LOCAL COORDINATES FOR COMPLEX AND QUATERNIONIC HYPERBOLIC PAIRS

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Abstract

Let $G(n) = \mathrm{Sp}(n, 1)$ or $\mathrm{SU}(n, 1)$. We classify conjugation orbits of generic pairs of loxodromic elements in $G(n)$. Such pairs, called 'nonsingular', were introduced by Gongopadhyay and Parsad for $\mathrm{SU}(3, 1)$. We extend this notion and classify $G(n)$-conjugation orbits of such elements in arbitrary dimension. For $n = 3$, they give a subspace that can be parametrized using a set of coordinates whose local dimension equals the dimension of the underlying group. We further construct twist-bend parameters to glue such representations and obtain local parametrization for generic representations of the fundamental group of a closed (genus $g \geq 2$) oriented surface into $G(3)$.

1. Introduction

Let $\mathbb{F} = \mathbb{H}$ or $\mathbb{C}$, where $\mathbb{H}$ denotes the division ring of Hamilton's quaternions. Let $G(n)$, or simply G, denote the group $\mathrm{SU}(n, 1; \mathbb{F})$ that acts as the isometry group of the $\mathbb{F}$-hyperbolic space $\mathbf{H}_{\mathbb{F}}^n$. Usually we denote $\mathrm{SU}(n, 1; \mathbb{C}) = \mathrm{SU}(n, 1)$ and $\mathrm{SU}(n, 1; \mathbb{H}) = \mathrm{Sp}(n, 1)$. This paper concerns the problem of classifying G-conjugation orbits of loxodromic pairs in $G \times G$. The G-conjugation orbit space can be identified with the character variety or the deformation space $\mathfrak{X}(\mathrm{F}_2, G) = \mathrm{Hom}(\mathrm{F}_2, G)/G$, where G acts on $\mathrm{Hom}(\mathrm{F}_2, G)$ by inner automorphisms and $\mathrm{F}_2 = \langle x, y \rangle$ is the free group with generators x and y. In [GK2], we obtained a local parametrization of a representation $\rho : \mathrm{F}_2 \to \mathrm{Sp}(n, 1)$, where both $\rho(x)$ and $\rho(y)$ are semisimple. When $G = \mathrm{SU}(n, 1)$, for loxodromic pairs such a local parametrization is available from the work [GP18B]. A main idea used in these works was to project fixed points of a pair of loxodromic

elements onto the moduli space of G-congruence classes of an ordered tuple of points on $\partial \mathbf{H}_{\mathbb{F}}^n$. Counting eigenvalues without multiplicities, a loxodromic element of G has precisely two null eigenspaces and $n-1$ lines spanned by eigenvectors of positive norm. In [GK2, GP18B], the $n-1$ lines spanned by these positive-definite eigenvectors were projected to the boundary $\partial \mathbf{H}_{\mathbb{F}}^n$. This associated tuple of points on $\partial \mathbf{H}_{\mathbb{F}}^n$ along with the spectrum data essentially classified the pair. The difficulty to generalize the work from the complex hyperbolic isometries to the quaternionic hyperbolic setup arose due to the fact that the eigenvalues of an element in $\mathrm{Sp}(n,1)$ are not uniquely defined, but they appear in similarity classes. So, the conjugacy invariants available in $\mathrm{Sp}(n,1)$ are not well behaved unlike their complex counterpart. We avoided this difficulty by associating a combination of spatial and numerical invariants to obtain the local parametrizations in [GK2, GK1].

Following the classical construction of the Fenchel–Nielsen coordinates on the Teichmüller space, especially for the loxodromic representations in low dimensions, one may like to have the local (real) dimension (or the 'degrees of freedom') of the coordinates to add up to the dimension of $\mathfrak{X}(\mathrm{F}_2, G)$, which is the same as the (real) dimension of the Lie group G. We call such a parameter system as being of 'Fenchel–Nielsen type'. The coordinate systems obtained in [GK2, GP18B], however, do not add up to the dimension of the underlying group even for $n=3$. In general, it is unlikely that a Fenchel–Nielsen-type parameter system can be obtained for arbitrary pairs as shown in [GL17]. However, it may be possible to associate Fenchel–Nielsen-type coordinates (at least locally) to special subsets of the character variety. Parker and Platis obtained such a parameter system for irreducible loxodromic representations $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(2,1))$. In [GK1], we obtained Fenchel–Nielsen-type coordinates for irreducible loxodromic representations in $\mathfrak{X}(\mathrm{F}_2, \mathrm{Sp}(2,1))$. For generic loxodromic representations in $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(3,1))$, called 'nonsingular', such a system of parameters is obtained from the work [GP18A]. In [GP18B, Section 7.2], a version of nonsingularity was defined for generic loxodromic pairs in $\mathrm{SU}(n,1)$. It was proved that such a pair projects to a unique point on the moduli space of $\mathrm{SU}(n,1)$-congruence classes of ordered tuples of boundary points.

In this paper, we extend the notion of nonsingular pairs to $\mathrm{SU}(n,1;\mathbb{F})$ and classify such pairs by associating a system of parameters. The associated numerical invariants are comparable to the complex cross ratios used in [CuG12]. These invariants are obtained directly from the spectrum data of the pairs. However, in the quaternionic setting, the quaternionic versions of the cross ratios are not enough to classify such pairs. A set of spatial parameters, called 'projective points', needs to be associated. When one fixes the numerical invariants, these spatial parameters come from the fiber over the space of the numerical invariants. This generalizes the parametrization obtained in [GK1, Corollary 1.5], though, unlike the $\mathrm{Sp}(2,1)$ case, we do not know the precise domains of the numerical invariants. Restricting the classification to $\mathrm{SU}(3,1;\mathbb{F})$, we obtain a Fenchel–Nielsen-type parameter system for generic loxodromic representations in $\mathfrak{X}(\mathrm{F}_2, \mathrm{SU}(3,1;\mathbb{F}))$. As an application, we obtain local parametrization for generic representations of a closed genus-g surface

group into $\mathrm{Sp}(3,1)$, where $g \geq 2$. This extends the work in [GP18A] over the quaternions.

