\mathrm{d}\wp}{\mathrm{d}z} = \wp', \qquad \frac{\mathrm{d}\wp'}{\mathrm{d}z} = 6\wp^2 - \tfrac{1}{2}g_2. \tag{10.6}$$

Therefore, we may treat the classical Weierstrassian relation between $\wp$ and $\wp'$,

$$g_3(\wp, \wp') = 4\wp^3 - g_2\wp - \wp'^2,$$

as the *algebraic* (*polynomial*) *integral* of (10.6) or as the *surface of a constant level* for the dynamical system with variable coefficients, i.e. (10.5).

Another corollary of this system is the fact that each of the functions ζ, $\wp$, $\wp'$ satisfies the ordinary τ-differential equation of second order with variable coefficients g_2, g_3, η, and function σ satisfies an equation of third order. These equations are too large to display here. The function $\boldsymbol{Z} = \zeta(z|\tau) - z\eta(\tau)$, as an example, solves an equation obtainable by elimination of the variable $\wp$ from the two polynomials that follow from (10.5):

$$\{\pi\mathrm{i}\boldsymbol{Z}_\tau + 2(\wp+\eta)\boldsymbol{Z}\}^2 - 4\wp^3 + g_2\wp + g_3,$$
$$\frac{\pi^2}{8}\frac{\boldsymbol{Z}_{\tau\tau}}{\boldsymbol{Z}} + \mathrm{i}\frac{\pi}{2}(\boldsymbol{Z}^2 + \wp - 2\eta)\frac{\boldsymbol{Z}_\tau}{\boldsymbol{Z}} + (\wp+\eta)\boldsymbol{Z}^2 - \wp^2 + \eta\wp - \eta^2 + \tfrac{1}{4}g_2.$$

These polynomials are understood to be equal to zero. They do not explicitly contain the variable z. We do not consider here the θ, θ'-analogs of these equations.

We observe that Hitchin's solution is a function of the quantities $\zeta(A\tau + B|\tau)$, $\wp(A\tau + B|\tau)$ and $\wp'(A\tau + B|\tau)$. Thus, these three functions define a system that is dynamical in its own right and follows from equations (10.5) and (10.6).

PROPOSITION 10.2. *The Hitchin case of Painlevé equation (10.1) is equivalent to the dynamical system*

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}\tau} = \frac{\mathrm{i}}{\pi}\{\boldsymbol{\wp}' + 2(\boldsymbol{\wp} + \eta)\boldsymbol{\zeta}\}, \qquad \frac{\mathrm{d}\boldsymbol{\wp}}{\mathrm{d}\tau} = -\frac{\mathrm{i}}{\pi}\{2\boldsymbol{\zeta}\boldsymbol{\wp}' + 4(\boldsymbol{\wp} - \eta)\boldsymbol{\wp} - \tfrac{2}{3}g_2\}, \tag{10.7}$$

with variable coefficients $\eta = \eta(\tau)$, $g_2 = g_2(\tau)$, $g_3 = g_3(\tau)$. Here,

$$\boldsymbol{\wp}' := \sqrt{4\boldsymbol{\wp}^3 - g_2\boldsymbol{\wp} - g_3}$$

and equations (10.7) may be supplemented by their corollary

$$\frac{\mathrm{d}\boldsymbol{\wp}'}{\mathrm{d}\tau} \equiv -\frac{\mathrm{i}}{\pi}\{6(\boldsymbol{\wp} - \eta)\boldsymbol{\wp}' + (12\boldsymbol{\wp}^2 - g_2)\boldsymbol{\zeta}\}.$$

The general integral of the equations reads as

$$\left.\begin{aligned} \boldsymbol{\zeta} &= \zeta(A\tau + B|\tau) - A\eta'(\tau) - B\eta(\tau), \\ \boldsymbol{\wp} &= \wp(A\tau + B|\tau). \end{aligned}\right\} \tag{10.8}$$

In other words, we may view the sixth Painlevé transcendent under Hitchin's parameters as a pair of 'τ-equations' (10.7), i.e. as an integrable case of the τ-representation (10.2).

