1
Analytical Methods

1.1 Setting and basic terminology

We will deal with maps

$$x \mapsto f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, i.e., has all required continuous partial derivatives with respect to its arguments.[1] To simplify our presentation, we assume that f is a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, so that its inverse $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is globally defined and smooth. A sequence of points $x_n \in \mathbb{R}^n$ is called an *orbit* of (1.1) if

$$x_{k+1} = f(x_k), \qquad k \in \mathbb{Z}.$$

One says that $x_0 \in \mathbb{R}^n$ is a starting point of the orbit. In general, an orbit can be finite, i.e., undefined starting from some (positive or negative) k. The part of an orbit with $k \geq 0$ is called the *forward orbit*. If f is invertible, the *backward orbit* is uniquely defined.

A *fixed point* x_0 satisfies $f(x_0) = x_0$. The orbit starting at a fixed point x_0 is constant:

$$\ldots, x_0, x_0, x_0, \ldots \ .$$

A nonconstant K-periodic orbit $\{x_k\}$, i.e., such that

$$x_K = x_0,$$

where $K > 1$ is the minimal integer possible, is called a *cycle* with *period* K or *K-periodic orbit*. A cycle with period K defines a set of K distinct points,

$$C = \left\{x_0, f(x_0), f^{(2)}(x_0), \ldots, f^{(K-1)}(x_0)\right\},$$

[1] If f is only defined on an open region $U \subset \mathbb{R}^n$ and one is interested in studying dynamics generated by (1.1), then, usually, it is possible to extend f to the whole state space and study a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and restrict to U.

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with $x_0 = f^{(K)}(x_0)$. Here, $f^{(k)}$ denotes the composition of k copies of f, also called the kth *iterate* of f. Each point in C is a fixed point of $f^{(K)}$.

A subset $S \subset \mathbb{R}^n$ is said to be *invariant* if any orbit starting at $x_0 \in S$ is located in S, i.e., $f^{(k)}(x_0) \in S$ for all $k \in \mathbb{Z}$. Fixed points and cycles are the simplest invariant sets, but more complicated ones exist, e.g., *invariant manifolds* (closed curves, tori) and *fractal invariant sets*.

Let S be an invariant set of a diffeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$. The set

$$W^s(S) := \{x \in \mathbb{R}^n : f^{(k)}(x) \to S \text{ as } k \to \infty\}$$

is called the *stable set* of S. It is composed of all points converging to S under iteration of f. Similarly,

$$W^u(S) := \{x \in \mathbb{R}^n : f^{(-k)}(x) \to S \text{ as } k \to \infty\}$$

is called the *unstable set* of S.

A fixed point x_0 of (1.1) is called *hyperbolic* if the Jacobian matrix $A = f_x(x_0) := Df(x_0)$ is nonsingular and has no eigenvalues with $|\lambda| = 1$. If x_0 is hyperbolic, A has n_s *stable eigenvalues* with $|\lambda| < 1$ and n_u *unstable eigenvalues* with $|\lambda| > 1$ with $n_s + n_u = n$. Denote by E^s (E^u) the generalized invariant eigenspace of A corresponding to the union of its stable (unstable) eigenvalues.

Theorem 1.1 (Local Stable and Unstable Invariant Manifolds (Palis and de Melo, 1982)) *Near a hyperbolic fixed point x_0, the map* (1.1) *has two smooth embedded invariant manifolds $W^s(x_0)$ and $W^u(x_0)$ that are tangent at x_0 to the eigenspaces E^s and E^u, respectively.*

The next key notion is that of the *equivalence* of maps. We introduce another map

$$x \mapsto g(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $g: \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth. The maps (1.1) and (1.2) are *topologically equivalent* if there is a homeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ that maps orbits of (1.1) onto orbits of (1.2). Analytically, this means that

$$f(x) = h^{-1}(g(h(x)), \quad x \in \mathbb{R}^n,$$

or, equivalently, but easier in practice,

$$h(f(x)) = g(h(x)), \quad x \in \mathbb{R}^n.$$

The number and stability of invariant sets are the same for both maps. If the homeomorphism h is a diffeomorphism, we call the two maps *smoothly equivalent*. One can consider two smoothly equivalent maps as one map written in two different coordinate systems. If we restrict our attention to an open neighborhood U of a fixed point or a cycle, we say that the corresponding equivalence is *local*.

Theorem 1.2 (Grobman–Hartman) *Consider a smooth map*

$$x \mapsto Ax + F(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where A is an $n \times n$ matrix and $F(x) = O(\|x\|^2)$. If $x = 0$ is a hyperbolic fixed point of (1.3)*, then* (1.3) *is locally topologically equivalent near this point to its linearization*

$$x \mapsto Ax, \quad x \in \mathbb{R}^n.$$

Consider now a family of maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^p, \tag{1.4}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is smooth. The parameter point $\alpha_0 \in \mathbb{R}^p$ is called a *bifurcation point* if arbitrarily close to it there is $\alpha \in \mathbb{R}^p$ such that (1.4) is not topologically equivalent to

$$x \mapsto f(x, \alpha_0), \quad x \in \mathbb{R}^n,$$

in some domain $U \subset \mathbb{R}^n$. The appearance of a topologically nonequivalent map under a variation of parameters is called a *bifurcation*. Our main goal in this book is to classify and study local bifurcations occurring in generic one- and two-parameter families of smooth maps, and to provide the necessary analytical and numerical tools to analyze these bifurcations in concrete maps. Here, "local" means happening in a small but fixed neighborhood of a fixed point. The minimal number of parameters required to meet a particular bifurcation in a generic family (1.4) is called the *codimension* of the bifurcation. Hence, we focus on a systematic study of local codim 1 and 2 bifurcations. It must be noted immediately that global bifurcations of codim 1 involving cycles and more complicated invariant sets may occur near local codim 2 bifurcation points. We treat the most important aspects of these global bifurcations.

It should also be clear that hyperbolic fixed points do not bifurcate. Indeed, in a smooth family (1.4), a hyperbolic fixed point can only move slightly under small parameter variations, and the local orbit structure near this point remains unchanged due to the Grobman–Hartman Theorem 1.2. Thus, only non-hyperbolic fixed points require further analysis.

1.2 Center manifold reduction

Consider a smooth map

$$x \mapsto Ax + F(x), \quad x \in \mathbb{R}^n, \tag{1.5}$$

where A is a nonsingular $n \times n$ matrix and $F(x) = O(\|x\|^2)$. This map has a fixed point $x = 0$ and we would like to study the orbit structure near the origin. Now, suppose that $x = 0$ is a nonhyperbolic fixed point, so that there are in general $n_c > 0$ *critical* eigenvalues of A satisfying $|\lambda| = 1$, n_s *stable* eigenvalues with $|\lambda| < 1$, and n_u *unstable* eigenvalues with $|\lambda| > 1$. Counting these eigenvalues with their algebraic multiplicities, we have $n_c + n_s + n_u = n$. Let E^c, E^s and E^u be the generalized invariant eigenspaces of A corresponding to the critical, stable, and unstable eigenvalues. The following direct-sum decomposition holds: $\mathbb{R}^n = E^c \oplus E^s \oplus E^u$.

It turns out that the map (1.5) possesses an invariant manifold near $x = 0$.

Theorem 1.3 (Center Manifold) *There exists an invariant manifold W_0^c locally defined near $x = 0$ for* (1.5) *with* $\dim W_0^c = n_c$ *that is tangent to E^c at $x = 0$ and has the same (finite) smoothness as F.*

The manifold W_0^c is called the *center manifold*. In general, it is not unique. The map (1.5) is smoothly (linearly) equivalent to the map

$$\begin{pmatrix} \xi \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} A_0\xi + F_0(\xi, u, v) \\ A_1 u + F_1(\xi, u, v) \\ A_2 v + F_2(\xi, u, v) \end{pmatrix}, \tag{1.6}$$

where the components of $\xi \in \mathbb{R}^{n_c}$ are coordinates in E^c, the components of $u \in \mathbb{R}^{n_s}$ are coordinates in E^s, and the components of $v \in \mathbb{R}^{n_u}$ are coordinates in E^u. According to Theorem 1.3, the center manifold W_0^c can be represented locally by a graph of a smooth mapping

$$H\colon \mathbb{R}^{n_c} \to \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}, \quad H(0) = 0, H_\xi(0) := DH(0) = 0$$

(see Figure 1.1). In this setting, we have the following theorem.

