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INEQUALITIES IN l_1 AND l_p AND APPLICATIONS TO GROUP ALGEBRAS

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In this note, we show that, if (a_n) in l_1 with $\sum|a_n|<2$ and $\sum|a_n|^2=1$, then $\max\{|a_i|+|a_j|: i\neq j\}\geqq 1$, but that the corresponding theorem for sequences in $l_p(1<p<2)$ fails—but only just! Applications to group algebras are given, when it is shown that elements in $l_1(G)$ with powers bounded by $\frac{1}{2}(1+\sqrt{3})$ are bounded away from the identity e of G, but that the corresponding result for $l_p(G)$ is false.

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1. The inequalities

Lemma 1.1. *Let (a_n) be a sequence of positive numbers with $a_1\geqq a_2\geqq\dots$ satisfying $\sum_{i=1}^{\infty}a_i=K$, $\sum_{i=1}^{\infty}a_i^2=1$, then*

(i) $a_1\geqq\dfrac{1}{K}$,

(ii) $a_2\geqq\dfrac{1-a_1^2}{K-a_1}$,

and, if $K\leqq 2$,

(iii) $a_1+a_2\geqq 1$.

Proof. For (i),

$$1=\sum_{i=1}^{\infty}a_i^2\leqq a_1\sum_{i=1}^{\infty}a_i=a_1K,$$

since $a_1=\max\{a_i\colon i\geqq 1\}$. Similarly, for (ii),

$$1-a_1^2=\sum_{i=2}^{\infty}a_i^2\leqq a_2\sum_{i=2}^{\infty}a_i=a_2(K-a_1),$$

since $a_2=\max\{a_i\colon i\geqq 2\}$. For (iii)

$$a_1+a_2 \geqq a_1+\frac{1-a_1^2}{K-a_1}=\frac{Ka_1+1-2a_1^2}{K-a_1}.$$

Thus it is sufficient to show that

$$Ka_1+1-2a_1^2 \geqq K-a_1$$

i.e. $2a_1^2-(K+1)a_1+K-1 \leqq 0$ for all a_1, $1/K \leqq a_1 \leqq 1$. But at $a_1=1/K$, the quadratic becomes

$$\frac{2}{K^2}-\frac{K+1}{K}+K-1=\frac{K^3-2K^2-K+2}{K^2}$$

$$=\frac{(K^2-1)(K-2)}{K^2},$$

which is $\leqq 0$, since $1 \leqq K < 2$. At $a_1=1$, $2a_1^2-(K+1)a_1+K-1=0$, and hence

$2a_1^2-(K+1)a_1+K-1 \leqq 0$ for all a_1, $\frac{1}{K} \leqq a_1 \leqq 1$.

In fact we can improve the inequalities in (i) and (iii) when $K<\sqrt{2}$, and it is this result that is used in the applications to group algebras.

Corollary 1.2. *Let (a_n) be a sequence of positive numbers with $a_1 \geqq a_2 \geqq \dots$ satisfying $\sum_{i=1}^{\infty} a_i = K < \sqrt{2}$, $\sum_{i=1}^{\infty} a_i^2 = 1$. Then*

(iv) $a_1 \geqq \frac{1}{2}(K+\sqrt{2-K^2})$,

(v) $(a_1+a_2)^2 \geqq 1+a_2$, with equality only when $a_1=1$.

Proof. For (iv), since

$$\sum_{i=2}^{\infty} a_n^2 \leqq \left(\sum_{i=2}^{\infty} a_n\right)^2,$$

we have

$$1-a_1^2 \leqq (K-a_1)^2$$

i.e.

$$2a_1^2-2Ka_1+K^2-1 \geqq 0$$

or

$$(a_1-K/2)^2 \geqq (2-K^2)/4$$

Thus

$$a_1 \geqq \frac{1}{2}(K+\sqrt{2-K^2}).$$

Note that $a_1 \geqq 1/K$ by Lemma 1.1, so that $a_1 \geqq \frac{1}{2}K$ since $1/K \geqq \frac{1}{2}K$.

For (v), we clearly have equality when $a_1 = 1$, for then $a_2 = 0$. Suppose that the result is false and that there exists $a_1 < 1$ with

$$(a_1+a_2)^2 \leqq 1+a_2.$$

Now $(a_1+a_2)^2 - a_2$ is quadratic in a_2, and since its derivative $2(a_1+a_2)-1$ is positive (Lemma 1.1 (iii)), the inequality will be satisfied when a_2 takes its least value $(1-a_1^2)/(K-a_1)$ (Lemma 1.1 (ii)). Therefore

$$\left(a_1+\frac{1-a_1^2}{K-a_1}\right)^2 \leqq \frac{1-a_1^2}{K-a_1}+1$$

or

$$\left(a_1+\frac{1-a_1^2}{K-a_1}-1\right)\left(a_1+\frac{1-a_1^2}{K-a_1}+1\right) \leqq \frac{1-a_1^2}{K-a_1}.$$