Now we define the 'generic' representations which are investigated in this paper and describe the results obtained. Let $\mathbb{F}^{n,1}$ be the vector space $\mathbb{F}^{n+1}$ equipped with a nondegenerate Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n,1)$. Then $\mathbf{H}^n_{\mathbb{F}}$ is the projectivization of the set of vectors v such that $\langle v, v \rangle < 0$. The boundary $\partial \mathbf{H}^n_{\mathbb{F}}$ is the projectivization of the null vectors. The projection of a vector $\mathbf{v}$ is denoted by v on the projective space. A *k-dimensional totally geodesic subspace* of $\mathbf{H}^n_{\mathbb{F}}$, which is also called an $\mathbb{F}^k$-*plane*, is the projectivization of a copy of $\mathbb{F}^{k,1}$ in $\mathbb{F}^{n,1}$. An $\mathbb{F}^1$-plane is simply called an $\mathbb{F}$-*line*, and an $\mathbb{F}^{n-1}$-plane is simply called an $\mathbb{F}$-*hyperplane*. The boundary of an $\mathbb{F}^k$-plane is called an $\mathbb{F}^k$-*chain*. A point v on the projective space is *polar* to an $\mathbb{F}^{n-1}$-plane C if the lift of C in $\mathbb{F}^{n,1}$ is the orthogonal complement of $\mathbf{v}$. In particular, we must have $\langle \mathbf{v}, \mathbf{v} \rangle > 0$. The positive vector $\mathbf{v}$ is polar to an $\mathbb{F}^{n-1}$-chain L if L is the boundary of an $\mathbb{F}^{n-1}$-plane C that is polar to v.

An element A in G is called hyperbolic (or loxodromic) if it has exactly two fixed points on $\partial \mathbf{H}^n_{\mathbb{F}}$. Such an A has two eigenvalue classes represented by $re^{i\theta}$, $r^{-1}e^{i\theta}$, $r < 1$, $\theta \in [-\pi, \pi]$, and the rest of the $n-1$ classes are represented by $e^{i\phi_1}, \ldots, e^{i\phi_{n-1}}$, $\phi_i \in [-\pi, \pi]$. An element A in G is *regular* if the eigenvalue classes are mutually disjoint.

Let A be a regular hyperbolic element. We denote by $\mathbf{a}_A$ and $\mathbf{r}_A$ the null eigenvectors of A corresponding to the classes $re^{i\theta}$ and $r^{-1}e^{i\theta}$, respectively. Let $\mathbf{x}_{j,A}$, $1 \leq j \leq n-1$, be the eigenvector to $e^{i\phi_j}$. The eigenvector $\mathbf{x}_{j,A}$ is positive-definite, that is, $\langle \mathbf{x}_{j,A}, \mathbf{x}_{j,A} \rangle > 0$ for $1 \leq j \leq n-1$. Note that A fixes $x_{j,A}$ on $\mathbb{FP}^n$. For a hyperbolic (or loxodromic) element A in $\mathrm{SU}(n,1)$, the characteristic polynomial determines the conjugacy class, and the traces $\mathrm{tr}(A^j)$, $1 \leq j \leq [(n+1)/2]$, determine the coefficients of the characteristic polynomial. For $A \in \mathrm{Sp}(n,1)$, there is a natural complex representation $A_{\mathbb{C}}$ of A in $\mathrm{GL}(2(n+1), \mathbb{C})$. The tuple of the coefficients of the characteristic polynomial of $A_{\mathbb{C}}$ gives the *real trace* of A, denoted by $\mathrm{tr}_{\mathbb{R}}(A)$.

In this paper we use the following definition.

DEFINITION 1.1. An element $A \in \mathrm{Sp}(n,1)$ is *loxodromic* if it is hyperbolic and has no real eigenvalue.

For a loxodromic element A in $\mathrm{Sp}(n,1)$, the real trace $\mathrm{tr}_{\mathbb{R}}(A)$ is an element of $\mathbb{R}^{n+1}$. Marché and Will in [MW12] have used flags in $\mathbf{H}^2_{\mathbb{C}} \cup \partial \mathbf{H}^2_{\mathbb{C}}$ to give a set of local coordinates to generic elements on the $\mathrm{PU}(2,1)$ character variety of the fundamental group of a punctured oriented surface. Taking motivation from their work, we use certain flags to define the generic pairs that we investigate in this paper.

DEFINITION 1.2. A *flag* is a triple (p, C, Π), where p is a point on $\Pi \cap \partial \mathbf{H}^n_{\mathbb{F}}$, C is an $\mathbb{F}$-line containing p on the boundary of C, Π is an $\mathbb{F}$-hyperplane and $C \subset \Pi$.

Thus, a positive point x on $\mathbb{FP}^n$ along with a boundary point p and an $\mathbb{F}$-line C define a flag.

DEFINITION 1.3. Given a loxodromic element A, we associate canonical flags to A given by $F_{j,A} = (a_A, L_A, W_{j,A})$, $1 \le j \le n-1$, where L_A is the line joining a_A and r_A, and $W_{j,A}$ is the projectivization of $\mathbf{x}_{j,A}^\perp$.

DEFINITION 1.4. Two flags (p, C, Π) and (p', C', Π') are said to form a *generic pair* if the following holds.

(i) p does not belong to the boundary of C' and p' does not belong to the boundary of C.

(ii) ∂C is disjoint from $\partial \Pi'$ and $\partial C'$ is disjoint from $\partial \Pi$.

DEFINITION 1.5. Let A, B be two loxodromic elements in $\mathrm{SU}(n, 1; \mathbb{F})$. The pair (A, B) is called *weakly nonsingular* if:

1. A and B do not have a common fixed point;
2.   the elements A and B are regular;
3. $n-2$ of the canonical flags of A form generic pairs with $n-2$ of the canonical flags of B.

DEFINITION 1.6. A pair (A, B) of loxodromic elements in $\mathrm{SU}(n, 1; \mathbb{F})$ is called *nonsingular* if it is weakly nonsingular and the null fixed points of A and B do not belong to the boundary of the same proper totally geodesic hyperplane. We note that the last condition of nonsingularity implies that (A, B) is necessarily irreducible, that is, $\langle A, B \rangle$ neither fixes a point nor preserves a proper $\mathbb{F}^k$-plane.

