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YET MORE VERSIONS OF THE FUGLEDE–PUTNAM THEOREM*

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Abstract. We give two types of generalisation of the well-known Fuglede–Putnam theorem. The paper is 'spiced up' with some examples and applications.

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1. Introduction. The Fuglede–Putnam theorem (first proved by B. Fuglede [**6**] and then by C. R. Putnam [**14**] in a more general version) plays a major role in the theory of bounded (and unbounded) operators thanks to its numerous applications. Many authors have worked on it since the papers of Fuglede and Putnam. M. Rosenblum [**17**] gave a simple proof of that theorem using Liouville's theorem. S. Berberian [**2**] showed with a nice matrix operator trick that the Fuglede theorem was actually equivalent to that of Putnam. Then there were various generalizations to non-normal operators (e.g. hyponormal, subnormal, etc; see [**8**] for their definitions). There is a vast literature on this from which we only cite [**3**, **7**, **15**, **20**].

It is also worth mentioning that the author, in a previous work [**11**], gave a generalization of the Fuglede–Putnam theorem where all the operators involved were unbounded.

The classical and most known form of the Fuglede–Putnam theorem is the following.

Theorem A. *If A, N and M are bounded operators such that M and N are normal, then*

$$AN = MA \Longrightarrow AN^* = M^*A.$$

The proof may be found in many textbooks (see e.g. [**4**, Chap. IX, Theorem 6.7], [**8**, p. 67] or [**9**, Problem 152]).

Although there have been many generalizations, most of them went into the same direction, i.e. relaxing the normality hypotheses on M and N. So, the purpose of this paper is to generalize the Fuglede–Putnam theorem, but in two different ways. The first one is to have a fourth operator involved in the equation (this was actually an open question in [**12**]).

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The second one is to remove the normality hypothesis on the operators M and N (as done, for instance, by Okuyama and Watanabe in [**13**]).

The paper is 'spiced up' with some examples and applications.

All operators considered in this paper are assumed to be linear, bounded and defined on a complex Hilbert space $\mathcal{H}$.

An operator A is called self-adjoint if it coincides with its adjoint A^*, normal if it commutes with its adjoint, unitary if $AA^* = A^*A = I$, an isometry if $A^*A = I$, and a co-isometry if A^* is an isometry, i.e. $AA^* = I$.

In the end of this paper the notions of M-hyponormal or dominant operators will be used the definitions of which we give for the sake of convenience of the reader. An operator T is called M-hyponormal if for some constant $M \geq 1$ and all $\lambda \in \mathbb{C}$ one has $(T-\lambda)(T-\lambda)^* \leq M^2(T-\lambda)^*(T-\lambda)$; it is called dominant if for all complex numbers λ there exists a number $M_\lambda \geq 1$ such that $(T-\lambda)(T-\lambda)^* \leq M_\lambda^2(T-\lambda)^*(T-\lambda)$.

Finally, we also recall some known properties which may be found in [**21**].

Definition. Let N be a contraction i.e. $\|N\| \leq 1$. The Julia operator $J(N)$ is defined by

$$J(N) = \begin{pmatrix} (1-NN^*)^{\frac{1}{2}} & N \\ -N^* & (1-N^*N)^{\frac{1}{2}} \end{pmatrix}.$$

Theorem B. *If N is a contraction, then $J(N)$ is unitary.*

Any other property or definition which will be used in this paper will be assumed to be known by the reader. Some general references are [**4**, **8**, **18**].

2. Negative Results.

Without any condition on the operators A and B it seems hopeless that such generalizations hold.

Claim 1. Assume N and M are normal and that A and B are self-adjoint such that $AN = MB$. Then $AN^* = M^*B$.

False. Take

$$N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},\ M = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case A and B are self-adjoint. We do have $AN = MB$ but we do not have $AN^* \neq M^*B$. However, we obtain $BN^* = M^*A$.

Claim 2. Assume N and M are unitary and that A and B are self-adjoint such that $AN = MB$. Then $AN^* = M^*B$.

False again. Take

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Again $AN = MB$, $AN^* \neq M^*B$ but $BN^* = M^*A$.

The 'sad' fact about the previous example is that the operators A and B are also unitary, i.e. the equation $AN^* = M^*B$ does not hold even if all operators involved are unitary.

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All this has made us think that if there has to be a fourth operator involved in these equations, then the order of A and B must be interchanged in the conclusion.

3. Positive Results: First Generalization.

THEOREM 1. *If M is an isometry and N is a co-isometry. If A and B are such that $AN = MB$, then $BN^* = M^*A$.*

Proof. We have

$$A = ANN^* = (AN)N^* = (MB)N^* = M(BN^*).$$

Then

$$M^*A = M^*MBN^* = BN^*. \qquad \square$$

REMARK. Even if the order of the operators A and B does not look as one would have hoped for, but it is actually a generalization since setting $A = B$ allows us to get back to the known version.

