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Cardinal Invariants of Analytic P-Ideals

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Abstract. We study the cardinal invariants of analytic P-ideals, concentrating on the ideal $\mathcal{Z}$ of asymptotic density zero. Among other results we prove $\min\{\mathfrak{b}, \operatorname{cov}(\mathcal{N})\} \leq \operatorname{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \operatorname{non}(\mathcal{N})\}$.

Introduction

Analytic P-ideals and their quotients have been extensively studied in recent years. The first step to better understanding the structure of the quotient forcings $\mathcal{P}(\omega)/\mathcal{I}$ is to understand the structure of the ideal itself. Significant progress in understanding the way in which the structure of an ideal affects the structure of its quotient has been done by I. Farah [Fa1, Fa2, Fa3, Fa4, Fa5].

Typically (but not always) the quotients $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is an analytic P-ideal, are proper and weakly distributive. For some special ideals these quotients have been identified: $\mathcal{P}(\omega)/\mathcal{Z}$ is as forcing notion equivalent to $\mathcal{P}(\omega)/\mathrm{fin} * \mathbb{B}(2^\omega)$ [Fa5], and $\mathcal{P}(\omega)/\operatorname{tr}(\mathcal{N})$ [HZ1] is as forcing notion equivalent to the iteration of $\mathbb{B}(\omega)$ followed by an $\aleph_0$-distributive forcing (see the definitions below).

A secondary motivation comes from the problem of which ideals can be destroyed by a weakly distributive forcing. Even for the class of analytic P-ideals only partial results are known (see Section 3).

In this note we contribute to this line of research by investigating cardinal invariants of analytic P-ideals, comparing them to other, standard, cardinal invariants of the continuum.

In the first section we introduce cardinal invariants of ideals on ω, along the lines of the cardinal invariants contained in the Cichoń's diagram. We also recall the definitions of standard orderings on ideals on ω (Rudin–Keisler, Tukey, Katětov) and their impact on the cardinal invariants of the ideals. Basic theory of analytic P-ideals on ω and examples are also reviewed here. Known results on additivity and cofinality of analytic P-ideals are summarized in the second section.

The main part of the paper is contained in the third section. There we study the order of Katětov restricted to analytic P-ideals, giving a detailed description of how the summable and density ideals are placed in the Katětov order. For the rest of the section, we focus on the ideal of asymptotic density zero and compare its covering number to standard cardinal invariants of the continuum. We prove that $\min\{\mathfrak{b}, \operatorname{cov}(\mathcal{N})\} \leq \operatorname{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \operatorname{non}(\mathcal{N})\}$ and mention some consistency results. We introduce the notion of a totally bounded analytic P-ideal and show that

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all analytic P-ideals which are not totally bounded can be destroyed by a weakly distributive forcing.

In the last section we study the separating number of analytic P-ideals, an invariant closely related to the Laver and Mathias–Prikry type forcings associated with the ideal.

Two major problems remain open here: (1) Is $\mathrm{add}^*(\mathcal{I}) = \mathrm{add}(\mathcal{N})$ for every tall analytic P-ideal $\mathcal{I}$? (2) Can every analytic P-ideal be destroyed by a weakly distributive forcing? What about $\mathcal{Z}$?

We assume knowledge of the method of forcing as well as the basic theory of cardinal invariants of the continuum as covered in [BJ]. Our notation is standard and follows [Ku, Je, BJ]. In particular, c_0, ℓ_1 and ℓ_∞ denote the standard Banach spaces of sequences of reals. For A, B infinite subsets of ω, we say that A is *almost contained* in B ($A \subseteq^* B$) if $A \setminus B$ is finite. The symbol $A =^* B$ means that $A \subseteq^* B$ and $B \subseteq^* A$. For functions $f, g \in \omega^\omega$ we write $f \leq^* g$ to mean that there is some $m \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq m$. The *bounding number* $\mathfrak{b}$ is the least cardinal of an $\leq^*$-unbounded family of functions in ω^ω. The *dominating number* $\mathfrak{d}$ is the least cardinal of a $\leq^*$-cofinal family of functions in ω^ω. Recall that a family of subsets of ω has the *strong finite intersection property* if any finite subfamily has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the minimal size of a family of subsets of ω with the strong finite intersection property but without an infinite pseudointersection (*i.e.,* without a common lower bound in the $\subseteq^*$ order). A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a *splitting family* if for every infinite $A \subseteq \omega$ there is an $S \in \mathcal{S}$ such that $S \cap A$ and $A \setminus S$ are infinite. The *splitting number* $\mathfrak{s}$ is the minimal size of a splitting family in $\mathcal{P}(\omega)$.

The set 2^ω is equipped with the product topology, that is, the topology with basic open sets of the form $[s] = \{x \in 2^\omega : s \subseteq x\}$, where $s \in 2^{<\omega}$. The topology of $\mathcal{P}(\omega)$ is that obtained via the identification of each subset of ω with its characteristic function.

An *ideal* on X is a family of subsets of X closed under taking finite unions and subsets of its members. We assume throughout the paper that all ideals contain all singletons $\{x\}$ for $x \in X$. An ideal $\mathcal{I}$ on ω is called *P-ideal* if for any sequence $X_n \in \mathcal{I}$, $n \in \omega$, there exists $X \in \mathcal{I}$ such that $X_n \subseteq^* X$ for all $n \in \omega$. An ideal $\mathcal{I}$ on ω is *analytic* if it is analytic as a subspace of $\mathcal{P}(\omega)$ with the above topology. Recall that an ideal on ω is *tall* (or *dense*) if every infinite set of ω contains an infinite set from the ideal. If $\mathcal{I}$ is an ideal on ω and $Y \subseteq \omega$ is an infinite set, then we denote by $\mathcal{I}\restriction Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$; note that the underlying set of the ideal $\mathcal{I}\restriction Y$ is not the underlying set of $\mathcal{I}$ but Y. For an ideal $\mathcal{I}$ on ω, $\mathcal{I}^*$ denotes the dual filter, $\mathcal{M}$ denotes the ideal of meager subsets of $\mathbb{R}$, and $\mathcal{N}$ the ideal of Lebesgue null subsets of $\mathbb{R}$. Given an ideal $\mathcal{I}$ on a set X, the following are standard cardinal invariants associated with $\mathcal{I}$:

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} \notin \mathcal{I}\},$$
$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} = X\},$$
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\},$$
$$\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}.$$

The provable relationships between the cardinal invariants of $\mathcal{M}$ and $\mathcal{N}$ are summed up in the following diagram:

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

Cichoń's Diagram

Regarding our forcing terminology, $p \le q$ means that p is a stronger condition than q. By $\mathbb{B}(\kappa)$ we denote the measure algebra of Maharam type κ (the algebra of Baire subsets of 2^κ modulo the ideal $\mathcal{N}_\kappa$ of Haar measure zero subsets of 2^κ). Recall that a forcing $\mathbb{P}$ is *weakly distributive* (ω^ω-bounding) if every new function in ω^ω is pointwise dominated by a ground model function. A partial order $\mathbb{P}$ satisfies *Axiom A* if there is a sequence $\langle \le_n : n \in \omega \rangle$ of orderings on $\mathbb{P}$ such that

(A1) $p \le_0 q$ if $p \le q$ for every $p, q \in \mathbb{P}$.
(A2) $p \le_{n+1} q \Rightarrow p \le_n q$ for every $p, q \in \mathbb{P}$.
(A3) If $\{p_n : n \in \omega\}$ is such that if $p_{n+1} \le_n p_n$ for every $n \in \omega$, then there is a $p \in \mathbb{P}$ such that $p \le_n p_n$ for every $n \in \omega$.
(A4) For every maximal antichain $\mathcal{A}$ in $\mathbb{P}$, for every $p \in \mathbb{P}$ and $n \in \omega$, there is a $q \le_n p$ such that $\{r \in \mathcal{A} : r \text{ is compatible with } q\}$ is countable.

For more on forcing, *e.g.*, the definition of proper forcing, consult [Ku, BJ]. Even though we do not define properness, we remind the reader that every Axiom A forcing is proper.

1 Analytic P-Ideals and Their Cardinal Invariants

Definition 1.1 Let $\mathcal{I}$ be a tall ideal on ω containing the ideal of finite sets. Define the following cardinals associated with $\mathcal{I}$:

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in \mathcal{I})(\exists A \in \mathcal{A})(A \not\subseteq^* X)\},$$

$$\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^{\aleph_0})(\exists A \in \mathcal{A})(|A \cap X| = \aleph_0)\},$$

$$\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\},$$

$$\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\}.$$

These cardinals have been studied, some of them not in the context of ideals on ω, but as cardinals associated to the dual filters.[1] The notation for $\mathrm{add}^*(\mathcal{I})$ is taken

[1]Brendle and Shelah [BS] introduced cardinal invariants $\mathfrak{p}(\mathcal{F})$ and $\pi\mathfrak{p}(\mathcal{F})$ associated with an (ultra)filter $\mathcal{F}$. For tall ideal $\mathcal{I}$, $\mathrm{add}^*(\mathcal{I}) = \mathfrak{p}(\mathcal{I}^*)$, $\mathrm{cov}^*(\mathcal{I}) = \pi\mathfrak{p}(\mathcal{I}^*)$, $\mathrm{non}^*(\mathcal{I}) = \pi\chi(\mathcal{I}^*)$ and $\mathrm{cof}^*(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) = \chi(\mathcal{I}^*)$.

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from [Ba], other authors use $\mathfrak{p}(\mathcal{I}^*)$, $\mathfrak{b}(\mathcal{I}, \subseteq^*)$ or $\mathfrak{b}(\mathcal{I})$. We justify our preference for the names chosen here by the next proposition.

For every tall ideal $\mathcal{I}$ on ω, there is a natural ideal of Borel subsets of $\mathcal{P}(\omega)$ associated with $\mathcal{I}$ defined as $\widehat{\mathcal{I}} = \{\mathcal{X} \subseteq \mathcal{P}(\omega) : (\exists I \in \mathcal{I})(\mathcal{X} \subseteq \widehat{I})\}$, where $\widehat{I} = \{X \subseteq \omega : |X \cap I| = \aleph_0\}$. One can easily check that $I \subseteq^* J$ if and only if $\widehat{I} \subseteq \widehat{J}$. Hence, $\mathcal{J} \subseteq \mathcal{P}(\omega)$ is a P-ideal if and only if $\widehat{\mathcal{J}}$ is a σ-ideal.