The above definition generalizes the 'nonsingular' pairs defined in [GP18A]. The terminology 'nonsingularity' in [GP18A] was motivated from the property that the mixed cross ratios were nonzero for such a pair. Similar considerations are implicit in the above definition as well.

Corresponding to the boundary fixed points of (A, B), we already have the conjugacy invariants given by the cross ratios and the angular invariants. We recall here that for four distinct points z_1, z_2, z_3 and z_4 in $\partial \mathbf{H}_{\mathbb{F}}^n$, the *usual cross ratio* is defined by

$$\mathbb{X}(z_1, z_2, z_3, z_4) = \langle \mathbf{z_3}, \mathbf{z_1} \rangle \langle \mathbf{z_3}, \mathbf{z_2} \rangle^{-1} \langle \mathbf{z_4}, \mathbf{z_2} \rangle \langle \mathbf{z_4}, \mathbf{z_1} \rangle^{-1}, \tag{1-1}$$

where $\mathbf{z_i}$ is a lift of z_i in $\mathbb{F}^{n,1}$. These cross ratios were introduced by Korányi and Reimann for points on $\partial \mathbf{H}_{\mathbb{C}}^n$ in [KR87]; also see [Gol99]. Platis has investigated quaternionic versions of these cross ratios in [Pla14]. The complex cross ratios are independent of the chosen lifts of z_i and are conjugacy invariants. However, the quaternionic cross ratios are not independent of the chosen lifts of the points; therefore, they are not well-defined conjugacy invariants. But similarity classes of the cross ratios are independent of the chosen lifts. Accordingly, $\mathfrak{R}(\mathbb{X})$ and $|\mathbb{X}|$ are the conjugacy invariants associated to the quaternionic cross ratios. Also, unlike the complex case, quaternionic cross ratios do not classify a quadruple of boundary points up to $\mathrm{Sp}(n, 1)$-congruence.

It can be seen that modulo the symmetric group action on the four boundary fixed points of (A, B), only three such cross ratios are needed to determine the others under the permutation. We denote these cross ratios by

$$\mathbb{X}_1(A, B) = \mathbb{X}(a_A, r_A, a_B, r_B), \ \mathbb{X}_2(A, B) = \mathbb{X}(a_A, r_B, a_B, r_A), \ \mathbb{X}_3(A, B) = \mathbb{X}(r_A, r_B, a_B, a_A).$$

Platis proved in [Pla14] that for $n \geq 3$, the set of cross ratios $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ of a quadruple of points on $\partial \mathbf{H}_{\mathbb{F}}^n$ forms a five-dimensional semi-algebraic subset of $\mathbb{R}^5$.

In the quaternionic setup, Cartan's angular invariant associated to a triple (z_1, z_2, z_3) on $\mathbf{H}_{\mathbb{H}}^n \cup \partial \mathbf{H}_{\mathbb{H}}^n$ is given by the following expression, see [AK07, Cao16],

$$\mathbb{A}(z_1, \ z_2, \ z_3) = \arccos \frac{\mathfrak{R}(-\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle)}{|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle|}, \tag{1-2}$$

where $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 \rangle = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \langle \mathbf{z}_2, \mathbf{z}_3 \rangle \langle \mathbf{z}_3, \mathbf{z}_1 \rangle$. The quaternionic angular invariants are independent of the chosen lifts of z_i and are conjugacy invariants. So, there are angular invariants that correspond to the quadruple of the boundary fixed points. We denote these angular invariants by

$$\mathbb{A}_1(A, B) = \mathbb{A}(a_A, r_A, a_B), \ \mathbb{A}_2(A, B) = \mathbb{A}(a_A, r_A, r_B), \ \mathbb{A}_3(A, B) = \mathbb{A}(r_A, a_B, r_B).$$

In [Cao16], Cao proved that an ordered quadruple of points on $\partial \mathbf{H}_{\mathbb{H}}^n$ is determined up to $\mathrm{Sp}(n, 1)$-congruence by the similarity classes of the cross ratios and the above angular invariants.

In order to classify a weakly nonsingular pair (A, B), we would require more invariants. For this, we extend the above definition of the cross ratio by taking one (or more) of the points z_i to be points on $\mathbb{FP}^n$ corresponding to the positive-definite eigenvectors of A and B. We call such invariants *generalized cross ratios*. We also define generalized Goldman's eta invariants that correspond to two boundary points and a hyperplane; see [Gol99, Section 7.3.1]. The set of numerical invariants considered here comes from the Gram matrix associated to the pair (A, B). For (A, B) in $\mathrm{Sp}(n, 1)$, it is the similarity classes of these numerical quantities which are conjugacy invariants. So, the real parts and the moduli of the quantities are the conjugacy invariants associated to the $\mathrm{Sp}(n, 1)$ conjugation orbit of (A, B). However, these numerical invariants do not classify the pair (A, B) completely. Rather, there is a whole fiber of points that corresponds to a fixed tuple of numerical invariants. These fibered elements correspond to the product of copies of $\mathbb{CP}^1$ that we call *projective points* of (A, B). Each of these $\mathbb{CP}^1$ represents an eigenspace of A or B, and a point on the given $\mathbb{CP}^1$ corresponds to an 'eigenset'. We note here that corresponding to a regular loxodromic element, there are n projective points, one each for the $n - 1$ space-like eigenvectors and one for the null eigenvectors. With these terminologies, we have the following theorem, where we refer to Section 4.2 for the precise list of the numerical invariants mentioned here.

THEOREM 1.7. *Let* $\rho : \mathrm{F}_2 \to \mathrm{Sp}(n, 1)$ *be a representation such that* $(\rho(x), \rho(y))$ *is weakly nonsingular. Then* ρ *is determined uniquely in the character variety by*

$\text{tr}_{\mathbb{R}}(\rho(x))$, $\text{tr}_{\mathbb{R}}(\rho(y))$, *the angular invariant* $\mathbb{A}(a_{\rho(x)}, r_{\rho(x)}, a_{\rho(y)})$, *the projective points and the* $\text{Sp}(1)$-*conjugation orbit of the (unordered) tuple consisting of the usual cross ratios, the generalized cross ratios and Goldman's eta invariants.*

The following theorem follows by restricting the proof of the above theorem over complex numbers.

