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PRICING VULNERABLE EUROPEAN OPTIONS WITH STOCHASTIC CORRELATION

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In this paper, we present a new pricing model for vulnerable options, with time-varying variances for each asset described by Generalized Autoregressive Conditional Heteroscedasticity processes and correlated with the return of the asset. By connecting the underlying asset and the counterparty's assets through the market factor channel, the proposed model also captures stochastic correlation between the underlying asset return and the return of the counterparty's assets. The correlation depends on the levels of the variances of both assets and the market index as well. In the proposed framework, the closed-form solution for vulnerable options is derived and numerical results are presented to investigate the impact of counterparty default risk.

Keywords: GARCH models, stochastic correlation, stochastic volatility, vulnerable options

1. INTRODUCTION

Since the pioneering work on option pricing by Black and Scholes [2] and Merton [23], an extensive empirical literature has documented the empirical biases of the option valuation model and much research has modified the Black–Scholes model in order to incorporate stochastic volatility. More precisely, if the Black–Scholes model is correct, the implied volatility should be constant as in the Black–Scholes model, but it has been widely recognized that it is not always the case.

Continuous-time stochastic volatility models (see, e.g., Heston [15] and Hull and White [17]) have been proposed to take stochastic volatility into consideration. In a parallel development, the discrete-time Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models (see, e.g., Bollerslev [3], Duan [10], Heston and Nandi [16], and Ritchken and Trevor [25]) have also been investigated. Especially, the continuous-time stochastic volatility models, including Heston [15], Hull and White [17] and Scott [26] can in fact be approximated by GARCH processes (see, e.g., Duan [11] and Nelson [24]). GARCH processes are considered to take into account the volatility clustering phenomenon by Bollerslev [3] and are used to price options by Duan [10]. Ritchken and Trevor [25] develop an efficient lattice algorithm to price European and American options under GARCH processes. The authors also show that the algorithm is easily extended to price options under generalized GARCH processes, with many of the existing stochastic volatility bivariate diffusion models

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appearing as limiting cases. Duan, Gauthier, and Simonato [12] provide a series approximation. Heston and Nandi [16] derive a closed-form solution for European option prices in a GARCH model, and empirical analysis on S&P 500 index options shows the out-of-sample valuation errors from the GARCH model are substantially lower than the ad hoc Black–Scholes model studied in Dumas, Fleming, and Whaley [13], where a separate implied volatility for each option is used to fit to the smile in implied volatilities. Christoffersen et al. [6] present a new GARCH model for the valuation of European options, in which the volatility of returns consists of two components: a fully persistent long-run component, and a short-run component with a zero mean. In a successive paper, Christoffersen, Jacobs, and Ornthanalai [5] propose an interesting and useful class of GARCH-jump models, and find very strong support for time-varying jump intensities using the time-series of S&P 500 returns. However, GARCH-jump models are difficult to implement and test (see, e.g., Christoffersen et al. [5] and Durham Geweke, and Ghosh [14]).

In recent years, the over-the-counter (OTC) derivatives market has experienced tremendous development. With the immense development of the OTC derivatives market (The statistics in the ISDA survey, published in January 2015, show that the total OTC derivatives notional outstanding approximated US$691.5 trillion at the end of June 2014.), the significant counterparty default risk (Counterparty default risk is the risk in a financial contract that one counterparty defaults and fails to make the agreed payments. Default risk has been one of the risks participants in the OTC market have to face, see, for example, Arora, Gandhi, and Longstaff [1]). should by no means be ignored especially after the harrowing experience of the subprime mortgage crisis in 2007–2008. As a consequence, the issue of credit exposure must be taken seriously when valuing credit-sensitive OTC contracts such as credit default swaps (CDS), forwards and European options. Actually, counterparty default risk has been considered when pricing credit derivatives (see, e.g., Brigo, Capponi, and Pallavicini [4] and Crépey ([8,9])) and valuing European options (see, e.g., Johnson and Stulz [18], Tian et al. [28] and Wang [30]). In addition, European options with counterparty default risk are called as vulnerable options. Adopting structural approaches to describe credit risk, Johnson and Stulz [18] first incorporate credit risk into the option pricing model with the assumption that the option is the sole liability of the counterparty and derive the prices of vulnerable options. Klein [20] extends the result of Johnson and Stulz [18] by allowing the option writer to hold other liabilities, which rank equally with payments under the option. Based on the framework of Johnson and Stulz [18] and Klein [20], many other factors such as stochastic interest rate, rare shocks, stochastic volatility, and stochastic default barriers, are considered to illustrate the impacts of these factors on the prices of vulnerable options. For instance, Liao and Huang [22] investigate the stochastic interest rate case and provide a closed-form valuation formula for vulnerable options. Tian et al. [28] incorporate jump processes to describe the discontinuous changes in the asset prices and investigate the impact of jump risk on vulnerable option prices. Wang and Wang [29] provide a pricing model for vulnerable options with stochastic volatility risk, in which the volatility of returns consists of two components: long-term volatility assumed to be constant and short-term volatility described by a mean-reverting process. Klein and Inglis [21] extend the model of Klein [20] by incorporating the potential liability of the written option into the default boundary.

In this paper, we consider the pricing issue of vulnerable options under GARCH models. We start by specifying the dynamic of the market index, representing the market risk, and connect the dynamics of the underlying asset and the counterparty's assets through the market factor channel. In this way, the returns of the underlying asset and the counterparty's assets are correlated with each other. Compared with the previous literature on vulnerable options, this paper has the following characteristics. First, we use GARCH processes to describe the variances of all asset prices and the proposed model captures stochastic nature of volatility. There are few papers focusing on stochastic volatility when valuing vulnerable options. To our best knowledge, Yang, Lee, and Kim [33] and Wang and Wang [29] are the exceptions, where the authors adopt different kinds of continuous-time stochastic volatility models. Especially, Yang et al. [33] obtain an analytic approximation formula for vulnerable option prices using multiscale asymptotic analysis and Wang and Wang [29] derive a pricing formula of vulnerable options in a special case of their model. Different from these two studies, we work under GARCH models and get the closed-form solution for vulnerable options. In addition, stochastic correlation is also considered in this paper. Wang [31] presents a pricing model for vulnerable options in a GARCH model, where counterparty credit risk is considered in a reduced form model. Second, the proposed model is much easier to implement using historical asset prices, but it is not possible to exactly filter a volatility variable from asset prices in a continuous-time stochastic volatility model. In the numerical part, we implement the model using daily closing prices for the S&P 500 index, Microsoft Corporation stock and Bank of America Corporation stock for the period from January 3, 1995 to December 31, 2009, and illustrate the impact of counterparty default risk. Microsoft Corporation stock is one of the most popular S&P 500 stocks held by 50 largest mutual funds and hedge funds in America (For more details, please visit the website http://www.forbes.com/sites/liyanchen/2015/11/30/google-and-microsoft-top-the-most-popular-mutual-fund-stocks/#481966c5185e). Bank of America Corporation stock is also the S&P 500 index component. Here we take these two companies as an example and for any other stocks we can obtain numerical results similarly. Third, the proposed model captures stochastic nature of correlation between returns and volatility for each asset. Fourth, the proposed model is flexible in modeling stochastic correlation between the underlying asset return and the return of the counterparty's assets. Lastly, the closed-form solution for vulnerable options is also derived in the proposed GARCH model.

The remainder of this paper is organized as follows. In Section 2, GARCH models with stochastic correlation between the underlying asset return and the return of the counterparty's assets are proposed. Moreover, we derive an explicit pricing formula for vulnerable European options. Section 3 presents numerical results to illustrate the impact of counterparty default risk. Finally, concluding remarks are contained in Section 4. The detailed proofs are shown in the Appendix.

2. THE MODEL

In this section, we deal with the pricing issue of vulnerable European options under GARCH models. To connect the dynamics of the underlying asset and the counterparty's assets through the market factor channel, here we begin by specifying the values of the market index, representing a common risk factor. Besides retaining the attractive features of GARCH models, our formulation captures the time-varying correlation between the underlying asset return and the return of the counterparty's assets.

Assume that the uncertainty of the economy is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with an information flow $\{\mathscr{F}_t\}_{t\geq 0}$, where $\mathbb{P}$ is a real-world probability measure. Suppose that the dynamic of the market index follows a GARCH process,

$$\begin{cases} 0 \ln M(t) = \ln M(t-1) + r + \left(\lambda_m - \dfrac{1}{2}\right) h_m(t) + \sqrt{h_m(t)} Z_m(t), \\ h_m(t) = w_m + b_m h_m(t-1) + a_m \Big(Z_m(t-1) - c_m \sqrt{h_m(t-1)} \Big)^2, \end{cases} \tag{2.1}$$

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where $M(t)$ denotes the value of the market index at the close of day t, r is the risk-free interest rate and λ_m denotes the market price of risk. Shocks to the returns are generated by a standard normal disturbance $Z_m(t)$, and $h(t)$ is the conditional variance, which is known at the end of day $t-1$. This GARCH process is first used by Heston and Nandi [16] to investigate index options and has been extended by Christoffersen et al. [6] and Christoffersen et al. [5]. Here we focus on the version of the Heston and Nandi model, and the results can be applied to the extended version in Christoffersen et al. [6].

Based on the assumption of the market index, we suppose that the underlying asset price follow:

$$\begin{cases} \ln S(t) = \ln S(t-1) + r + \left(\lambda_s - \frac{1}{2}\right) h_{\mathrm{s}}(t) + \sqrt{h_{\mathrm{s}}(t)} Z_s(t) \\ \qquad + \left(\beta_{\mathrm{s}} \lambda_m - \frac{1}{2}\beta_{\mathrm{s}}^2\right) h_m(t) + \beta_{\mathrm{s}} \sqrt{h_m(t)} Z_m(t), \\ h_{\mathrm{s}}(t) = w_s + b_s h_{\mathrm{s}}(t-1) + a_s \left(Z_s(t-1) - c_s \sqrt{h_{\mathrm{s}}(t-1)}\right)^2, \end{cases} \tag{2.2}$$

where $S(t)$ denotes the underlying asset price at the close of day t and r is the risk-free interest rate. Shocks to the returns of the underlying asset consist of two parts: idiosyncratic shocks, corresponding to $Z_s(t)$, and common shocks, corresponding to $Z_m(t)$, where $Z_s(t)$ and $Z_m(t)$ are independent standard normal variables. The conditional variance corresponding to idiosyncratic shocks is denoted by $h_{\mathrm{s}}(t)$, which is also known at the end of day $t-1$, and λ_s denotes the market price of risk, stemming from idiosyncratic shocks. The parameter β_{s} describes the effect of common shocks on the underlying asset return. Motivated by the capital asset pricing model (CAPM), we also adopt the beta to represent the underlying asset's sensitivity to systematic risk or market risk, and the value of β_{s} can be represented by

$$\begin{aligned} \frac{\mathrm{Cov}_{t-1}\left(\ln \frac{M(t)}{M(t-1)}, \ln \frac{S(t)}{S(t-1)}\right)}{\mathrm{Var}_{t-1}\left(\ln \frac{M(t)}{M(t-1)}\right)} &= \frac{\mathrm{Cov}_{t-1}\left(\sqrt{h_m(t)} Z_m(t), \sqrt{h_{\mathrm{s}}(t)} Z_s(t) + \beta_{\mathrm{s}} \sqrt{h_m(t)} Z_m(t)\right)}{\mathrm{Var}_{t-1}\left(\sqrt{h_m(t)} Z_m(t)\right)} \\ &= \frac{\mathrm{Cov}_{t-1}\left(\sqrt{h_m(t)} Z_m(t), \beta_{\mathrm{s}} \sqrt{h_m(t)} Z_m(t)\right)}{\mathrm{Var}_{t-1}\left(\sqrt{h_m(t)} Z_m(t)\right)} \\ &= \beta_{\mathrm{s}}, \end{aligned} \tag{2.3}$$

where we have used the fact that $Z_s(t)$ and $Z_m(t)$ are independent. The form of β_{s} is consistent with that in the CAPM. Shocks to the returns of the underlying asset include idiosyncratic shocks and common ones, and the total conditional variance of $\ln S(t)$ is given by $h_{\mathrm{s}}(t) + \beta_{\mathrm{s}}^2 h_m(t)$, also consisting of two parts.