Proposition 1.2 *The following equalities hold:*

$$\operatorname{add}(\widehat{\mathcal{I}}) = \operatorname{add}^*(\mathcal{I}), \qquad \operatorname{cof}(\widehat{\mathcal{I}}) = \operatorname{cof}^*(\mathcal{I}),$$
$$\operatorname{non}(\widehat{\mathcal{I}}) = \operatorname{non}^*(\mathcal{I}), \qquad \operatorname{cov}(\widehat{\mathcal{I}}) = \operatorname{cov}^*(\mathcal{I}).$$

Proof The facts that $\operatorname{add}(\widehat{\mathcal{I}}) = \operatorname{add}^*(\mathcal{I})$ and $\operatorname{cof}(\widehat{\mathcal{I}}) = \operatorname{cof}^*(\mathcal{I})$ follow directly from the observation that $I \subseteq^* J$ if and only if $\widehat{I} \subseteq \widehat{J}$. To see $\operatorname{cov}^*(\mathcal{I}) = \operatorname{cov}(\widehat{\mathcal{I}})$, observe that if $\{I_\alpha : \alpha < \operatorname{cov}^*(\mathcal{I})\}$ is a family witnessing the definition of $\operatorname{cov}^*(\mathcal{I})$ and $X \in [\omega]^\omega$, then there is some $\alpha < \operatorname{cov}^*(\mathcal{I})$ such that $|X \cap I| = \aleph_0$, so $X \in \widehat{I_\alpha}$. On the other hand, if $\kappa < \operatorname{cov}^*(\mathcal{I})$ and $\{\mathcal{X}_\alpha : \alpha < \kappa\} \subseteq \widehat{\mathcal{I}}$, for every $\alpha < \kappa$ choose $I_\alpha \in \mathcal{I}$ so that $\mathcal{X}_\alpha \subseteq \widehat{I_\alpha}$. As $\kappa < \operatorname{cov}^*(\mathcal{I})$, there is some $X \in [\omega]^\omega$ such that $|X \cap I_\alpha| < \aleph_0$ for all $\alpha < \kappa$. Thus $\bigcup_{\alpha<\kappa} \mathcal{X}_\alpha \neq [\omega]^\omega$. ■

The inequalities holding among these cardinals are summarized in the following diagram:

$$\begin{tikzcd}
 & & \operatorname{non}^*(\mathcal{I}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \operatorname{add}^*(\mathcal{I}) \arrow[ur] \arrow[dr] & & \operatorname{cof}^*(\mathcal{I}) \arrow[r] & 2^{\aleph_0} \\
 & & \operatorname{cov}^*(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

It follows directly from the definition that $\operatorname{cov}^*(\mathcal{I}) \geq \mathfrak{p}$ for any tall ideal $\mathcal{I}$.

The main result about the structure of analytic P-ideals is due to Solecki [Fa1, Theorem 1.2.5]. Recall that a function $\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if $\varphi(\varnothing) = 0$, $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \in \mathcal{P}(\omega)$, and $\varphi(\{n\}) < \infty$ for every $n \in \omega$. A submeasure on ω is called *lower semicontinuous* if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for every $A \subseteq \omega$. There are two ideals associated with lower semicontinuous submeasures on ω:

$$\operatorname{Exh}(\varphi) = \{X \in \mathcal{P}(\omega) : \lim_{m\to\infty} \varphi(X \setminus m) = 0\},$$

and

$$\operatorname{Fin}(\varphi) = \{X \in \mathcal{P}(\omega) : \varphi(X) < \infty\}.$$

Theorem 1.3 *Let $\mathcal{I}$ be an ideal on ω. Then*

(i) (Mazur) *$\mathcal{I}$ is an F_σ ideal if and only if $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some lower semicontinuous submeasure φ.*
(ii) (Solecki) *$\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ.*
(iii) (Solecki) *$\mathcal{I}$ is an F_σ P-ideal if and only if $\mathcal{I} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure φ.*

We will use the following simple fact several times in the paper.

Lemma 1.4 *Let $\mathcal{I} = \mathrm{Exh}(\varphi)$, where φ is lower semicontinuous submeasure. Then $\mathcal{I}$ is a tall ideal if and only if $\lim_{n\to\infty} \varphi(\{n\}) = 0$.*

Proof If $\lim_{n\to\infty} \varphi(\{n\}) \neq 0$, then the set $E = \{n \in \omega : \varphi(\{n\}) \geq \varepsilon\}$ is infinite for some $\varepsilon > 0$, and $\lim_{n\to\infty} \varphi(I \setminus n) \neq 0$ for all infinite $I \subseteq E$. Thus $\mathcal{I}$ is not a tall ideal.

On the other hand, if $\lim_{n\to\infty} \varphi(\{n\}) = 0$ and $X \subseteq \omega$ is infinite, find an increasing sequence of $k_n \in X$ such that $\varphi(\{m\}) \leq \frac{1}{2^n}$ for all $m \geq k_n$. Then $I = \{k_n : n \in \omega\} \in \mathcal{I}$ since $\varphi(I \setminus k_m) \leq \sum_{n=m}^{\infty} \frac{1}{2^n}$ and $\sum_{n=m}^{\infty} \frac{1}{2^n}$ converges to zero. ■

Orderings on Ideals We shall take advantage of the (pre-)orderings on the family of ideals. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Recall:

(i) (Rudin–Keisler order) $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there exists a function $f\colon \omega \to \omega$ such that $I \in \mathcal{I}$ if and only if $f^{-1}(I) \in \mathcal{J}$ for every $I \subseteq \omega$.
(ii) (Rudin–Blass order) $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if there exists a finite-to-one function $f\colon \omega \to \omega$ such that $I \in \mathcal{I}$ if and only if $f^{-1}(I) \in \mathcal{J}$ for every $I \subseteq \omega$.
(iii) (Katětov order) $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there exists a function $f\colon \omega \to \omega$ such that $f^{-1}(I) \in \mathcal{J}$ for every $I \in \mathcal{I}$.
(iv) (Tukey order[2]) $\mathcal{I} \leq_{\mathrm{T}}^{*} \mathcal{J}$ if there exists a function $f\colon \mathcal{I} \to \mathcal{J}$ such that for every $\subseteq^*$-bounded set $X \subseteq \mathcal{J}$, $f^{-1}(X)$ is $\subseteq^*$-bounded in $\mathcal{I}$.

These orderings have a deep impact on the cardinal invariants of related ideals. Note that if $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$, and that if $X \in \mathcal{I}^+$ then $\mathcal{I}\restriction X \leq_{\mathrm{T}}^{*} \mathcal{I}$ while $\mathcal{I}\restriction X \geq_{\mathrm{K}} \mathcal{I}$. Notice also that $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ and for P-ideals $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \Rightarrow \mathcal{I} \leq^{*}_{\mathrm{T}} \mathcal{J}$.

Any $f\colon \omega \to \omega$ witnessing $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ is called a *Katětov reduction*. Also, $\mathcal{I}$ and $\mathcal{J}$ are Katětov equivalent if $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathrm{K}} \mathcal{I}$, which we denote by $\mathcal{I} \cong_{\mathrm{K}} \mathcal{J}$. Similarly for the other orderings. We use $\mathcal{I} \cong \mathcal{J}$ to denote that there is a permutation ρ of ω such that $I \in \mathcal{I}$ if and only if $\rho[I] \in \mathcal{J}$.[3]

[2] See Appendix
[3] According to [Fa1, 1.2.7] two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω are isomorphic if there is a partial one-to-one function $f\colon \omega \to \omega$ such that $\omega \setminus \mathrm{ran}(f) \in \mathcal{J}, \omega \setminus \mathrm{dom}(f) \in \mathcal{I}$ and $A \in \mathcal{I} \Leftrightarrow f[A] \in \mathcal{J}$ for all $A \subseteq \omega$. As the two definitions are equivalent for tall ideals, we choose the simpler one.

Some Ideals on ω With one exception, all ideals in which we are interested are tall.

$$\emptyset \times \mathrm{Fin} = \bigl\{ A \subseteq \omega \times \omega : (\forall n \in \omega)\bigl(\{m \in \omega : \langle n, m\rangle \in A\} \text{ is finite}\bigr)\bigr\}.$$

For $A \subseteq 2^{<\omega}$, let $\pi(A) = \{f \in 2^\omega : (\exists^\infty n \in \omega)(f\restriction n \in A)\}$. The *trace of the null ideal* is $\mathrm{tr}(\mathcal{N}) = \{A \subseteq 2^{<\omega} : \mu(\pi(A)) = 0\}$, where μ denotes the standard product measure on 2^ω. This ideal is, of course, very much related to the null ideal $\mathcal{N}$ on the reals.

The *eventually different ideal* is defined as

$$\mathcal{ED} = \bigl\{ A \subseteq \omega \times \omega : (\exists m, n \in \omega)(\forall k > n)\bigl(|\{l : \langle k, l\rangle \in A\}| \leq m\bigr)\bigr\},$$

i.e., it is the ideal generated by vertical sections and graphs of functions. Define $\mathcal{ED}_{\mathrm{fin}} = \mathcal{ED}\restriction\Delta$, where $\Delta = \{\langle m, n\rangle \in \omega \times \omega : n \leq m\}$. The ideals $\mathcal{ED}$ and $\mathcal{ED}_{\mathrm{fin}}$ are the only ideals mentioned in the paper which are not P-ideals. They are both F_σ.

The following will be proved in a forthcoming paper.[4]

Proposition 1.5

$$\mathrm{cov}^*(\mathcal{ED}) = \mathrm{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{\mathrm{fin}})\},$$
$$\mathrm{non}^*(\mathcal{ED}) = \omega, \quad \mathrm{non}^*(\mathcal{ED}_{\mathrm{fin}}) = \mathrm{cov}(\mathcal{M}).$$

Given $f\colon \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{n\in\omega} f(n) = \infty$, the *summable ideal* corresponding to f is the ideal

$$\mathcal{I}_f = \{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\}.$$

The ideal $\mathcal{I}_f$ is tall if and only if $\lim_{n\to\infty} f(n) = 0$. The lower semicontinuous submeasure on ω corresponding to $\mathcal{I}_f$ is $\varphi_f(A) = \sum_{n\in A} f(n)$. By definition, $\mathcal{I}_f = \mathrm{Fin}(\varphi_f)$. So, summable ideals are F_σ. A typical example of a summable ideal is the ideal

$$\mathcal{I}_{1/n} = \{A \subseteq \omega : \sum_{n\in A} \frac{1}{n} < \infty\}.$$

An *Erdős–Ulam ideal* is an ideal associated to a function $f\colon \mathbb{N} \to \mathbb{R}$. Then $\mathcal{EU}_f$ is the ideal of all subsets of f-density zero, *i.e.,* sets A such that

$$\lim_{n\to\infty} \frac{\sum_{i\in A\cap(n+1)} f(i)}{\sum_{i=0}^{n} f(i)} = 0.$$

The lower semicontinuous submeasure corresponding to $\mathcal{EU}_f$ is given by

$$\varphi_f(A) = \sup_{n\in\omega} \frac{\sum_{i\in A\cap(n+1)} f(i)}{\sum_{i=0}^{n} f(i)}.$$

A class of analytic P-ideals further extending the class of Erdős–Ulam ideals is the class of density ideals. For a submeasure φ, let $\mathrm{supp}(\varphi) = \{n \in \omega : \varphi(\{n\}) \neq 0\}$. Submeasures φ and ψ are *orthogonal* if they have disjoint supports.