Another consequence of the result above is that we can construct the Painlevé–Hitchin integral τ-calculus. Indeed, the first of equations (10.5) suggests that the quantity $\wp - (\zeta - z\eta)^2$ is integrable with respect to τ and expressible through a logarithm of the σ-function. Namely, that

$$-\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sigma = \wp - (\zeta - z\eta)^2 + (\eta^2 - \tfrac{1}{12}g_2)z^2 - 2\eta. \tag{10.9}$$

Owing to (10.7) and the fact that Hitchin's solution has a z-argument of the form $z = A\tau + B$, we can extend (10.9) and consider the objects (10.8), i.e. $\boldsymbol{\wp} - \boldsymbol{\zeta}^2$. Therefore,

$$-\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sigma(A\tau + B|\tau) = \boldsymbol{\wp} - \boldsymbol{\zeta}^2 + f(\tau),$$

where the unknown function $f(\tau)$ is independent of $\boldsymbol{\zeta}$ and $\boldsymbol{\wp}$. It is readily determined by a differentiation followed by the use of differential connection (2.13) between $\boldsymbol{\eta}$ and η. Applying an antiderivative to the last equation, we obtain the nice indefinite integral

$$\frac{\mathrm{i}}{\pi}\int^{\tau}(\boldsymbol{\wp} - \boldsymbol{\zeta}^2)\,\mathrm{d}\tau = \ln\theta_1(\tfrac{1}{2}A\tau + \tfrac{1}{2}B|\tau) - \ln\boldsymbol{\eta}(\tau) + \tfrac{1}{4}\mathrm{i}\pi A^2\tau;$$

this can be checked by a straightforward differentiation and converting everything to ϑ, θ-functions. This integral is a nontrivial Painlevé τ-analogue of the Weierstrassian relation $\int\zeta\,\mathrm{d}z = \ln\sigma$ between the meromorphic ζ-function and the entire function σ.

10.2. θ-function forms of Hitchin's solution

In addition to (10.4), Hitchin also suggested its θ-function form. We reproduce it here in the original notation [49, p. 33] as

$$y(x)=\frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)}+\frac{1}{3}\biggl(1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\biggr)$$
$$+\frac{\vartheta_1'''(\nu)\vartheta_1(\nu)-2\vartheta_1''(\nu)\vartheta_1'(\nu)+4\pi\mathrm{i}c_1(\vartheta_1''(\nu)\vartheta(\nu)-\vartheta_1'^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu)+2\pi\mathrm{i}c_1\vartheta_1(\nu))},$$

where $\nu=c_1\tau+c_2$. The differential properties of θ-functions or conversion formulae like (4.4) and (4.5) suggest that the availability of the higher θ-derivatives here is excessive and we can simplify this solution. Doing so, we obtain the very simple formula

$$y=\frac{\sqrt{x}}{\theta_1^2}\biggl\{\frac{\pi\vartheta_2^2\theta_2\theta_3\theta_4}{\theta_1'+2\pi A\theta_1}-\theta_2^2\biggr\},$$

where the symbols ϑ, θ_1', θ are understood to be equal to

$$\theta_1'=\theta_1'\biggl(A\frac{\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}+B\biggl|\frac{\mathrm{i}\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}\biggr),$$
$$\theta_k=\theta_k\biggl(A\frac{\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}+B\biggl|\frac{\mathrm{i}\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}\biggr),$$
$$\vartheta_2=\vartheta_2\biggl(\frac{\mathrm{i}\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}\biggr).$$

The elliptic integrals K and K' were introduced above. They give an inversion of the first formula in (10.3), that is, an equivalent of Jacobi's formula (8.4)

$$\tau=\mathrm{i}\frac{\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}.$$

Further reading of Hitchin's solution is related to the fact that all solutions to the Painlevé equations are meromorphic functions with fixed branch points. For (10.1) these are the three points $x=\{0,1,\infty\}$, and the general theory of Painlevé equations guarantees availability of what is called the $\boldsymbol{\tau}$-representation [26, p. 165],

$$y\sim x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{\boldsymbol{\tau}_1}{\boldsymbol{\tau}_2}. \tag{10.10}$$

Here, the 'bold tau' is a traditional tau-function notation having nothing in common with modulus τ or modulus $\boldsymbol{\tau}$ in §§ 7.2 and 9. Equations (10.5) lead to

$$\frac{\pi}{2\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\{\zeta(z|\tau)-z\eta(\tau)\}=\wp(z|\tau)+\frac{1}{2}\frac{\wp'(z|\tau)}{\zeta(z|\tau)-z\eta(\tau)}+\eta(\tau).$$

Comparing this property with (10.4), we observe the total logarithmic derivative

$$\wp(z|\tau)=\frac{\pi}{2\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{\zeta(A\tau+B|\tau)-A\eta'(\tau)-B\eta(\tau)}{\boldsymbol{\eta}^2(\tau)}.$$