THEOREM 1.8. *Let* $\rho : \text{F}_2 \to \text{SU}(n, 1)$ *be a representation such that* $(\rho(x), \rho(y))$ *is weakly nonsingular. Then* ρ *is determined uniquely in the character variety by* $\text{tr}(\rho(x)^j)$, $\text{tr}(\rho(y)^j)$, $1 \le j \le [(n + 1)/2]$, *the angular invariant* $\mathbb{A}(a_{\rho(x)}, r_{\rho(x)}, a_{\rho(y)})$, *the usual cross ratios, the generalized cross ratios and Goldman's eta invariants.*

Theorem 1.8 is implicit in the work [GP18B] and the above statement was noted in an older version: arXiv: 1705.10469v2.

However, the degrees of freedom of the parameters in the above classification do not add up to the dimension of the group even in lower dimensions. We would like to further obtain a smaller subfamily of invariants that might be sufficient for the classification. First, we consider the group $\text{SU}(n, 1)$. In the following, we use a method that is similar to the one used in [GP18A]. We only need the following generalized cross ratios to classify a nonsingular pair. For $1 \le k \le n - 2$, let

$$\alpha_k(A, B) = \mathbb{X}(a_A, r_A, a_B, x_{k,B}), \quad \beta_k(A, B) = \mathbb{X}(a_B, r_B, a_A, x_{k,A}).$$

By the definition of nonsingularity, the above quantities are nonzero and well defined. In the case of $\text{SU}(n, 1)$, Cunha and Gusevskii proved in [CuG10] that the moduli space of an ordered quadruple of points (p_1, p_2, p_3, p_4) on $\partial\mathbf{H}_{\mathbb{C}}^n$ is determined by a point on a five-dimensional subspace of $\mathbb{R}^5$ that consists of the points $(\mathbb{A}(p_1, p_2, p_3), \mathbb{X}_1(p_1, p_2, p_3, p_4), \mathbb{X}(p_1, p_4, p_3, p_2))$ satisfying some semi-algebraic equation. We use a point on this 'Cunha–Gusevskii variety'. We have the following result in this set up that generalizes [GP18A, Theorem 1.1].

THEOREM 1.9. *Let* (A, B) *be a nonsingular pair in* $\text{SU}(n, 1)$. *Then the* $\text{SU}(n, 1)$ *conjugation orbit of* (A, B) *is uniquely determined by the following parameters:*

- $\text{tr}(A^j)$, $\text{tr}(B^j)$, $1 \le j \le [(n + 1)/2]$;
- *the cross ratios* $\mathbb{X}_k(A, B)$, $k = 1, 2$;
- *the angular invariant* $\mathbb{A}(a_A, r_A, a_B)$;
- *the* α-*invariants* $\alpha_k(A, B)$ *and the* β-*invariants* $\beta_k(A, B)$, $1 \le k \le n - 2$.

Restating the above theorem in terms of representations, we have the following result.

THEOREM 1.10. *Let* $\rho : \text{F}_2 \to \text{SU}(n, 1)$ *be a representation such that* $(\rho(x), \rho(y))$ *is nonsingular. Then the point* ρ *in* $\mathfrak{X}(\text{F}_2, \text{SU}(n, 1))$ *is uniquely determined by the following parameters:*

- $\text{tr}(\rho(x)^j)$, $\text{tr}(\rho(y)^j)$, $1 \le j \le [(n + 1)/2]$;

- *the cross ratios $\mathbb{X}_k(\rho(x), \rho(y))$, $k = 1, 2$;*
- *the angular invariant $\mathbb{A}(a_{\rho(x)}, r_{\rho(x)}, a_{\rho(y)})$;*
- *the α-invariants $\alpha_k(\rho(x), \rho(y))$ and the β-invariants $\beta_k(\rho(x), \rho(y))$, $1 \leq k \leq n - 2$.*

Thus, the local dimension of the coordinates adds up to at most $6n - 1$: for the traces at most $n + 1$ contributing at most $2n + 2$; for the point on the cross ratio variety 5; for the α- and β-invariants $4(n - 2) = 2 \times (n - 2)$-$\alpha$-invariants $+ 2 \times (n - 2)$-β-invariants; the total adds up to $2n + 2 + 5 + 4(n - 2) = 6n - 1$.

A particularly interesting case appears when $n = 3$. In this case, the traces of loxodromics form a real three-dimensional family and the above parameters add up to 15, the dimension of SU(3, 1).

COROLLARY 1.11 [GP18A, Theorem 1.1]. *Let $\rho : F_2 \to SU(3, 1)$ be a representation such that $(\rho(x), \rho(y))$ is nonsingular. Then the point ρ in $\mathfrak{X}(F_2, SU(n, 1))$ is uniquely determined by the following 15-dimensional parameter system:*

- $\mathrm{tr}(\rho(x))$, $\mathrm{tr}(\rho(y))$*;*
- $\sigma(\rho(x))$, $\sigma(\rho(y))$*;*
- $\mathbb{X}_k(\rho(x), \rho(y))$, $k = 1, 2, 3$*;*
- $\alpha_1(\rho(x), \rho(y))$, $\beta_1(\rho(x), \rho(y))$,

where for an element $g \in SU(3, 1)$, $\sigma(g) = (\mathrm{tr}^2(g) - \mathrm{tr}(g^2))/2$.

However, for $n \geq 4$, the local dimension of the above parameter system is less than the dimension of the underlying group. With larger n, the upper bound $6n - 1$ of the dimension of the parameter system becomes smaller in comparison to the dimension of SU(n, 1), which is $n^2 + 2n$.