[4] M. Hrušák, *Katětov order.* In preparation.

Definition 1.6 For a sequence $\vec{\mu} = \{\mu_i\}_{i\in\omega}$ of orthogonal measures on ω each of which concentrates on some finite set, define the submeasure $\varphi_{\vec{\mu}}$ by

$$\varphi_{\vec{\mu}} = \sup_{i\in\omega} \mu_i.$$

Then

$$\mathcal{Z}_{\vec{\mu}} = \operatorname{Exh}(\varphi_{\vec{\mu}}) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi_{\vec{\mu}}(A \setminus n) = 0\}$$

is the *density ideal* generated by the sequence of measures.

The proof of the following result can be consulted in [Fa1, 1.13.3].

Theorem 1.7 *Every Erdős–Ulam ideal is equal to some density ideal $\mathcal{Z}_{\vec{\mu}}$, where each μ_n is a probability measure.*

The most common of the density ideals is the ideal $\mathcal{Z}$ of subsets of ω of *asymptotic density zero*, that is:

$$\mathcal{Z} = \{A \subseteq \omega : \lim_{n\to\infty} \tfrac{|A\cap n|}{n} = 0\}.$$

Equivalently, $A \in \mathcal{Z}$ if and only if

$$\lim_{n\to\infty} \frac{|A \cap [2^n, 2^{n+1})|}{2^n} = 0.$$

For more on analytic P-ideals and historical notes consult [Fa1].

2 Additivity and Cofinality

In this section we review some known results about additivity and cofinality of analytic P-ideals. The basic tool for studying these cardinal invariants is the Tukey ordering. This is largely due to the following observation. Although it is well known [Fr1, 1J(a)], we include its short proof for the sake of completeness.

Proposition 2.1 $\mathcal{I} \leq^*_{\mathrm{T}} \mathcal{J} \Rightarrow \operatorname{add}^*(\mathcal{I}) \geq \operatorname{add}^*(\mathcal{J})$ *and* $\operatorname{cof}^*(\mathcal{I}) \leq \operatorname{cof}^*(\mathcal{J})$.

Proof Suppose that $f : \mathcal{I} \to \mathcal{J}$ witnesses the definition of $\leq^*_{\mathrm{T}}$. Let $\mathcal{A} \subseteq \mathcal{I}$ be a family with $|\mathcal{A}| < \operatorname{add}^*(\mathcal{J})$. There is a set $B \in \mathcal{J}$ such that $f[A] \subseteq^* B$ for every $A \in \mathcal{A}$. As f maps unbounded sets to unbounded sets, it follows that $\mathcal{A}$ must be bounded.

Let $\mathcal{B} \subseteq \mathcal{J}$ be cofinal. For every $B \in \mathcal{B}$, there is an $A_B \in \mathcal{I}$ such that if $f[I] \subseteq^* B$ then $I \subseteq^* A_B$. It follows that the family $\{A_B : B \in \mathcal{B}\}$ is a cofinal subset of $\mathcal{I}$. ■

The following theorem summarizes the known results.

Theorem 2.2

(i) $\operatorname{add}^*(\mathcal{I}_{1/n}) = \operatorname{add}^*(\operatorname{tr}(\mathcal{N})) = \operatorname{add}(\mathcal{N})$.

(ii) (Todorčević, [To]) $\varnothing \times \mathrm{Fin} \leq_{\mathrm{T}} \mathcal{I} \leq_{\mathrm{T}} \mathcal{I}_{1/n}$ *for every analytic P-ideal $\mathcal{I}$. In particular,* $\operatorname{add}(\mathcal{N}) \leq \operatorname{add}^*(\mathcal{I}) \leq \mathfrak{b}$ *for all analytic P-ideals $\mathcal{I}$.*

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(iii) (Fremlin, [Fr2, 526H]) $\mathrm{add}^*(\mathcal{Z}) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}^*(\mathcal{Z}) = \mathrm{cof}(\mathcal{N})$.
(iv) (Farah, [Fa1, 1.13.10]) *Every tall Erdős–Ulam ideal is Rudin–Blass equivalent to* $\mathcal{Z}$.
(v) *Every tall summable ideal is Tukey equivalent to* $\mathcal{I}_{1/n}$.
(vi) $\mathrm{add}^*(\mathcal{I}) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}^*(\mathcal{I}) = \mathrm{cof}(\mathcal{N})$, *for every tall ideal* $\mathcal{I}$ *which is either summable or a density ideal.*

Proof For (v) by Claim 1 in [Fa1, 1.12.14], given two summable ideals $\mathcal{I}_f$ and $\mathcal{I}_g$, there is an $X \in \mathcal{I}_g^+$ such that $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g \restriction X$. On the other hand, $\mathcal{I}_g \restriction X \leq_{\mathrm{T}} \mathcal{I}_g$, hence $\mathcal{I}_f$ and $\mathcal{I}_g$ are Tukey equivalent. (vi) for summable ideals follows directly from (i) and (v), and for tall density ideals it follows from (ii), (iii) and the fact that for every tall density ideal $\mathcal{I}$ there is an $X \in \mathcal{I}^+$ such that $\mathcal{I}\restriction X$ is Erdős–Ulam, [Fa1, 1.13.10]. ■

Louveau and Veličković [LV] showed that there are many $\leq_{\mathrm{T}}$ non-equivalent analytic P-ideals. On the other hand, assuming $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$, all tall analytic P-ideals are $\leq_{\mathrm{T}}^*$-equivalent. It is natural to ask:

Questions 2.3

(a) Are all tall analytic P-ideals $\leq_{\mathrm{T}}^*$-equivalent?
(b) Is at least $\mathrm{add}^*(\mathcal{I}) = \mathrm{add}(\mathcal{N})$ for every tall analytic P-ideal?

3 Covering and Uniformity

The Katětov ordering relates to covering and uniformity of ideals in an analogous way as the Tukey ordering relates to additivity and cofinality.

Proposition 3.1 $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \Rightarrow \mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cov}^*(\mathcal{J})$ *and* $\mathrm{non}^*(\mathcal{I}) \leq \mathrm{non}^*(\mathcal{J})$.

Proof Let $\mathcal{A} \subseteq \mathcal{I}$ witness the definition of $\mathrm{cov}^*(\mathcal{I})$, and let $f\colon \omega \to \omega$ be a Katětov reduction witnessing $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$. Then

$$\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\} \cup \{f^{-1}(F) : F \in [\omega]^{<\aleph_0}\}$$

witnesses the definition of $\mathrm{cov}^*(\mathcal{J})$. Indeed, if $X \subseteq \omega$ is infinite, then either $f[X]$ is infinite and hence, there is some $A \in \mathcal{A}$ such that for infinitely many $n \in X$, $f(n) \in A$; therefore $|X \cap f^{-1}(A)| = \aleph_0$. Or else, $f[X]$ is finite and hence $X \subseteq f^{-1}(f[X])$. In either case X has infinite intersection with some member of $\mathcal{B}$.

If $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ witnesses $\mathrm{non}^*(\mathcal{J})$, then $\{f[A] : A \in \mathcal{A}\}$ witnesses $\mathrm{non}^*(\mathcal{I})$, for if $I \in \mathcal{I}$, then $f^{-1}(I) \in \mathcal{J}$ and there is some $A \in \mathcal{A}$ such that $A \cap f^{-1}(I)$ is finite. Hence $f[A] \cap I$ must be finite. ■

First we investigate the behaviour of the Katětov order restricted to analytic P-ideals. It turns out that the ideal $\mathcal{ED}_{\mathrm{fin}}$ is a lower bound for all analytic P-ideals in the Katětov order.

Proposition 3.2 $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{K}} \mathcal{I}$ *for every tall analytic P-ideal.*

Proof Let φ be a lower semicontinuous submeasure such that $\mathcal{I} = \mathrm{Exh}(\varphi)$. By Lemma 1.4 there is a strictly increasing sequence $\langle a_n : n \in \omega\rangle$ such that $\varphi(\{m\}) < 2^{-n}$ for $m \geq a_n$. Let $g\colon \omega \to \Delta \subseteq \omega \times \omega$ be a one-to-one function mapping intervals $[a_n, a_{n+1})$ into distinct vertical sections of Δ, say $\{\langle b_n, i\rangle : i \leq b_n\}$. This function g is a Katětov reduction. Indeed, let $A \in \mathcal{ED}_{\mathrm{fin}}$. Then there is a $k \in \omega$ such that $|A \cap \{\langle b_n, i\rangle : i \leq b_n\}| \leq k$ for all $n \in \omega$. Now given $\varepsilon > 0$, $\varphi(A \cap [a_n, a_{n+1})) \leq k2^{-n}$ and therefore $\varphi(A \setminus n) \leq \varepsilon$ for large enough $n \in \omega$. ■

As $\mathcal{ED}_{\mathrm{fin}}$ is not a P-ideal, the relation $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{K}} \mathcal{I}$ cannot be strengthened to $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{RK}} \mathcal{I}$. In the proof of the next result we use the following lemma, which is probably folklore, but we could not find any reference for it.

Lemma 3.3 *Let $\varepsilon > 0$ and $\mathcal{U}$ be an infinite family of clopen subsets of 2^ω, each of measure at least ε. Then*

$$\mu\big(\{x \in 2^\omega : (\exists^\infty U \in \mathcal{U})(x \in U)\}\big) \geq \varepsilon.$$

Proof Suppose not. Then there is a compact set $K \subseteq 2^\omega$ disjoint with $\{x \in 2^\omega : (\exists^\infty U \in \mathcal{U})(x \in U)\}$ such that $\mu(K) > 1 - \varepsilon$. Let $\delta = \varepsilon + \mu(K) - 1 > 0$. Then $\mu(U \cap K) \geq \delta$ for each $U \in \mathcal{U}$. Write $K = \bigcup_{n\in\omega} A_n$, where

$$A_n = \{x \in K : |\{U \in \mathcal{U} : x \in U\}| = n\}.$$

Then $\mu(K) = \sum_{n\in\omega} \mu(A_n)$, so there is an $m \in \omega$ such that $\sum_{n=m}^{\infty} \mu(A_n) < \frac{\delta}{2}$. For each $n < m$ let $C_n \subseteq A_n$ be compact such that $\mu(A_n \setminus C_n) \leq \frac{\delta}{2m}$. Putting $C = \bigcup_{n<m} C_n$, $\mu(\bigcup_{n<m} A_n \setminus C) < \frac{\delta}{2}$.