Theorem 1.4 (Reduction Principle) *The map* (1.6) *is locally topologically equivalent near the origin to*

$$\begin{pmatrix} \xi \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} A_0\xi + F_0(\xi, H(\xi)) \\ A_1 u \\ A_2 v \end{pmatrix}. \tag{1.7}$$

This theorem states that dynamics along the stable and unstable subspaces are separated and are determined by the linear maps $u \mapsto A_1 u$ and $v \mapsto A_2 v$,

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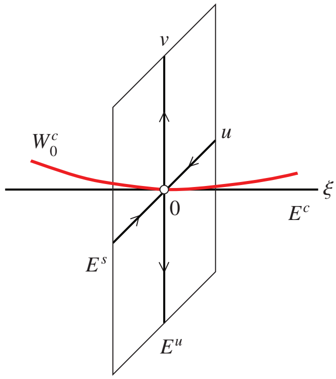

Figure 1.1 Critical center manifold W^c_0 for $n_c = n_s = n_u = 1$.

so that the center manifold is *normally hyperbolic*. These dynamics are trivial since all eigenvalues of A_1 satisfy $|\lambda| < 1$, while for those of A_2 we have $|\lambda| > 1$. The dynamics on the center manifold is governed by the nonlinear n_c-dimensional map $\xi \mapsto A_0\xi + f_0(\xi, H(\xi))$, where the linear part has all its n_c eigenvalues on the unit circle. This map is called the *restriction* of (1.6) to its center manifold W^c_0. While the center manifold may not be unique, all such manifolds are represented by functions H having coinciding Taylor expansions. This leads to restricted equations, which can only differ by "flat" functions.

Thus, the analysis of the map (1.5) reduces to that of its restriction to the center manifold. Since the number of critical eigenvalues is usually small, we achieve a considerable simplification.

For a smooth family of smooth maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^p, \tag{1.8}$$

where $f(x, 0) = Ax + F(x)$ as in (1.5), there exists a smooth continuation of W^c_0 for small $|\alpha|$, i.e., a family of locally defined invariant normally hyperbolic manifolds $W^c_\alpha \subset \mathbb{R}^n$, carrying all interesting local dynamics of $x \mapsto f(x, \alpha)$. This can be shown by considering the *extended* map

$$\begin{pmatrix} x \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} f(x, \alpha) \\ \alpha \end{pmatrix}, \quad (x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{1.9}$$

and applying Theorem 1.3 to this map. Indeed, for this map, the point $(x, \alpha) = (0, 0)$ is nonhyperbolic with $n_c + p$ eigenvalues on the unit circle. It has therefore a $(n_c + p)$-dimensional center manifold with n_c-dimensional α-slices defining W^c_α.

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1.3 Normal forms

A smooth map near a fixed point, e.g., the restriction of some map to a center manifold, can be simplified by nonlinear transformations. There is a systematic method to remove as many terms as possible from the Taylor expansion of the map. This method is called *Poincaré normalization*.

Let H_k be the linear space of vector-valued functions whose components are homogeneous polynomials of order k. Consider a smooth map

$$x \mapsto Ax + f^{(2)}(x) + f^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.10}$$

where $f^{(k)} \in H_k$ for $k \geq 2$. Introduce new coordinates $y \in \mathbb{R}^n$ by the substitution

$$x = y + h^{(m)}(y), \tag{1.11}$$

where $h^{(m)} \in H_m$ for some fixed $m \geq 2$. At this moment, $h^{(m)}$ is an arbitrary function from H_m. Notice that the substitution (1.11) is close to the identity near the origin and thus invertible there, and the inverse transformation

$$y = x - h^{(m)}(x) + O(\|x\|^{m+1}) \tag{1.12}$$

is also smooth. In the new coordinates y, the map (1.10) has the form

$$y \mapsto Ay + \sum_{k=2}^{m-1} f^{(k)}(y) + \left[f^{(m)}(y) - (M_A h^{(m)})(y)\right] + O(\|y\|^{m+1}), \tag{1.13}$$

where the linear operator M_A is defined by the formula

$$(M_A h)(y) := h(Ay) - Ah(y). \tag{1.14}$$

If $h \in H_m$, then $M_A h \in H_m$ for all $m \geq 2$.

Notice that all terms of order less than m in (1.13) are the same as in (1.10), while the terms of order m have changed and differ from $f^{(m)}(y)$ by $-(M_A h^{(m)})(y)$. Now, we define the *linear homological equation* in H_m:

$$M_A h^{(m)} = f^{(m)}. \tag{1.15}$$

If $f^{(m)}$ belongs to the *range* $M_A(H_m)$ of M_A, then there is a solution $h^{(m)}$ to (1.15), meaning that there is a transformation (1.11) that eliminates all homogeneous terms of order m in (1.10). In general, however, $f^{(m)} = g^{(m)} + r^{(m)}$, where $g^{(m)} \in M_A(H_m)$, while $r^{(m)}$ belongs to a *complement* $\widetilde{H}_m$ to $M_A(H_m)$ in H_m. Therefore, only the $g^{(m)}$ part of $f^{(m)}$ can be eliminated from (1.10) by a transformation (1.11). The remaining $r^{(m)}$ terms are called the *resonant terms* of order m. Since $\widetilde{H}_m$ is not uniquely defined, the same is true for the resonant terms.

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Applying the above elimination procedure recursively for $m = 2, 3, 4, \ldots$, one proves the following theorem going back to Poincaré.

Theorem 1.5 (Poincaré Normal Form) *There is a polynomial change of coordinates*

$$x = y + h^{(2)}(y) + h^{(3)}(y) + \cdots + h^{(m)}(y),\ h^{(k)} \in H_k,$$

that transforms a smooth map

$$x \mapsto Ax + f(x), \quad x \in \mathbb{R}^n, \tag{1.16}$$

with $f(x) = O(\|x\|^2)$ *into*

$$y \mapsto Ay + r^{(2)}(y) + r^{(3)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \tag{1.17}$$

where each $r^{(k)}$ *contains only resonant terms of order k, i.e.,* $r^{(k)} \in \widetilde{H}_k$ *for* $k = 2, 3, \ldots, m$.

If all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are real and different, one can assume that A is diagonal, while the standard unit vectors $\{e_j\}_{j=1,2,\ldots,n}$ are the corresponding eigenvectors. In the space H_m, the operator M_A then has eigenvalues $(\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n} - \lambda_j)$, where $m_1 + m_2 + \cdots + m_n = m$. In this case, the homogeneous vector-monomials

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\ e_j$$

are the eigenvectors of M_A in H_m. If a *resonance* occurs, i.e.,

$$\lambda_j = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}$$

with $m_j \geq 0,\ m \geq 2$, the corresponding vector-monomial is not in the range of M_A and thus defines a resonant term. This allows determining resonant terms without long computations.

Note that all formulated results are also valid in the complex case, when $x, y \in \mathbb{C}^n$ and the complex matrix A has n different eigenvalues.

System (1.17) is called the *Poincaré normal form* of (1.16). In Chapter 4 we will give an efficient method to find coefficients of the normal forms of maps restricted to center manifolds, that combines the Poincaré normalization with the computation of the center manifold.

When considering a family of maps (1.8) depending on parameters, two approaches to its parameter-dependent normal forms are possible. One can try to find a normalizing transformation in $\mathbb{R}^n$ with coefficients that smoothly depend on parameters. Alternatively, one can consider the extended map (1.9) in the (x, α)-space and apply a normalization there. The former approach works well if the critical fixed point has a smooth continuation for nearby parameter

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values, i.e., there is no eigenvalue 1. The latter approach is necessary if such an eigenvalue is present.

1.4 Approximating ODEs

When dealing with local codim 2 bifurcations, we will repeatedly use the approximation of maps near their fixed points by shifts along orbits of certain systems of autonomous ordinary differential equations (ODEs). This allows us to predict *global* bifurcations of closed invariant curves and tori happening in the maps near cyclic, homo-, and heteroclinic bifurcations of the approximating ODEs. Although the exact bifurcation structure is *different* for maps and approximating ODEs, they provide information that is hardly available by analysis of the maps alone.