Now we turn to describe the dynamic of the counterparty's assets. Analogously, suppose that the value of the counterparty's assets is driven by the following process:

$$\begin{cases} \ln V(t) = \ln V(t-1) + r + \left(\lambda_v - \frac{1}{2}\right) h_v(t) + \sqrt{h_v(t)} Z_v(t) \\ \qquad + \left(\beta_v \lambda_m - \frac{1}{2}\beta_v^2\right) h_m(t) + \beta_v \sqrt{h_m(t)} Z_m(t), \\ h_v(t) = w_v + b_v h_v(t-1) + a_v \left(Z_v(t-1) - c_v \sqrt{h_v(t-1)}\right)^2, \end{cases} \tag{2.4}$$

where $V(t)$ represents the value of the counterparty's assets and r is the risk-free interest rate. Similarly, shocks to the returns of the counterparty's assets also consist of idiosyncratic and common components, and $Z_v(t)$ is a standard normal variable independent of $Z_s(t)$ and $Z_m(t)$. The parameter β_v describes the effect of the common component on the value of the counterparty's assets, and captures the sensitivity of the counterparty's assets to systematic risk, stated as follows:

$$\beta_v = \frac{\mathrm{Cov}_{t-1}\Big(\ln \frac{M(t)}{M(t-1)}, \ln \frac{V(t)}{V(t-1)}\Big)}{\mathrm{Var}_{t-1}\Big(\ln \frac{M(t)}{M(t-1)}\Big)}. \tag{2.5}$$

In the proposed model, we use the same class of conditional variance processes for all asset prices. This kind of variance processes converge weakly to a square-root process of Cox, Ingersoll and Ross [7] (see Heston and Nandi [16] for more details), which is a continuous-time stochastic volatility process used in Heston [15] and many other studies in the option pricing literature. In addition, the proposed model can capture stochastic nature of variance, leverage effects, and stochastic correlation between the returns of the underlying asset and the counterparty's assets as shown in the following subsections.

2.1. Leverage Effects

From the dynamic of the market index in (2.1), using the values of the market index, we can observe disturbance processes $Z_m(t)$ as follows:

$$Z_m(t) = \frac{\ln \frac{M(t)}{M(t-1)} - r - (\lambda_m - \frac{1}{2})h_m(t)}{\sqrt{h_m(t)}}, \tag{2.6}$$

which in turn implies that the conditional variance $h_m(t+1)$ can be observed at time t,

$$\begin{aligned} h_m(t+1) &= w_m + b_m h_m(t) + a_m\Big(Z_m(t) - c_m\sqrt{h_m(t)}\Big)^2 \\ &= w_m + b_m h_m(t) + a_m\left(\frac{\ln \frac{M(t)}{M(t-1)} - r - (\lambda_m - \frac{1}{2})h_m(t)}{\sqrt{h_m(t)}} - c_m\sqrt{h_m(t)}\right)^2. \end{aligned}$$

Note that disturbance processes $Z_m(t)$ and the conditional variance $h_m(t+1)$ depends on the value of $\ln \frac{M(t)}{M(t-1)}$.

In contrast to the continuous-time stochastic volatility models, GARCH models have an obvious advantage that volatility or variance is observable from the history of asset prices. Moreover, the relation between the return and variance is described as follows:

$$\mathrm{Cov}_{t-1}(h_m(t+1), \ln M(t)) = -2a_m c_m h_m(t).$$

Positive values for a_m and c_m imply a negative correlation between returns and variance.

From the dynamic of the underlying asset in (2.2), the underlying asset price together with $Z_m(t)$ can also provide us with observable disturbance processes $Z_s(t)$:

$$Z_s(t) = \frac{\ln \frac{S(t)}{S(t-1)} - r - (\lambda_s - \frac{1}{2})h_s(t) - (\beta_s\lambda_m - \frac{1}{2}\beta_s^2)h_m(t)}{\sqrt{h_s(t)}} - \frac{\beta_s\sqrt{h_m(t)}Z_m(t)}{\sqrt{h_s(t)}},$$

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and the conditional variance for idiosyncratic shocks,

$$\begin{aligned}h_{\rm s}(t+1) &= w_s + b_s h_{\rm s}(t-1) + a_s\Big(Z_s(t) - c_s\sqrt{h_{\rm s}(t)}\Big)^2 \\ &= w_s + b_s h_{\rm s}(t-1) + a_s\left(\frac{\ln\frac{S(t)}{S(t-1)} - r - (\lambda_s - \frac{1}{2})h_{\rm s}(t) - (\beta_{\rm s}\lambda_m - \frac{1}{2}\beta_{\rm s}^2)h_m(t)}{\sqrt{h_{\rm s}(t)}}\right. \\ &\qquad \left. - \frac{\beta_{\rm s}\sqrt{h_m(t)}Z_m(t)}{\sqrt{h_{\rm s}(t)}} - c_s\sqrt{h_{\rm s}(t)}\right)^2.\end{aligned}$$

Analogously, the conditional variance for individual shocks is observable from the history of the counterparty's assets prices,

$$\begin{aligned}h_v(t+1) &= w_v + b_v h_v(t-1) + a_v\Big(Z_v(t-1) - c_v\sqrt{h_v(t-1)}\Big)^2 \\ &= w_v + b_v h_v(t-1) + a_v\left(\frac{\ln\frac{V(t)}{V(t-1)} - r - (\lambda_v - \frac{1}{2})h_v(t) - (\beta_{\rm s}\lambda_m - \frac{1}{2}\beta_{\rm s}^2)h_m(t)}{\sqrt{h_v(t)}}\right. \\ &\qquad \left. - \frac{\beta_{\rm s}\sqrt{h_m(t)}Z_m(t)}{\sqrt{h_v(t)}} - c_v\sqrt{h_v(t)}\right)^2.\end{aligned}$$

As illustrated above, the proposed model is much easier to implement using historical prices of the market index, the underlying asset and the counterparty's assets. It captures stochastic nature of volatility for each asset and correlation between returns and variance as well. Moreover, stochastic correlation between the underlying asset return and the return of the counterparty's assets is captured in the proposed model.

2.2. Stochastic Correlation

From the dynamics of the underlying asset and the counterparty's assets in (2.2) and (2.4), one gets that the covariance of the underlying asset return with the return of the counterparty's assets is given by

$$\begin{aligned}\mathrm{Cov}_t\left(\ln\frac{S(t+1)}{S(t)}, \ln\frac{V(t+1)}{V(t)}\right) &= \mathrm{Cov}_t(\beta_{\rm s}\sqrt{h_m(t+1)}Z_m(t+1), \beta_v\sqrt{h_m(t+1)}Z_m(t+1)) \\ &= \beta_{\rm s}\beta_v h_m(t+1),\end{aligned}$$

where we have used the fact that $Z_m(t+1)$, $Z_s(t+1)$ and $Z_v(t+1)$ are independent of each other.

The correlation between the underlying asset return and the return of the counterparty's assets is determined by $\beta_{\rm s}$ and β_v and depends on the current level of the variance of the market index. This implies that the correlation coefficient between two returns is given by

$$\begin{aligned}\rho_t &= \frac{\mathrm{Cov}_t(\ln\frac{S(t+1)}{S(t)}, \ln\frac{V(t+1)}{V(t)})}{\sqrt{\mathrm{Var}_t(\ln\frac{S(t+1)}{S(t)})}\sqrt{\mathrm{Var}_t(\ln\frac{V(t+1)}{V(t)})}} \\ &= \frac{\beta_{\rm s}\beta_v h_m(t+1)}{\sqrt{h_{\rm s}(t+1) + \beta_{\rm s}^2 h_m(t+1)}\sqrt{h_v(t+1) + \beta_v^2 h_m(t+1)}}.\end{aligned} \tag{2.7}$$

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Note that the correlation coefficient is time-varying and depends on current levels of the variances of both assets and the market index as well. Hence, the proposed model displays not only leverage effects of each asset but also stochastic correlation between two assets.

2.3. Equivalent Martingale Measures (EMMs)

To derive vulnerable option prices, we need to determine an EMM. To this end, define the following conditional Radon–Nikodym derivative (The form of the Radon–Nikodym derivative is motivated by the affine structure of the pricing kernel; see, e.g., Christoffersen et al. [5].),

$$
\begin{aligned}
L(t+1) &:= \frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} \\
&= \frac{\exp\left\{\begin{matrix}\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)\\ +\theta_v\sqrt{h_v(t+1)}Z_v(t+1)\end{matrix}\right\}}{\mathbb{E}_t\left[\exp\left\{\begin{matrix}\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)\\ +\theta_v\sqrt{h_v(t+1)}Z_v(t+1)\end{matrix}\right\}\right]},
\end{aligned}
\tag{2.8}
$$

where $Z_m(t+1)$, $Z_s(t+1)$ and $Z_v(t+1)$ are the normal shocks to returns of the market index, the underlying asset and the counterparty's assets, respectively. $h_i(t+1),\ i=m,s,v$ are the conditional variances, known at time t. To ensure that Q is an EMM, we should put the conditions on the Radon–Nikodym derivative. The following result gives a necessary and sufficient condition such that the martingale condition is satisfied.

PROPOSITION 2.1: *The martingale condition holds if and only if*

$$
\theta_m=-\lambda_m,\quad \theta_s=-\lambda_s,\quad \theta_v=-\lambda_v.
$$

Moreover, $Z_i(t)+\lambda_i\sqrt{h_i(t)}$ *is a standard normal distribution under the EMM* Q*, for* $i=m,s,v$.

PROOF: See the Appendix. ■

In the following, we identify the risk-neutral dynamics under the EMM Q and the result is as follows.

PROPOSITION 2.2: *The dynamic of the market index* $M(t)$ *admits the following form under* Q*,*

$$
\begin{cases}
\ln M(t)=\ln M(t-1)+r-\dfrac{1}{2}h_m(t)+\sqrt{h_m(t)}Z_m^*(t),\\[2ex]
h_m(t)=w_m+b_m h_m(t-1)+a_m\left(Z_m^*(t-1)-(c_m+\lambda_m)\sqrt{h_m(t-1)}\right)^2,
\end{cases}
\tag{2.9}
$$

where $Z_m^*(t):=Z_m(t)+\lambda_m\sqrt{h_m(t)}$ *is a standard normal variable under* Q*, shown in Proposition 2.1.*

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The dynamic of the underlying asset $S(t)$ takes the form as follows under Q,

$$\begin{cases} \ln S(t) = \ln S(t-1) + r - \frac{1}{2}h_s(t) + \sqrt{h_s(t)}Z_s^*(t) - \frac{1}{2}\beta_s^2 h_m(t) + \beta_s\sqrt{h_m(t)}Z_m^*(t), \\ h_s(t) = w_s + b_s h_s(t-1) + a_s\left(Z_s^*(t-1) - (c_s+\lambda_s)\sqrt{h_s(t-1)}\right)^2, \end{cases} \tag{2.10}$$

where $Z_s^*(t) := Z_s(t) + \lambda_s\sqrt{h_s(t)}$ is a standard normal variable under Q, shown in Proposition 2.1.