Now we consider the quaternionic case. An advantage of Theorem 1.7 is that the numerical invariants used there do not depend on the choices of the lifts of points of $\mathbb{HP}^n$ to $\mathbb{H}^{n,1}$, and they serve as well-defined conjugacy invariants. But the similarity classes of $\alpha_k(A, B)$ and $\beta_k(A, B)$ do not determine the Gram matrix of (A, B) uniquely. This calls for some adjustment in the choices of the invariants. One way to avoid this difficulty is to adopt the convention of fixing a frame of reference. We adopt the convention of fixing the lift of the attracting fixed points. We take the standard lift, see Section 2, of the attracting fixed point of A in the pair (A, B). After this restriction, the numerical quantities $\alpha_k(A, B)$ and $\beta_k(A, B)$ are well-defined invariants, and the usual cross ratios are uniquely assigned to (A, B). A comparable convention of fixing a frame of reference was used by Gou and Jiang in [GJ17] in their understanding of the moduli space of ordered quadruples on $\partial \mathbf{H}_{\mathbb{H}}^n$. In view of the chosen frame of reference, we have the following result.

THEOREM 1.12. *Let $\rho : F_2 \to Sp(n, 1)$ be a representation such that $(\rho(x), \rho(y))$ is nonsingular. We adopt the convention of taking the standard lift of the fixed point $a_{\rho(x)}$. Then the point ρ in $\mathfrak{X}(F_2, Sp(n, 1))$ is determined by the following parameters:*

- $\mathrm{tr}_{\mathbb{R}}(\rho(x))$, $\mathrm{tr}_{\mathbb{R}}(\rho(y))$*;*

- *the angular invariants* $\mathbb{A}_k(\rho(x), \rho(y))$;
- *the usual cross ratios* $\mathbb{X}_k(\rho(x), \rho(y))$, *k = 1, 2, 3;*
- *the α-invariants* $\alpha_k(\rho(x), \rho(y))$ *and the β-invariants* $\beta_k(\rho(x), \rho(y))$, $1 \le k \le n - 2$;
- *the projective points* $(p_1(\rho(x)), \ldots, p_n(\rho(x))), (p_1(\rho(y)), \ldots, p_n(\rho(y)))$.

The degrees of freedom of the above set of coordinates add up to at most $14n - 6$ (for each real trace $n + 1$, contributing $2 \times (n + 1) = 2(n + 1)$; for the point on the cross ratio variety 5; for three angular invariants 3; for the projective points $4n = 2 \times (2n$ projective points); for the α- and β-invariants $8(n - 2) = 2 \times 4(n - 2)$). For $n = 3$, the degrees of freedom add up to 36, which is the dimension of Sp(3, 1).

COROLLARY 1.13. *Let $\rho : F_2 \to \mathrm{Sp}(3, 1)$ be a representation such that $(\rho(x), \rho(y))$ is nonsingular. Then the point ρ in $\mathfrak{X}(F_2, \mathrm{Sp}(3, 1))$ is determined by the following parameters:*

- $\mathrm{tr}_{\mathbb{R}}(\rho(x))$, $\mathrm{tr}_{\mathbb{R}}(\rho(y))$, *k = 1, 2, 3;*
- *the angular invariants* $\mathbb{A}_k(\rho(x), \rho(y))$;
- *the usual cross ratios* $\mathbb{X}_k(\rho(x), \rho(y))$;
- $\alpha_1(\rho(x), \rho(y))$, $\beta_1(\rho(x), \rho(y))$;
- *the projective points* $(p_1(\rho(x)), p_2(\rho(x)), p_3(\rho(x))), (p_1(\rho(y)), p_2(\rho(y)), p_3(\rho(y)))$.

This motivates us to construct a gluing process to glue such a representation and associate coordinates to generic surface group representations into Sp(3, 1). Let Σ_g denote a closed, connected, orientable surface of genus $g \ge 2$. Let $\pi_1(\Sigma_g)$ denote the fundamental group of Σ_g. Choose $C = \{\gamma_j\}$, $j = 1, 2, \ldots, 3g - 3$, a maximal family of simple closed curves on Σ_g such that no γ_j is homotopically trivial and no two are homotopically equivalent. The homotopy types of the curves may be considered to be elements of $\pi_1(\Sigma_g)$. We also assume that g of the curves γ_j correspond to two boundary components of the same three-holed sphere. Consider discrete, faithful representations $\rho : \pi_1(\Sigma_g) \to \mathrm{SU}(3, 1; \mathbb{F})$ such that the $3g - 3$ group elements $\rho(\gamma_j)$ are loxodromic and each of the groups $\langle \rho(\gamma_k), \rho(\gamma_l) \rangle$ obtained from the given decomposition is nonsingular. We call such a representation *nonsingular*. We construct 'twist-bend' parameters to glue such a representation. Complex hyperbolic twist-bends for representations into SU(3, 1) were constructed in [GP18A]. However, the method in [GP18A] does not generalize to Sp(3, 1). Here, we generalize the approach used in [GK1] to construct the twist-bend parameters. We have noted the construction for representations into Sp(3, 1) for emphasizing the quaternionic hyperbolic case. The same method restricts to SU(3, 1) as well, thus providing an alternative approach to the construction of twist-bends in the complex hyperbolic case. Then, using standard arguments as in [PP08, GK2], we have the following result.

THEOREM 1.14. *Let Σ_g be a closed orientable surface of genus $g \ge 2$ with a simple curve system $C = \{\gamma_j\}$, $j = 1, 2, \ldots, 3g - 3$. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(3, 1)$ be a nonsingular representation of the surface group $\pi_1(\Sigma_g)$ into Sp(3, 1). There are $72g - 72$ real parameters that determine ρ in the character variety $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{Sp}(3, 1))/\mathrm{Sp}(3, 1)$.*

When considering the representations into $SU(3,1)$, we recover [GP18A, Theorem 1.3].