Consider a map having a fixed point $x = 0$:

$$x \mapsto f(x) = Ax + f^{(2)}(x) + f^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.18}$$

where A is the Jacobian matrix of f at $x = 0$, while each component of $f^{(k)} \in H_k$ is a homogeneous polynomial of order k, $f^{(k)}(x) = O(\|x\|^k)$:

$$f_i^{(k)}(x) = \sum_{j_1+j_2+\cdots+j_n=k} b^{(k)}_{i,j_1 j_2\cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}.$$

In addition, consider a system of differential equations of the *same* dimension as the map (1.18) having an equilibrium at the point $x = 0$:

$$\dot{x} = F(x) = \Lambda x + F^{(2)}(x) + F^{(3)}(x) + \cdots, \quad x \in \mathbb{R}^n, \tag{1.19}$$

where Λ is a matrix and the terms $F^{(k)}$ have the same properties as the corresponding $f^{(k)}$ above. Denote by $\varphi^t(x)$ the (local) flow associated with (1.19). An interesting question is whether it is possible to construct a system (1.19), whose *unit-time shift* φ^1 along orbits coincides with (or at least approximates) the map f given by (1.18).

The map (1.18) is said to be *approximated up to order k* by system (1.19) if its Taylor expansion coincides with that of the unit-time shift φ^1 along the orbits of (1.19) up to and including terms of order k:

$$f(x) = \varphi^1(x) + O(\|x\|^{k+1}).$$

System (1.19) is then called an *approximating ODE system*.

We can construct the Taylor expansion of $\varphi^t(x)$ with respect to x at $x = 0$ as follows using *Picard iterations*. Namely, set

$$x^{(1)}(t) = e^{\Lambda t} x.$$

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So, $x^{(1)}$ is the solution of the linear equation $\dot{x} = \Lambda x$ with initial condition x, and define the Picard iteration

$$x^{(k+1)}(t) = e^{\Lambda t}x + \int_0^t e^{\Lambda(t-\tau)} \left(F^{(2)}(x^{(k)}(\tau)) + \cdots + F^{(k+1)}(x^{(k)}(\tau))\right) d\tau. \quad (1.20)$$

Clearly, the $(k+1)$ iteration does not change $O(\|x\|^l)$ terms for any $l \le k$. Substituting $t = 1$ into $x^{(k)}(t)$ provides the correct Taylor expansion of $\varphi^1(x)$ up to and including terms of order k:

$$\varphi^1(x) = e^{\Lambda}x + g^{(2)}(x) + g^{(3)}(x) + \cdots + g^{(k)}(x) + O(\|x\|^{k+1}). \quad (1.21)$$

Next we require that the corresponding terms in (1.21) and (1.18) coincide:

$$e^{\Lambda} = A, \quad (1.22)$$

and

$$g^{(k)}(x) = f^{(k)}(x), \quad k = 2, 3, \ldots, \quad (1.23)$$

and then try to find Λ and the coefficients of $g^{(k)}$ (and, eventually, the coefficients of $F^{(k)}$) in terms of those of $f^{(k)}$, i.e., $b^{(k)}_{i,j_1 j_2 \cdots j_n}$. This is *not* always possible.

First of all, (1.22) does not always have a *real* solution matrix Λ, even if A is nonsingular. A sufficient condition for the solvability is that all eigenvalues of A are positive. Moreover, not all equations (1.23) may be solvable for the coefficients $b^{(k)}_{i,j_1 j_2 \cdots j_n}$ with $|j| := j_1 + j_2 + \cdots + j_n = k$. The corresponding conditions could be formulated explicitly in a rather general form. We will not do this, since in our cases we will verify the solvability explicitly. Actually, these conditions are always satisfied if the map (1.18) is close to identity. More results on the existence of the approximating vector field g can be found in Gramchev and Walcher (2005) and Takens (1974).

In the parameter-dependent case, one approximates the extended map by an extended flow, thus obtaining a parameter-dependent ODE system.

1.5 Simplest bifurcations of planar ODEs

In our analysis of bifurcations of maps, we will often encounter auxiliary smooth planar autonomous ODEs depending on one or two parameters. While the main purpose of these auxiliary vector fields is the study of global bifurcations, their local bifurcations are also useful. Therefore, for further reference, we summarize all necessary results without proof about bifurcations of such systems. Of course, this overview is not a substitute for a systematic study of this classical part of bifurcation theory.

At fixed parameter values, one defines for such an ODE system its *orbits* (oriented by the advance of time) and *phase portrait*. Two such systems are considered as *topologically equivalent* (in some domains of $\mathbb{R}^2$) if their phase portraits are homeomorphic, i.e., one can be obtained from the other by a continuous invertible deformation. Note that such a transformation maps orbits into orbits, but not necessarily solutions into solutions. An appearance of a topologically nonequivalent phase portrait is called a *bifurcation*. Since the number of equilibria, the number of periodic orbits, and their stability, as well as the presence of connecting orbits, are topological invariants, a bifurcation of the 2D system means a change of (some of) these properties.

All bifurcations can be divided into *local*, i.e., occurring in an arbitrary small fixed neighborhood of an equilibrium, and *global*. Each bifurcation is characterized by a number of *bifurcation conditions*. Similarly as for maps, this number is called *codimension* and is equal to the number of independent parameters needed to unfold this bifurcation in a generic system, i.e., systems without symmetries or integrals of motion. Bifurcation theory studies canonical unfoldings (*normal forms*) of bifurcations (if they exist) and provides techniques to find out which of the possible unfoldings actually occurs in the particular ODE system. One describes unfoldings by means of *bifurcation diagrams*, i.e., stratifications of the parameter space near a bifurcation point induced by the topological equivalence of phase portraits.

1.5.1 Generic one-parameter local bifurcations in 2D ODEs

Consider a smooth one-parameter planar ODE

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^2, \alpha \in \mathbb{R}. \tag{1.24}$$

Suppose $u_0 \in \mathbb{R}^2$ is an *equilibrium* of (1.24) at $\alpha_0 \in \mathbb{R}$, i.e., $f(u_0, \alpha_0) = 0$. An equilibrium u_0 is called *hyperbolic* if $\Re(\lambda) \neq 0$ for any eigenvalue $\lambda \in \mathbb{C}$ of its Jacobian matrix $A = f_u(u_0, \alpha_0)$. A hyperbolic equilibrium can be smoothly continued with respect to α near α_0, and the Grobman–Hartman Theorem for ODEs ensures that it does not exhibit any local bifurcations. Indeed, the equilibrium remains hyperbolic for parameter values close to α_0 and has a local phase portrait that is topologically equivalent to that of the linearized ODE. Thus, a local bifurcation can happen only for a nonhyperbolic equilibrium with $\Re(\lambda) = 0$.

In generic one-parameter planar ODEs, one can encounter only two types of nonhyperbolic equilibria, i.e., with either

(1) two real eigenvalues, with one eigenvalue $\lambda_1 = 0$; or

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(2) two purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$ with $\omega_0 > 0$.

Each condition indeed defines a codim 1 local bifurcation of generic planar ODEs. Case (1) leads to a *fold* (or *saddle-node*) *bifurcation*. Case (2) implies a *Hopf* (or *Andronov–Hopf*) *bifurcation*. To describe their canonical unfoldings, assume that $\alpha_0 = 0$ and $u_0 = 0$.