Replacing (A, B) with $(2A, 2B)$ and transforming the right-hand side of this equation into the θ-functions, we can rewrite the previous parametric form of the solution as

$$y = \frac{2\mathrm{i}}{\pi}\frac{1}{\vartheta_3^4(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{\theta_1'(A\tau+B|\tau)+2\pi\mathrm{i}A\theta_1(A\tau+B|\tau)}{\vartheta_2^2(\tau)\theta_1(A\tau+B|\tau)}, \qquad x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}.$$

The conversion of this result into the original x-representation (10.10) now becomes exercise, but not a simple one.

PROPOSITION 10.3. *The general solution to Hitchin's case of the Painlevé equation (10.1) in the tau-function form (10.10) is*

$$\begin{aligned} y &= x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{\begin{array}{l}\{\theta_1'(A\mathsf{K}(\sqrt{x})/\mathsf{K}'(\sqrt{x})+B|\mathrm{i}\mathsf{K}(\sqrt{x})/\mathsf{K}'(\sqrt{x}))\\ \qquad +2\pi A\theta_1(A\mathsf{K}(\sqrt{x})/K'(\sqrt{x})+B|\mathrm{i}\mathsf{K}(\sqrt{x})/K'(\sqrt{x}))\}^2\end{array}}{(1-x)\theta_1^2(A\mathsf{K}(\sqrt{x})/\mathsf{K}'(\sqrt{x})+B|\mathrm{i}\mathsf{K}(\sqrt{x})/\mathsf{K}'(\sqrt{x}))\mathsf{K}'^2(\sqrt{x})} \\ &= \frac{\mathsf{E}'(\sqrt{x})}{\mathsf{K}'(\sqrt{x})} + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\ln\left\{\frac{\theta_1'}{\theta_1}\left(A\frac{\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}+B\middle|\frac{\mathrm{i}\mathsf{K}(\sqrt{x})}{\mathsf{K}'(\sqrt{x})}\right)+2\pi A\right\}, \end{aligned}$$

where Legendre's elliptic integrals $(\mathsf{K}, \mathsf{K}', \mathsf{E}, \mathsf{E}')$ *[33, 93], as functions of* $\sqrt{x}$, *are differentially closed according to the rules*

$$\begin{aligned} 2\frac{\mathrm{d}\mathsf{K}}{\mathrm{d}x} &= \frac{\mathsf{E}}{x(1-x)} - \frac{\mathsf{K}}{x}, & 2\frac{\mathrm{d}\mathsf{K}'}{\mathrm{d}x} &= \frac{\mathsf{E}'}{x(x-1)} - \frac{\mathsf{K}'}{x-1}, \\ 2\frac{\mathrm{d}\mathsf{E}}{\mathrm{d}x} &= \frac{\mathsf{E}}{x} - \frac{\mathsf{K}}{x}, & 2\frac{\mathrm{d}\mathsf{E}'}{\mathrm{d}x} &= \frac{\mathsf{E}'}{x-1} - \frac{\mathsf{K}'}{x-1}. \end{aligned}$$

Verifying this form of the solution by a direct substitution in (10.1) is a good and rather non-trivial exercise. We do not go into further detail because comprehensive analysis of this $\boldsymbol{\tau}$-function form, including additional motivation, explanations and corollaries, has been detailed in [11], where the complete reference list can also be found.

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References

1 M. J. Ablowitz, S. Chakravarty and L. A. Takhtajan. A self-dual Yang–Mills hierarchy and its reduction to integrable systems in 1+1 and 2+1 dimensions. *Commun. Math. Phys.* **158** (1993), 289–314.

2 M. J. Ablowitz, S. Chakravarty and H. Hahn. Integrable systems and modular forms of level 2. *J. Phys.* A **39** (2006), 15341–15353.

3 M. Abramowitz and I. Stegan. *Handbook of mathematical functions: with formulas, graphs and mathematical tables*, Applied Mathematics Series, vol. 55 (New York: Courier Dover, 1964).

4 N. I. Akhiezer. *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs, vol. 79 (Providence, RI: American Mathematical Society, 1990).

5 N. M. Apostol. *Modular functions and Dirichlet series in number theory* (Springer, 1990).

6 J. V. Armitage and W. F. Eberlein. *Elliptic functions* (Cambridge University Press, 2006).

7 M. Atiyah and N. Hitchin. *The geometry and dynamics of magnetic monopoles* (Princeton, NJ: Princeton University Press, 1988).