The dynamic of the counterparty's assets $V(t)$ admits the following form under Q,

$$\begin{cases} \ln V(t) = \ln V(t-1) + r - \frac{1}{2}h_v(t) + \sqrt{h_v(t)}Z_v^*(t) - \frac{1}{2}\beta_v^2 h_m(t) + \beta_v\sqrt{h_m(t)}Z_m^*(t), \\ h_v(t) = w_v + b_v h_v(t-1) + a_v\left(Z_v^*(t-1) - (c_v+\lambda_v)\sqrt{h_v(t-1)}\right)^2, \end{cases} \tag{2.11}$$

where $Z_v^*(t) := Z_v(t) + \lambda_v\sqrt{h_v(t)}$ is a standard normal variable under Q, shown in Proposition 2.1.

PROOF: See the Appendix. ∎

Up to now, we have determined an EMM Q and got the risk-neutral dynamics. In the following, we will derive the values of vulnerable European options using the generating functions.

2.4. Vulnerable Option Prices

In this subsection, we derive the values of vulnerable European call options (Due to the similarity between call and put options, we only show specific calculations and numerical analysis for vulnerable European call options in this paper.) with strike price K and maturity T, based on the risk-neutral dynamics in Proposition 2.2. To this end, we first focus on the payoff of vulnerable options. In contrast to options without counterparty default risk, the payoff of vulnerable options depends on whether default occurs or not.

Following Klein [20], and Wang, Song, and Wang [32], we consider counterparty default risk with structural approaches, that is, a credit loss occurs if the market value of the assets of the counterparty, $V(T)$, is less than some amount D (It should be noted that the option is not often the only liability of the option issuer. Following Klein [20], we also allow the option issuer to have other liabilities, which rank equally with the option and denote by D the amount of all the outstanding claims. Default occurs when the option issuer fails to make the payment under the option or other liabilities. That is, the value of the option issuer's assets is less than the amount of all the outstanding claims D.). Here D is set to the amount of all the outstanding claims. Once a credit loss occurs at exercise time T, $\alpha V(T)$ is paid as the deadweight costs due to the bankruptcy or reorganization, and the remaining value of $(1-\alpha)V(T)$ is paid to the holders of the option and other liabilities. Then the recovery is $\frac{(1-\alpha)V(T)}{D}$, with the amount of all the outstanding claims D.

If there is no default at maturity T, it is equal to the payoff on a vanilla European call option. Mathematically, this part of the payoff equals

$$1_{\{V(T)>D\}}(S(T)-K)^+.$$

When a credit loss occurs at exercise time T, only a proportion of the payoff can be recovered, and the corresponding payoff is given by

$$1_{\{V(T)\leq D\}}\frac{(1-\alpha)V(T)}{D}(S(T)-K)^+.$$

Denote the value of a vulnerable option by C^*, which is represented by,

$$C^* = e^{-r(T-t)}\mathbb{E}_t^Q\Big[(S(T)-K)^+\Big(1(V(T)\geq D)+\frac{(1-\alpha)V(T)}{D}1(V(T)<D)\Big)\Big], \quad \textbf{(2.12)}$$

where $\mathbb{E}_t^Q[\ ]$ denotes the conditional expectation given the information at time t.

Now we turn to derive the closed-form solution for the generating function of $S(T)$ and $V(T)$ and use it to calculate option prices in (2.12). Let $f(t;T,\phi_1,\phi_2)$ denote the conditional generating function of $S(T)$ and $V(T)$,

$$f(t;T,\phi_1,\phi_2) = \mathbb{E}_t^Q[S(T)^{\phi_1}V(T)^{\phi_2}].$$

Let $x(t) = \ln S(t)$ and $y(t) = \ln V(t)$, and $f(t;T,\phi_1,\phi_2)$ is also the conditional moment generating function of $x(T)$ and $y(T)$, that is,

$$f(t;T,\phi_1,\phi_2) = \mathbb{E}_t^Q[e^{\phi_1 x(T)+\phi_2 y(T)}].$$

The explicit expression of $f(t;T,\phi_1,\phi_2)$ is given below.

PROPOSITION 2.3: *The moment generating function of $x(T)$ and $y(T)$, with the notations $x(T) = \ln S(T)$ and $y(T) = \ln V(T)$, admits the following form,*

$$\begin{aligned} f(t;T,\phi_1,\phi_2) = \exp\Big\{&\phi_1 x(t)+\phi_2 y(t)+A(t;T,\phi_1,\phi_2)+B_1(t;T,\phi_1,\phi_2)h_s(t+1) \\ &+ B_2(t;T,\phi_1,\phi_2)h_v(t+1)+B_3(t;T,\phi_1,\phi_2)h_m(t+1)\Big\}, \end{aligned}$$

where $A(t;T,\phi_1,\phi_2)$, $B_1(t;T,\phi_1,\phi_2)$, $B_2(t;T,\phi_1,\phi_2)$, and $B_3(t;T,\phi_1,\phi_2)$ are given by

$$\begin{aligned} A(t;T,\phi_1,\phi_2) &= (\phi_1+\phi_2)r + w_s B_1(t+1;T,\phi_1,\phi_2) + w_v B_2(t+1;T,\phi_1,\phi_2) \\ &\quad + w_m B_3(t+1;T,\phi_1,\phi_2) + A(t+1;T,\phi_1,\phi_2) \\ &\quad - \frac{1}{2}\ln(1-2a_s B_1(t+1;T,\phi_1,\phi_2)) - \frac{1}{2}\ln(1-2a_v B_2(t+1;T,\phi_1,\phi_2)) \\ &\quad - \frac{1}{2}\ln(1-2a_m B_3(t+1;T,\phi_1,\phi_2)), \\ B_1(t;T,\phi_1,\phi_2) &= b_s B_1(t+1;T,\phi_1,\phi_2) - \frac{1}{2}\phi_1 + \phi_1(c_s+\lambda_s) - \frac{1}{2}(c_s+\lambda_s)^2 \\ &\quad + \frac{(1/2)(\phi_1-(c_s+\lambda_s))^2}{1-2a_s B_1(t+1;T,\phi_1,\phi_2)}, \end{aligned}$$

$$B_2(t;T,\phi_1,\phi_2) = b_v B_2(t+1;T,\phi_1,\phi_2) - \frac{1}{2}\phi_2 + \phi_2(c_v+\lambda_v) - \frac{1}{2}(c_v+\lambda_v)^2 + \frac{(1/2)(\phi_2-(c_v+\lambda_v))^2}{1-2a_v B_2(t+1;T,\phi_1,\phi_2)},$$

$$B_3(t;T,\phi_1,\phi_2) = b_m B_3(t+1;T,\phi_1,\phi_2) - \frac{1}{2}\phi_1\beta_{\mathrm{s}}^2 - \frac{1}{2}\phi_2\beta_v^2 + (\phi_1\beta_{\mathrm{s}}+\phi_2\beta_v)(c_m+\lambda_m) - \frac{1}{2}(c_m+\lambda_m)^2 + \frac{(1/2)(\phi_1\beta_{\mathrm{s}}+\phi_2\beta_v-(c_m+\lambda_m))^2}{1-2a_m B_3(t+1;T,\phi_1,\phi_2)},$$

and these coefficients can be obtained recursively using the terminal conditions,

$$A(T;T,\phi_1,\phi_2) = B_1(T;T,\phi_1,\phi_2) = B_2(T;T,\phi_1,\phi_2) = B_3(T;T,\phi_1,\phi_2) = 0.$$

PROOF: See the Appendix. ■

Based on the expression of the moment generating function $f(t;T,\phi_1,\phi_2)$, we can have the characteristic function of the logarithm of asset prices, which can be used to calculate the density and derive option prices. In particular, the vulnerable option price in (2.12) is obtained.

PROPOSITION 2.4: *The price of vulnerable options with strike price K and maturity T is given by*

$$C^* = e^{-r(T-t)}\Big[f(t;T,1,0)\Pi_1(t;T) - K\Pi_2(t;T) + \frac{1-\alpha}{D}f(t;T,1,1)\Pi_3(t;T) - \frac{1-\alpha}{D}Kf(t;T,0,1)\Pi_4(t;T)\Big], \tag{2.13}$$

where the closed form of $f(t;T,\phi_1,\phi_2)$ is derived in Proposition 2.3 and $\Pi_1(t;T)$, $\Pi_2(t;T)$, $\Pi_3(t;T)$ and $\Pi_4(t;T)$ are given in (A.7)–(A.10).

PROOF: See the Appendix. ■

Thanks to the explicit expression of the generating function, we have obtained the closed-form solution for vulnerable options under the proposed GARCH model. In the proposed framework, the market index is used to connect the dynamics of the underlying asset and the assets of the counterparty, and the model captures stochastic nature of the correlation between the returns of these assets.

3. NUMERICAL RESULTS

In this section, we investigate the impact of counterparty default risk on vulnerable option prices. The derived formula in (2.13) is used to obtain the values of vulnerable options. In addition, we contrast vulnerable option prices with the values of vanilla options (Vanilla option prices can be obtained by taking $D \to 0$, that is, there is no possibility of default. Alternatively, we can obtain the pricing formula using the moment generating function of $x(T) = \ln S(T)$.) to show the impact of counterparty default risk.

TABLE 1. MLEs using an estimation sample of daily returns, 1995–2009. We use daily closing prices for the period from January 3, 1995 to December 31, 2009. The interest rate is approximated by daily yields of 3-month Treasury bills. Robust standard errors are obtained using the outer product of the gradient at the optimum parameter values.

	Parameter	Estimate	Standard error
S&P 500 Index	λ_m	1.576	1.574
	w_m	3.000×10^{-15}	8.576×10^{-9}
	b_m	8.500×10^{-1}	6.348×10^{-4}
	a_m	3.921×10^{-6}	1.726×10^{-8}
	c_m	1.755×10^{2}	1.534
Microsoft Corporation	λ_s	1.017	8.808×10^{-1}
	w_s	9.319×10^{-11}	6.487×10^{-9}
	b_s	9.497×10^{-1}	2.310×10^{-5}
	a_s	1.874×10^{-5}	1.052×10^{-8}
	c_s	4.385×10^{-4}	8.804×10^{-1}
Bank of America Corporation	λ_v	3.862×10^{-1}	1.203
	w_v	3.483×10^{-12}	1.120×10^{-9}
	b_v	9.058×10^{-1}	9.342×10^{-6}
	a_v	3.051×10^{-5}	3.169×10^{-9}
	c_v	2.665×10^{-3}	1.203

GARCH process parameters are obtained by empirically estimating (2.1), (2.2), and (2.4). We use daily closing prices for the S&P 500 index, Microsoft Corporation stock and Bank of America Corporation stock for the period from January 3, 1995 to December 31, 2009. Additionally, the daily time-series of 3-month Treasury bills are used as our proxy for the interest rate. The market betas β_s and β_v in (2.3) and (2.5) are estimated directly using returns data, that is, $\beta_s = 1.950$ and $\beta_v = 1.276$. Table 1 presents maximum likelihood estimates (MLEs) of the physical model parameters. For each parameter, we also report its robust standard error obtained using the outer product of the gradient at the optimum parameter value.