Fold (saddle-node) bifurcation in the plane

By a linear invertible change of variables, the critical system $\dot{u} = f(u, 0)$ can be transformed near $u = 0$ into

$$\begin{cases} \dot{x} &= ax^2 + bxy + cy^2 + O(\|(x,y)\|^3), \\ \dot{y} &= \lambda_2 y + O(\|(x,y)\|^2), \end{cases}$$

where $(x, y) \in \mathbb{R}^2$, $a, b, c \in \mathbb{R}$ and $\lambda_2 \neq 0$ is the second (real) eigenvalue of A. The variables (x, y) are coordinates in the directions of the eigenvectors of the Jacobian matrix $A = f_u(0, 0)$ corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Let $q, p \in \mathbb{R}^2$ be nonzero vectors satisfying

$$Aq = A^T p = 0$$

and normalized such that $\langle p, q \rangle = 1$. Then a can be computed as the quadratic coefficient in the Taylor expansion

$$\langle p, f(\xi q, 0)\rangle = a\xi^2 + O(\xi^3),$$

i.e.,

$$a = \frac{1}{2}\frac{d^2}{d\xi^2}\langle p, f(\xi q, 0)\rangle\bigg|_{\xi=0}.$$

Theorem 1.6 *If $a \neq 0$ and $\lambda_2 \neq 0$, then the system* (1.24) *is locally topologically equivalent near the fold bifurcation to*

$$\begin{cases} \dot{x} &= \beta(\alpha) + ax^2, \\ \dot{y} &= \lambda_2 y, \end{cases}$$

where $\beta = \beta(\alpha)$ is a smooth function with $\beta(0) = 0$.

If $\beta'(0) \neq 0$, we can use β as the new unfolding parameter and visualize the bifurcation diagram of the canonical unfolding

$$\begin{cases} \dot{x} &= \beta + ax^2, \\ \dot{y} &= \lambda_2 y \end{cases} \tag{1.25}$$

(see Figure 1.2). In this *topological normal form*, two equilibrium points

$$O_{1,2} = \left(\mp\sqrt{-\frac{\beta}{a}}, 0\right)$$

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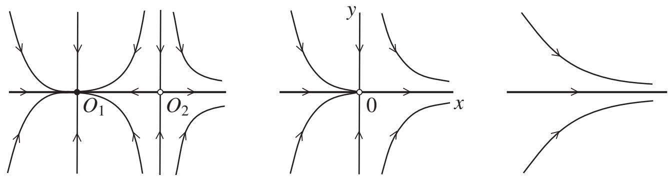

$\beta < 0$ $\beta = 0$ $\beta > 0$

Figure 1.2 Planar fold bifurcation in the topological normal form (1.25): $a > 0, \lambda_2 < 0$.

collide and disappear when β changes sign. This is called a *fold* (or *saddle-node*) bifurcation. In the original coordinates (u_1, u_2), the same topological transition happens in system (1.24), with deformed phase portraits.

Remark 1.7 Notice that all essential rearrangements in system (1.25) occur on the line $y = 0$ that is exponentially stable or unstable, depending on the sign of λ_2. In the original system, this line becomes a smooth (parameter-dependent) curve W^c_α, which is a local *center manifold* of (1.24) near the fold bifurcation.

(Andronov–)Hopf bifurcation in the plane

By a linear invertible change of variables, the critical system $\dot{u} = f(u, 0)$ can be transformed near $u = 0$ into

$$\begin{cases} \dot{x} &= -\omega_0 y + R(x, y), \\ \dot{y} &= \omega_0 x + S(x, y), \end{cases}$$

where $R(x, y) = O(\|(x, y)\|^2)$ and $S(x, y) = O(\|(x, y)\|^2)$ are smooth functions. Introducing $z = x + iy \in \mathbb{C}$ and $\bar{z} = x - iy$, this system can be written as one complex ODE

$$\dot{z} = i\omega_0 z + g(z, \bar{z}), \tag{1.26}$$

where

$$g(z, \bar{z}) = R\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iS\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = \sum_{j+k\geq 2} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k .$$

One can directly compute the function $g(z, \bar{z})$ in (1.26) using the original coordinates (u_1, u_2). Let $A = f_u(0, 0)$ be the Jacobian matrix of (1.24) at $(u_0, \alpha_0) = (0, 0)$. Introduce $q, p \in \mathbb{C}^2$, such that

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p,$$

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and $\langle p, q \rangle = \bar{p}^T q = 1$. Then

$$g(z, \bar{z}) = \langle p, f(zq + \bar{z}\bar{q}) \rangle,$$

so that

$$g_{jk} = \left. \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k} \right|_{z=\bar{z}=0} \langle p, f(zq + \bar{z}\bar{q}) \rangle,$$

where z and $\bar{z}$ should be considered as independent variables.

There exists a polynomial change of variable

$$z = w + \frac{1}{2} h_{20} w^2 + h_{11} w \bar{w} + \frac{1}{2} h_{02} \bar{w}^2 + \frac{1}{6} h_{30} w^3 + \frac{1}{2} h_{12} w \bar{w}^2 + \frac{1}{6} h_{03} \bar{w}^3,$$

such that (1.26) will take the *Poincaré normal form*

$$\dot{w} = i\omega_0 w + c_1 w |w|^2 + O(|w|^4),$$

where $c_1 \in \mathbb{C}$. Define the *first Lyapunov coefficient*

$$l_1 := \frac{1}{\omega_0} \Re(c_1).$$

One can show that

$$l_1 = \frac{1}{2\omega_0^2} \Re(i g_{20} g_{11} + \omega_0 g_{21}). \tag{1.27}$$

Theorem 1.8 *If* $l_1 \neq 0$ *and* $\omega_0 > 0$*, then* (1.24) *is locally topologically equivalent near the Hopf bifurcation to the following system in polar coordinates*

$$\begin{cases} \dot{\rho} &= \rho(\beta(\alpha) + l_1 \rho^2), \\ \dot{\varphi} &= 1, \end{cases}$$

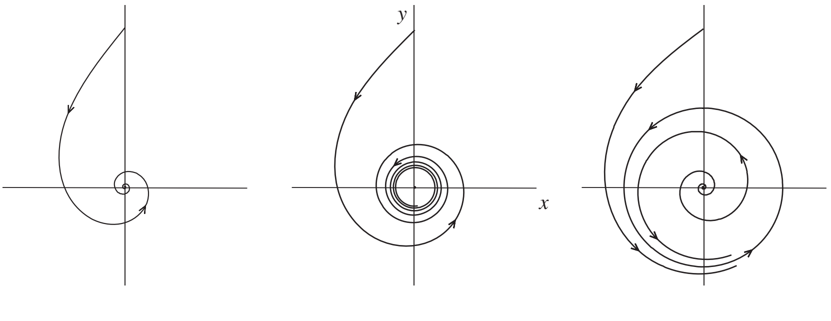

Figure 1.3 Supercritical Hopf bifurcation: $l_1 < 0$.

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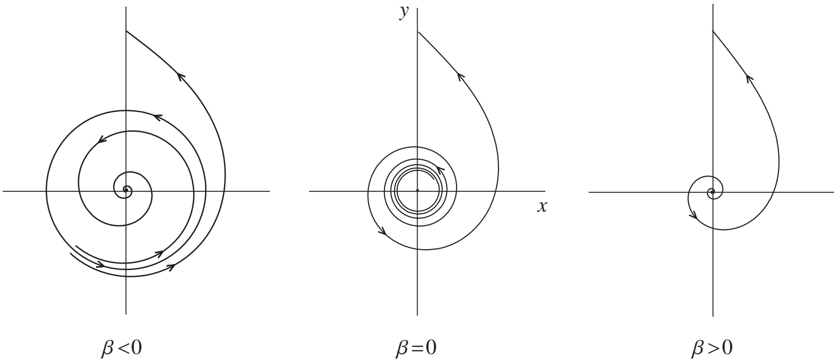

Figure 1.4 Subcritical Hopf bifurcation: $l_1 > 0$.

where $\beta = \beta(\alpha)$ *is a smooth function with* $\beta(0) = 0$.

If $\beta'(0) \neq 0$, we can use β as the new unfolding parameter and consider the bifurcation diagram of the topological normal form

$$\begin{cases} \dot{\rho} &=& \rho(\beta + l_1\rho^2), \\ \dot{\varphi} &=& 1. \end{cases} \tag{1.28}$$

A *limit cycle* of radius $\rho_0 = \sqrt{-\frac{\beta}{l_1}} > 0$ appears or disappears, while the focus at the origin changes stability, (Figures 1.3 and 1.4). This phenomenon is called the planar *(Andronov–)Hopf bifurcation*. In the original system (1.24), a deformed limit cycle bifurcates (with the period approaching $2\pi/\omega_0$).