8 A. Baruch. *Ueber die Differentialrelationen zwischen den Thetafunktionen eines Arguments* (Goettingen: Kaestner, 1910).

9 E. D. Belokolos, A. I. Bobenko, V. Z. Enol′skii, A. R. Its and V. B. Matveev. *Algebro-geometric approach to nonlinear integrable equations* (Springer, 1994).

10 G. G. A. Biermann. Probelemata quaedam mechanica functionum ellipticarum ope soluta. Dissertation, Friedrich-Wilhelms Universität Berlin (1865).

11 Yu. V. Brezhnev. A $\boldsymbol{\tau}$-function solution of the sixth Painlevé transcendent. *Theoret. Math. Phys.* **161** (2009), 1616–1633.

12 Yu. V. Brezhnev. Transcendental trace formulas for finite-gap potentials. *Theoret. Math. Phys.* **164** (2010), 920–928.

13 Yu. V. Brezhnev. Spectral/quadrature duality: Picard–Vessiot theory and finite-gap potentials. *Contemp. Math.* **563** (2012), 1–31.

14 Yu. V. Brezhnev, S. L. Lyakhovich and A. A. Sharapov. Dynamical systems defining Jacobi's ϑ-constants. *J. Math. Phys.* **52** (2011), 112704.

15 C. Briot and C. Bouquet. *Theorie des fonctions elliptiques* (Paris: Gauthier–Villars, 1875).

16 J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier. *The* 1-2-3 *of modular forms* (Springer, 2008).

17 H. Bruns. Ueber die Perioden der elliptischen Integrale erster und zweiter Gattung. *Math. Ann.* **XXVII** (1886), 234–252.

18 V. M. Buchstaber, D. V. Leykin and M. V. Pavlov. Egorov hydrodynamic chains, the Chazy equation and $\mathrm{SL}(2,\mathbb{C})$. *Funct. Analysis Applic.* **3** (2003), 251–262.

19 A. Cayley. *An elementary treatise on elliptic functions* (London: George Bell and Sons, 1895).

20 A. Cayley. *The collected mathematical papers*, vol. X (Cambridge University Press, 1896).

21 S. Chakravarty and M. J. Ablowitz. Integrability, monodromy, evolving deformations, and self-dual Bianchi IX systems. *Phys. Rev. Lett.* **76** (1996), 857–860.

22 S. Chakravarty, M. J. Ablowitz and P. A. Clarkson. Reductions of self-dual Yang–Mills fields and classical systems. *Phys. Rev. Lett.* **65** (1990), 1085–1087.

23 K. Chandrasekharan. *Elliptic functions* (Springer, 1985).

24 J. Chazy. Sur les équations différentielles dont l'intégrale générale possède une coupure essentielle mobile. *C. R. Acad. Sci. Paris Sér. I* **150** (1910), 456–458.

25 P. A. Clarkson and P. L. Olver. Symmetry and the Chazy equation. *J. Diff. Eqns* **124** (1996), 225–246.

26 R. Conte. *The Painlevé property: one century later*, CRM Series in Mathematical Physics (Springer, 1999).

27 D. Cox, J. Little and D. O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra* (Springer, 2007).

28 H. Durège. *Theorie der elliptischen Funktionen* (Leipzig: Teubner, 1908).

29 J. C. Eilbeck and V. Z. Enol′skii. Bilinear operators and power series for the Weierstrass σ-function. *J. Phys.* A **33** (2000), 791–794.

30 G. Eisenstein. *Mathematische Abhandlungen: Besonders aus dem Gebiete der höheren Arithmetik und der elliptischen Functionen* (Hildesheim: Olms, 1967).

31 A. Enneper. *Elliptische Functionen: Theorie und Geschichte* (Halle: Louis Nebert, 1876).

32 A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher transcendental functions, I, the hypergeometric function, Legendre functions* (New York: McGraw–Hill, 1953).

33 A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher Transcendental Functions, III, elliptic and automorphic functions* (New York: McGraw–Hill, 1955).

34 H. Farkas and I. Kra. *Theta constants, Riemann surfaces and the modular group*, Graduate Studies in Mathematics, vol. 37 (Providence, RI: American Mathematical Society, 2001).