Therefore

$$(1-a_1^2)\left(\frac{1+a_1}{K-a_1}-1\right)\left(\frac{1-a_1}{K-a_1}+1\right) \leqq \frac{1-a_1^2}{K-a_1}.$$

Since $a_1 < 1$,

$$\left(\frac{1+a_1}{K-a_1}-1\right)\left(\frac{1-a_1}{K-a_1}+1\right) \leqq \frac{1}{K-a_1}.$$

or

$$\frac{1-a_1^2}{(K-a_1)^2}+\frac{2a_1}{K-a_1}-1 \leqq \frac{1}{K-a_1}.$$

or

$$1-a_1^2+2a_1(K-a_1)-(K-a_1)^2 \leqq K-a_1$$

i.e.

$$4a_1^2-(4K+1)a_1+(K^2+K-1) \geqq 0.$$

This is quadratic in a_1 and so the inequality must be satisfied when a_1 takes an extreme value. Since $K < \sqrt{2}$, we have $1/K > \frac{1}{2}K$. Thus by Lemma 1.1 (i), a_1 lies between $\frac{1}{2}K$ and 1.

Therefore, either

$$K^2 - \frac{1}{2}(4K+1)K + (K^2+K-1) \geqq 0,$$

which is equivalent to $K \geqq 2$, and is false, or

$$4-(4K+1)+(K^2+K-1)\geqq 0,$$

which is equivalent to $(K-1)(K-2)\geqq 0$, contradicting $1<K<2$. This completes the proof.

In the l_p case, parts (i) and (ii) have equivalent formulations which are true, but (iii) fails.

Lemma 1.3. *Let* $1\leqq p<2$ *and* (a_n) *be a sequence of positive numbers with* $a_1\geqq a_2\geqq \dots$ *satisfying* $(\sum_{i=1}^{\infty} a_i^p)^{1/p}=K$, $\sum_{i=1}^{\infty} a_i^2=1$, *then*

(i) $a_1\geqq \dfrac{1}{K^{p/(2-p)}}$

(ii) $a_2\geqq \left(\dfrac{1-a_1^2}{K^p-a_1^p}\right)^{1/(2-p)}$

Proof. Let $1/P+1/q=1$. For (i),

$$\begin{aligned}1=\sum_{i=1}^{\infty} a_i^2 &\leqq \left(\sum_{i=1}^{\infty} a_i^q\right)^{1/q}\left(\sum_{i=1}^{\infty} a_i^p\right)^{1/p}\\ &\leqq \left(a_1^{q-p}\sum_{i=1}^{\infty} a_i^p\right)^{1/q}\left(\sum_{i=1}^{\infty} a_i^p\right)^{1/p}\\ &=a_1^{(q-p)/q}\left(\sum_{i=1}^{\infty} a_i^p\right)^{(1/q+1/p)}\\ &=a_1^{(2-p)}K^p.\end{aligned}$$

Hence $a_1\geqq 1/K^{p/(2-p)}$.

For (ii),

$$\begin{aligned}1-a_1^2=\sum_{i=2}^{\infty} a_i^2 &\leqq \left(\sum_{i=2}^{\infty} a_i^q\right)^{1/q}\left(\sum_{i=2}^{\infty} a_i^p\right)^{1/p}\\ &\leqq \left(a_2^{q-p}\sum_{i=2}^{\infty} a_i^p\right)^{1/q}\left(\sum_{i=2}^{\infty} a_i^p\right)^{1/p}\\ &=a_2^{(q-p)/q}\left(\sum_{i=2}^{\infty} a_i^p\right)^{(1/p+1/q)}\\ &=a_2^{(2-p)}(K^p-a_1^p).\end{aligned}$$

Hence $a_2 \geqq ((1-a_1^2)/(K^p-a_1^p))^{p/(2-p)}$.

Example 1.4. For $1<p<2$ and any $K>1$, there exists a positive sequence (a_n) with $a_1 \geqq a_2 \geqq \dots$ satisfying $(\sum_{i=1}^{\infty} a_i^p)^{1/p} < K$, $\sum_{i=1}^{\infty} a_i^2 = 1$ and $a_1 + a_2 < 1$.