Based on the estimated parameters, we now have the risk-neutral dynamics of the market index and two assets in (2.9)–(2.11). For the initial variances of the S&P 500 index and each asset, we set them to the ones calculated from the returns data, respectively. The annualized variance for the S&P 500 index is 7.596×10^{-3}, corresponding to annualized volatility of 0.087. The annualized total variances for two assets are 3.384×10^{-2} and 2.395×10^{-2}, respectively. To obtain the value of vulnerable options, the values for default boundary D and deadweight cost α are also needed. These two parameters are crucial in observing the impact of counterparty default risk, and we examine different situations by setting the default boundary D to a simple n multiple of its initial asset value $V(0) = 100$ (Actually, the default boundary affects the value of vulnerable option only through the ratio of default boundary to its initial asset value.). We set $n = 0.4,\ 0.5,\ 0.6$ and $\alpha = 0.3,\ 0.4,\ 0.5$ to investigate the impact of counterparty default risk.

Without loss of generality, we set the current price of the underlying asset to 100 and the interest rate to 0.02. We consider three moneyness cases by setting strike prices to $K = 90,\ 100,\ 110$, corresponding to in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM) options, respectively, and four maturity cases, being 0.5, 1.0, 2.0, and

Table 2. The difference between option prices without counterparty default risk and vulnerable option prices. We use the MLEs in Table 1 to price options. Panel A shows the differences when the default boundary is 40% of the counterparty' asset value. Panels B and C correspond to the ratios of 50 and 60%, respectively. The values in the column labeled 'Prices' are vanilla option prices.

			Panel A, $D = 40$			Panel B, $D = 50$			Panel C, $D = 60$		
Maturiy	Strike	Prices	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.5	90	20.22	0.0007	0.0008	0.0010	0.0105	0.0132	0.0159	0.0712	0.0892	0.1071
0.5	100	15.14	0.0004	0.0005	0.0006	0.0061	0.0077	0.0093	0.0438	0.0549	0.0660
0.5	110	11.11	0.0002	0.0003	0.0004	0.0035	0.0045	0.0054	0.0267	0.0335	0.0403
1.0	90	26.91	0.0250	0.0310	0.0369	0.1251	0.1534	0.1818	0.3854	0.4691	0.5527
1.0	100	22.40	0.0175	0.0216	0.0258	0.0902	0.1108	0.1315	0.2855	0.3480	0.4104
1.0	110	18.59	0.0123	0.0152	0.0181	0.0654	0.0804	0.0954	0.2121	0.2587	0.3054
2.0	90	36.07	0.2472	0.2959	0.3446	0.6343	0.7530	0.8717	1.2475	1.4695	1.6915
2.0	100	32.21	0.1997	0.2393	0.2789	0.5209	0.6191	0.7172	1.0372	1.2233	1.4094
2.0	110	28.81	0.1625	0.1949	0.2273	0.4305	0.5121	0.5937	0.8669	1.0236	1.1802
5.0	90	52.89	1.6551	1.8894	2.1237	2.6923	3.0533	3.4144	3.8655	4.3592	4.8530
5.0	100	50.08	1.4942	1.7070	1.9198	2.4444	2.7744	3.1044	3.5251	3.9785	4.4319
5.0	110	47.51	1.3557	1.5499	1.7441	2.2296	2.5324	2.8352	3.2284	3.6463	4.0642

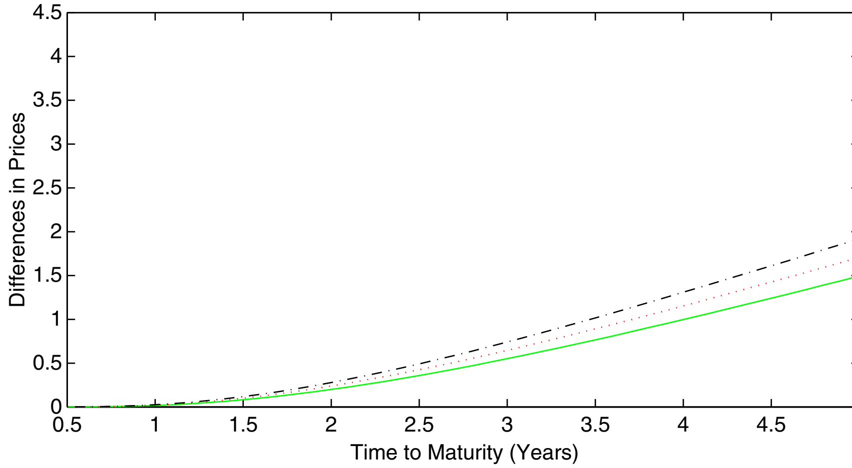

FIGURE 1. The difference between option prices without counterparty default risk and vulnerable option prices with alternative maturities. We use the MLEs in Table 1 and assume $K = 100$ and $D = 40$ to price options. The solid, dotted, and dot-dashed lines correspond to $\alpha = 0.3$, $\alpha = 0.4$, and $\alpha = 0.5$, respectively.

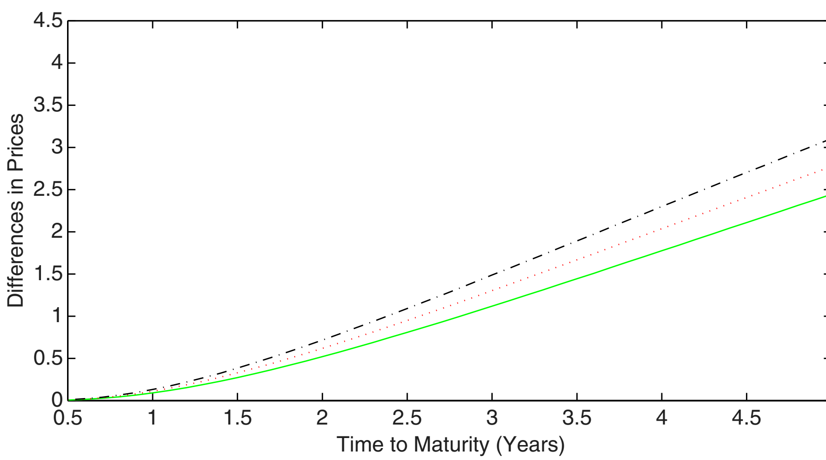

FIGURE 2. The difference between option prices without counterparty default risk and vulnerable option prices with alternative maturities. We use the MLEs in Table 1 and assume $K = 100$ and $D = 50$ to price options. The solid, dotted, and dot-dashed lines correspond to $\alpha = 0.3$, $\alpha = 0.4$, and $\alpha = 0.5$, respectively.

5.0 years. Table 2 reveals the difference between option prices without counterparty default risk and vulnerable option prices for each parameter combination. The values in the column labeled 'Prices' are those of the options without counterparty default risk. From Table 2, we can see that default risk has little impact when maturity is quite short, and observe that the impact of counterparty default risk becomes more and more pronounced as the life of the option increases, especially when the default boundary is higher. These impacts can be observed more clearly in Figures 1–3. The counterparty defaults more likely with a longer

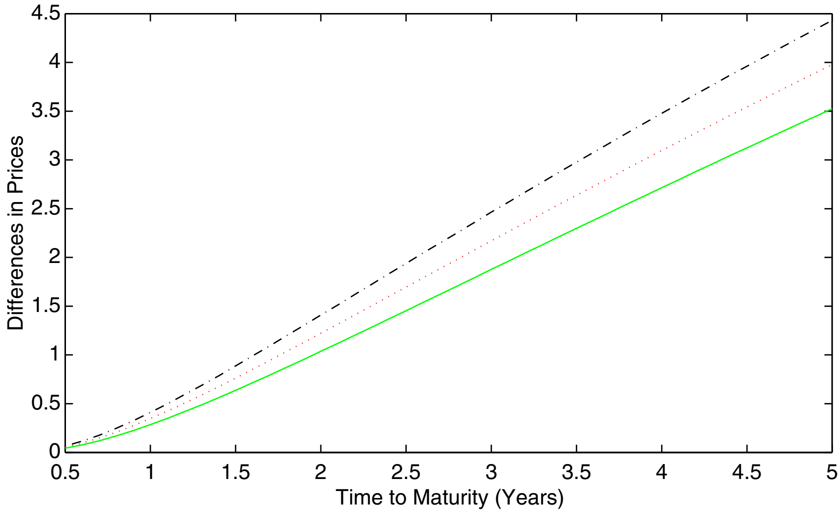

FIGURE **3.** The difference between option prices without counterparty default risk and vulnerable option prices with alternative maturities. We use the MLEs in Table 1 and assume $K = 100$ and $D = 60$ to price options. The solid, dotted, and dot-dashed lines correspond to $\alpha = 0.3$, $\alpha = 0.4$, and $\alpha = 0.5$, respectively.

maturity, reducing vulnerable option prices and hence corresponding to wide differences. For instance, the difference is 0.4104 for an ATM option with a maturity of 1.0 year, while the one is 4.4319 for 5.0-year options, with $D = 60$ and $\alpha = 0.5$ (Panel C). In addition, the values of the default boundary D and deadweight cost α have significant effects on option prices intuitively, since they affect directly the loss once default events occur. The quantitative impact of counterparty default risk should be taken into account when the long-term option is issued by a highly leveraged firm.

Up to now, we have focused on the case with initial variances calculated from the returns data. In the following, we investigate in detail differences between option prices without counterparty default risk and vulnerable option prices with alternative initial variances. Table 3 shows the differences with alternative initial total variances for the underlying asset. Intuitively, a higher initial total variance leads to a higher vanilla option value and a larger loss once default occurs. Hence, the difference increases when the initial total variance of the underlying asset rises. Similarly, a higher initial total variance of the counterparty's assets corresponds to larger default risk and an increase of the differences, as shown in Table 4.

In Table 5, we consider the case with alternative initial variances of the S&P 500 index and the constant total variances of two assets. It is interesting to find that the difference rises with an increase of the initial variance of the S&P 500 index. Increasing the initial variance of the S&P 500 index does not change the total risk for the underlying asset and the counterparty's assets (since the total variances of two assets are kept constants), but indeed increases the proportion of market risk, which corresponds to a higher correlation coefficient between two assets. The positive correlation coefficient ensures that the values of two assets move in the same direction more likely, corresponding to a smaller effect of credit risk on vulnerable options and hence a smaller difference. In a word, the impact of counterparty default risk becomes more significant when increasing idiosyncratic risk of the

Table 3. The differences between option prices without counterparty default risk and vulnerable option prices with alternative initial variance for the underlying asset. We use the MLEs in Table 1 and take $D = 60$ to price options. Panel A shows the differences for the initial variance being 90% of the one calculated from the returns data. Panels B and C correspond to the ratios of 100% and 115%, respectively.