35 A. R. Forsyth. *Theory of functions of a complex variable* (Cambridge University Press, 1893).

36 R. Fricke. Elliptische Funktionen. In *Encyklopädie der mathematischen Wissenschaften*, vol. 2, pp. 181–348 (Leipzig: Teubner, 1913).

37 R. Fricke. *Die elliptischen Funktionen und ihre Anwendungen*, vol. I (Leipzig: Teubner, 1916).

38 R. Fricke. *Die elliptischen Funktionen und ihre Anwendungen*, vol. II (Leipzig: Teubner, 1922).

39 R. Fricke. *Die elliptischen Funktionen und ihre Anwendengen*, vol. III (ed. C. Adelmann, J. Elstrodt and E. Klimenko). (Springer, 2012).

40 F. Frobenius and L. Stickelberger. Ueber die Differentiation der elliptischen Functionen nach den Perioden und Invarianten. *J. Reine Angew. Math.* **XCII** (1882), 311–337.

41 Ch. Gudermann. *Theorie der Modular-Functionen und der Modular-Integrale* (Berlin: Reiner, 1844).

42 G.-H. Halphen. *Traité des fonctions elliptiques et de leurs applications*, vol. I (Paris: Gauthier–Villars, 1886).

43 G.-H. Halphen. *Traité des fonctions elliptiques et de leurs applications*, vol. II (Paris: Gauthier–Villars, 1886).

44 G.-H. Halphen. *Traité des fonctions elliptiques et de leurs applications*, vol. III (Paris: Gauthier–Villars, 1891).

45 H. Hancock. *Lectures on the theory of elliptic functions*, vol. I (London: Chapman and Hall, 1910).

46 J. Harnad. Picard–Fuchs equations, hauptmodulus and integrable systems. In *Integrability: the Seilberg–Witten and Witham equations, Edinburgh, 1998*, Ch. 8, pp. 137–151 (2000).

47 J. Harnad and J. McKay. Modular solutions to equations of generalized Halphen type. *Proc. R. Soc. Lond.* A **456** (2000), 261–294.

48 C. Hermite. *Œuvres de Charles Hermite*, vol. II (Paris: Gauthier–Villars, 1908).

49 N. Hitchin. Twistor spaces, Einstein metrics and isomonodromic deformations. *J. Diff. Geom.* **42** (1995), 30–112.

50 A. Hurwitz. Grundlagen einer independenten Theorie der elliptischen Modulfunctionen und Theorie der Multiplicatorgleichungen erster Stufe. *Math. Ann.* **XVIII** (1881), 528–592.

51 A. Hurwitz and R. Courant. *Vorlesungen üeber algemeine funktionentheorie und elliptische funktionen* (Springer, 1964).

52 C. G. J. Jacobi. *Gesammelte Werke*, vol. I (Berlin: Reimer, 1881).

53 C. G. J. Jacobi. *Gesammelte Werke*, vol. II (Berlin: Reimer, 1882).

54 F. Klein. *Gesammelte Mathematische Abhandlungen*, vol. III (Springer, 1923).

55 F. Klein and R. Fricke. *Vorlesungen über die Theorie der elliptishen Modulfunktionen, I, Grundlegung der Theorie* (Leipzig: Teubner, 1890).

56 F. Klein and R. Fricke. *Vorlesungen über die Theorie der elliptishen Modulfunktionen, II, Fortbildung und Anwendung der Theorie* (Leipzig: Teubner, 1892).

57 L. Koenigsberger. *Vorlesungen über die theorie der Elliptischen Functionen*, vol. II (Leipzig: Teubner, 1874).

58 L. Koenigsberger. *Zur Geschichte der Theorie der elliptischen Transzendenten in den Jahren* 1826–29 (Leipzig: Teubner, 1879).

59 M. Krause. *Theorie der Doppeltperiodischen Functionen*, vol. I (Leipzig: Teubner, 1895).

60 A. Krazer. *Lehrbuch der Thetafunktionen* (Leipzig: Teubner, 1903).

61 D. F. Lawden. *Elliptic functions and applications*, Applied Mathematical Sciences, vol. 80 (Springer, 1989).

62 A. Levin. A geometric interpretation of an infinite product for the Lemniscate constant. *Am. Math. Mon.* **113** (2006), 510–520.

63 R. S. Maier. Nonlinear differential equations satisfied by certain classical modular forms. *Manuscr. Math.* **134** (2011), 1–42.