Now it is not hard to see that $\{U \in \mathcal{U} : U \cap C \neq \varnothing\}$ is finite. Since $\mu(K \setminus C) < \delta$, $\mathcal{U}$ itself must be finite. ■

Theorem 3.4 $\mathcal{I}_{1/n} \leq_{\mathrm{K}} \mathrm{tr}(\mathcal{N}) \leq_{\mathrm{K}} \mathcal{Z}$.

Proof It is easy to see that $\mathcal{Z} \cong \{A \subseteq 2^{<\omega} : \lim_{n\to\infty} \frac{|A\cap 2^n|}{2^n} = 0\}$ and $\mathcal{I}_{1/n} \cong \{A \subseteq 2^{<\omega} : \sum_{n\in\omega} \frac{|A\cap 2^n|}{2^n} < \infty\}$. To see the latter, first note that $\sum_{n\in A} \frac{1}{n} = \sum_{n\in\omega} \sum\{\frac{1}{m} : m \in A \cap [2^n, 2^{n+1})\}$ and then the isomorphism can be deduced observing that

$$\begin{aligned} \frac{1}{2}\sum_{n\in\omega} \frac{|A \cap [2^n, 2^{n+1})|}{2^n} &\leq \sum_{n\in\omega} \sum_{m\in A\cap[2^n,2^{n+1})} \frac{1}{2^{n+1}} \leq \sum_{n\in\omega} \sum_{m\in A\cap[2^n,2^{n+1})} \frac{1}{m} \\ &\leq \sum_{n\in\omega} \sum_{m\in A\cap[2^n,2^{n+1})} \frac{1}{2^n} \leq \sum_{n\in\omega} \frac{|A \cap [2^n, 2^{n+1})|}{2^n}. \end{aligned}$$

For $\mathcal{I}_{1/n} \leq_{\mathrm{K}} \mathrm{tr}(\mathcal{N})$, just note that if $A \subseteq 2^{<\omega}$ is such that $\sum_{n\in\omega} \frac{|A\cap 2^n|}{2^n} < \infty$, then $\pi(A) \in \mathcal{N}$. So, $\{A \subseteq 2^{<\omega} : \sum_{n\in\omega} \frac{|A\cap 2^n|}{2^n} < \infty\} \subseteq \mathrm{tr}(\mathcal{N})$.

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To see that $\operatorname{tr}(\mathcal{N}) \leq_K \mathcal{Z}$, it suffices to prove that

$$A \in \operatorname{tr}(\mathcal{N}) \Rightarrow \lim_{n\to\infty} \frac{|2^n \cap A|}{2^n} = 0.$$

If it were not the case, then for some $A \in \operatorname{tr}(\mathcal{N})$ and for some $\varepsilon > 0$, the set $J = \{n \in \omega : \frac{|2^n \cap A|}{2^n} \geq \varepsilon\}$ would be infinite. As $A \in \operatorname{tr}(\mathcal{N})$, $\pi(A) \in \mathcal{N}$, so there is a compact $K \subseteq 2^\omega$ such that $\mu(K) > 1 - \frac{\varepsilon}{3}$ and $K \cap \pi(A) = \varnothing$. For $n \in J$, let $U_n = \bigcup_{s \in 2^n \cap A} [s]$. Then $\mu(U_n \cap K) \geq \frac{2\varepsilon}{3}$. By the previous lemma there is an $x \in K \cap \pi(A)$, contradicting the choice of K. ■

Proposition 3.5 $\mathcal{I} \leq_K \operatorname{tr}(\mathcal{N})$ *for all summable ideals* $\mathcal{I}$.

Proof Let $\mathcal{I}$ be a summable ideal. By [Fa1, Lemma 1.12.4], it follows that there is an $X \in \mathcal{I}^+_{1/n}$ such that $\mathcal{I} \leq_K \mathcal{I}_{1/n}\restriction X$. Fix such $X \in \mathcal{I}^+_{1/n}$. It suffices to show that $\mathcal{I}_{1/n}\restriction X \leq_K \operatorname{tr}(\mathcal{N})$.

Identify $\mathcal{I}_{1/n}$ with $\{A \subseteq 2^{<\omega} : \sum_{n\in\omega} \frac{|A\cap 2^n|}{2^n} < \infty\}$. So, $\sum_{n\in\omega} \frac{|X\cap 2^n|}{2^n} = \infty$. It may happen that $X \in \operatorname{tr}(\mathcal{N})$. However, this problem is easily fixable. Recursively choose an increasing sequence of integers $\langle n_i : i \in \omega\rangle$ such that $\sum_{n\in[n_i,n_{i+1})} \frac{|A\cap 2^n|}{2^n} \geq 1$. There is a bijection $g\colon 2^{<\omega} \to 2^{<\omega}$ such that

(i) $g[2^n] = 2^n$, for every $n \in \omega$ and
(ii) for each $s \in 2^{n_i+1}$ there is an $m \in [n_i, n_{i+1})$ such that $s\restriction m \in g[X]$.

Note that for every $I \subseteq X, I \in \mathcal{I}_{1/n}$ if and only if $g[I] \in \mathcal{I}_{1/n}$. So without loss of generality we can assume that g is the identity function, and hence $\pi[X] = 2^\omega$. In particular, $X \notin \operatorname{tr}(\mathcal{N})$ and $\mathcal{I}_{1/n}\restriction X \subseteq \operatorname{tr}(\mathcal{N})\restriction X$.

To prove that $\operatorname{tr}(\mathcal{N})\restriction X \leq_K \operatorname{tr}(\mathcal{N})$, define a function $h\colon 2^{<\omega} \to X$ as follows: for every $s \in 2^{<\omega}$, let $m_s = \max\{k \in \operatorname{dom}(s) : s\restriction k \in X\}$. This m_s is well defined for almost all $s \in 2^{<\omega}$ by (ii). Fix $t \in X$ and let

$$h(s) = \begin{cases} s\restriction m_s & \text{if } m_s \text{ is defined,} \\ t & \text{otherwise.} \end{cases}$$

To finish the proof, it suffices to notice that $\pi[A] \subseteq \pi[h[A]]$ and hence h is a Katětov reduction witnessing $\operatorname{tr}(\mathcal{N})\restriction X \leq_K \operatorname{tr}(\mathcal{N})$. ■

Proposition 3.6

(i) (Farah) *Every Erdős–Ulam ideal is Rudin–Blass equivalent to* $\mathcal{Z}$.
(ii) *If* $\mathcal{I}$ *is a density ideal then* $\mathcal{I} \leq_K \mathcal{Z}$.

Proof (i) follows directly from [Fa1, p. 46] as any two Erdős–Ulam ideals $\mathcal{EU}_f$ and $\mathcal{EU}_g$ are even Rudin–Blass equivalent.

For (ii), it suffices to note that for every tall density ideal $\mathcal{I}$, there is a positive set X such that $\mathcal{I}\restriction X$ is Erdős–Ulam. ■

From the above propositions, it follows that the density ideal $\mathcal{Z}$ is Katětov maximal among all ideals considered so far. On the other hand, the summable ideals are very low in the Katětov order. In particular, for any tall analytic P-ideal $\mathcal{I}$, there is an $X \in \mathcal{I}_{1/n}^{+}$ such that $\mathcal{I}_{1/n} \restriction X \leq_{\mathrm{K}} \mathcal{I}$. To see this, fix a lower semicontinuous submeasure φ such that $\mathcal{I} = \operatorname{Exh}(\varphi)$ and note that $\mathcal{I}_f \subseteq \mathcal{I}$ for f defined by $f(n) = \varphi(\{n\})$ as $\varphi_f \geq \varphi$. By [Fa1, p. 36] $\mathcal{I}_f \leq_{\mathrm{K}} \mathcal{I}_{1/n} \restriction X$ for some $\mathcal{I}_{1/n}$-positive set X.

However, the summable ideals are highly non-homogeneous and we conjecture that they have consistently distinct covering numbers.

Using Proposition 3.1 we can summarize the impact of the Katětov order on analytic P-ideals as follows.

Theorem 3.7

(i) *For every tall analytic P-ideal $\mathcal{I}$,* $\operatorname{add}(\mathcal{N}) \leq \operatorname{cov}^*(\mathcal{I}) \leq \operatorname{non}(\mathcal{M})$ *and* $\operatorname{cov}(\mathcal{M}) \leq \operatorname{non}^*(\mathcal{I}) \leq \operatorname{cof}(\mathcal{N})$.

(ii) *For every tall summable ideal $\mathcal{I}$,* $\operatorname{cov}(\mathcal{N}) \leq \operatorname{cov}^*(\mathcal{I})$ *and* $\operatorname{non}^*(\mathcal{I}) \leq \operatorname{non}(\mathcal{N})$.

(iii) *For every Erdős–Ulam ideal $\mathcal{I}$,* $\operatorname{cov}^*(\mathcal{I}) = \operatorname{cov}^*(\mathcal{Z})$ *and* $\operatorname{non}^*(\mathcal{I}) = \operatorname{non}^*(\mathcal{Z})$.

(iv) *For every tall density or summable ideal $\mathcal{I}$,* $\operatorname{cov}^*(\mathcal{Z}) \leq \operatorname{cov}^*(\mathcal{I})$ *and* $\operatorname{non}^*(\mathcal{I}) \leq \operatorname{non}^*(\mathcal{Z})$.

The next proposition resembles the well-known characterization of the splitting number (see [Va])

$$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \ell_\infty \wedge (\forall X \in [\omega]^{\aleph_0})(\exists s \in \mathcal{S})(s \restriction X \text{ is not convergent})\}$$

and may be of independent interest.

Proposition 3.8

$$\mathfrak{b} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq c_0 \wedge (\forall X \in [\omega]^{\aleph_0})(\exists s \in \mathcal{S})(s \restriction X \notin \ell_1)\}.$$

Proof Denote by ν the cardinal on the right-hand side of the equation. We first show that $\mathfrak{b} \leq \nu$. In order to do this, assume that $\kappa < \mathfrak{b}$ and show that $\kappa < \nu$. Let $\mathcal{S} \in [c_0]^\kappa$. For each $s \in \mathcal{S}$ define $f_s : \omega \to \omega$ by letting

$$f_s(n) = \min\Big\{\, m \in \omega : (\forall k \geq m)\big(|s(k)| < \tfrac{1}{2^k}\big) \,\Big\}.$$

There is a $g \in \omega^\omega$ such that $(\forall s \in \mathcal{S})(f_s \leq^* g)$. We can assume that g is strictly increasing. Put $X = \operatorname{ran}(g)$. If $s \in \mathcal{S}$, then there is an $m \in \omega$ such that $f_s(k) \leq g(k)$ for all $k \geq m$ and hence

$$\sum_{x \in X \setminus g(m)} |s(x)| = \sum_{n=m}^{\infty} |s(g(n))| \leq \sum_{n=m}^{\infty} \frac{1}{2^n} < \infty.$$

Therefore, $(\forall s \in \mathcal{S})(s \restriction X \in \ell_1)$ and hence $\kappa < \nu$.