The direction of the cycle bifurcation is determined by the sign of the first Lyapunov coefficient l_1. Notice that the cycle stability is the same as that of the critical equilibrium ("weak focus").

Remark 1.9 The saddle-node and Hopf bifurcations occur also in smooth parameter-dependent n-dimensional ODEs

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

Without loss of generality, we assume that the critical equilibrium is $u = 0$ and the bifurcation takes place at $\alpha = 0$.

At the *fold bifurcation*, the Jacobian matrix $A = f_u(0, 0)$ has a simple zero eigenvalue $\lambda_1 = 0$ and no other eigenvalues with $\Re(\lambda) = 0$. In this case, there exists a smooth parameter-dependent invariant curve W^c_α on which the system is locally topologically equivalent to the x-equation in (1.25) with $\beta = \beta(\alpha)$,

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i.e.,

$$\dot{x} = \beta + ax^2.$$

Thus, generically, two equilibrium points in W^c_α collide and disappear.

The normal form coefficient a can be computed as

$$a = \frac{1}{2}\langle p, B(q,q)\rangle, \tag{1.29}$$

where $q, p \in \mathbb{R}^2$ satisfy $Aq = A^T p = 0$, $\langle q,q\rangle = \langle p,q\rangle = 1$ and

$$B_i(q,r) = \sum_{j,k\in\{1,2,\ldots,n\}} \frac{\partial^2 f_i(0,0)}{\partial u_j \partial u_k} q_j r_k, \quad i = 1,2,\ldots,n. \tag{1.30}$$

At the *(Andronov–)Hopf bifurcation*, the Jacobian matrix $A = f_u(0,0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$ and no other eigenvalues with $\Re(\lambda) = 0$. In this case, there exists a smooth parameter-dependent invariant surface W^c_α on which the system is locally topologically equivalent to (1.28). Hence, a limit cycle bifurcates in W^c_α from a focus that changes stability.

The first Lyapunov coefficient can be computed by the following formula

$$l_1 = \frac{1}{2\omega_0}\Re\left[\langle p, C(q,q,\bar{q}) - 2B(q, A^{-1}B(q,\bar{q})) + B(\bar{q}, (2i\omega_0 I_n - A)^{-1}B(q,q))\rangle\right], \tag{1.31}$$

where $p, q \in \mathbb{C}^n$ satisfy $Aq = i\omega_0 q$, $A^T p = -i\omega_0 p$ and $\langle q,q\rangle = \langle p,q\rangle = 1$ with $\langle p,q\rangle := \bar{p}^T q$. The components of the multilinear form B have been defined above, while those of C are given by

$$C_i(q,r,s) = \sum_{j,k,l\in\{1,2,\ldots,n\}} \frac{\partial^3 f_i(0,0)}{\partial u_j \partial u_k \partial u_l} q_j r_k s_l, \quad i = 1,2,\ldots,n. \tag{1.32}$$

Note that (1.31) is valid for $n \geq 2$. However, for $n = 2$ one may prefer to use formula (1.27), as that does not involve solving any linear system or inverting a matrix.

1.5.2 Generic two-parameter local bifurcations in 2D ODEs

Consider a smooth two-parameter planar ODE

$$\dot{u} = f(u,\alpha), \quad u \in \mathbb{R}^2, \alpha \in \mathbb{R}^2. \tag{1.33}$$

In such planar ODEs, only three types of doubly degenerate equilibrium points can be encountered generically, i.e., either with

(1) one simple eigenvalue $\lambda_1 = 0$ and $a = 0$, i.e., the normal form coefficient (1.29) of the fold vanishes;

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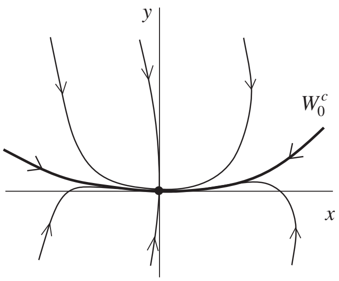

Figure 1.5 Local critical center manifold at cusp bifurcation.

(2) a double zero non-semisimple eigenvalue $\lambda_{1,2} = 0$; or
(3) two purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$ and $l_1 = 0$, i.e., the first Lyapunov coefficient (1.31) vanishes.

Each condition indeed defines a codim 2 local bifurcation of generic planar ODEs. Case (1) corresponds to a *cusp* bifurcation. Case (2) implies a *Bogdanov–Takens* bifurcation. Case (3) leads to a *generalized Hopf* (or *Bautin*) bifurcation. We now describe their canonical unfoldings. We will assume that the bifurcation occurs at $\alpha_0 = 0$ and the corresponding critical equilibrium is $u_0 = 0$.

Cusp bifurcation

By a linear invertible change of variables, the critical system $\dot{u} = f(u, 0)$ at the cusp bifurcation can be transformed into

$$\begin{cases} \dot{x} &= p_{11}xy + \frac{1}{2}p_{02}y^2 + \frac{1}{6}p_{30}x^3 + \frac{1}{2}p_{21}x^2y + \frac{1}{2}p_{12}xy^2 + \frac{1}{6}p_{03}y^3 + O(\|(x,y)\|^4), \\ \dot{y} &= \lambda_2 y + \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + O(\|(x,y)\|^3). \end{cases}$$

As in the fold case, the variables (x, y) are coordinates in the directions of the eigenvectors of the Jacobian matrix $A = f_u(0, 0)$ corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 \neq 0$. It has an invariant center manifold W_0^c that is locally given by the graph of the smooth function

$$y = \frac{1}{2}w_2x^2 + O(x^3), \quad w_2 = -\frac{q_{20}}{\lambda_2},$$

so that the restriction of the critical ODE to W_0^c can be written as

$$\dot{x} = cx^3 + O(x^4),$$

where

$$c = \frac{1}{6}\left(p_{30} - \frac{3}{\lambda_2}q_{20}p_{11}\right).$$

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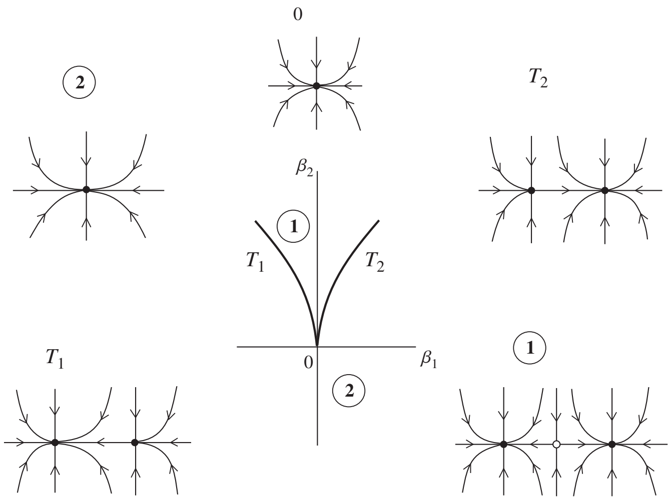

Figure 1.6 Bifurcation diagram of the topological normal form for cusp bifurcation: $s = \sigma = -1$.

Theorem 1.10 *If* $c \neq 0$, *then* (1.33) *is locally topologically equivalent near the cusp bifurcation to the system*

$$\begin{cases} \dot{x} &= \beta_1(\alpha) + \beta_2(\alpha)x + sx^3, \\ \dot{y} &= \sigma y, \end{cases}$$

where $\beta = \beta(\alpha)$ *is a smooth vector-valued function with* $\beta_1(0) = \beta_2(0) = 0$, *while* $s = \text{sign}(c) = \pm 1$ *and* $\sigma = \text{sign}(\lambda_2) = \pm 1$.

If the 2D mapping $\alpha \mapsto \beta(\alpha)$ is regular at $\alpha = 0$, i.e., its Jacobian matrix $\beta_\alpha(0)$ is nonsingular, then (β_1, β_2) can be used as the new unfolding parameters. The bifurcation diagram of the topological normal form

$$\begin{cases} \dot{x} &= \beta_1 + \beta_2 x + sx^3, \\ \dot{y} &= \sigma y, \end{cases} \tag{1.34}$$

contains a fold bifurcation curve $T = T_1 \cup T_2$ that delimits a narrow wedge. For parameter values chosen inside the wedge three equilibrium points exist, while outside the wedge only one equilibrium exists.