Maturity (Years)	Strike	Panel A, 90%			Panel B, 100%			Panel C, 115%		
		$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.5	90	0.0711	0.0890	0.1069	0.0712	0.0892	0.1071	0.0714	0.0894	0.1073
0.5	100	0.0437	0.0548	0.0659	0.0438	0.0549	0.0660	0.0439	0.0551	0.0662
0.5	110	0.0266	0.0334	0.0402	0.0267	0.0335	0.0403	0.0268	0.0336	0.0404
1.0	90	0.3852	0.4688	0.5523	0.3854	0.4691	0.5527	0.3858	0.4695	0.5533
1.0	100	0.2853	0.3477	0.4101	0.2855	0.3480	0.4104	0.2859	0.3484	0.4109
1.0	110	0.2118	0.2585	0.3051	0.2121	0.2587	0.3054	0.2124	0.2591	0.3059
2.0	90	1.2471	1.4691	1.6910	1.2475	1.4695	1.6915	1.2481	1.4702	1.6923
2.0	100	1.0369	1.2229	1.4089	1.0372	1.2233	1.4094	1.0378	1.2239	1.4101
2.0	110	0.8666	1.0232	1.1798	0.8669	1.0236	1.1802	0.8675	1.0242	1.1809
5.0	90	3.8651	4.3588	4.8525	3.8655	4.3592	4.8530	3.8661	4.3560	4.8537
5.0	100	3.5247	3.9781	4.4314	3.5251	3.9785	4.4319	3.5257	3.9792	4.4326
5.0	110	3.2280	3.6459	4.0638	3.2284	3.6463	4.0642	3.2290	3.6470	4.0650

Table 4. The differences between option prices without counterparty default risk and vulnerable option prices with alternative initial variance for the counterparty's assets. We use the MLEs in Table 1 and take $D = 60$ to price options. Panel A shows the differences for the initial variance being 90% of the one calculated from the returns data. Panels B and C correspond to the ratios of 100% and 115%, respectively.

Maturity (Years)	Strike	Panel A, 90%			Panel B, 100%			Panel C, 115%		
		$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.5	90	0.0710	0.0889	0.1067	0.0712	0.0892	0.1071	0.0716	0.0896	0.1076
0.5	100	0.0436	0.0547	0.0658	0.0438	0.0549	0.0660	0.0440	0.0552	0.0663
0.5	110	0.0266	0.0334	0.0401	0.0267	0.0335	0.0403	0.0268	0.0337	0.0405
1.0	90	0.3850	0.4686	0.5521	0.3854	0.4691	0.5527	0.3861	0.4699	0.5536
1.0	100	0.2852	0.3476	0.4100	0.2855	0.3480	0.4104	0.2861	0.3486	0.4111
1.0	110	0.2117	0.2583	0.3050	0.2121	0.2587	0.3054	0.2125	0.2592	0.3060
2.0	90	1.2469	1.4689	1.6908	1.2475	1.4695	1.6915	1.2483	1.4705	1.6926
2.0	100	1.0367	1.2227	1.4087	1.0372	1.2233	1.4094	1.0380	1.2241	1.4103
2.0	110	0.8665	1.0231	1.1797	0.8669	1.0236	1.1802	0.8676	1.0243	1.1811
5.0	90	3.8649	4.3586	4.8522	3.8655	4.3592	4.8530	3.8665	4.3602	4.8540
5.0	100	3.5245	3.9779	4.4312	3.5251	3.9785	4.4319	3.5259	3.9794	4.4329
5.0	110	3.2278	3.6457	4.0636	3.2284	3.6463	4.0642	3.2292	3.6472	4.0652

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Table 5. The differences between option prices without counterparty default risk and vulnerable option prices with alternative initial variance for the S & P500 index. We use the MLEs in Table 1 and take $D = 60$ to price options. Panel A shows the differences for the initial variance being 90% of the one calculated from the returns data. Panels B and C correspond to the ratios of 100% and 115%, respectively.

Maturity (Years)	Strike	Panel A, 90%			Panel B, 100%			Panel C, 115%		
		$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.5	90	0.0715	0.0895	0.1075	0.0712	0.0892	0.1071	0.0708	0.0886	0.1064
0.5	100	0.0440	0.0552	0.0663	0.0438	0.0549	0.0660	0.0435	0.0545	0.0655
0.5	110	0.0268	0.0337	0.0405	0.0267	0.0335	0.0403	0.0265	0.0332	0.0340
1.0	90	0.3862	0.4700	0.5537	0.3854	0.4691	0.5527	0.3844	0.4678	0.5512
1.0	100	0.2861	0.3487	0.4112	0.2855	0.3480	0.4104	0.2847	0.3470	0.4092
1.0	110	0.2125	0.2593	0.3061	0.2121	0.2587	0.3054	0.2114	0.2579	0.3044
2.0	90	1.2485	1.4707	1.6929	1.2475	1.4695	1.6915	1.2459	1.4677	1.6895
2.0	100	1.0381	1.2244	1.4106	1.0372	1.2233	1.4094	1.0358	1.2217	1.4075
2.0	110	0.8678	1.0246	1.1813	0.8669	1.0236	1.1802	0.8657	1.0222	1.1786
5.0	90	3.8668	4.3607	4.8545	3.8655	4.3592	4.8530	3.8636	4.3571	4.8506
5.0	100	3.5263	3.9799	4.4334	3.5251	3.9785	4.4319	3.5232	3.9765	4.4297
5.0	110	3.2295	3.6476	4.0657	3.2284	3.6463	4.0642	3.2266	3.6444	4.0621

underlying asset or the counterparty's assets. Increasing the proportion of market risk drops the impact of counterparty default risk.

4. CONCLUSION

In this paper, we propose a new pricing model for vulnerable options, where the variances of all asset prices are described by GARCH processes and the dynamics of the underlying asset and the counterparty's assets are connected through the market factor channel. The proposed model captures stochastic nature of volatility for each asset and correlation between returns and volatility. Moreover, the correlation between the underlying asset return and the return of the counterparty's assets is time-varying and depends on the level of the variance of the market index. We derive the closed-form solution for vulnerable options and present the numerical results to show the impact of counterparty default risk on option prices.

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APPENDIX

Proof of Proposition 2.1: Under the EMM Q, the expected return of $M(t)$, $S(t)$, and $V(t)$ from time t to $t+1$ must be equal to risk-free interest rate, that is,

$$E_t\left[L(t+1)\frac{M(t+1)}{M(t)}\right] = e^r, \tag{A.1}$$

$$E_t\left[L(t+1)\frac{S(t+1)}{S(t)}\right] = e^r, \tag{A.2}$$

$$E_t\left[L(t+1)\frac{V(t+1)}{V(t)}\right] = e^r, \tag{A.3}$$

where $L(t)$ is the Radon–Nikodym derivative defined in (2.8).
Substituting the expression of $L(t+1)$ in (2.8) into (A.1) yields,

$$\begin{aligned}
&\mathbb{E}_t\left[L(t+1)\frac{M(t+1)}{M(t)}\right]\\
&= \mathbb{E}_t\left[\frac{e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\right]}\frac{M(t+1)}{M(t)}\right]\\
&= \frac{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\frac{M(t+1)}{M(t)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\right]}\\
&= \frac{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}\frac{M(t+1)}{M(t)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}\right]},
\end{aligned}$$

where we have used the fact that $Z_s(t+1)$ and $Z_v(t+1)$ are independent of $Z_m(t+1)$, given the information at time t. Recall the dynamics of the market index in (2.1) and $Z_m(t+1)$ is a standard normal variable, the form of (A.1) continues to be

$$\begin{aligned}
&\mathbb{E}_t\left[L(t+1)\frac{M(t+1)}{M(t)}\right]\\
&\quad=\frac{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+r+(\lambda_m-\frac{1}{2})h_m(t+1)+\sqrt{h_m(t+1)}Z_m(t+1)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}\right]}\\
&\quad=\frac{\mathbb{E}_t\left[e^{(\theta_m+1)\sqrt{h_m(t+1)}Z_m(t+1)+r+(\lambda_m-\frac{1}{2})h_m(t+1)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}\right]}\\
&\quad=\frac{e^{\frac{1}{2}(\theta_m+1)^2h_m(t+1)+r+(\lambda_m-\frac{1}{2})h_m(t+1)}}{e^{\frac{1}{2}\theta_m^2h_m(t+1)}}\\
&\quad=e^{(\theta_m+\lambda_m)h_m(t+1)+r},
\end{aligned}$$

which implies that $\theta_m=-\lambda_m$.

Similarly, substituting the expression of $L(t+1)$ in (2.8) into (A.2), one gets that

$$\begin{aligned}
&\mathbb{E}_t\left[L(t+1)\frac{S(t+1)}{S(t)}\right]\\
&\quad=\mathbb{E}_t\left[\frac{e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\right]}\frac{S(t+1)}{S(t)}\right]\\
&\quad=\frac{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\frac{S(t+1)}{S(t)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)+\theta_v\sqrt{h_v(t+1)}Z_v(t+1)}\right]}\\
&\quad=\frac{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)}\frac{S(t+1)}{S(t)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)}\right]},
\end{aligned}$$

where we have used the fact that $Z_v(t+1)$ is independent of $Z_m(t+1)$ and $Z_s(t+1)$, given the information at time t. Substituting the expression of $S(t+1)$ implies

$$\begin{aligned}
&\mathbb{E}_t\left[L(t+1)\frac{S(t+1)}{S(t)}\right]\\
&\quad=\frac{\mathbb{E}_t\left[e^{(\theta_m+\beta_s)\sqrt{h_m(t+1)}Z_m(t+1)+(\theta_s+1)\sqrt{h_s(t+1)}Z_s(t+1)+r+(\lambda_s-\frac{1}{2})h_s(t+1)+(\beta_s\lambda_m-\frac{1}{2}\beta_s^2)h_m(t+1)}\right]}{\mathbb{E}_t\left[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_s\sqrt{h_s(t+1)}Z_s(t+1)}\right]}\\
&\quad=\frac{e^{\frac{1}{2}(\theta_m+\beta_s)^2h_m(t+1)+\frac{1}{2}(\theta_s+1)^2h_s(t+1)+r+(\lambda_s-\frac{1}{2})h_s(t+1)+(\beta_s\lambda_m-\frac{1}{2}\beta_s^2)h_m(t+1)}}{e^{\frac{1}{2}\theta_m^2h_m(t+1)+\frac{1}{2}\theta_s^2h_s(t+1)}}\\
&\quad=e^{(\theta_m+\lambda_m)h_m(t+1)+(\theta_s+\lambda_s)h_s(t+1)+r}\\
&\quad=e^{(\theta_s+\lambda_s)h_s(t+1)+r},
\end{aligned}$$

where the fact that $Z_m(t+1)$ and $Z_s(t+1)$ are independent standard normal variables has been used. The above expression implies that $\theta_s=-\lambda_s$. Analogously, we can obtain that $\theta_v=-\lambda_v$.

Girsanov's Theorem immediately gives us that $Z_i^*(t):=Z_i(t)+\lambda_i\sqrt{h_i(t)}$ is a standard normal variable under the EMM Q, for $i=m,s,v$. ■

Proof of Proposition 2.2: Recall that $Z_i^*(t) := Z_i(t) + \lambda_i \sqrt{h_i(t)}$ is a standard normal variable under the EMM Q, for $i = m, s, v$. Hence, rewrite the spot return of the value of the market index in the following form:

$$\begin{aligned}\ln M(t) &= \ln M(t-1) + r + \left(\lambda_m - \frac{1}{2}\right) h_m(t) + \sqrt{h_m(t)} Z_m(t) \\ &= \ln M(t-1) + r - \frac{1}{2} h_m(t) + \sqrt{h_m(t)}(Z_m(t) + \lambda_m \sqrt{h_m(t)}) \\ &= \ln M(t-1) + r - \frac{1}{2} h_m(t) + \sqrt{h_m(t)} Z_m^*(t).\end{aligned}$$

For the variance process of the market index, we have

$$\begin{aligned}h_m(t) &= w_m + b_m h_m(t-1) + a_m \left(Z_m(t-1) - c_m \sqrt{h_m(t-1)}\right)^2 \\ &= w_m + b_m h_m(t-1) + a_m \left(Z_m^*(t-1) - (c_m + \lambda_m)\sqrt{h_m(t-1)}\right)^2.\end{aligned}$$

Therefore, under the EMM Q, the dynamic of the market index is given by

$$\begin{cases} \ln M(t) = \ln M(t-1) + r - \dfrac{1}{2} h_m(t) + \sqrt{h_m(t)} Z_m^*(t), \\ h_m(t) = w_m + b_m h_m(t-1) + a_m \left(Z_m^*(t-1) - (c_m + \lambda_m)\sqrt{h_m(t-1)}\right)^2, \end{cases} \tag{A.4}$$

where $Z_m^*(t)$ is a standard normal variable under the EMM Q.