REMARK. We also observe that strong conditions are to be imposed if one wants to keep the wanted order of A and B (e.g. some commutativity hypothesis).

REMARK. The hypotheses M being an isometry and N being a co-isometry cannot be dropped.

For if one takes U to be the unilateral shift defined on ℓ^2, then by setting

$$M = N = A = B = U \text{ (and hence } N \text{ is not a co-isometry)},$$

one sees that $AN = MB$ while $BN^* \neq M^*A$.

And if one sets

$$M = N = A = B = U^* \text{ (and hence } M \text{ is not an isometry)},$$

then $AN = MB$ whereas $BN^* \neq M^*A$.

COROLLARY 1. *If M is an isometry and A is an operator such that $A^*M = M^*A^*$, then $AM^* = MA$.*

Proof. Apply Theorem 1 and take the adjoint of the equation obtained. $\square$

REMARK. The previous corollary constitutes in some sense a generalization of Barría's Lemma (see Lemma 2 in [**1**]).

The coming result will be needed in order to state more consequences of the main theorem in this section.

LEMMA A (M. R. EMBRY [**5**]). *If A is such that $0 \notin W(A)$ ($W(A)$ is its numerical range) or $\sigma(A) \cap \sigma(-A) = \emptyset$ ($\sigma(A)$ being the spectrum of A) and $AE = -EA$, where either A or E is normal, then $E = 0$.*

Another application of the previous theorem is as follows.

COROLLARY 2. *Suppose M is an isometry and N is a co-isometry. Assume further that A is a self-adjoint operator such that $0 \notin W(A)$ or that $\sigma(A) \cap \sigma(-A) = \emptyset$. If $AN = MA$, then $N = M$.*

Proof. Since $AN = MA$, then by Theorem 1 we obtain $AN^* = M^*A$. Also by taking the adjoint of the first equation we get $N^*A = AM^*$. Combining these two equations gives us

$$A(N-M)^* = -(N-M)^*A.$$

By Lemma A we conclude that $N = M$. □

Another application concerns an operator equation.

COROLLARY 3. *Assume X is such that $0 \notin W(X)$ (or $\sigma(X) \cap \sigma(-X) = \emptyset$). Let U be the unilateral shift. Then the operator equation*

$$XU = U^*X$$

has no non-zero normal solution X on ℓ^2.

Proof. Assume X is normal. Since U is an isometry,

$$XU = U^*X \Longrightarrow XU^* = UX \text{ and hence } X(U - U^*) = -(U - U^*)X.$$

Lemma A implies that U is self-adjoint which is wrong, establishing the result. □

Obviously a unitary operator is an isometry. Conversely, a self-adjoint (or normal) isometry is unitary. Here we give an answer to a similar question (and the method of proof is similar to that in [**11**]).

PROPOSITION 1. *Assume that A and B are two self-adjoint operators such that $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$. If AB is an isometry, then it is self-adjoint.*

REMARK. Under the same hypothesis as in the previous proposition, one has

$$AB \text{ isometry} \iff AB \text{ unitary}.$$

Proof. Let $N = AB$. Since A and B are self-adjoint,

$$BN = BAB = (BA)B = N^*B.$$

Now as N is an isometry, Theorem 1 gives us

$$BN^* = NB, \text{ i.e. } B^2A = AB^2.$$

Since $\sigma(B) \cap \sigma(-B) \subseteq \{0\}$, applying the spectral mapping theorem to f (the function $x^2 \mapsto x$ defined on $\sigma(B^2)$) allows us to obtain

$$BA = AB, \text{ i.e. } N = N^*,$$

completing the proof. □

COROLLARY 4. *Let A be a projection onto a closed subspace $\mathcal{M}$. If B is self-adjoint, then*

$$AB \text{ isometry} \Longrightarrow B \text{ reduces } \mathcal{M}.$$

Proof. Since AB is an isometry, then the previous result yields $AB = BA$ and hence B reduces $\mathcal{M}$. □

4. Positive Results: Second Generalization. Before giving another generalization, we wish to recall the following standard lemma.

LEMMA 1. *Assume that N is unitary and that A and B are two bounded operators. Then*

$$NA = BN \Longrightarrow NA^* = B^*N.$$

Now we wish to drop the 'unitarity' hypothesis on N. We use a trick of matrix operators via the Julia operator which was defined in the introduction.