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To see that $\nu \le \mathfrak{b}$, proceed as follows. Consider a $\le^*$-unbounded family of strictly increasing functions, $\{f_\alpha : \alpha < \mathfrak{b}\}$, such that $f_\alpha(0) > 0$, for each $\alpha < \mathfrak{b}$. For all $\alpha < \mathfrak{b}$ and $n \in \omega$, let

$$s_\alpha(n) = \begin{cases} \frac{1}{m+1} & \text{if } f_\alpha(m) \le n < f_\alpha(m+1), \\ 1 & \text{if } n < f_\alpha(0). \end{cases}$$

We claim that $\mathcal{S} = \{s_\alpha : \alpha < \mathfrak{b}\}$ witnesses the definition of ν. To see this, let $X \in [\omega]^{\aleph_0}$ and let $\{x_n : n \in \omega\}$ be the increasing enumeration of X. There is an $\alpha < \mathfrak{b}$ such that $(\exists^\infty n \in \omega)(x_n < f_\alpha(n))$. Define a sequence of natural numbers recursively by setting $n_{k+1} = \min\{m > n_k : x_m < f_\alpha(m)\}$. To see that $s_\alpha \restriction X \notin \ell_1$, note that

$$\begin{aligned}\sum_{n=0}^{\infty} s_\alpha(x_n) &\ge n_0 \cdot \frac{1}{n_0} + (n_1 - n_0) \cdot \frac{1}{n_1} + \cdots + (n_{k+1} - n_k) \cdot \frac{1}{n_k} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \Big(1 - \frac{n_k}{n_{k+1}}\Big).\end{aligned}$$

By the next lemma, the last series never converges, which completes the proof of the proposition. We thank M. Garaev for supplying a short proof of the lemma. ■

Lemma 3.9 *If $\langle x_n : n \in \omega\rangle$ is a strictly increasing sequence of natural numbers, then $\sum_{n=0}^{\infty}(1 - \frac{x_n}{x_{n+1}})$ diverges.*

Proof Without loss of generality $x_0 = 0$. For every $n \in \omega$, let $y_n = x_{n+1} - x_n$. Then $y_n \ge 1$ and $x_{n+1} = y_0 + \cdots + y_n$ for each $n \in \omega$. So, $1 - \frac{x_n}{x_{n+1}} = \frac{y_n}{y_0 + \cdots + y_n}$.

Assume that $\sum_{n=0}^{\infty} \frac{y_n}{y_0 + \cdots + y_n}$ converges. If $\langle y_n : n \in \omega\rangle$ is bounded by some b, then

$$\sum_{n=0}^{m} \frac{y_n}{y_0 + \cdots + y_n} \ge \frac{1}{b} \sum_{n=0}^{m} \frac{1}{n+1},$$

which is a contradiction. So, assume that $\langle y_n : n \in \omega\rangle$ is unbounded. There is a $c \in \omega$ such that $\sum_{n=c+1}^{\infty} \frac{y_n}{y_0 + \cdots + y_n} < \frac{1}{10}$. Choose $N \in \omega$ so that $y_N > y_0 + \cdots + y_c$. Then

$$\frac{1}{10} > \sum_{n=c+1}^{N} \frac{y_n}{y_0 + \cdots + y_n} > \frac{\frac{1}{2}(y_{c+1} + \cdots + y_N) + \frac{1}{2} y_N}{y_0 + \cdots + y_N} > \frac{1}{2},$$

which, too, is a contradiction. ■

For the rest of this section we will concentrate on cardinal invariants of the density ideal $\mathcal{Z}$, trying to locate them among the standard cardinal invariants of the continuum.

Theorem 3.10 *The following inequalities hold:*

$$\min\{\mathrm{cov}(\mathcal{N}),\mathfrak{b}\} \leq \mathrm{cov}^*(\mathcal{Z}) \leq \mathrm{non}(\mathcal{M}) \tag{1}$$

and

$$\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d},\mathrm{non}(\mathcal{N})\}. \tag{2}$$

Proof The facts that $\mathrm{cov}^*(\mathcal{Z}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}^*(\mathcal{Z})$ follow directly from Theorem 3.7(i).

To show that $\min\{\mathfrak{b},\mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}^*(\mathcal{Z})$, let $\kappa < \min\{\mathfrak{b},\mathrm{cov}(\mathcal{N})\}$. We show that $\kappa < \mathrm{cov}^*(\mathcal{Z})$. Let $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{Z}$. To each A_α we associate a null set. To that end, we work on $\prod_{n\in\omega} 2^n$ endowed with μ^*, the natural product of probability measures (each factor has the counting measure multiplied by $\frac{1}{2^n}$, so that $\mu(2^n) = 1$). Let $A_{\alpha,n} = \{m < 2^n : 2^n + m \in A_\alpha\}$. Then $\lim_{n\to\infty} \frac{|A_{\alpha,n}|}{2^n} = 0$, for each $\alpha < \kappa$, and since $\kappa < \mathfrak{b}$ and by Proposition 3.8 there is an $X \in [\omega]^{\aleph_0}$ such that

$$(\forall\alpha < \kappa)\Bigl(\sum_{n\in X} \frac{|A_{\alpha,n}|}{2^n} < \varepsilon\Bigr). \tag{3}$$

Then let $N_\alpha = \{h \in \prod_{n\in X} 2^n : (\exists^\infty n \in \omega)(h(n) \in A_{\alpha,n})\}$. The relation (3) shows that $\mu^*(N_\alpha) = 0$, so $N_\alpha \in \mathcal{N}$, for every $\alpha < \kappa$. Moreover, as $\kappa < \mathrm{cov}(\mathcal{N})$, there is an $h \in \prod_{n\in X} 2^n \setminus \bigcup\{N_\alpha : \alpha < \kappa\}$. Let Z be the set $\{2^n + h(n) : n \in X\}$. Then $Z \in \mathcal{Z}$ and $Z \cap A_\alpha =^* \varnothing$, for every $\alpha < \kappa$. So, $\kappa < \mathrm{cov}^*(\mathcal{Z})$.

To conclude the proof, we show that $\mathrm{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d},\mathrm{non}(\mathcal{N})\}$. Fix a $\leq^*$-dominating family $\{g_\alpha : \alpha < \mathfrak{d}\}$ consisting of strictly increasing functions. For every $\alpha < \mathfrak{d}$, choose non-null sets $Y_\alpha \subseteq \prod_{n\in\omega} 2^{g_\alpha(n)}$ of cardinality $\mathrm{non}(\mathcal{N})$ and enumerate each of them as $\{f_{\alpha,\beta} : \beta < \mathrm{non}(\mathcal{N})\}$. For $\alpha < \mathfrak{d}$ and $\beta < \mathrm{non}(\mathcal{N})$, let

$$L_{\alpha,\beta} = \{2^{g_\alpha(n)} + f_{\alpha,\beta}(n) : n \in \omega\} \in \mathcal{Z}.$$

We claim that $\mathcal{L} = \{L_{\alpha,\beta} : \alpha < \mathfrak{d} \wedge \beta < \mathrm{non}(\mathcal{N})\}$ witnesses the definition of $\mathrm{non}^*(\mathcal{Z})$. To see this, fix $Z \in \mathcal{Z}$. There is a function $f_Z\colon \omega \to \omega$ such that whenever $g \geq^* f_Z$ then $\sum_{n=0}^\infty \frac{|Z\cap[2^{g(n)},2^{g(n+1)})|}{2^{g(n)}} < \infty$. Let $\alpha < \mathfrak{d}$ be such that $f_Z \leq^* g_\alpha$. Then the set

$$N_\alpha = \{f \in \prod_{n\in\omega} 2^{g_\alpha(n)} : (\exists^\infty m \in \omega)(2^{g_\alpha(m)} + f(m) \in Z)\}$$

has measure zero, so there is a $\beta < \mathrm{non}(\mathcal{N})$ such that $f_{\alpha,\beta} \notin N_\alpha$. Therefore,

$$|L_{\alpha,\beta} \cap Z| < \aleph_0$$

as required. ■

For $A \subseteq \omega$, the asymptotic density of A is defined as $d(A) = \lim_{n\to\infty} \frac{|A\cap n|}{n}$, whenever the limit exists. Define a probability measure on ω by $\mu(\{n\}) = \frac{1}{2^{n+1}}$. Given an infinite set B consider the product measure μ_B on ω^B. For our next result we will use the following direct consequence of the law of large numbers.

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Proposition 3.11 *Let B be any countable set. Then*

$$\mu_B\big(\{f \in \omega^B : (\forall n \in \omega)(d(f^{-1}(n)) = \tfrac{1}{2^{n+1}})\}\big) = 1.$$

Theorem 3.12

(i) $\mathrm{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$,[5]
(ii) $\mathrm{non}^*(\mathcal{Z}) \leq \min\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$.

Proof For (i), let $A = \{f \in \omega^\omega : (\forall n \in \omega)(d(f^{-1}(n)) = \frac{1}{2^{n+1}})\}$. By Proposition 3.11 $\mu_\omega(A) = 1$. Fix a μ_ω-non-null set $X \subseteq A$ and an unbounded family $\mathcal{D} \subseteq \omega^\omega$. For $f \in \mathcal{D}$ and $g \in X$, let $I_{g,f} = \bigcup_{n\in\omega} g^{-1}(n) \cap f(n)$.

We claim that $I_{g,f} \in \mathcal{Z}$. To see this, note that $\{g^{-1}(n) : n \in \omega\}$ is a partition of ω such that $d(g^{-1}(n)) = \frac{1}{2^{n+1}}$ for each $n \in \omega$. The set $I_{g,f}$ has only finite intersection with each element of the partition, hence $I_{g,f} \in \mathcal{Z}$.

To finish the proof it suffices to verify that $\{I_{g,f} : x \in X,\ f \in D\}$ is a covering family for $\mathcal{Z}$. Indeed, if $B \subseteq \omega$ is an infinite set, put

$$F_B = \{f \in A : (\exists n \in \omega)(f^{-1}(n) \cap B) = \varnothing\}.$$

We claim that $\mu_\omega(F_B) = 0$. By Proposition 3.11,

$$\mu_B\big(\{f \in \omega^B : (\forall n \in \omega)(d(f^{-1}(n)) = \tfrac{1}{2^{n+1}})\}\big) = 1.$$

Let $\pi\colon \omega^\omega \to \omega^B$ be the natural projection. Then

$$\mu_\omega\big(\pi^{-1}\big[\{f \in \omega^B : (\forall n \in \omega)(d(f^{-1}(n)) = \tfrac{1}{2^{n+1}})\}\big]\big) = 1$$

and $F_B \cap \{f \in \omega^B : (\forall n \in \omega)(d(f^{-1}(n)) = \frac{1}{2^{n+1}})\} = \varnothing$.