Similarly, rewrite the spot return of the underlying asset as follows:

$$\begin{aligned}\ln S(t) &= \ln S(t-1) + r + \left(\lambda_s - \frac{1}{2}\right) h_s(t) + \sqrt{h_s(t)} Z_s(t) \\ &\quad + \left(\beta_s \lambda_m - \frac{1}{2}\beta_s^2\right) h_m(t) + \beta_s \sqrt{h_m(t)} Z_m(t) \\ &= \ln S(t-1) + r - \frac{1}{2} h_s(t) + \sqrt{h_s(t)}(Z_s(t) + \lambda_s \sqrt{h_s(t)}) \\ &\quad - \frac{1}{2}\beta_s^2 h_m(t) + \beta_s \sqrt{h_m(t)}(Z_m(t) + \lambda_m \sqrt{h_m(t)}) \\ &= \ln S(t-1) + r - \frac{1}{2} h_s(t) + \sqrt{h_s(t)} Z_s^*(t) - \frac{1}{2}\beta_s^2 h_m(t) + \beta_s \sqrt{h_m(t)} Z_m^*(t),\end{aligned}$$

where we have used the fact that $Z_m^*(t) := Z_m(t) + \lambda_m \sqrt{h_m(t)}$ and $Z_s^*(t) := Z_s(t) + \lambda_s \sqrt{h_s(t)}$. For the variance process of the underlying asset, one gets that

$$\begin{aligned}h_s(t) &= w_s + b_s h_s(t-1) + a_s \left(Z_s(t-1) - c_s \sqrt{h_s(t-1)}\right)^2 \\ &= w_s + b_s h_s(t-1) + a_s \left(Z_s^*(t-1) - (c_s + \lambda_s)\sqrt{h_s(t-1)}\right)^2.\end{aligned}$$

Therefore, we have obtained the dynamic of the underlying asset under Q,

$$\begin{cases} \ln S(t) = \ln S(t-1) + r - \dfrac{1}{2} h_s(t) + \sqrt{h_s(t)} Z_s^*(t) - \dfrac{1}{2}\beta_s^2 h_m(t) + \beta_s \sqrt{h_m(t)} Z_m^*(t), \\ h_s(t) = w_s + b_s h_s(t-1) + a_s \left(Z_s^*(t-1) - (c_s + \lambda_s)\sqrt{h_s(t-1)}\right)^2, \end{cases} \tag{A.5}$$

where $Z_m^*(t)$ and $Z_s^*(t)$ are standard normal variables under the EMM Q.

Analogously, the risk-neutral dynamic of counterparties' assets is given below:

$$\begin{cases} \ln V(t) = \ln V(t-1) + r - \dfrac{1}{2} h_v(t) + \sqrt{h_v(t)} Z_v^*(t) - \dfrac{1}{2}\beta_v^2 h_m(t) + \beta_v \sqrt{h_m(t)} Z_m^*(t), \\ h_v(t) = w_v + b_v h_v(t-1) + a_v \left(Z_v^*(t-1) - (c_v + \lambda_v)\sqrt{h_v(t-1)}\right)^2, \end{cases} \tag{A.6}$$

where $Z_m^*(t)$ and $Z_v^*(t)$ are independent standard normal variables under Q. ■

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PROOF OF PROPOSITION 2.3: Let $x(t) = \ln S(t)$ and $y(t) = \ln V(t)$. Denote by $f(t; T, \phi_1, \phi_2)$ the conditional moment generating function of $x(T)$ and $y(T)$ or equivalently the conditional generating function of $S(T)$ and $V(T)$. By definition of the moment generating function, one gets that

$$f(t; T, \phi_1, \phi_2) = \mathbb{E}_t\Big[e^{\phi_1 x(T)+\phi_2 y(T)}\Big].$$

In the following, we show that the moment generating function has the log-linear form as follows:

$$\begin{aligned} f(t; T, \phi_1, \phi_2) = \exp\Big\{&\phi_1 x(t) + \phi_2 y(t) + A(t; T, \phi_1, \phi_2) + B_1(t; T, \phi_1, \phi_2) h_{\mathrm{s}}(t+1) \\ &+ B_2(t; T, \phi_1, \phi_2) h_v(t+1) + B_3(t; T, \phi_1, \phi_2) h_m(t+1)\Big\}. \end{aligned}$$

For convenience, we use the more parsimonious notation $f(t)$ to indicate $f(t; T, \phi_1, \phi_2)$, and similarly for $A(t)$, $B_1(t)$, $B_2(t)$, and $B_3(t)$.

At time T, $x(T)$ and $y(T)$ are known and we have that $f(T) = \exp\{\phi_1 x(T) + \phi_2 y(T)\}$, which gives us the terminal conditions

$$A(T) = B_1(T) = B_2(T) = B_3(T) = 0.$$

Applying the law of iterated expectations to $f(t)$, we get that

$$\begin{aligned} f(t) &= \mathbb{E}_t\Big[e^{\phi_1 x(T)+\phi_2 y(T)}\Big] \\ &= \mathbb{E}_t\Big[\mathbb{E}_{t+1}\Big[e^{\phi_1 x(T)+\phi_2 y(T)}\Big]\Big] \\ &= \mathbb{E}_t\Big[f(t+1)\Big] \\ &= \mathbb{E}_t\Big[\exp\Big\{\phi_1 x(t+1) + \phi_2 y(t+1) + A(t+1) + B_1(t+1) h_{\mathrm{s}}(t+2) \\ &\quad + B_2(t+1) h_v(t+2) + B_3(t+1) h_m(t+2)\Big\}\Big]. \end{aligned}$$

Substituting the dynamics of $x(t+1)$, $y(t+1)$, $h_{\mathrm{s}}(t+2)$, $h_v(t+2)$ and $h_m(t+2)$ gives

$$\begin{aligned} f(t) = \mathbb{E}_t\Big[\exp\Big\{&\phi_1 x(t) + \phi_1 r - \frac{1}{2}\phi_1 h_{\mathrm{s}}(t+1) + \phi_1\sqrt{h_{\mathrm{s}}(t+1)} Z_s^*(t+1) \\ &- \frac{1}{2}\phi_1 \beta_{\mathrm{s}}^2 h_m(t+1) + \phi_1 \beta_{\mathrm{s}} \sqrt{h_m(t+1)} Z_m^*(t+1) \\ &+ \phi_2 y(t) + \phi_2 r - \frac{1}{2}\phi_2 h_v(t+1) + \phi_2 \sqrt{h_v(t+1)} Z_v^*(t+1) \\ &- \frac{1}{2}\phi_2 \beta_v^2 h_m(t+1) + \phi_2 \beta_v \sqrt{h_m(t+1)} Z_m^*(t+1) + A(t+1) \\ &+ B_1(t+1)\Big(w_s + b_s h_{\mathrm{s}}(t+1) + a_s (Z_s^*(t+1) - (c_s + \lambda_s)\sqrt{h_{\mathrm{s}}(t+1)})^2\Big) \\ &+ B_2(t+1)\Big(w_v + b_v h_v(t+1) + a_v (Z_v^*(t+1) - (c_v + \lambda_v)\sqrt{h_v(t+1)})^2\Big) \\ &+ B_3(t+1)\Big(w_m + b_m h_m(t+1) + a_m (Z_m^*(t+1) - (c_m + \lambda_m)\sqrt{h_m(t+1)})^2\Big)\Big\}\Big]. \end{aligned}$$

Define Ψ_s, Ψ_v and Ψ_m in the following forms:

$$\begin{aligned} \Psi_s &= \phi_1 \sqrt{h_{\mathrm{s}}(t+1)} Z_s^*(t+1) + a_s B_1(t+1)\Big(Z_s^*(t+1) - (c_s + \lambda_s)\sqrt{h_{\mathrm{s}}(t+1)}\Big)^2, \\ \Psi_v &= \phi_2 \sqrt{h_v(t+1)} Z_v^*(t+1) + a_v B_2(t+1)\Big(Z_v^*(t+1) - (c_v + \lambda_v)\sqrt{h_v(t+1)}\Big)^2, \\ \Psi_m &= (\phi_1 \beta_{\mathrm{s}} + \phi_2 \beta_v)\sqrt{h_m(t+1)} Z_m^*(t+1) + a_m B_3(t+1)\Big(Z_m^*(t+1) - (c_m + \lambda_m)\sqrt{h_m(t+1)}\Big)^2. \end{aligned}$$

Then rearranging terms implies that

$$
\begin{aligned}
f(t) &= \mathbb{E}_t\Big[\exp\Big\{\phi_1 x(t) + \phi_1 r - \frac{1}{2}\phi_1 h_s(t+1) - \frac{1}{2}\phi_1\beta_s^2 h_m(t+1) \\
&\quad + \phi_2 y(t) + \phi_2 r - \frac{1}{2}\phi_2 h_v(t+1) - \frac{1}{2}\phi_2\beta_v^2 h_m(t+1) + A(t+1) \\
&\quad + B_1(t+1)\Big(w_s + b_s h_s(t+1)\Big) + B_2(t+1)\Big(w_v + b_v h_v(t+1)\Big) \\
&\quad + B_3(t+1)\Big(w_m + b_m h_m(t+1)\Big) + \Psi_s + \Psi_v + \Psi_m\Big\}\Big] \\
&= \mathbb{E}_t\Big[\exp\Big\{\phi_1 x(t) + \phi_2 y(t) + (\phi_1+\phi_2)r + w_s B_1(t+1) + w_v B_2(t+1) + w_m B_3(t+1) + A(t+1) \\
&\quad + \left(b_s B_1(t+1) - \frac{1}{2}\phi_1\right) h_s(t+1) + \left(b_v B_2(t+1) - \frac{1}{2}\phi_2\right) h_v(t+1) \\
&\quad + \left(b_m B_3(t+1) - \frac{1}{2}\phi_1\beta_s^2 - \frac{1}{2}\phi_2\beta_v^2\right) h_m(t+1) + \Psi_s + \Psi_v + \Psi_m\Big\}\Big] \\
&= \exp\Big\{\phi_1 x(t) + \phi_2 y(t) + (\phi_1+\phi_2)r + w_s B_1(t+1) + w_v B_2(t+1) + w_m B_3(t+1) + A(t+1) \\
&\quad + \left(b_s B_1(t+1) - \frac{1}{2}\phi_1\right) h_s(t+1) + \left(b_v B_2(t+1) - \frac{1}{2}\phi_2\right) h_v(t+1) \\
&\quad + \left(b_m B_3(t+1) - \frac{1}{2}\phi_1\beta_s^2 - \frac{1}{2}\phi_2\beta_v^2\right) h_m(t+1)\Big\}\mathbb{E}_t\Big[\exp\{\Psi_s + \Psi_v + \Psi_m\}\Big].
\end{aligned}
$$