THEOREM 1.15. *For $g \geq 2$, let Σ_g be a closed orientable surface of genus g with a simple curve system $C = \{\gamma_j\}$, $j = 1, 2, \ldots, 3g - 3$. Let $\rho : \pi_1(\Sigma_g) \to SU(3,1)$ be a nonsingular representation of the surface group $\pi_1(\Sigma_g)$ into $SU(3,1)$. There are $30g - 30$ real parameters that determine ρ in the character variety $\mathrm{Hom}(\pi_1(\Sigma_g), SU(3,1))/SU(3,1)$.*

Structure of the paper. In Section 2, we briefly recall basic notions and notation. We follow similar notation as in our earlier papers [GK1, GK2]. We recall and re-interpret the projective points in Section 3. In Section 4, we prove Theorem 1.7. In Section 5, we prove Theorem 1.9 and Theorem 1.12. The twist-bend parameters are constructed in Section 6 and a sketch of the proof of Theorem 1.15 is given in Section 6.

2. Preliminaries

2.1. Matrices over the quaternions. Let $\mathbb{V}$ be a right vector space over $\mathbb{H}$ and T be a right linear transformation of $\mathbb{V}$. After choosing a basis of $\mathbb{V}$, such a linear transformation can be represented with an $n \times n$ matrix M_T over $\mathbb{H}$, where $n = \dim \mathbb{V}$. The map T is invertible if and only if M_T is invertible. Suppose that $\lambda \in \mathbb{H}^*$ is a (right) eigenvalue of T. Let v be an eigenvector to λ. Note that for $\mu \in \mathbb{H}^*$,

$$T(v\mu) = T(v)\mu = (v\lambda)\mu = (v\mu)\mu^{-1}\lambda\mu.$$

Thus, the eigenvalues of T occur in similarity classes and, if v is a λ-eigenvector, then $v\mu \in v\mathbb{H}$ is a $\mu^{-1}\lambda\mu$-eigenvector. Thus, the eigenspace $v\mathbb{H}$ is not uniquely assigned to a single eigenvalue, but to the similarity class of λ. So, the similarity classes of eigenvalues are conjugacy invariants over the quaternions, and the notion of characteristic or minimal polynomial is not well defined. Each similarity class of eigenvalues contains a unique pair of complex-conjugate numbers. We choose one of these complex numbers, $re^{i\theta}$, $\theta \in [0, \pi]$, to be the representative of its similarity class. We may refer to a similarity class representative as 'the eigenvalue of T', though it should be understood that our reference is to the similarity class. At places where we need to distinguish between the similarity class and a representative, we denote the similarity class of an eigenvalue representative λ by $[\lambda]$.

2.2. The hyperbolic space. Let $\mathbb{F} = \mathbb{H}$ or $\mathbb{C}$. Let $\mathbb{V} = \mathbb{F}^{n,1}$ be the n-dimensional right vector space over $\mathbb{F}$ equipped with the Hermitian form of signature $(n, 1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z} = \bar{w}_{n+1} z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n + \bar{w}_1 z_{n+1},$$

where $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where I_{n-1} is the identity matrix of rank $n-1$. We consider the following subspaces of $\mathbb{H}^{n,1}$:

$$\mathbb{V}_- = \{\mathbf{z} \in \mathbb{F}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}, \ \mathbb{V}_+ = \{\mathbf{z} \in \mathbb{F}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\},$$

$$\mathbb{V}_0 = \{\mathbf{z} \in \mathbb{F}^{n,1} \setminus \{\mathbf{0}\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}.$$

A vector $\mathbf{z}$ in $\mathbb{F}^{n,1}$ is called *positive, negative* or *null* depending on whether $\mathbf{z}$ belongs to $\mathbb{V}_+$, $\mathbb{V}_-$ or $\mathbb{V}_0$. Let $\mathbb{P} : \mathbb{F}^{n,1} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{FP}^n$ be the right projection onto the quaternionic projective space. The image of a vector $\mathbf{z}$ is denoted by z. The quaternionic hyperbolic space $\mathbf{H}_\mathbb{F}^n$ is defined to be $\mathbb{P}(\mathbb{V}_-)$. The ideal boundary $\partial \mathbf{H}_\mathbb{F}^n$ is defined to be $\mathbb{P}(\mathbb{V}_0)$. So, we can write $\mathbf{H}_\mathbb{F}^n = \mathbb{P}(\mathbb{V}_-)$ as

$$\mathbf{H}_\mathbb{F}^n = \{(w_1, \ldots, w_n) \in \mathbb{H}^n : 2\mathfrak{R}(w_1) + |w_2|^2 + \cdots + |w_n|^2 < 0\},$$

where, for a point $\mathbf{z} = \begin{bmatrix} z_1 & z_2 & \ldots & z_{n+1} \end{bmatrix}^T \in \mathbb{V}_- \cup \mathbb{V}_0$, $w_i = z_i z_{n+1}^{-1}$ for $i = 1, \ldots, n$. This is the Siegel domain model of $\mathbf{H}_\mathbb{F}^n$. Similarly, one can define the ball model by replacing H with an equivalent Hermitian form H' given by the diagonal matrix: $H' = \text{diag}(-1, 1, \ldots, 1)$. We mostly use the Siegel domain model here.

There are two distinguished points in $\mathbb{V}_0$, which we denote by $\mathbf{o}$ and ∞, given by

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we can write $\partial \mathbf{H}_\mathbb{H}^n = \mathbb{P}(\mathbb{V}_0)$ as

$$\partial \mathbf{H}_\mathbb{F}^n \setminus \{\infty\} = \{(z_1, \ldots, z_n) \in \mathbb{H}^n : 2\mathfrak{R}(z_1) + |z_2|^2 + \cdots + |z_n|^2 = 0\}.$$

Note that $\overline{\mathbf{H}_\mathbb{F}^n} = \mathbf{H}_\mathbb{F}^n \cup \partial \mathbf{H}_\mathbb{F}^n$.

Given a point z of $\overline{\mathbf{H}_\mathbb{F}^n} \setminus \{\infty\} \subset \mathbb{FP}^n$, we may lift $z = (z_1, \ldots, z_n)$ to a point $\mathbf{z}$ in $\mathbb{V}$, called the *standard lift* of z. It is represented in projective coordinates by

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}.$$

The Bergman metric in $\mathbf{H}_\mathbb{F}^n$ is defined in terms of the Hermitian form given by

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{bmatrix}.$$

If z and w in $\mathbf{H}_\mathbb{F}^n$ correspond to vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{V}_-$, then the Bergman metric is also given by the distance ρ:

$$\cosh^2 \left(\frac{\rho(z, w)}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}.$$

More information on the basic formalism of the quaternionic hyperbolic space may be found in [ChGr].