For $N>2$, let $a_1=(N-2)/N$, $a_i = 1/N$ for $i=2,3,\dots 4N-3$, and $a_i=0$ for $i>4N-3$. Then clearly

$$a_1+a_2=\frac{N-1}{N}<1,$$

$$\sum_{i=1}^{\infty} a_i^2=\left(\frac{N-2}{N}\right)^2+(4N-4)\left(\frac{1}{N}\right)^2=1$$

and

$$\left(\sum_{i=1}^{\infty} a_i^p\right)=\left(\frac{N-2}{N}\right)^p+(4N-4)\left(\frac{1}{N}\right)^p$$

$$=\frac{(N-2)^p+(4N-4)}{N^p}$$

$$\to 1 \text{ as } N\to\infty.$$

Thus, for N large enough, $(\sum_{i=1}^{\infty} a_i^p)^{1/p} < K$.

The examples indicate that the function f defined by

$$f(x)=x+\left(\frac{1-x^2}{K^p-x^p}\right)^{1/(2-p)}$$

when $p>1$, satisfies $f(1)=1$ and $f'(1)<0$, so that $f(x)<1$ for $x<1$ and close enough to 1.

Remark 1.5. Just how close x has to be to 1 can be illustrated as follows: when $p=1.4$ and $K<\frac{1}{2}(1+3^{p/2})^{1/p}$,

$$f(x)<1 \text{ implies } 0.996<x<1$$

2. Applications to group algebras

In [4] measures with bounded powers on abelian groups were characterised by the size of the bound. In particular, it was shown that if G is a locally compact abelian group and μ is a measure on G with $\|\mu^n\| < \frac{1}{2}(1+\sqrt{3})$, for all n in $\mathbb{Z}$, then μ has the form $\alpha\delta_x$ for a group element x, and $|\alpha|=1$. In fact, this result follows from [**1**, Theorem 2.6]. Here is the equivalent result for non-abelian groups.

Theorem 2.1. *Let G be a discrete group with identity e, and $\mu\in l_1(G)$ be of the form $\mu=\alpha e+f$ with $|\alpha|>1/K$. If μ satisfies $\|\mu^n\|\leqq K<\frac{1}{2}(1+\sqrt{3})$ for all n in $\mathbb{Z}$, then $f=0$ and $\mu=\alpha e$ with $|\alpha|=1$.*

We need the following lemma, which is essentially proved in Lemma 2.4 of [**5**].

Lemma 2.2. *Under the hypothesis of Theorem* 2.1, *for each n, the largest coefficient in μ^n is that of e.*

Proof. We will show that e has the largest coefficient in μ^2 when μ is self-adjoint, and then it is clear how Lemma 2.4 of [**5**] is used for μ^n in the non-self-adjoint case. We have $\mu^{-1}=\bar{\alpha}e+f^*$, and, since $e=\mu*\mu^{-1}$, $|\alpha|^2+\|f\|_2^2=1$.

Suppose that the group element $u\neq e$ has the largest coefficient in μ^2. Let $\mu=\alpha e+\beta u+g$ and so the coefficient of u in μ^2 has modulus less than

$$\begin{aligned}2|\alpha\beta|+\|g\|^2&\leqq 2|\alpha|(K-|\alpha|-\|g\|)+\|g\|^2\\&<2|\alpha|K-2|\alpha|^2,\text{ since }|\alpha|>\|g\|\\&=2|\alpha|(K-|\alpha|)\end{aligned}$$

which decreases with $|\alpha|$.

By Corollary 1.2 (iv), $|\alpha|\geqq\frac{1}{2}(K+\sqrt{2-K^2})$, and so the coefficient of u in μ^2 has modulus less than

$$\frac{1}{2}(K+\sqrt{2-K^2})(K-\sqrt{2-K^2})=K^2=1.$$

But this, being the largest coefficient, must be $\geqq\frac{1}{2}(K+\sqrt{2-K^2})$, again by corollary 1.2 (iv). Therefore $\frac{1}{2}(K+\sqrt{2-K^2})<K^2-1$, or

$$(2-K^2)<(2K^2-K-2)^2=4K^4-4K^3-7K^2+4K+4,$$

i.e.

$$2(K^2-1)(2K^2-2K-1)>0.$$

But this contradicts $1\leqq K<\frac{1}{2}(1+\sqrt{3})$.

In the self-adjoint case, Theorem 2.1 now follows from Corollary 1.2. For suppose that $\mu=\alpha e+\beta v+g$ where $|\alpha|\geqq|\beta|\geqq$ all coefficients of g and let $\mu^2=\alpha_1 e+\beta_1 v+h$ (with α_1 the largest coefficient in μ^2). Then

$$\begin{aligned}|\beta_1| &> 2|\alpha\beta| - \|g\|_2^2 \\ &> 2|\alpha\beta| - (1-|\alpha|^2+|\beta|^2) \\ &= (|\alpha|+|\beta|)^2 \\ &> |\beta|, \text{ by Corollary 1.2 (v).}\end{aligned}$$

By continuity, and since Corollary 1.2 (v) is not tight at $\frac{1}{2}(1+\sqrt{3})$, there exists ε such that $|\beta_1| > |\beta| + \varepsilon$. Since this is true for all powers of μ, this gives a contradiction to the boundedness of the powers of μ. Hence $\beta = 0$.