In the following, we focus on $\mathbb{E}_t\Big[\exp\{\Psi_s + \Psi_v + \Psi_m\}\Big]$, which in turn gives us the form of $f(t)$. Note that $Z_s^*(t+1)$ is a standard normal variable, we have that

$$
\begin{aligned}
\mathbb{E}_t\left[\exp\{\Psi_s\}\right] &= \mathbb{E}_t\left[\exp\left\{\phi_1\sqrt{h_s(t+1)}Z_s^*(t+1) + a_s B_1(t+1)\Big(Z_s^*(t+1) - (c_s+\lambda_s)\sqrt{h_s(t+1)}\Big)^2\right\}\right] \\
&= \mathbb{E}_t\Big[\exp\Big\{a_s B_1(t+1)\Big(Z_s^*(t+1) - \Big(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\Big)\sqrt{h_s(t+1)}\Big)^2 \\
&\quad + a_s B_1(t+1)\Big((c_s+\lambda_s)^2 - \Big(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\Big)^2\Big)h_s(t+1)\Big\}\Big] \\
&= \exp\left\{a_s B_1(t+1)\left((c_s+\lambda_s)^2 - \Big(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\Big)^2\right) h_s(t+1)\right\} \\
&\quad \times \mathbb{E}_t\left[\exp\left\{a_s B_1(t+1)\left(Z_s^*(t+1) - \left(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\right)\sqrt{h_s(t+1)}\right)^2\right\}\right] \\
&= \exp\{a_s B_1(t+1)\Big((c_s+\lambda_s)^2 - \Big(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\Big)^2\Big)h_s(t+1)\} \\
&\quad \times \exp\left\{-\frac{1}{2}\ln(1 - 2a_s B_1(t+1)) + \frac{a_s B_1(t+1)\left(c_s + \lambda_s - \frac{\phi_1}{2a_s B_1(t+1)}\right)^2 h_s(t+1)}{1 - 2a_s B_1(t+1)}\right\},
\end{aligned}
$$

where in the last equality we have used the fact

$$
Ee^{a(Z+b)^2} = e^{-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}},
$$

with Z being a standard normal variable.

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Completing some algebra shows that the coefficient of $h_s(t+1)$ becomes

$$a_s B_1(t+1)\left((c_s+\lambda_s)^2-\left(c_s+\lambda_s-\frac{\phi_1}{2a_sB_1(t+1)}\right)^2\right)+\frac{a_sB_1(t+1)\left(c_s+\lambda_s-\frac{\phi_1}{2a_sB_1(t+1)}\right)^2}{1-2a_sB_1(t+1)}$$
$$=a_sB_1(t+1)(c_s+\lambda_s)^2+\frac{2a_s^2B_1(t+1)^2\left(c_s+\lambda_s-\frac{\phi_1}{2a_sB_1(t+1)}\right)^2}{1-2a_sB_1(t+1)}$$
$$=a_sB_1(t+1)(c_s+\lambda_s)^2+\frac{2((c_s+\lambda_s)a_sB_1(t+1)-\frac{\phi_1}{2})^2}{1-2a_sB_1(t+1)}$$
$$=\frac{a_sB_1(t+1)(1-2a_sB_1(t+1))(c_s+\lambda_s)^2+2\left((c_s+\lambda_s)a_sB_1(t+1)-\frac{\phi_1}{2}\right)^2}{1-2a_sB_1(t+1)}$$
$$=\frac{a_sB_1(t+1)(c_s+\lambda_s)^2-2a_s^2B_1(t+1)^2(c_s+\lambda_s)^2+2\left((c_s+\lambda_s)a_sB_1(t+1)-\frac{\phi_1}{2}\right)^2}{1-2a_sB_1(t+1)}$$
$$=\frac{a_sB_1(t+1)(c_s+\lambda_s)^2-2\phi_1a_sB_1(t+1)(c_s+\lambda_s)+\frac{1}{2}\phi_1^2}{1-2a_sB_1(t+1)}$$
$$=\phi_1(c_s+\lambda_s)-\frac{1}{2}(c_s+\lambda_s)^2+\frac{\frac{1}{2}(\phi_1-(c_s+\lambda_s))^2}{1-2a_sB_1(t+1)}.$$

Therefore, we have that

$$\mathbb{E}_t[\exp\{\Psi_s\}]=\exp\left\{-\frac{1}{2}\ln(1-2a_sB_1(t+1))\right.$$
$$\left.+\left(\phi_1(c_s+\lambda_s)-\frac{1}{2}(c_s+\lambda_s)^2+\frac{\frac{1}{2}(\phi_1-(c_s+\lambda_s))^2}{1-2a_sB_1(t+1)}\right)h_s(t+1)\right\}.$$

Similarly, one gets that

$$\mathbb{E}_t[\exp\{\Psi_v\}]=\exp\left\{-\frac{1}{2}\ln(1-2a_vB_2(t+1))\right.$$
$$\left.+\left(\phi_2(c_v+\lambda_v)-\frac{1}{2}(c_v+\lambda_v)^2+\frac{\frac{1}{2}(\phi_2-(c_v+\lambda_v))^2}{1-2a_vB_2(t+1)}\right)h_v(t+1)\right\},$$
$$\mathbb{E}_t[\exp\{\Psi_m\}]=\exp\left\{-\frac{1}{2}\ln(1-2a_mB_3(t+1))\right.$$
$$+\left((\phi_1\beta_s+\phi_2\beta_v)(c_m+\lambda_m)-\frac{1}{2}(c_m+\lambda_m)^2\right.$$
$$\left.\left.+\frac{\frac{1}{2}(\phi_1\beta_s+\phi_2\beta_v-(c_m+\lambda_m))^2}{1-2a_mB_3(t+1)}\right)h_m(t+1)\right\}.$$

Hence, $A(t)$, $B_1(t)$, $B_2(t)$, and $B_3(t)$ are given by

$$A(t)=(\phi_1+\phi_2)r+w_sB_1(t+1)+w_vB_2(t+1)+w_mB_3(t+1)+A(t+1)-\frac{1}{2}\ln(1-2a_sB_1(t+1))$$
$$-\frac{1}{2}\ln(1-2a_vB_2(t+1))-\frac{1}{2}\ln(1-2a_mB_3(t+1)),$$
$$B_1(t)=b_sB_1(t+1)-\frac{1}{2}\phi_1+\phi_1(c_s+\lambda_s)-\frac{1}{2}(c_s+\lambda_s)^2+\frac{\frac{1}{2}(\phi_1-(c_s+\lambda_s))^2}{1-2a_sB_1(t+1)},$$
$$B_2(t)=b_vB_2(t+1)-\frac{1}{2}\phi_2+\phi_2(c_v+\lambda_v)-\frac{1}{2}(c_v+\lambda_v)^2+\frac{\frac{1}{2}(\phi_2-(c_v+\lambda_v))^2}{1-2a_vB_2(t+1)},$$

$$B_3(t) = b_m B_3(t+1) - \frac{1}{2}\phi_1\beta_s^2 - \frac{1}{2}\phi_2\beta_v^2 + (\phi_1\beta_s + \phi_2\beta_v)(c_m + \lambda_m) - \frac{1}{2}(c_m + \lambda_m)^2 + \frac{\frac{1}{2}(\phi_1\beta_s + \phi_2\beta_v - (c_m + \lambda_m))^2}{1 - 2a_m B_3(t+1)}.$$

Up to now, we have obtained

$$f(t;T,\phi_1,\phi_2) = \exp\Big\{\phi_1 x(t) + \phi_2 y(t) + A(t) + B_1(t)h_s(t+1) + B_2(t)h_v(t+1) + B_3(t)h_m(t+1)\Big\}.$$

These coefficients can be obtained recursively using the terminal conditions,

$$A(T) = B_1(T) = B_2(T) = B_3(T) = 0.$$

■

Proof of Proposition 2.4: Recall the moment generating function of $x(T)$ and $y(T)$ in Proposition 2.3, with the notations $x(t) = \ln S(t)$ and $y(t) = \ln V(t)$,

$$f(t;T,\phi_1,\phi_2) = \mathbb{E}_t\Big[e^{\phi_1 x(T) + \phi_2 y(T)}\Big].$$

In the following, we derive vulnerable option prices using the characteristic function $f(t;T,i\phi_1,i\phi_2)$. Given the information at t, denote the joint density function of $x(T) = \ln S(T)$ and $y(T) = \ln V(T)$ by $g(x,y)$, and rewrite the form of option prices C^* in (2.12) as follows:

$$\begin{aligned}
C^* &= e^{-r(T-t)}\mathbb{E}_t^Q\Big[(S(T)-K)^+\Big(1(V(T)\ge D) + \frac{(1-\alpha)V(T)}{D}1(V(T)<D)\Big)\Big]\\
&= e^{-r(T-t)}\mathbb{E}_t^Q\Big[(e^{x(T)}-K)^+\Big(1(y(T)\ge \ln D) + \frac{1-\alpha}{D}e^{y(T)}1(y(T)<\ln D)\Big)\Big]\\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^x-K)^+\Big(1(y\ge \ln D) + \frac{1-\alpha}{D}e^{y}1(y<\ln D)\Big)g(x,y)\mathrm{d}y\mathrm{d}x\\
&= e^{-r(T-t)}\int_{\ln K}^{\infty}\int_{-\infty}^{\infty}(e^x-K)\Big(1(y\ge \ln D) + \frac{1-\alpha}{D}e^{y}1(y<\ln D)\Big)\Big]g(x,y)\mathrm{d}y\mathrm{d}x\\
&:= e^{-r(T-t)}\Big[A_1 + A_2 + A_3 + A_4\Big],
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \int_{\ln K}^{\infty}\int_{\ln D}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x,\\
A_2 &= -K\int_{\ln K}^{\infty}\int_{\ln D}^{\infty} g(x,y)\mathrm{d}y\mathrm{d}x,\\
A_3 &= \frac{1-\alpha}{D}\int_{\ln K}^{\infty}\int_{-\infty}^{\ln D} e^{x+y} g(x,y)\mathrm{d}y\mathrm{d}x,\\
A_4 &= -\frac{1-\alpha}{D}K\int_{\ln K}^{\infty}\int_{-\infty}^{\ln D} e^{y} g(x,y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Now we calculate A_1–A_4 respectively, which then give us the closed-form solution for vulnerable options. First, let us focus on the term A_1. Rewrite it as follows:

$$\begin{aligned}
A_1 &= \int_{\ln K}^{\infty}\int_{\ln D}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x,\\
&= \frac{\int_{\ln K}^{\infty}\int_{\ln D}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x\\
&= \frac{\int_{\ln K}^{\infty}\int_{\ln D}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x} f(t;T,1,0)\\
&:= \Pi_1(t;T) f(t;T,1,0),
\end{aligned}$$

where $\Pi_1(t;T) \in [0,1]$. Define a new probability measure Q_1 by the following Radon-Nikodym derivative:

$$\frac{\mathrm{d}Q_1}{\mathrm{d}Q} = \frac{e^{x(T)}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x},$$

and the characteristic function of $x(T)$ and $y(T)$ under Q_1 is given by

$$\begin{aligned} f_1(t;T,i\phi_1,i\phi_2) &= \frac{1}{f(t;T,1,0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x \times e^{i\phi_1 x + i\phi_2 y} g(x,y)\mathrm{d}y\mathrm{d}x \\ &= \frac{f(t;T,i\phi_1+1,\phi_2)}{f(t;T,1,0)}, \end{aligned}$$

where we have used the definition of $f(t;T,\phi_1,\phi_2)$. Utilizing the Radon–Nikodym derivative implies

$$\begin{aligned} \Pi_1(t;T) &= \frac{\int_{\ln K}^{\infty}\int_{\ln D}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x} \\ &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)1(x \geq \ln K, y \geq \ln D)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x g(x,y)\mathrm{d}y\mathrm{d}x} \\ &= Q_1(x(T) \geq \ln K, y(T) \geq \ln D). \end{aligned}$$