Pick $g \in X \setminus F_B$. Then $B \cap g^{-1}(n) \neq \varnothing$ for every $n \in \omega$. Define $f_B\colon \omega \to \omega$ by

$$f_B(n) = \min(B \cap g^{-1}(n)).$$

There is an $f \in \mathcal{D}$ such that $f_B <^* f$. Then $|I_{x,f} \cap B| = \aleph_0$, and the proof is done.

For (ii) dualize the argument for (i). Let $\kappa < \min\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ and fix $\{X_\alpha : \alpha < \kappa\} \subseteq [\omega]^{\aleph_0}$. As each F_{X_α} is null, there is a $g \in A \setminus \bigcup_{\alpha<\kappa} F_{X_\alpha}$. That is, each X_α intersects $g^{-1}(n)$ for all $n \in \omega$. For $\alpha < \kappa$, let $f_\alpha(n) = \min(g^{-1}(n) \cap X_\alpha) + 1$. As $\kappa < \mathfrak{d}$, there is an $f \in \omega^\omega$ not dominated by any f_α. Let $Z = \bigcup_{n\in\omega} g^{-1}(n) \cap f(n)$. Then $Z \in \mathcal{Z}$, and $Z \cap X_\alpha$ is infinite for each $\alpha < \kappa$. So, $\kappa < \mathrm{non}^*(\mathcal{Z})$. ■

Next we mention some consistency results.

Theorem 3.13 *The following are consistent with ZFC:*

(i) $\mathrm{cov}(\mathcal{N}) < \mathrm{cov}^*(\mathcal{I})$ *for all tall analytic P-ideals,*
(ii) $\mathrm{cov}^*(\mathcal{I}) < \mathrm{add}(\mathcal{M})$ *for all tall analytic P-ideals.*

[5] Hrušák and Zapletal [HZ1] proved a general fact from which it follows that $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}^*(\mathrm{tr}(\mathcal{N})) \leq \max\{\mathrm{cov}(\mathcal{N}), \mathfrak{d}\}$. In particular, $\mathrm{cov}^*(\mathcal{Z}) \leq \max\{\mathrm{cov}(\mathcal{N}), \mathfrak{d}\}$.

Proof (i) holds in any model where $\mathrm{cov}(\mathcal{N}) < \mathfrak{p}$ (see [BJ]).

(ii) holds in the Hechler model. To see this, it suffices to note that under CH every tall P-ideal has a cofinal family which forms a strictly increasing tower. By a result of Baumgartner and Dordal [BD], the iteration of Hechler forcing preserves towers and hence the cofinal family from V still witnesses $\mathrm{cov}^*(\mathcal{Z}) = \omega_1$ in the extension. It is well known that $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ in the Hechler model (see [BJ]). ■

A forcing $\mathbb{P}$ *destroys* (diagonalizes) an ideal $\mathcal{I}$ if it introduces a set $A \subseteq \omega$ such that $|A \cap I| < \aleph_0$ for every $I \in \mathcal{I} \cap V$. The covering number of an ideal can be viewed as a measure of how difficult it is to diagonalize it. In particular, it is of interest to identify those ideals which can be diagonalized by an ω^ω-bounding forcing. The results of the previous section together with the following result of Brendle and Yatabe [BY] tell us which (analytic P-)ideals are destroyed by the random forcing (see also [HZ1]).

Theorem 3.14 (Brendle-Yatabe) *Let $\mathcal{I}$ be a tall ideal on ω. Then the random forcing destroys $\mathcal{I}$ if and only if $\mathcal{I} \leq_K \mathrm{tr}(\mathcal{N})$.*

Theorem 3.15 *It is relatively consistent with ZFC that*

$$\mathrm{cov}^*(\mathcal{Z}) < \mathrm{cov}(\mathcal{N}) = \mathrm{cov}^*(\mathrm{tr}(\mathcal{N})) = \mathrm{cov}(\mathcal{I})$$

for all summable ideals $\mathcal{I}$.

Proof The model for this is the random real model. In this model, $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \omega_1 < \mathrm{cov}(\mathcal{N}) = \mathfrak{c}$ (see [BJ]). By Theorem 3.7, $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{I})$ for all summable ideals $\mathcal{I}$ and by Theorem 3.12 $\mathrm{cov}^*(\mathcal{Z}) = \omega_1$. ■

Definition 3.16 An analytic ideal $\mathcal{I}$ on ω is said to be *totally bounded* if whenever φ is a lower semicontinuous submeasure on $\mathcal{P}(\omega)$ for which $\mathcal{I} = \mathrm{Exh}(\varphi)$, then $\varphi(\omega) < \infty$.

The following lemma provides an easy criterion for checking that an ideal is totally bounded. Recall that the splitting number $\mathfrak{s}(\mathbb{B})$ of a Boolean algebra is defined as the minimal size of a family $\mathcal{S}$ of non-zero elements of $\mathbb{B}$ such that for every non-zero $b \in \mathbb{B}$ there is an $s \in \mathcal{S}$ such that $b \wedge s \neq 0 \neq b - s$.

Lemma 3.17 *If $\mathcal{I}$ is an analytic P-ideal such than $\mathfrak{s}(\mathcal{P}(\omega)/\mathcal{I}) = \omega$, then $\mathcal{I}$ is totally bounded.*

Proof Consider a submeasure φ on $\mathcal{P}(\omega)$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$ and $\varphi(\omega) = \infty$. Fix any family $\{A_n : n \in \omega\}$ of $\mathcal{I}$-positive sets. To show that this family is not a splitting family, note that either A_0 or else $\omega \setminus A_0$ has infinite submeasure. Without loss of generality assume A_0 has. Choose then a finite $F_0 \subseteq A_0$ with $\varphi(F_0) \geq 1$. Now, either $(A_0 \setminus F_0) \cap A_1$ or else $(A_0 \setminus F_0) \setminus A_1$ has infinite submeasure. Once again, without loss of generality, assume $(A_0 \setminus F_0) \cap A_1$ has infinite submeasure and choose some finite $F_1 \subseteq (A_0 \setminus F_0) \cap A_1$ such that $\varphi(F_1) \geq 1$. Now proceed alike with $(A_0 \cup A_1 \setminus (F_0 \cup F_1)) \cap A_2$ and $(A_0 \cup A_1 \setminus (F_0 \cup F_1)) \setminus A_2$ and so on.

Let $B = \bigcup_{n\in\omega} F_n$. Then $B \notin \mathcal{I}$ and it is clear that B is not split by any element of the family $\{A_n\}_{n\in\omega}$. ■

Proposition 3.18 *The ideals* $\mathcal{Z}$ *and* $\operatorname{tr}(\mathcal{N})$ *are totally bounded.*

Proof By the previous lemma, it suffices to show that there are countable splitting families in the respective algebras. For $n > 0$ and $k < n$, let $S_{n,k} = \{k+mn : m \in \omega\}$. It is easily seen that $\mathcal{S} = \{[S_{n,k}] : n, k \in \omega\}$ is a splitting family in $\mathcal{P}(\omega)/\mathcal{Z}$. For $s \in 2^{<\omega}$, let $\langle s\rangle = \{t \in 2^{<\omega} : s \subseteq t\}$. It is just as easy to see that $\mathcal{S} = \{[\langle s\rangle] : s \in 2^{<\omega}\}$ is a splitting family for $\mathcal{P}(2^{<\omega})/\operatorname{tr}(\mathcal{N})$. ■

If an analytic P-ideal is not totally bounded, then it is contained in a proper F_σ-ideal by Theorem 1.3. An ω^ω-bounding forcing for diagonalizing F_σ-ideals was discovered by C. Laflamme [La]. We briefly review an incarnation of Laflamme's forcing. Let $\mathcal{I}$ be an F_σ-ideal. Then $\mathcal{I} = \operatorname{Fin}(\varphi)$ for some lower semicontinuous submeasure φ by Theorem 1.3. Define the poset $\mathbb{P}_\varphi$ as the set of all trees $T \subseteq \omega^{<\omega}$ with stem s and such that:

(i) $\operatorname{succ}_T(t) = \{n \in \omega : t^\frown n \in T\}$ is finite for every $t \in T$, and
(ii) $\lim_{t\in T} \varphi(\operatorname{succ}_T(t)) = \infty$.

ordered by inclusion.

Call a tree $T \in \mathbb{P}_\varphi$ a *k-tree with stem s* if

(i) $(\forall t \in T)(t \subseteq s \text{ or } t \supseteq s)$ and
(ii) $(\forall t \in T)(s \subseteq t \Rightarrow \phi(\operatorname{succ}_T(t)) \geq k)$.

If $n \in \omega$ and $T, T' \in \mathbb{P}_\varphi$, we say that $T \leq_n T'$ if

$$(\forall t \in T)\ \varphi(\operatorname{succ}_{T'}(t)) \leq n \Rightarrow \operatorname{succ}_T(t) = \operatorname{succ}_{T'}(t),$$
$$\varphi(\operatorname{succ}_{T'}(t)) \geq n \Rightarrow \varphi(\operatorname{succ}_T(t)) \geq n.$$

Lemma 3.19 *The forcing* $\mathbb{P}_\varphi$ *is proper and* ω^ω*-bounding.*

Proof We actually show that $\mathbb{P}_\varphi$ together with the sequence $\langle \leq_n : n \in \omega\rangle$ satisfies Axiom A. (A1)–(A3) are easy to verify. To show (A4) and the fact that $\mathbb{P}_\varphi$ is ω^ω-bounding, it suffices to prove [BJ, Lemma 7.2.10]:

Claim 3.20 If $n \in \omega$, $T \in \mathbb{P}_\varphi$ and $\dot{a}$ is a $\mathbb{P}_\varphi$-name for an ordinal, then there exist $S \leq_n T$ and a finite set of ordinals F such that $S \Vdash \dot{a} \in F$.

Fix T, n and $\dot{a}$. Define a rank function on the nodes of $T \in \mathbb{P}_\varphi$ by

(i) $\operatorname{rk}(t) = 0$, if there is an n-tree T' with stem t which decides $\dot{a}$, and
(ii) $\operatorname{rk}(t) \leq i+1$, if $\operatorname{rk}(t) \not\leq i$ and $\varphi(\{t' \in \operatorname{succ}_T(t) : \operatorname{rk}(t') \leq i\}) \geq \frac{1}{2}\varphi(\operatorname{succ}_T(t))$.