2.3. Isometries. Let $U(n, 1; \mathbb{F})$ be the isometry group of the Hermitian form $\langle \cdot, \cdot \rangle$. Each matrix A in $U(n, 1; \mathbb{F})$ satisfies the relation $A^{-1} = H^{-1} A^* H$, where A^* is the conjugate transpose of A. The isometry group of $\mathbf{H}_{\mathbb{F}}^n$ is the projective unitary group $PU(n, 1; \mathbb{F})$, the group $U(n, 1)$ modulo the center. We denote $U(n, 1; \mathbb{C}) = U(n, 1)$ and $U(n, 1; \mathbb{H}) = Sp(n, 1)$.

2.4. Hyperbolic elements in $SU(n, 1; \mathbb{F})$. Let A be hyperbolic in $SU(n, 1; \mathbb{F})$. Let $a_A \in \partial \mathbf{H}_{\mathbb{F}}^n$ be the *attracting fixed point* of A that corresponds to the eigenvalue $re^{i\theta}$, $r < 1$, and let $r_A \in \partial \mathbf{H}_{\mathbb{F}}^n$ be the *repelling fixed point* corresponding to the eigenvalue $r^{-1}e^{i\theta}$. Let a_A and r_A lift to eigenvectors $\mathbf{a}_A$ and $\mathbf{r}_A$, respectively. Let $\mathbf{x}_{j,A}$ be an eigenvector corresponding to $e^{i\phi_j}$, $j = 1, \ldots, n-1$. The points $x_{j,A}$, $j = 1, \ldots, n-1$ on $\mathbb{P}(\mathbb{V}_+)$ are the space-like (or positive-definite) projective fixed points of A. Define $E_A(r, \theta, \phi_1, \ldots, \phi_{n-1})$ as

$$E_A(r, \theta, \phi_1, \ldots, \phi_{n-1}) = \begin{bmatrix} re^{i\theta} & 0 & \ldots & 0 & 0 \\ 0 & e^{i\phi_1} & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & e^{i\phi_{n-1}} & 0 \\ 0 & 0 & \ldots & 0 & r^{-1}e^{i\theta} \end{bmatrix}. \tag{2-1}$$

Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_{1,A} & \ldots & \mathbf{x}_{n-1,A} & \mathbf{r}_A \end{bmatrix}$ be the matrix corresponding to the eigenvectors. We can choose C_A to be an element of $Sp(n, 1)$ by normalizing the eigenvectors:

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = 1, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1, \ i = 1, \ldots, n-1.$$

Then $A = C_A E_A(r, \theta, \phi_1, \ldots, \phi_{n-1}) C_A^{-1}$.

LEMMA 2.1 [ChGr]. *Two hyperbolic elements in $SU(n, 1; \mathbb{F})$ are conjugate if and only if they have the same similarity classes of eigenvalues.*

DEFINITION 2.2. Let A be a hyperbolic element in $SU(n, 1; \mathbb{F})$. Let λ represent an eigenvalue from the similarity class of eigenvalues $[\lambda]$ of A. Let $\mathbf{x}$ be a λ-eigenvector. Then $\mathbf{x}$ defines a point x on $\mathbb{FP}^n$ that is either a point on $\partial \mathbf{H}_{\mathbb{F}}^n$ or a point in $\mathbb{P}(\mathbb{V}_+)$. The lift of x in $\mathbb{F}^{n,1}$ is the quaternionic line $\mathbf{x}\mathbb{F}$. We call x a *projective fixed point* of A corresponding to $[\lambda]$. If A is regular, it fixes exactly $n + 1$ points on $\mathbb{P}(\mathbb{V})$ and thus it has $n + 1$ projective fixed points.

REMARK 2.3. We emphasize here that the projective fixed points of A are not the same as the projective points of A. The notion of the projective points of A is elaborated in Section 3.

LEMMA 2.4. *The group $Sp(n, 1)$ can be embedded in the group $GL(2n + 2, \mathbb{C})$.*

PROOF. Write $\mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$. For $A \in \mathrm{Sp}(n, 1)$, express $A = A_1 + \mathbf{j}A_2$, where $A_1, A_2 \in M_{n+1}(\mathbb{C})$. The correspondence $A \mapsto A_\mathbb{C}$, where

$$A_\mathbb{C} = \begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix}, \tag{2-2}$$

embeds $\mathrm{Sp}(n, 1)$ into $\mathrm{GL}(2n + 2, \mathbb{C})$.   □

The following lemma is a special case of [GP13, Theorem 3.1].

LEMMA 2.5. *Let A be an element in $\mathrm{Sp}(n, 1)$. Let $A_\mathbb{C}$ be the corresponding element in $\mathrm{GL}(2n + 2, \mathbb{C})$. The characteristic polynomial of $A_\mathbb{C}$ is of the form*

$$\chi_A(x) = \sum_{j=0}^{2n+2} a_j x^{2(n+1)-j},$$

where $a_0 = 1 = a_{2n+2}$ and, for $1 \le j \le n + 1$, $a_j = a_{2(n+1)-j}$. Write $\chi_A(x) = x^{n+1}g(x + x^{-1})$. Let Δ be the negative of the discriminant of the polynomial $g_A(t) = g(x + x^{-1})$. Then A is regular loxodromic if and only if $\Delta > 0$ and $\sum_{j=0}^{n} a_j \ne -\frac{1}{2}a_{n+1} \ne \sum_{j=0}^{n}(-1)^{n+1-j}a_j$. The conjugacy class of A is determined by the real numbers a_j, $1 \le j \le n + 1$.