This is not true for the l_p norm when $p > 1$.

Example 2.3. Let G be a group with finite subgroups of arbitrary large order (e.g. G the circle group), then for $p>1$ and all $K>1$ and $\varepsilon>0$, there exists μ in $l_p(G)$ with powers bounded by K such that $\|\mu - e\| < \varepsilon$.

If H is a subgroup of order n, let $f = 1/n\sum\{x : x \in H\}$, then $f * f = f$ and so $\mu = e - 2f$ is an element of order 2 in $l_p(G)$ with bounded powers. Also

$$\begin{aligned}\|\mu\|_p = \|e-2f\|_p &= \left[\left(\frac{n-2}{n}\right)^p + (n-1)\left(\frac{2}{n}\right)^2\right]^{1/p} \\ &= \frac{1}{n}[(n-2)^p + (n-1)2^p]^{1/p} \\ &\to 1 \text{ as } n \to \infty.\end{aligned}$$

Also

$$\begin{aligned}\|\mu - e\|_p = \|2f\|_p &= \left(n\left(\frac{2}{n}\right)^p\right)^{1/p} \\ &= \frac{2}{n^{(1-1/p)}} \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus for n large enough, $\|\mu\|_p < K$, and $\|\mu - e\|_p < \varepsilon$.

Proof of Theorem 2.1. We have that $\mu = \alpha e + f$ with $|\alpha| > 1/K$. Let $\mu^2 = \alpha_1 e + f_1$ with $|\alpha_1| > 1/K$. Then

$$\begin{aligned}\|f_1\| &\geqq \|2\alpha f\| - \|f\|^2 \\ &\geqq \|f\|(2|\alpha| - \|f\|) \\ &\geqq \|f\|(3|\alpha| - K),\end{aligned}$$

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since $|\alpha| + \|f\| < K$. But $|\alpha| > \frac{1}{2}(K+\sqrt{2-K^2})$, and so

$$3|\alpha| - K > \frac{3}{2}(K+\sqrt{2-K^2}) - K = \frac{1}{2}(K+3\sqrt{2-K^2});$$

but $\frac{1}{2}(K+3\sqrt{2-K^2}) > 1$ is equivalent to

$$2(5K-7)(K+1) < 0,$$

which is true as $K < \frac{7}{5}$.

Since $K < \frac{1}{2}(1+\sqrt{3})$, there exists ε, depending only on K, such that

$$\|f\| > (1+\varepsilon)\|f\|.$$

Repeated squaring now contradicts the boundedness of $\|\mu^n\|$.

Example 2.3 has implications for isomorphism theorems for l_p group algebras. For $p>1$, $l_p(G)$ is not an algebra, but it contains enough algebra structure to reflect the group structure. Indeed, in [2], Parrott showed that, for locally compact groups G_1 and G_2, if T is a linear isometry from $L^p(G_1)$ onto $L^p(G_2)$ which satisfies $T(f*g) = Tf*Tg$ whenever f, g and $f*g$ in $L^p(G_1)$, then G_1 and G_2 are isomorphic. The corresponding theorem when T is norm-decreasing is proved in [3]. There is reason to believe that, in the case of discrete groups, the following generalisation is true.

Conjecture 2.4. *For $1 \leqq p < \infty$, if T is a linear isomorphism from $l_p(G_1)$ onto $l_p(G_2)$ with $\|T\| < \frac{1}{2}(1+3^{p/2})^{1/p}$ which satisfies $T(f*g) = Tf*Tg$ whenever f, g and $f*g$ in $L_p(G_1)$, then G_1 and G_2 are isomorphic.*

This is true for all discrete groups if $p=1$ (see [5]) and can be proved for $1.31 < p < 2$. It is true for *all* $p \neq 2$ when G_1 and G_2 are finite. The proof uses the fact that any algebra isomorphism between the group algebras of finite groups preserves the coefficient of the identity. This is not true when the groups are infinite.

In the proof of Conjecture 2.4 when $p=1$, the restriction of the norm of T means that, if x is not the identity of G_1, the coefficient of the identity in Tx cannot be too big—it certainly cannot be the largest (see [5]). In fact in the l_1 case, this is true when T is assumed to be a monomorphism rather than an isomorphism. Now the measures constructed in Example 2.3, show that this is *not* true when $p>1$. In fact

$$T\delta_n = \mu^n$$

defines a monomorphism of $l_1(Z)$ into $l_1(G)$, and for a suitable choice of μ, $\|T\|$ and the coefficient of the identity in Tx can both be made arbitrarily close to one. Details of these results on isomorphisms of l_p group algebras will appear later.

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