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DERIVED SUBSPACES OF METRIC SPACES

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ABSTRACT. It is shown that the boundary of the set of accumulation points of a metrizable space X is compact iff X has a compatible metric d such that $d(A, B) > 0$ whenever A and B are disjoint closed subsets of X, each of which is disjoint from the set of accumulation points.

Let acc(X) denote the set of all accumulation points of a topological space X. Metrizable spaces X for which acc(X) is compact have quite interesting properties and have been studied in all the references at the end of the paper. For example, in [5] Nagata showed that the finest uniformity for a space X is metric iff X is metrizable and acc(X) is compact, and in [7] Willard showed that every Hausdorff quotient of X is metrizable iff X is metrizable and acc(X) is compact. Using Theorems 3 and 4 of [5], it is not difficult to show that for a metrizable space X, acc(X) is compact iff X has a compatible metric d such that $d(F, H) > 0$ whenever F and H are disjoint closed subsets of X. The purpose of this note is to establish a similar characterization of those metrizable spaces X for which the boundary of acc(X) is compact.

THEOREM. Let X be a metrizable space and let $A = \mathrm{acc}(X)$. The following are equivalent.

(i) $Bd(A)$ is compact.

(ii) There exists a compatible metric d for X such that if F and H are disjoint closed subsets of $X - \mathrm{int}(A)$, then $d(F, H) > 0$.

(iii) There exists a compatible metric d for X such that if F and H are disjoint closed subsets of X, each of which are also disjoint from A, then $d(F, H) > 0$.

Proof. Let $B = Bd(A)$ and assume that B is compact. We shall show that (ii) holds.

Suppose that $B = \varnothing$. Let s be any metric on A which is bounded by 1 and define a metric d for X by letting $d(x, y) = 1$ if either x or y belong to $X - A$ and $x \neq y$ and letting $d(x, y) = s(x, y)$ if x and y belong to A. It is easy to see that d is a compatible metric for X which satisfies condition (ii).

Now consider the case in which the boundary of A is not empty. Let t be any compatible metric for the space X. For each point x in $X - A$, choose a point

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$b(x)$ in B such that $t(x, b(x))<2t(x, B)$. Define a metric d on X in the following way. If x and y are distinct points in $X-A$, let

$$d(x, y)=t(x, b(x))+t(b(x), b(y))+t(b(y), y).$$

If x is a point in $X-A$ and y is a point in A, let

$$d(x, y)=t(x, b(x))+t(b(x), y).$$

If x and y are both points in A, let $d(x, y)=t(x, y)$. It is not difficult to show that d is a metric for the set X. We shall show that d is compatible with the topology for X by showing that d and t are equivalent metrics.

Since $t(x, y)\le d(x, y)$ for all x and y in X, if $d(y_n, y)\to 0$ for a point y and sequence $\{y_n\}$, then $t(y_n, y)\to 0$ also. Now assume that $\{x_n\}$ is a sequence in X and x a point such that $t(x_n, x)\to 0$. We shall show that $d(x_n, x)\to 0$.

If x is in $X-A$, then $\{x\}$ is open and since $x_n\to x$, necessarily $x_n=x$ for all sufficiently large values of n, so that $d(x_n, x)\to 0$.

Now suppose that x is in A. If x_n is in A for all n, then $d(x_n, x)\to 0$ since $d(x_n, x)=t(x_n, x)$ for all n.

Suppose x is in A and that x_n is in $X-A$ for all n. Then

$$d(x, x_n)=t(x, b(x_n))+t(b(x_n), x_n).$$

Since $t(x_n, x)\to 0$, the point x must belong to B in this case. Given any point x_n, the point $b(x_n)$ was chosen so that $t(x_n, b(x_n))<2t(x_n, B)$. In particular, $t(x_n, b(x_n))<2t(x_n, x)$ since $x\in B$. Since $t(x_n, x)\to 0$, this implies that $t(x_n, b(x_n))\to 0$ as $n\to\infty$. Since $t(x, b(x_n))\le t(x, x_n)+t(x_n, b(x_n))$ and since both $t(x, x_n)\to 0$ and $t(x_n, b(x_n))\to 0$, we also have $t(x, b(x_n))\to 0$. It follows that $d(x, x_n)\to 0$ as $n\to\infty$.

Now in the general case in which $x\in A$ and $t(x_n, x)\to 0$, the sequence $\{x_n\}$ splits into the subsequence $\{y_n\}$ of those terms which are in A and $\{z_n\}$ of those terms which are in $X-A$. By the arguments given above, we have $d(y_n, x)\to 0$ and $d(z_n, x)\to 0$, so that $d(x_n, x)\to 0$, completing the proof that d and t are equivalent metrics for X.

Now let F and H be disjoint closed subsets of X with F and H both disjoint from $\operatorname{int}(A)$. Assume that $d(F, H)=0$. Since B is compact, $d(F\cap B, H\cap B)>0$, so there exists a sequence $\{x_n\}$ in $F-A$ and a sequence $\{y_n\}$ in $H-A$ such that $d(x_n, y_n)\to 0$. By the definition of the metric d it follows that $t(x_n, b(x_n))\to 0$, $t(b(x_n), b(y_n))\to 0$ and $t(b(y_n), y_n)\to 0$. Since B is compact and $b(x_n)$ and $b(y_n)$ belong to B for all n, the sequences $\{b(x_n)\}$ and $\{b(y_n)\}$ must have a common cluster point, say b. But $t(x_n, b(x_n))\to 0$ then implies that b is also a cluster point of $\{x_n\}$, that is, that $b\in F$. Similarly, $t(y_n, b(y_n))\to 0$ implies that $b\in H$. This contradicts the fact that F and H are disjoint and completes the proof that (i) implies (ii).

That (ii) implies (iii) is immediate.

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To show that (iii) implies (i), assume that B is not compact. Choose a discrete sequence $\{b_n\}$ in B such that $b_n \neq b_m$ when $n \neq m$; the set $B' = \{b_n : n = 1, 2, \ldots\}$ is closed in X. Let d be any compatible metric for X. Choose sequences $\{x_n\}$ and $\{y_n\}$ in $X - A$ such that $d(x_n, b_n) \to 0$, $d(y_n, b_n) \to 0$ and such that $x_i \neq y_j$ for all natural numbers i and j. Let $F = \{x_n : n = 1, 2, \ldots\}$ and $H = \{y_n : n = 1, 2, \ldots\}$. The sets F and H are disjoint closed subsets of X which are also disjoint from A, and $d(F, H) = 0$, so that (iii) does not hold, completing the proof.

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