Remark 1.11 As in the fold case, all essential rearrangements in system (1.34) occur on the line $y = 0$ that is exponentially stable or unstable, depending on the sign of λ_2. In the original system, this line becomes a smooth

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(parameter-dependent) curve W^c_α which is a local *center manifold* of (1.33) near the cusp bifurcation.

Bogdanov–Takens bifurcation

By a linear invertible change of variables, the critical system $\dot{u} = f(u, 0)$ at the Bogdanov–Takens (BT) bifurcation can be transformed to

$$\begin{cases} \dot{x} = y + \frac{1}{2}p_{20}x^2 + p_{11}xy + \frac{1}{2}p_{02}y^2 + O(\|(x,y)\|^3) =: P(x,y), \\ \dot{y} = \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}x^2 + O(\|(x,y)\|^3). \end{cases}$$

The variables (x, y) are coordinates in the directions of the eigenvector and the generalized eigenvector of the Jacobian matrix $A = f_u(0, 0)$ corresponding to its double non-semisimple eigenvalue $\lambda_1 = 0$. The local smooth invertible change of variables

$$\begin{cases} \xi = x, \\ \eta = P(x,y) \end{cases}$$

reduces this system near the origin to

$$\begin{cases} \dot{\xi} = \eta, \\ \dot{\eta} = a\xi^2 + b\xi\eta + c\eta^2 + O(\|(\xi,\eta)\|^3), \end{cases}$$

where

$$a = \frac{1}{2}q_{20}, \quad b = p_{20} + q_{11}.$$

Theorem 1.12 *If* $ab \neq 0$, *then* (1.33) *is locally topologically equivalent near the Bogdanov–Takens bifurcation to*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy, \end{cases}$$

where $\beta = \beta(\alpha)$ *is a smooth vector-valued function with* $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \text{sign}(ab) = \pm 1$.

As in the cusp case, if the 2D mapping $\alpha \mapsto \beta(\alpha)$ is regular at $\alpha = 0$, then (β_1, β_2) can be used as the new unfolding parameters. The bifurcation diagram of the topological normal form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \beta_1 + \beta_2 x + x^2 + sxy \end{cases} \tag{1.35}$$

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is presented in Figure 1.7 for $s < 0$. It includes several bifurcation curves near the origin:

- *fold* $T = T_- \cup T_+ : \beta_1 = \frac{1}{4}\beta_2^2$;
- *Andronov–Hopf* $H : \beta_1 = 0,\ \beta_2 < 0$;
- *saddle homoclinic* $P : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2),\ \beta_2 < 0$.

A unique limit cycle appears at the Andronov–Hopf bifurcation curve H and disappears via the saddle homoclinic bifurcation at curve P. The last bifurcation is global. Crossing the curve P, the limit cycle approaches a *homoclinic orbit* that connects a saddle point with itself, and its period tends to infinity. Having located and analyzed the Bogdanov–Takens bifurcation, it is also possible to predict saddle homoclinic orbits by purely algebraic tools.

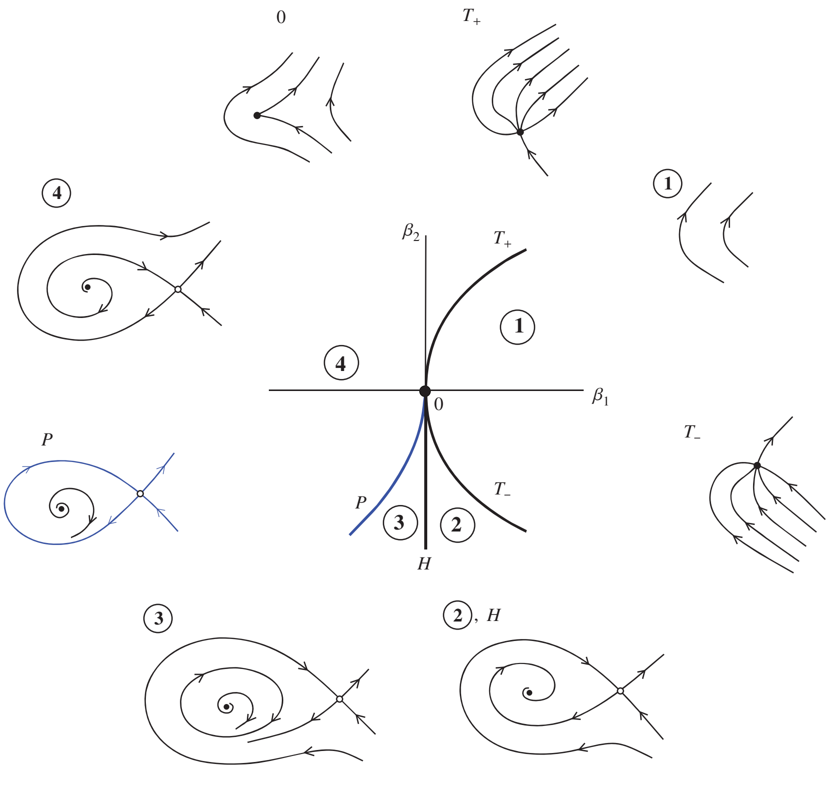

Figure 1.7 Bifurcation diagram of the topological normal form for Bogdanov–Takens bifurcation: $s = -1$.

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Generalized Hopf bifurcation

As in the codim 1 Hopf case, the critical system $\dot{u} = f(u, 0)$ at the generalized Hopf bifurcation can be transformed to the complex form

$$\dot{z} = i\omega_0 z + \sum_{2\leq j+k\leq 5} \frac{1}{j!k!} g_{jk} z^k \bar{z}^j + O(|z|^6),$$

which can locally be reduced by a polynomial change of variables to the *Poincaré normal form*

$$\dot{w} = i\omega w + c_1 w|w|^2 + c_2 w|w|^4 + O(|w|^6),$$

where the first Lyapunov coefficient vanishes: $l_1 = \frac{1}{\omega_0}\Re(c_1) = 0$. Now, we define the *second Lyapunov coefficient*

$$l_2 := \frac{1}{\omega_0}\Re(c_2).$$

There is an explicit formula for l_2 when $l_1 = 0$ in terms of g_{jk} (see (Kuznetsov, 2004)).

Theorem 1.13 *If $l_2 \neq 0$, then* (1.33) *is locally topologically equivalent near the generalized Hopf bifurcation to the following system in polar coordinates:*

$$\begin{cases} \dot{\rho} = \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4), \\ \dot{\varphi} = 1, \end{cases}$$

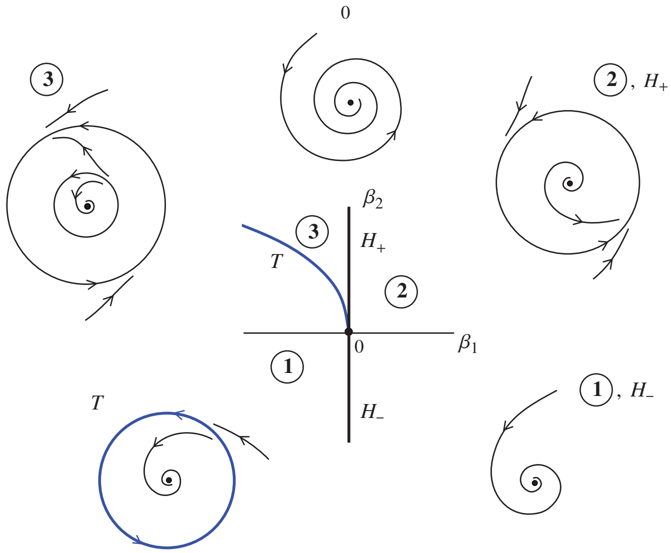

Figure 1.8 Bifurcation diagram of the topological normal form for generalized Hopf bifurcation: $s = -1$.

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where $\beta = \beta(\alpha)$ *is a smooth vector-valued function with* $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \mathrm{sign}(l_2) = \pm 1$.