From standard probability theory (see, e.g., Kendall and Stuart [19] and Shephard [27]), one obtains that the distribution function $F_1(x(T),y(T);x,y)$ corresponding to the joint characteristic function $f_1(t;T,i\phi_1,i\phi_2)$ is given by

$$\begin{aligned} F_1(x(T),y(T);x,y) = &-\frac{1}{4} + \frac{1}{2}F_1(x(T);x) + \frac{1}{2}F_1(y(T);y) \\ &- \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty}\left(\mathrm{Re}\left[\frac{e^{-i\phi_1 x - i\phi_2 y} f_1(t;T,i\phi_1,i\phi_2)}{\phi_1\phi_2}\right]\right. \\ &\left. - \mathrm{Re}\left[\frac{e^{-i\phi_1 x + i\phi_2 y} f_1(t;T,i\phi_1,-i\phi_2)}{\phi_1\phi_2}\right]\right)\mathrm{d}\phi_1\mathrm{d}\phi_2, \end{aligned}$$

where Re[ ] denotes the real part of a complex number, $F_1(x(T);x)$ and $F_1(y(T);y)$ are the marginal distributions for $x(T)$ and $y(T)$,

$$F_1(x(T);x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_1 x} f_1(t;T,i\phi_1,0)}{i\phi_1}\Big]\mathrm{d}\phi_1,$$

and

$$F_1(y(T);y) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_2 x} f_1(t;T,0,i\phi_2)}{i\phi_2}\Big]\mathrm{d}\phi_2.$$

Hence, we have that

$$\begin{aligned} \Pi_1(t;T) &= Q_1(x(T) \geq \ln K, y(T) \geq \ln D) \\ &= 1 - F_1(x(T);\ln K) - F_1(y(T);\ln D) + F_1(x(T),y(T);\ln K,\ln D), \end{aligned}$$

that is,

$$\begin{aligned} \Pi_1(t;T) = &\frac{1}{4} + \frac{1}{2\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K} f_1(t;T,i\phi_1,0)}{i\phi_1}\Big]\mathrm{d}\phi_1 \\ &+ \frac{1}{2\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_2\ln D} f_1(t;T,0,i\phi_2)}{i\phi_2}\Big]\mathrm{d}\phi_2 \\ &- \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty}\Big(\mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K - i\phi_2\ln D} f_1(t;T,i\phi_1,i\phi_2)}{\phi_1\phi_2}\Big] \\ &- \mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K + i\phi_2\ln D} f_1(t;T,i\phi_1,-i\phi_2)}{\phi_1\phi_2}\Big]\Big)\mathrm{d}\phi_1\mathrm{d}\phi_2. \end{aligned} \quad \textbf{(A.7)}$$

As for A_2, we can rewrite it as follows:

$$A_2 = -K\int_{\ln K}^{\infty}\int_{\ln D}^{\infty} g(x,y)\mathrm{d}y\mathrm{d}x, \\ = -KQ(x(T)\geq \ln K, y(T)\geq \ln D).$$

Since we have obtained the characteristic function $f(t;T,i\phi_1,i\phi_2)$ of $x(T)$ and $y(T)$ under Q, the corresponding distribution function $F(x(T),y(T);x,y)$ can be derived and hence it holds that

$$\begin{aligned}\Pi_2(t;T) &:= Q(x(T)\geq \ln K, y(T)\geq \ln D)\\ &= 1 - F(x(T);\ln K) - F(y(T);\ln D) + F(x(T),y(T);\ln K,\ln D)\\ &= \frac{1}{4} + \frac{1}{2\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K}f(t;T,i\phi_1,0)}{i\phi_1}\Big]\mathrm{d}\phi_1\\ &\quad + \frac{1}{2\pi}\int_0^{\infty}\mathrm{Re}\Big[\frac{e^{-i\phi_2\ln D}f(t;T,0,i\phi_2)}{i\phi_2}\Big]\mathrm{d}\phi_2\\ &\quad - \frac{1}{2\pi^2}\int_0^{\infty}\int_0^{\infty}\Big(\mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K - i\phi_2\ln D}f(t;T,i\phi_1,i\phi_2)}{\phi_1\phi_2}\Big]\\ &\quad - \mathrm{Re}\Big[\frac{e^{-i\phi_1\ln K + i\phi_2\ln D}f(t;T,i\phi_1,-i\phi_2)}{\phi_1\phi_2}\Big]\Big)\mathrm{d}\phi_1\mathrm{d}\phi_2. \end{aligned} \tag{A.8}$$

Similarly, we can deal with A_3.

$$\begin{aligned}A_3 &= \frac{1-\alpha}{D}\int_{\ln K}^{\infty}\int_{-\infty}^{\ln D} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x\\ &= \frac{1-\alpha}{D}\frac{\int_{\ln K}^{\infty}\int_{-\infty}^{\ln D} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x}f(t;T,1,1)\\ &:= \frac{1-\alpha}{D}\Pi_3(t;T)f(t;T,1,1)\\ &= \frac{1-\alpha}{D}f(t;T,1,1)Q_3(x(T)\geq \ln K, -y(T)\geq -\ln D)\end{aligned}$$

where

$$\frac{\mathrm{d}Q_3}{\mathrm{d}Q} = \frac{e^{x(T)+y(T)}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x}.$$

Next, we derive the characteristic function $f_3(t;T,i\phi_1,i\phi_2)$ of $x(T)$ and $-y(T)$ under Q_3, that is,

$$\begin{aligned}f_3(t;T,i\phi_1,i\phi_2) &= \mathbb{E}_t^{Q_3}\left[e^{i\phi_1 x(T)+i\phi_2(-y(T))}\right]\\ &= \mathbb{E}_t^{Q}\left[\frac{e^{x(T)+y(T)}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x}e^{i\phi_1 x(T)+i\phi_2(-y(T))}\right]\\ &= \frac{1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y}g(x,y)\mathrm{d}y\mathrm{d}x}\mathbb{E}_t^{Q}\left[e^{(i\phi_1+1)x(T)+(1-i\phi_2)y(T)}\right]\\ &= \frac{f(t;T,i\phi_1+1,1-i\phi_2)}{f(t;T,1,1)}.\end{aligned}$$

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Denote by $F_3(x(T), -y(T); x, y)$ the corresponding distribution function of of $x(T)$ and $-y(T)$ under Q_3. Then one gets that

$$\begin{aligned}\Pi_3(t;T) &:= Q_3(x(T) \geq \ln K, -y(T) \geq -\ln D)\\ &= 1 - F_3(x(T); \ln K) - F_3(-y(T); -\ln D) + F_3(x(T), -y(T); \ln K, -\ln D)\\ &= \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f_3(t;T,i\phi_1,0)}{i\phi_1}\right]\mathrm{d}\phi_1\\ &\quad + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi_2 \ln D} f_3(t;T,0,i\phi_2)}{i\phi_2}\right]\mathrm{d}\phi_2\\ &\quad - \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty \left(\mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K + i\phi_2 \ln D} f_3(t;T,i\phi_1,i\phi_2)}{\phi_1\phi_2}\right]\right.\\ &\quad \left. -\mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K - i\phi_2 \ln D} f_3(t;T,i\phi_1,-i\phi_2)}{\phi_1\phi_2}\right]\right)\mathrm{d}\phi_1\mathrm{d}\phi_2. \qquad \textbf{(A.9)}\end{aligned}$$

Analogously, it holds that

$$\begin{aligned}A_4 &= -\frac{1-\alpha}{D}K\int_{\ln K}^\infty\int_{-\infty}^{\ln D} e^y g(x,y)\mathrm{d}y\mathrm{d}x\\ &= -\frac{1-\alpha}{D}K\frac{\int_{\ln K}^\infty\int_{-\infty}^{\ln D} e^y g(x,y)\mathrm{d}y\mathrm{d}x}{\int_{-\infty}^\infty\int_{-\infty}^{\infty} e^y g(x,y)\mathrm{d}y\mathrm{d}x} f(t;T,0,1)\\ &:= -\frac{1-\alpha}{D}K\Pi_4(t;T)f(t;T,0,1)\\ &= -\frac{1-\alpha}{D}Kf(t;T,0,1)Q_4(x(T) \geq \ln K, -y(T) \geq -\ln D),\end{aligned}$$

where

$$\frac{\mathrm{d}Q_4}{\mathrm{d}Q} = \frac{e^{y(T)}}{\int_{-\infty}^\infty\int_{-\infty}^{\infty} e^y g(x,y)\mathrm{d}y\mathrm{d}x}.$$

Therefore, the characteristic function $f_4(t;T,i\phi_1,i\phi_2)$ of $x(T)$ and $-y(T)$ under Q_4 is given by

$$\begin{aligned}f_4(t;T,i\phi_1,i\phi_2) &= \mathbb{E}_t^{Q_4}\left[e^{i\phi_1 x(T) + i\phi_2(-y(T))}\right]\\ &= \mathbb{E}_t^{Q}\left[\frac{e^{y(T)}}{\int_{-\infty}^\infty\int_{-\infty}^{\infty} e^y g(x,y)\mathrm{d}y\mathrm{d}x} e^{i\phi_1 x(T) + i\phi_2(-y(T))}\right]\\ &= \frac{1}{\int_{-\infty}^\infty\int_{-\infty}^{\infty} e^y g(x,y)\mathrm{d}y\mathrm{d}x}\mathbb{E}_t^{Q}\left[e^{\phi_1 x(T) + (1-i\phi_2)y(T)}\right]\\ &= \frac{f(t;T,i\phi_1,1-i\phi_2)}{f(t;T,0,1)},\end{aligned}$$

and one gets that

$$\begin{aligned}\Pi_4(t;T) &:= Q_4(x(T) \geq \ln K, -y(T) \geq -\ln D)\\ &= \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f_4(t;T,i\phi_1,0)}{i\phi_1}\right]\mathrm{d}\phi_1\\ &\quad + \frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi_2 \ln D} f_4(t;T,0,i\phi_2)}{i\phi_2}\right]\mathrm{d}\phi_2\\ &\quad - \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty \left(\mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K + i\phi_2 \ln D} f_4(t;T,i\phi_1,i\phi_2)}{\phi_1\phi_2}\right]\right.\\ &\quad \left. -\mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K - i\phi_2 \ln D} f_4(t;T,i\phi_1,-i\phi_2)}{\phi_1\phi_2}\right]\right)\mathrm{d}\phi_1\mathrm{d}\phi_2. \qquad \textbf{(A.10)}\end{aligned}$$

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Up to now, we have obtained the closed-form for vulnerable option prices C^* in (2.12) as follows:

$$\begin{aligned} C^* &= e^{-r(T-t)}\Big[A_1 + A_2 + A_3 + A_4\Big] \\ &= e^{-r(T-t)}\Big[f(t;T,1,0)\Pi_1(t;T) - K\Pi_2(t;T) \\ &\quad + \frac{1-\alpha}{D} f(t;T,1,1)\Pi_3(t;T) - \frac{1-\alpha}{D} K f(t;T,0,1)\Pi_4(t;T)\Big], \end{aligned}$$

where $\Pi_1(t;T)$, $\Pi_2(t;T)$, $\Pi_3(t;T)$, and $\Pi_4(t;T)$ are given in (A.7)–(A.10). ■

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