64 D. Mumford. *Tata lectures on theta*, vol. I, Progress in Mathematics, vol. 28 (Birkhäuser, 1983).

65 D. Mumford. *Tata lectures on theta*, vol. II, Progress in Mathematics, vol. 43 (Birkhäuser, 1984).

66 C. Neumann. *Vorlesungen über Riemann's Theorie der Abeleschen Integrale* (Leipzig: Teubner, 1865).

67 Y. Ohyama. Differential relations of theta functions. *Osaka J. Math.* **32** (1995), 431–450.

68 P. Painlevé. Sur les équations différentialles du second ordre à points critiques fixes. *C. R. Acad. Sci. Paris Sér. I* **143** (1906), 1111–1117.

69 H. Rademacher and E. Grosswald. *Dedekind sums*, The Carus Mathematical Monographs, vol. 16 (The Mathematical Association of America, 1972).

70 S. Ramanujan. On certain arithmetical functions. *Trans. Camb. Phil. Soc.* **22** (1916), 159–184.

71 R. A. Rankin. *Modular forms and functions* (Cambridge University Press, 1977).

72 H. E. Rauch and A. Lebowitz. *Elliptic functions, theta functions and Riemann surfaces* (Baltimore, MD: Williams and Wilkins, 1973).

73 B. Riemann. *Elliptische Functionen, Herausgegeben von H. Stahl* (Leipzig: Teubner, 1899).

74 L. Rosati and M. C. Nucci. A Lie symmetry connection between Jacobi's modular differential equation and Schwarzian differential equation. *J. Nonlin. Math. Phys.* **12** (2005), 144–161.

75 W. G. Scott. Nucleon structure, duality and elliptic theta functions. Technical Report, RAL–TR/1999-087 (Rutherford Appleton Laboratory, 1999).

76 I. Somov. *Foundations for the theory of elliptic functions* (St. Petersburg: Imperial Academy of Sciences, 1850). (In old Russian.)

77 L. A. Takhtajan. A simple example of modular forms as tau-functions for integrable equations. *Theoret. Math. Phys.* **93** (1992), 330–341.

78 J. Tannery and J. Molk. *Éléments de la théorie des fonctions elliptiques*, vol. I (Paris: Gauthier–Villars, 1893).

79 J. Tannery and J. Molk. *Éléments de la théorie des fonctions elliptiques*, vol. II (Paris: Gauthier–Villars, 1896).

80 J. Tannery and J. Molk. *Éléments de la théorie des fonctions elliptiques*, vol. III (Paris: Gauthier–Villars, 1898).

81 J. Tannery and J. Molk. *Éléments de la théorie des fonctions elliptiques*, vol. IV (Paris: Gauthier–Villars, 1902).

82 J. Thomae. *Sammlung von Formeln und Sätzen aus dem Gebiete der elliptischen Funktionen nebst Anwendungen* (Leipzig: Teubner, 1905).

83 M. A. Tikhomandritskiĭ. A note on introduction of Θ-functions to the theory of elliptic functions. In *Communications and proceedings of the mathematical society of the imperial university of Khar′kov*, vol. I, pp. 47–67 (1883). (In Russian.)

84 M. A. Tikhomandritskiĭ. *Theory of elliptic integrals and elliptic functions* (Khar′kov: Silberberg Typography, 1895). (In Russian.)

85 K. P. Tod. Self-dual Einstein metrics from the Painlevé VI equation. *Phys. Lett.* A **190** (1994), 221–224.

86 J. Todd. The lemniscate constants. *Commun. Assoc. Comp. Machinery* **18** (1975), 14–19.

87 P. V. Verhulst. *Traité élémentaire des fonctions elliptiques* (Bruxelles: Hayez, 1841).

88 H. Weber. *Lehrbuch der Algebra, III, Elliptische Funktionen und algebraische Zahlen* (Braunschweig: Vieweg, 1908).

89 K. Weierstrass. *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen* (Göttingen: Kaestner, 1885).

90 K. Weierstrass. *Mathematische Werke*, vol. II (Berlin: Mayer and Müller, 1895).

91 K. Weierstrass. *Mathematische Werke*, vol. V (Berlin: Mayer and Müller, 1904).

92 A. Weil. *Elliptic functions according to Eisenstein and Kronecker* (Springer, 1976).

93 E. T. Whittaker and G. N. Watson. *A course of modern analysis: an introduction to the general theory of infinite processes and of analytic functions, with an account of the principal transcendental functions* (Cambridge University Press, 1996).

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