As usual, if the 2D mapping $\alpha \mapsto \beta(\alpha)$ is regular at $\alpha = 0$, then (β_1, β_2) can be used as the new unfolding parameters. The bifurcation diagram of the topological normal form

$$\begin{cases} \dot{\rho} &=& \rho(\beta_1 + \beta_2\rho^2 + s\rho^4), \\ \dot{\varphi} &=& 1 \end{cases} \tag{1.36}$$

is presented in Figure 1.8 for $s = -1$. It includes two bifurcation curves near the origin:

- *Andronov–Hopf* $H: \beta_1 = 0$;
- *fold of limit cycles* $T: \beta_1 = -\frac{1}{4}\beta_2^2,\ \beta_2 > 0$.

At the branch H^- of H with $\beta_2 < 0$, the supercritical Hopf bifurcation happens that generates a *stable* limit cycle. On the contrary, at the branch H^+ of H with $\beta_2 > 0$, an *unstable* limit cycle bifurcates via the subcritical Hopf bifurcation. These two cycles collide and disappear at the global bifurcation curve T. For parameter values at this curve, the normal form has a degenerate (nonhyperbolic) limit cycle that is stable from one side and unstable from the other. This cycle has a nontrivial multiplier $+1$.

Remark 1.14 The cusp, Bogdanov–Takens and generalized Hopf bifurcations occur also in smooth n-dimensional ODEs, depending on two parameters

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^2.$$

As usual, assume that the critical equilibrium is $u = 0$ and the bifurcation takes place at $\alpha = 0$.

At the *cusp bifurcation*, the Jacobian matrix $A = f_u(0, 0)$ has a simple zero eigenvalue $\lambda_1 = 0$ and no other eigenvalues with $\Re(\lambda) = 0$. In this case, generically, there exists a smooth parameter-dependent invariant curve W^c_α on which the system is locally topologically equivalent to

$$\dot{x} = \beta_1 + \beta_2 x + s x^3,$$

where $s = \mathrm{sign}(c)$ and $\beta = \beta(\alpha)$. Thus, generically, the n-dimensional ODE system has a parametric portrait that is locally equivalent to that of system (1.34). The normal form coefficient c is given by

$$c = \frac{1}{6}\langle p, C(q, q, q) + 3B(q, h_2)\rangle, \tag{1.37}$$

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where $q, p \in \mathbb{R}^2$ satisfy $Aq = A^T p = 0$, $\langle q, q\rangle = \langle p, q\rangle = 1$ and $h_2 \in \mathbb{R}^n$ can be computed by solving the nonsingular linear system

$$\begin{pmatrix} A & q \\ p^T & 0 \end{pmatrix}\begin{pmatrix} h_2 \\ r \end{pmatrix} = \begin{pmatrix} -B(q,q) \\ 0 \end{pmatrix}.$$

The expressions for the multilinear forms B and C were given in (1.30) and (1.32).

At the *Bogdanov–Takens bifurcation*, the Jacobian matrix $A = f_u(0,0)$ has a double non-semisimple zero eigenvalue $\lambda_{1,2} = 0$ and no other eigenvalues with $\Re(\lambda) = 0$. In this case, generically, there exists a smooth parameter-dependent invariant surface W^c_α on which the system is locally topologically equivalent to (1.35). The normal form coefficients a and b can be computed as

$$a = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle, \quad b = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle,$$

where $q_{0,1}, p_{0,1} \in \mathbb{R}^n$ satisfy

$$Aq_0 = 0, \; Aq_1 = q_0, \; A^T p_1 = 0, \; A^T p_0 = p_1$$

and are normalized such that $\langle q_0, q_0\rangle = \langle p_0, q_0\rangle = \langle p_1, p_1\rangle = \langle p_1, q_1\rangle = 1$ and $\langle p_0, q_1\rangle = \langle p_1, q_0\rangle = 0$.

At the *generalized Hopf bifurcation*, the Jacobian matrix $A = f_u(0,0)$ has a pair of simple purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$ and no other eigenvalues with $\Re(\lambda) = 0$. In this case, there exists a smooth parameter-dependent invariant surface W^c_α on which the system is locally topologically equivalent to (1.36). The second Lyapunov coefficient l_2 can be computed by the formula

$$l_2 = \frac{1}{\omega_0}\Re(c_2),$$

where

$$\begin{aligned} c_2 = \frac{1}{12}\langle p, & E(q,q,q,\bar{q},\bar{q}) \\ & + D(q,q,q,\bar{h}_{20}) + 3D(q,\bar{q},\bar{q},h_{20}) + 6D(q,q,\bar{q},h_{11}) \\ & + C(\bar{q},\bar{q},h_{30}) + 3C(q,q,\bar{h}_{21}) + 6C(q,\bar{q},h_{21}) + 3C(q,\bar{h}_{20},h_{20}) \\ & + 6C(q,h_{11},h_{11}) + 6C(\bar{q},h_{20},h_{11}) \\ & + 2B(\bar{q},h_{31}) + 3B(q,h_{22}) + B(\bar{h}_{20},h_{30}) + 3B(\bar{h}_{21},h_{20}) + 6B(h_{11},h_{21})\rangle. \end{aligned}$$

Here, $q, p \in \mathbb{C}^n$ satisfy $Aq = i\omega_0 q$, $A^T p = -i\omega_0 p$ and are normalized according to $\langle q, q\rangle = \langle p, q\rangle = 1$.

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The vectors $h_{20}, h_{11}, h_{30} \in \mathbb{C}^n$ are given by

$$\begin{aligned} h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1} B(q,\bar{q}), \\ h_{30} &= (3i\omega_0 I_n - A)^{-1} [C(q,q,q) + 3B(q,h_{20})], \end{aligned}$$

while $h_{21} \in \mathbb{C}^n$ can be found by solving the nonsingular linear system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ r \end{pmatrix} = \begin{pmatrix} C(q,q,\bar{q}) + B(\bar{q},h_{20}) + 2B(q,h_{11}) - 2c_1 q \\ 0 \end{pmatrix},$$

where

$$c_1 = \frac{1}{2}\langle p, C(q,q,\bar{q}) + B(\bar{q},(2i\omega_0 I_n - A)^{-1} B(q,q)) - 2B(q, A^{-1}B(q,\bar{q}))\rangle.$$

Recall that c_1 is purely imaginary at the generalized Hopf point.

Finally, we have

$$\begin{aligned} h_{31} = (2i\omega_0 I_n - A)^{-1}[&D(q,q,q,\bar{q}) + 3C(q,q,h_{11}) + 3C(q,\bar{q},h_{20}) \\ &+ 3B(h_{20},h_{11}) + B(\bar{q},h_{30}) + 3B(q,h_{21}) - 6c_1 h_{20}], \\ h_{22} = -A^{-1}[&D(q,q,\bar{q},\bar{q}) + 4C(q,\bar{q},h_{11}) + C(\bar{q},\bar{q},h_{20}) + C(q,q,\bar{h}_{20}) \\ &+ 2B(h_{11},h_{11}) + 2B(q,\bar{h}_{21}) + 2B(\bar{q},h_{21}) + B(\bar{h}_{20},h_{20})]. \end{aligned}$$

In the above formulas, the multilinear forms B and C should be computed via (1.30) and (1.32), while

$$\begin{aligned} D_i(q,r,z,v) &= \sum_{j,k,l,m \in \{1,2,\dots,n\}} \frac{\partial^4 f_i(0,0)}{\partial u_j \partial u_k \partial u_l \partial u_m} q_j r_k z_l v_m, \\ E_i(q,r,z,v,w) &= \sum_{j,k,l,m,s \in \{1,2,\dots,n\}} \frac{\partial^5 f_i(0,0)}{\partial u_j \partial u_k \partial u_l \partial u_m \partial u_s} q_j r_k z_l v_m w_s, \end{aligned}$$

for $i = 1, 2, \dots, n$.