THEOREM 2. *Let A and B be two bounded operators. Suppose N is a contraction such that*

$$(1 - N^*N)^{\frac{1}{2}}A = B(1 - NN^*)^{\frac{1}{2}} = (1 - N^*N)^{\frac{1}{2}}A^* = B^*(1 - NN^*)^{\frac{1}{2}} = 0.$$

Then

$$NA = BN \Longrightarrow NA^* = B^*N.$$

REMARK. Before giving the proof, we would like to draw the attention of the readers that there do exist non-unitary operators N and operators A and B satisfying the hypotheses of the previous theorem. Take for instance $A = I$,

$$B = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & \ddots & \\ & & 0 & 0 & 1 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & \ddots & \\ & 0 & 1 & 0 & \ddots & \\ 0 & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

One can check that N is only an isometry and that all other hypotheses are fulfilled.

Proof. Consider the matrix operators defined on $\mathcal{H} \oplus \mathcal{H}$ as

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{N} = J(N) = \begin{pmatrix} (1 - NN^*)^{\frac{1}{2}} & N \\ -N^* & (1 - N^*N)^{\frac{1}{2}} \end{pmatrix}.$$

Then

$$\tilde{N}\tilde{A} = \begin{pmatrix} 0 & NA \\ 0 & (1 - N^*N)^{\frac{1}{2}}A \end{pmatrix} = \begin{pmatrix} 0 & NA \\ 0 & 0 \end{pmatrix} \text{ by hypothesis.}$$

We also have

$$\tilde{B}\tilde{N} = \begin{pmatrix} B(1 - NN^*)^{\frac{1}{2}} & BN \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & BN \\ 0 & 0 \end{pmatrix} \text{ by hypothesis.}$$

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Then $\tilde{B}\tilde{N} = \tilde{N}\tilde{A}$.

But $\tilde{N}$ is unitary so that the previous lemma gives us

$$\tilde{B}^*\tilde{N} = \begin{pmatrix} B^*(1-NN^*)^{\frac{1}{2}} & BN \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & NA^* \\ 0 & (1-N^*N)^{\frac{1}{2}}A^* \end{pmatrix} = \tilde{N}\tilde{A}^*.$$

The remaining unused two hypotheses allow us to get $B^*N = NA^*$ and this completes the proof. □

Corollary 5. *Let A and B be two bounded operators. If N is an isometry such that*

$$B(1-NN^*)^{\frac{1}{2}} = B^*(1-NN^*)^{\frac{1}{2}} = 0, \tag{1}$$

then

$$BN = NA \Longrightarrow B^*N = NA^*.$$

Remark. The condition $B(1-NN^*)^{1/2} = B^*(1-NN^*)^{1/2} = 0$ cannot be completely eliminated in the previous corollary. For instance, if one takes again the unilateral shift U on ℓ^2 and sets $N = B = U$, then N is an isometry and one can check that it does not verify equation 1. If we also set $A = U$, then

$$BN = U^2 = NA \text{ while } B^*N = U^*U \neq UU^* = NA^*.$$

We now give an example satisfying the hypotheses of the previous corollary and not satisfied by any other known version of the Fuglede–Putnam theorem.

Example 1. Consider the infinite matrices

$$N = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & 0 & 0 & & \\ 1 & 0 & \ddots & \ddots & \\ 0 & 1 & 0 & \ddots & \\ & 0 & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots & \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & & & & 0 \\ & 0 & 0 & 0 & 0 & & & \\ & & 0 & 0 & 2 & 0 & & \\ & & & 0 & 0 & 1 & 0 & \\ & & & & 0 & 0 & 1 & \ddots \\ 0 & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & \ddots \\ & & 0 & 0 & 0 & \ddots \\ & & & 0 & \ddots & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}.$$

Then condition (1) is satisfied. Also, we easily have $BN = NA$ and hence $NA^* = B^*N$.

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However, one can also check that B is not normal and neither is A. In fact, B is not hyponormal. It is not even dominant, hence our operators are not covered by the Fuglede–Putnam theorem versions of Radjabalipour [**15**] or Stampfli and Wadhwa [**19**].

Before we give the last remark in this paper, we would like to recall the following theorem.

THEOREM C (Okuyama and Watanabe [**13**]). *Let A and B be two bounded linear operators. Let N be a partial isometry. If*

(1) $NA = BN$,
(2) $\|A\| \geq \|B\|$,
(3) $(N^*N)A = A(N^*N)$ *and*
(4) $N(\|A\|^2 - AA^*)^{\frac{1}{2}} = 0$,

then $NA^* = B^*N$.

REMARK. Okuyama and Watanabe [**13**] gave in their paper an example which satisfied their theorem and did not satisfy any of the Fuglede–Putnam theorem versions. So our example (the foregoing one), in its turn, satisfies all hypotheses (in Theorem C) but (4) and the Fuglede–Putnam conclusion still holds.

A Question. An interesting question is the following: Since (as alluded to in the introduction) there have been many generalizations of the Fuglede–Putnam theorem to non-normal operators, then can we prove it for posinormal operators? This notion appeared in [**10**, **16**].

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