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Note that, for every $t \in T$, there is an $i \in \omega$ such that $\mathrm{rk}(t) = i$. For if not, then there is a $t \in T$ with undefined rank and one can recursively construct a tree $T' \subseteq T$ with stem t such that for every $s \in T'$, if $s \geq t$, then

$$\mathrm{succ}_{T'}(s) = \{s' \in \mathrm{succ}_T(s) : \mathrm{rk}(s') \text{ is undefined}\}$$

and

$$\varphi(\mathrm{succ}_{T'}(s)) \geq \tfrac{1}{2}\varphi(\mathrm{succ}_T(s)).$$

Then $T' \in \mathbb{P}_\varphi$ and no extension of T' decides $\dot{a}$, which is a contradiction.

Now, $E = \{t \in T : \mathrm{rk}(t) > 0\}$ is finite as T is finitely branching and rk is strictly decreasing on T. So, there is a $k \in \omega$ such that $E \subseteq \{t \in T : |t| < k\}$. In particular, for every $t \in T$ with $|t| = k$, there is an n-tree S_t with stem t and an ordinal α_t such that $S_t \Vdash \dot{a} = \alpha_t$. Let

$$S = \bigcup_{t \in \omega^k \cap T} S_t \quad \text{and} \quad F = \{\alpha_t : t \in \omega^k \cap T\}.$$

Then $S \leq_n T$ and $S \Vdash \dot{a} \in F$. ■

Lemma 3.21 $\mathbb{P}_\varphi$ *destroys* $\mathcal{I}$.

Proof Let $\dot{A}_{\mathrm{gen}}$ be the canonical name for the subset of ω coded by a generic filter in $\mathbb{P}_\varphi$. We show that if $I \in \mathcal{I}$ and $T \in \mathbb{P}_\varphi$, there exist $S \leq T$ and $m \in \omega$ such that $S \Vdash \dot{A}_{\mathrm{gen}} \cap (I \setminus m) = \varnothing$. Fix $I \in \mathcal{I}$ and $T \in \mathbb{P}_\varphi$. As $\mathcal{I} = \mathrm{Fin}(\varphi)$, there is an $n \in \omega$ such that $\varphi(I) < n$. Let $T' \leq T$ be a $2n$-tree with stem s. Then for every $t \in T'$ such that $s \subseteq t$, $\varphi(\mathrm{succ}_{T'}(t) \setminus I) \geq \varphi(\mathrm{succ}_{T'}(t)) - n$. Define $S \leq T'$ recursively by:

(i) $t \subseteq s \Rightarrow t \in S$,

(ii) $t \in S$ and $t \supseteq s \Rightarrow \mathrm{succ}_S(t) = \mathrm{succ}_{T'}(t) \setminus I$.

Let $m = \max(\mathrm{ran}(s)) + 1$. Then $S \in \mathbb{P}_\varphi$ and $S \leq T$ and $S \Vdash \dot{A}_{\mathrm{gen}} \cap (I \setminus m) = \varnothing$. ■

Theorem 3.22 *It is relatively consistent with ZFC that* $\omega_1 = \mathfrak{d} < \mathrm{cov}^*(\mathcal{I})$ *for all analytic P-ideals which are not totally bounded.*

Proof Start with a model of CH and iterate forcings of the type $\mathbb{P}_\varphi$ with countable support of length ω_2, where each $\mathbb{P}_\varphi$ appears cofinally. Then $\mathrm{cov}^*(\mathcal{I}) = \mathfrak{c} = \omega_2$ as $\mathbb{P}_\varphi$ destroys $\mathcal{I} = \mathrm{Fin}(\varphi)$, and hence also $\mathrm{Exh}(\varphi)$. On the other hand, $\mathfrak{d} = \omega_1$ in the resulting model as countable support iteration of ω^ω-bounding forcings is ω^ω-bounding [Sh]. ■

The advantage of our forcings over those of Laflamme is the simplicity of their definition. On the other hand, it seems to be more difficult to decide stronger properties of the forcings $\mathbb{P}_\varphi$. We do not even know whether a forcing $\mathbb{P}_\varphi$ can add random reals. Forcings of the type $\mathbb{P}_\varphi$ are still not quite well understood and deserve further investigation. For instance, it would be nice to know if or when they preserve P-points and if or when they preserve positive outer measure. As the forcings of the type $\mathbb{P}_\varphi$ are in general not homogeneous, it is reasonable to expect that the answers to these questions may differ depending on the choice of the generic filter.

There are several natural open problems concerning cardinal invariants of analytic P-ideals. We explicitly mention at least four.

Questions 3.23

(a) Is $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{d}$ (or even $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{b}$)? [6]
(b) Is $\mathrm{cov}^*(\mathcal{Z})$ minimal among the covering numbers of analytic P-ideals?
(c) Is $\mathrm{cov}^*(\mathcal{I}) = \mathrm{cov}^*(\mathcal{I}_{1/n})$ for all tall summable ideals $\mathcal{I}$?
(d) Is $\mathrm{cov}^*(\mathcal{I}) = \mathrm{cov}^*(\mathcal{J})$ for all density ideals $\mathcal{I}$ and $\mathcal{J}$ which are not Erdős–Ulam?

4 The Separating Number

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{G} \subseteq \mathcal{I}$, $\mathcal{H} \subseteq \mathcal{I}^+$ and $A \subseteq \omega$. The set A *separates* $\mathcal{G}$ from $\mathcal{H}$ if $|A \cap I| < \aleph_0$ for every $I \in \mathcal{G}$, and $|A \cap X| = \aleph_0$ for every $X \in \mathcal{H}$.

A forcing $\mathbb{P}$ *separates* an ideal $\mathcal{I}$ if it introduces a set $A \subseteq \omega$ such that A separates $\mathcal{I} \cap V$ from $\mathcal{I}^+ \cap V$. That is, $\mathcal{I} \cap V$ is contained in $\{B \subseteq \omega : A \cap B \text{ is finite}\}$ while $\mathcal{I}^+ \cap V \cap \{B \subseteq \omega : A \cap B \text{ is finite}\}$ is empty. So, $\mathbb{P}$ separates $\mathcal{I}$ and $\mathcal{I}^+$ (as subsets of $\mathcal{P}(\omega)$) by a very simple F_σ-set. In other words, separation is a strong kind of diagonalization (see [HZ1]).

Definition 4.1 For an ideal $\mathcal{I}$ on ω, the *separating number* of $\mathcal{I}$ is

$$\mathrm{sep}(\mathcal{I}) = \min\big\{ |\mathcal{G}| + |\mathcal{H}| : \mathcal{G} \subseteq \mathcal{I}, \mathcal{H} \subseteq \mathcal{I}^+ \text{ and } (\forall A \subseteq \omega)(A \text{ does not separate } \mathcal{G} \text{ from } \mathcal{H})\big\}.$$

Just like $\mathrm{cov}^*(\mathcal{I})$ measures destructibility of the ideal $\mathcal{I}$, the separating number measures how difficult it is to separate $\mathcal{I}$. It is easily seen that

$$\mathrm{add}^*(\mathcal{I}) \leq \mathrm{sep}(\mathcal{I}) \leq \mathrm{cov}^*(\mathcal{I})$$

for every tall ideal $\mathcal{I}$. For maximal ideals $\mathcal{I}$, $\mathrm{sep}(\mathcal{I}) = \mathrm{cov}^*(\mathcal{I})$.

The Rudin–Keisler order relates to sep in the same way that the Tukey order relates to add^* and the Katětov order relates to cov^*.

Proposition 4.2 $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \Rightarrow \mathrm{sep}(\mathcal{J}) \leq \mathrm{sep}(\mathcal{I})$.

Proof Fix $f\colon \omega \to \omega$ witnessing that $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$. Let $\mathcal{G} \subseteq I$ and $\mathcal{H} \subseteq \mathcal{I}^+$, each of cardinality less than $\kappa \leq \mathrm{sep}(\mathcal{J})$. Without loss of generality assume that $f^{-1}(G) \in \mathcal{G}$ for every finite $G \subseteq \omega$. Then for $\mathcal{G}^{\leftarrow} = \{f^{-1}(G) : G \in \mathcal{G}\}$ and $\mathcal{H}^{\leftarrow} = \{f^{-1}(H) : H \in \mathcal{H}\}$ there is an $A \subseteq \omega$ such that $|A \cap G| < \aleph_0$ for every $G \in \mathcal{G}^{\leftarrow}$ and $|A \cap H| = \aleph_0$

[6] By Theorem 3.12 and the result of [HZ1] $\mathrm{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$ as well as $\mathrm{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$. Consequently, if $\mathcal{Z}$ can be destroyed by an ω^ω-bounding forcing $\mathbb{P}$, then $\mathbb{P}$ must add random reals as well as make the ground model reals of measure zero.

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for every $H \in \mathcal{H}^{\leftarrow}$. Then $f[A] \cap G$ is finite for every $G \in \mathcal{G}$ and $f[A] \cap H$ is infinite for every $H \in \mathcal{H}$. Indeed, if $f[A] \cap H$ were finite for some $H \in \mathcal{H}$, then $f^{-1}(f[A] \cap H) \in \mathcal{G}^{-1}$ and thus $A \cap f^{-1}(H)$ is also finite, contradicting the choice of A. Therefore $\kappa \leq \mathrm{sep}(\mathcal{I})$. ■

To highlight the importance of the separating number we mention the following theorem. Recall that $\mathbb{L}_{\mathcal{I}^*}$ is the standard Laver type forcing for diagonalizing (in fact, separating) of the ideal $\mathcal{I}$. Let $\mathfrak{m}(\mathbb{P})$ denote the minimum cardinal κ for which $\mathsf{MA}_\kappa(\mathbb{P})$ fails. Recall also that a free ultrafilter $\mathcal{U}$ on ω is *nowhere dense* (nwd) if for every $f\colon \omega \to \mathbb{R}$ there is $U \in \mathcal{U}$ such that $f[U]$ is nowhere dense in $\mathbb{R}$.

The following will appear in a forthcoming paper.[7]

Theorem 4.3 *Let $\mathcal{I}$ be an ideal on ω. Then*

$$\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) = \begin{cases} \min\{\mathfrak{b}, \mathrm{sep}(\mathcal{I})\} & \text{if } \mathcal{I}^* \text{ is an nwd-ultrafilter,} \\ \min\{\mathrm{add}(\mathcal{M}), \mathrm{sep}(\mathcal{I})\} & \text{otherwise.} \end{cases}$$

The forcing $\mathbb{L}_{\mathcal{I}^*}$ always adds a dominating real. Another standard forcing for diagonalizing an ideal $\mathcal{I}$ is the *Mathias–Prikry* forcing $\mathbb{M}_{\mathcal{I}^*}$ defined as follows: $\mathbb{M}_{\mathcal{I}^*} = \{\langle s, F\rangle : s \in [\omega]^{<\omega} \text{ and } F \in \mathcal{I}^*\}$ ordered by $\langle s, F\rangle \leq \langle t, G\rangle$ if and only if $s \supseteq t$, $F \subseteq G$ and $s \setminus t \subseteq G$. A straightforward density argument shows that (just like $\mathbb{L}_{\mathcal{I}^*}$) the forcing $\mathbb{M}_{\mathcal{I}^*}$ not only diagonalizes $\mathcal{I}$, it separates $\mathcal{I}$. It is not known in general when $\mathbb{M}_{\mathcal{I}^*}$ adds a dominating real, however, if $\mathcal{I}$ is F_σ, then $\mathbb{M}_{\mathcal{I}^*}$ does not add dominating reals even when iterated (see [Br, 3.2 Case 1] for a proof).