1.6 Pontryagin–Melnikov theory

Consider a planar Hamiltonian system

$$\dot{x} = J\nabla H(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.38}$$

where the *Hamiltonian function* $H : \mathbb{R}^2 \to \mathbb{R}$ is smooth and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla H(x) = \begin{pmatrix} H_{x_1}(x) \\ H_{x_2}(x) \end{pmatrix},$$

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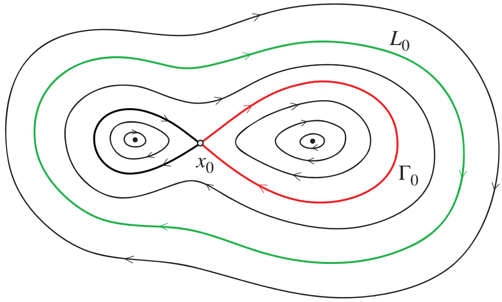

Figure 1.9 Phase portrait of a planar Hamiltonian system.

so that

$$J\nabla H(x) = \begin{pmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{pmatrix}.$$

It is well known that periodic orbits in Hamiltonian systems appear in continuous families as closed level curves of H. Generically, such families approach either a center or an orbit homoclinic to a hyperbolic saddle or extend to infinity (see Figure 1.9). We want to study limit cycles and homoclinic orbits in one- and two-parameter generic smooth perturbations of (1.38).

First consider the following one-parameter planar ODE:

$$\dot{x} = J\nabla H(x) + \varepsilon f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.39}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function. For $\varepsilon = 0$, the system (1.39) reduces to the Hamiltonian system (1.38). Since we are interested in non-Hamiltonian perturbations, we assume that $\operatorname{div} f$ does not vanish.

We want to study hyperbolic limit cycles of the perturbed system (1.39). Such cycles branch off from special cycles of the unperturbed system (1.38), as the following theorem ensures (see, e.g., (Andronov et al., 1973; Guckenheimer and Holmes, 1990)).

Theorem 1.15 (Pontryagin, 1934) *Let L_0 be a clockwise-oriented cycle of* (1.39) *for* $\varepsilon = 0$ *corresponding to a periodic solution* $\varphi(t)$ *with* (*minimal*) *period* T_0. *If*

$$M_0 := \int_{L_0} f_2(x)\, dx_1 - f_1(x)\, dx_2 = 0,$$

while

$$M_1 := \int_0^{T_0} \operatorname{div} f(\varphi(t))\, dt \neq 0,$$

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then

(1) *there exists an annulus around* L_0 *in which the system* (1.39) *has, for all sufficiently small* $\varepsilon > 0$, *a unique hyperbolic limit cycle* L_ε, *such that* $L_\varepsilon \to L_0$ *as* $\varepsilon \to 0$;

(2) *this cycle* L_ε *is stable for* $\varepsilon M_1 < 0$ *and unstable for* $\varepsilon M_1 > 0$.

The theorem is illustrated in Figure 1.10(a), where a stable cycle L_ε is shown.

Notice that Green's Theorem implies

$$M_0 = \int_{\Omega_0} \operatorname{div} f(x)\, dx,$$

where $\Omega_0 \subset \mathbb{R}^2$ is the domain inside the cycle L_0.

Let us now consider perturbations of a saddle homoclinic orbit. Suppose that the Hamiltonian system (1.38) has an orbit Γ_0 that is homoclinic to a hyperbolic saddle point x_0 (see Figure 1.9). Let $H(x_0) = h_0$, so that $\Gamma_0 \subset \{x \in \mathbb{R}^2 : H(x) = h_0\}$.

Introduce now the following two-parameter perturbation of (1.38):

$$\dot{x} = J\nabla H(x) + \varepsilon f(x, \mu), \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.40}$$

where $\varepsilon, \mu \in \mathbb{R}$ are parameters, and $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a smooth function with nonvanishing div f. The reason for introducing the second parameter will become clear later. Finally, suppose for simplicity that $f(x_0, \mu) = 0$ for $\mu \in \mathbb{R}$. This assumption implies that x_0 is an equilibrium for all values of both parameters. Then the following result holds (see e.g., (Guckenheimer and Holmes, 1990; Sanders and Verhulst, 1985)).

Theorem 1.16 (Melnikov, 1963) *Let* Γ_0 *be an orbit homoclinic to a saddle equilibrium of* (1.40) *for* $\varepsilon = 0$. *Suppose that for some* $\mu = \mu_0$

$$\int_{\Gamma_0} f_2(x, \mu_0)\, dx_1 - f_1(x, \mu_0)\, dx_2 = 0,$$

while

$$\int_{\Gamma_0} \frac{\partial f_2}{\partial \mu}(x, \mu_0)\, dx_1 - \frac{\partial f_1}{\partial \mu}(x, \mu_0)\, dx_2 \neq 0.$$

Then there exists a unique function $\mu_H(\varepsilon)$ *with* $\mu_H(0) = \mu_0$, *and an annulus around* Γ_0 *in which the system* (1.40) *has, for all sufficiently small* ε *and* $\mu = \mu_H(\varepsilon)$, *a homoclinic to* x_0 *orbit* $\Gamma_\varepsilon \to \Gamma_0$ *as* $\varepsilon \to 0$.

The theorem is illustrated in Figure 1.10(b), where a perturbed phase portrait with homoclinic orbit Γ_ε existing when $\mu = \mu_H(\varepsilon)$ is shown.

For some combination of parameters (ε, μ), the system (1.40) can also have nonhyperbolic cycles with multiplier +1. Such degenerate cycles bifurcate

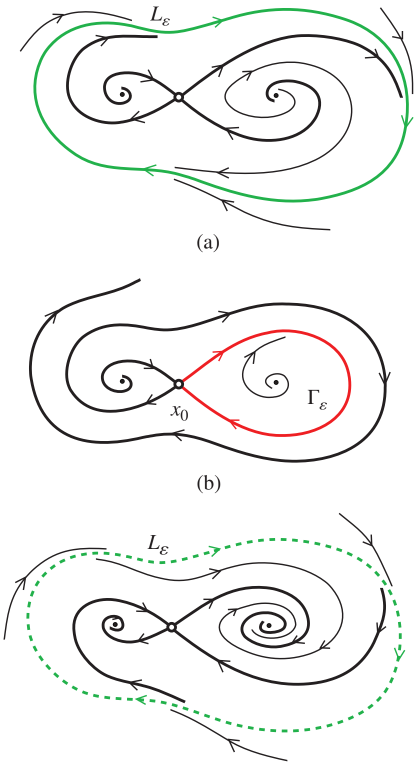

Figure 1.10 Phase portraits of a perturbed Hamiltonian system with small $\varepsilon > 0$: (a) L_ε is a stable limit cycle in (1.39); (b) Γ_ε is a homoclinic orbit to x_0 in (1.40) for $\mu = \mu_H(\varepsilon)$; (c) L_ε is a nonhyperbolic limit cycle in (1.40) for $\mu = \mu_C(\varepsilon)$.

from those cycles of the unperturbed system (1.38), for which M_1 defined in Theorem 1.15 vanishes. Namely, the following result holds.

Theorem 1.17 *Let L_0 be a clockwise-oriented cycle of* (1.40) *for $\varepsilon = 0$ corresponding to a periodic solution $\varphi(t)$ with the* (*minimal*) *period T_0 and let $\Omega_0 \subset \mathbb{R}^2$ denote the domain inside the cycle L_0. Suppose that for some $\mu = \mu_0$*

$$\int_{\Omega_0} \operatorname{div} f(x, \mu_0)\, dx = 0$$

and

$$\int_0^{T_0} \operatorname{div} f(\varphi(t), \mu_0)\, dt = 0.$$

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Then generically there exists a unique function $\mu_C(\varepsilon)$ *with* $\mu_C(0) = \mu_0$*, and an annulus around* L_0 *in which the system* (1.40) *has, for all sufficiently small* $\varepsilon > 0$ *and* $\mu = \mu_C(\varepsilon)$*, a nonhyperbolic cycle* $L_\varepsilon \to L_0$ *as* $\varepsilon \to 0$*.*

The theorem is illustrated in Figure 1.10(c), where a perturbed phase portrait with a nonhyperbolic limit cycle L_ε existing when $\mu = \mu_C(\varepsilon)$ is shown.

The genericity conditions include

$$\int_{L_0} \frac{\partial f_2}{\partial \mu}(x, \mu_0)\, dx_1 - \frac{\partial f_1}{\partial \mu}(x, \mu_0)\, dx_2 \neq 0,$$

as well as one more integral condition ensuring the nonvanishing of the quadratic part of the Poincaré map of the nonhyperbolic cycle for small $\varepsilon > 0$.

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