Proposition 4.4

(i) *If $\mathcal{I}$ is a Erdős–Ulam ideal, then* $\mathrm{sep}(\mathcal{I}) = \mathrm{sep}(\mathcal{Z})$.

(ii) *If $\mathcal{I}$ is a tall density ideal, then* $\mathrm{sep}(\mathcal{I}) \leq \mathrm{sep}(\mathcal{Z})$.

(iii) *If $\mathcal{I}_f$ and $\mathcal{I}_g$ are tall summable ideals, then* $\mathrm{sep}(\mathcal{I}_f) = \mathrm{sep}(\mathcal{I}_g)$.

Proof First note that $\mathrm{sep}(\mathcal{I}) \leq \mathrm{sep}(\mathcal{I}\restriction X)$ for every ideal $\mathcal{I}$ and $X \in \mathcal{I}^+$. (i) follows directly from Proposition 4.2 and [Fa1, 1.13.10]. (ii) follows from (i) and the fact for every tall density $\mathcal{I}$ there is a $X \in \mathcal{I}^+$ such that $\mathcal{I}\restriction X$ is Erdős–Ulam.

Using the observation at the beginning of the proof, (iii) follows from Proposition 4.2 and [Fa1, Claim 1 of Lemma 1.12.4]. ■

Theorem 4.5

(i) $\mathrm{sep}(\mathcal{I}) \leq \mathfrak{b}$ *for all analytic P-ideals $\mathcal{I}$ which are not F_σ.*

(ii) *It is consistent with ZFC that* $\mathrm{sep}(\mathcal{I}) > \mathfrak{b}$ *for all F_σ P-ideals $\mathcal{I}$.*

[7] F. Hernández and M. Hrušák, *Covering separable spaces by nowhere dense sets.* In preparaton.

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Proof Let $\mathcal{I}$ be an analytic P-ideal which is not F_σ. By a result of Solecki [So, Theorem 3.4], there is a partition $\langle A_n : n \in \omega\rangle$ of ω into $\mathcal{I}$-positive sets such that every $I \subseteq \omega$ intersecting each A_n in a finite set is in $\mathcal{I}$. Let $\kappa < \mathrm{sep}(\mathcal{I})$ and fix a family of functions $\{f_\alpha \in \omega^\omega : \alpha < \kappa\}$. Enumerate each A_n as $\{a^n_i : i \in \omega\}$ and let

$$B_\alpha = \{a^n_m : m \leq f_\alpha(n)\}.$$

As $\kappa < \mathrm{sep}(\mathcal{I})$, there is an $X \subseteq \omega$ which intersects every A_n in an infinite set and contains only finitely many elements of each B_α, $\alpha < \kappa$. Define $f\colon \omega \to \omega$ by

$$f(n) = \min\{k \in A_n : (A_n \setminus k) \cap X = \varnothing\}.$$

Then f dominates f_α for every $\alpha < \kappa$.

To prove (ii), start with a model of CH and iterate forcings of the type $\mathbb{M}_{\mathcal{I}^*}$, where $\mathcal{I}$ is an F_σ P-ideal, with finite support of length ω_2, where each $\mathbb{M}_{\mathcal{I}^*}$ appears cofinally. Then $\mathrm{sep}(\mathcal{I}) = \mathfrak{c} = \omega_2$ as $\mathbb{M}_{\mathcal{I}^*}$ separates $\mathcal{I}$. On the other hand, $\mathfrak{b} = \omega_1$ in the resulting model as finite support iteration of forcings of type $\mathbb{M}_{\mathcal{I}^*}$, for $\mathcal{I}$ F_σ, does not add dominating reals. ■

A Appendix

The classical definition of Tukey order for partially ordered sets is as follows:

Definition A.1 ([Fr1]) Let P and Q be partially ordered sets. Then

(i) $P \leq_{\mathrm{T}} Q$ if there is a function $f\colon P \to Q$ such that $\{p \in P : f(p) \leq q\}$ is bounded for every $q \in Q$.

(ii) $P \leq_\omega Q$ if there is a function $f\colon P \to Q$ such that $\{p \in P : f(p) \leq q\}$ is countably dominated for every $q \in Q$.

When restricted to ideals, most authors, *e.g.* [LV, Fr1, Fa1], define the Tukey order using $(\mathcal{I}, \subseteq)$ rather than $(\mathcal{I}, \subseteq^*)$. Following them we denote by $\leq_{\mathrm{T}}$ the standard Tukey order on ideals. Even though there does not seem to be any relation between $\leq_{\mathrm{T}}$ and $\leq^*_{\mathrm{T}}$ for ideals in general, for P-ideals $\leq_{\mathrm{T}}$ is finer than $\leq^*_{\mathrm{T}}$.

Proposition A.2 *Let $\mathcal{I}$ and $\mathcal{J}$ be P-ideals. Then*

(i) *$\mathcal{I} \leq_\omega \mathcal{J}$ if and only if $\mathcal{I} \leq^*_{\mathrm{T}} \mathcal{J}$,*

(ii) *$\mathcal{I} \leq_{\mathrm{T}} \mathcal{J}$ implies $\mathcal{I} \leq^*_{\mathrm{T}} \mathcal{J}$.*

Proof (i) For the forward implication, first observe that $\mathcal{I} \leq^*_{\mathrm{T}} \mathcal{J}$ if and only if there is a function $g\colon \mathcal{I} \to \mathcal{J}$ such that for every $J \in \mathcal{J}$ there is an $I \in \mathcal{I}$ such that $A \subseteq^* I$ whenever $A \in \mathcal{I}$ and $g(A) \subseteq^* J$.

Take an f witnessing $\mathcal{I} \leq_\omega \mathcal{J}$ and fix $J \in \mathcal{J}$. Consider the family $\{J_n : n \in \omega\}$ of all finite modifications of J. Then $\{A \in \mathcal{I} : f(A) \subseteq^* J\} = \bigcup_{n\in\omega}\{A \in \mathcal{I} : f(A) \subseteq J_n\}$. Since f witnesses $\mathcal{I} \leq_\omega \mathcal{J}$, there are families $\{I_{n,m} : m \in \omega\} \subseteq \mathcal{I}$, for every $n \in \omega$, such that whenever $f(A) \subseteq J_n$ there is an $m \in \omega$ such that $A \subseteq I_{n,m}$. As $\mathcal{I}$ is a P-ideal there is an $I \in \mathcal{I}$ such that $I_{n,m} \subseteq^* I$ for every $m, n \in \omega$. Then $A \subseteq^* I$ whenever $f(A) \subseteq^* J$, that is, f witnesses $\mathcal{I} \leq_{\mathrm{T}}^* \mathcal{J}$.

For the reverse implications let $f\colon \mathcal{I} \to \mathcal{J}$ be a witness to $\mathcal{I} \leq_{\mathrm{T}}^* \mathcal{J}$. That is, given $J \in \mathcal{J}$, there is an $I \in \mathcal{I}$ such that $f(A) \subseteq^* J$ implies $A \subseteq^* I$. Then all finite modifications of I countably dominate $\{A \in \mathcal{I} : f(A) \subseteq J\}$.

(ii) follows from (i) and the trivial fact that $\mathcal{I} \leq_{\mathrm{T}} \mathcal{J}$ implies $\mathcal{I} \leq_{\omega} \mathcal{J}$. ∎

References

[Ba] T. Bartoszyński, *Invariants of Measure and Category* (1999). Available at arXiv:math.LO/9910015.

[BJ] T. Bartoszyński and H. Judah, *Set theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[BD] J. E. Baumgartner and P. Dordal, *Adjoining dominating functions.* J. Symbolic Logic, **50**(1985), no. 1, 94–101.

[Br] J. Brendle, *Mob families and mad families.* Arch. Math. Logic **37**(1997), no. 3, 183–197.

[BS] J. Brendle and S. Shelah, *Ultrafilters on ω—their ideals and their cardinal characteristics.* Trans. Amer. Math. Soc. **351**(1999), no. 7, 2643–2674.

[BY] J. Brendle and S. Yatabe, *Forcing indestructibility of MAD families.* Ann. Pure Appl. Logic **132**(2005), no. 2-3, 271–312.

[Fa1] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers.* Mem. Amer. Math. Soc. **148**(2000), no. 702.

[Fa2] ———, *Analytic Hausdorff gaps.* In: Set Theory, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 58, American Mathematical Society, Providence, RI, 2002, pp. 65–72.

[Fa3] ———, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?* Illinois J. Math. **46**(2002), no. 4, 999–1033.

[Fa4] ———, *Luzin gaps.* Trans. Amer. Math. Soc. **356**(2004), no. 6, 2197–2239 (electronic).

[Fa5] Ilijas Farah, *Analytic Hausdorff gaps. II. The density zero ideal.* Israel J. Math. **154**(2006), 235–246.

[Fr1] D. H. Fremlin, *The partially ordered sets of measure theory and Tukey's ordering.* Note Mat. **11**(1991), 177–214.

[Fr2] David H. Fremlin, *Measure theory. Set theoretic measure theory.* Torres Fremlin, Colchester, England, 2004. Available at http://www/essex.ac.uk/maths/staff/fremlin/my.htm.

[HZ1] M. Hrušák and J. Zapletal, *Forcing with quotients.* To appear in Arch. Math. Logic.

[Je] T. Jech, *Set theory*. Second edition. Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1997.

[Ku] K. Kunen, *Set Theory. An Introduction to Independence Proofs.* Studies in Logic and the Foundations of Mathematics 102, North-Holland, Amsterdam, 1980.

[La] C. Laflamme, *Zapping small filters.* Proc. Amer. Math. Soc. **114**(1992), no. 2, 535–544.

[LV] A. Louveau and B. Veličković, *Analytic ideals and cofinal types.* Ann. Pure Appl. Logic **99**(1999), no. 1-3, 171–195.

[Sh] S. Shelah, *Proper and Improper Forcing*. Second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.

[So] S. Solecki, *Analytic ideals and their applications.* Ann. Pure Appl. Logic **99**(1999), no. 1-3, 51–72.

[To] S. Todorčević, *Analytic gaps.* Fund. Math. **150**(1996), no. 1, 55–66.

[Va] J. E. Vaughan, *Small uncountable cardinals and topology. With an appendix by S. Shelah.* In: Open Problems in Topology. North-Holland, Amsterdam, 1990, pp. 195–218.

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