PROOF. Note that $g(x + x^{-1}) = \sum_{j=0}^{n}(x^{n+1-j} + x^{-(n+1-j)}) + a_{n+1}$. It is proved in [GP13, Theorem 3.1] that A is regular hyperbolic if and only if $\Delta > 0$. Now A has no eigenvalue ± 1 if and only if $g(\pm 2) \ne 0$, that is, $a_{n+1} + 2\sum_{j=0}^{n} a_j \ne 0 \ne a_{n+1} + 2\sum_{j=0}^{n}(-1)^{n+1-j}a_j$.   □

DEFINITION 2.6. Let A be a regular loxodromic element in $\mathrm{Sp}(n, 1)$. The $(n + 1)$-tuple of real numbers $(a_1, \ldots, a_{n+1})$ as in Lemma 2.5 is called the *real trace* of A and we denote it by $tr_\mathbb{R}(A)$.

2.5. Useful results. We use the following result by Cao [Cao16] that determines quadruples of points on $\partial\mathbf{H}^n_\mathbb{H}$. We refer to [Cao16, AK07] for the basic notions of angular invariants. For the notation used in the following statement, see [GK2, Section 2].

THEOREM 2.7 [Cao16]. *Let $Z = (z_1, z_2, z_3, z_4)$ and $W = (w_1, w_2, w_3, w_4)$ be two quadruples of pairwise distinct points in $\partial\mathbf{H}^n_\mathbb{H}$. Then there exists an isometry $h \in \mathrm{Sp}(n, 1)$ such that $h(z_i) = w_i$, $i = 1, 2, 3, 4$, if and only if the following conditions hold.*

1. *For $j = 1, 2, 3$, $\mathbb{X}_j(z_1, z_2, z_3, z_4)$ and $\mathbb{X}_j(w_1, w_2, w_3, w_4)$ belong to the same similarity class.*
2. $\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(w_1, w_2, w_3)$, $\mathbb{A}(z_1, z_2, z_4) = \mathbb{A}(w_1, w_2, w_4)$ *and* $\mathbb{A}(z_2, z_3, z_4) = \mathbb{A}(w_2, w_3, w_4)$.

Cao also proved that, for $n \ge 3$, the moduli space of $\mathrm{Sp}(n, 1)$-congruence classes of points is homeomorphic to a semi-algebraic subspace of $\mathbb{C}^3 \times \mathbb{R} \times \mathbb{R}$ defined by these invariants.

In the complex hyperbolic set up, the moduli space of ordered quadruples of points was obtained by Cunha and Gusevskii. We recall their result.

THEOREM 2.8 [CuG10]. *Let* $Z = (z_1, z_2, z_3, z_4)$ *and* $W = (w_1, w_2, w_3, w_4)$ *be two quadruples of pairwise distinct points in* $\partial \mathbf{H}_{\mathbb{C}}^n$. *Then there exists an isometry* $h \in \mathrm{SU}(n, 1)$ *such that* $h(z_i) = w_i$, $i = 1, 2, 3, 4$, *if and only if the following conditions hold.*

1. $\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(w_1, w_2, w_3)$.
2. $\mathbb{X}(z_1, z_2, z_3, z_4) = \mathbb{X}(w_1, w_2, w_3, w_4)$ *and* $\mathbb{X}(z_1, z_4, z_2, z_3) = \mathbb{X}(w_1, w_4, w_2, w_3)$.

Further, these invariants $(\mathbb{X}(z_1, z_2, z_3, z_4), \mathbb{X}(z_1, z_4, z_2, z_3), \mathbb{A}(z_1, z_2, z_3))$ *form a semi-algebraic subset of* $\mathbb{C}^2 \setminus \{\{0\} \times \mathbb{R}\}$ *that is homeomorphic to the moduli space.*

3. Projective points

3.1. Projective points. We recall the concept of projective points from [GK1]. Let T be an invertible matrix over $\mathbb{H}$. Let $\lambda \in \mathbb{H} \setminus \mathbb{R}$ be a chosen eigenvalue of T in the similarity class $[\lambda]$. Identify the $[\lambda]$-eigenspace with $\mathbb{H}$. Consider the λ-*eigenset*: $S_\lambda = \{x \in V \mid Tx = x\lambda\}$. Note that this set is $xZ(\lambda)$ that is a copy of $\mathbb{C}$ in $\mathbb{H}$. Now identify $\mathbb{H}$ with $\mathbb{C}^2$. Two nonzero quaternions q_1 and q_2 are equivalent if $q_2 = q_1 c, c \in \mathbb{C} \setminus 0$. This equivalence relation projects $\mathbb{H}$ to the one-dimensional complex projective space $\mathbb{CP}^1$, the $[\lambda]$-eigensphere. Since $[\lambda]$ is a conjugacy invariant of T, so also is the $[\lambda]$-eigensphere $\mathbb{CP}^1$.

Let v be the projection of the $[\lambda]$-eigenspace. Then, for each point on $\mathbb{CP}^1$, there is a choice of the lift $\mathbf{v}$ of v that spans a complex line in $\mathbf{v}\mathbb{H}$. This choice of $\mathbf{v}$ corresponds to the eigenset of the eigenvalue λ of $\mathbf{v}$, and the corresponding point on the eigensphere $\mathbb{CP}^1$ is called a *projective point* of $[\lambda]$.

3.2. Projective points and loxodromic elements. Now suppose that A is a regular loxodromic element in $\mathrm{Sp}(n, 1)$. If a_A and r_A are the fixed points of A, then we can determine the projective point corresponding to r_A if we know the projective point corresponding to a_A on $\mathbb{CP}^1$. So, we require a single projective point corresponding to the pair (a_A, r_A) on $\mathbb{CP}^1$. Here we have used the fact that $Z(\lambda) = Z(\bar{\lambda}^{-1})$. Similarly, the projective points of $\mathbf{x}_{1,A}, \ldots, \mathbf{x}_{n-1,A}$ correspond to the centralizers $Z(\mu_1), \ldots, Z(\mu_{n-1})$, respectively.

The following classification of loxodromic elements in $\mathrm{Sp}(n, 1)$ follows from [GK2, Section 4.1].

LEMMA 3.1. *Let* A *and* A' *be regular loxodromic elements in* $\mathrm{Sp}(n, 1)$. *Then* $A = A'$ *if and only if they have the same projective fixed points, the same real trace and the same projective points.*

The above lemma may be interpreted as follows. Let C be the $\mathrm{Sp}(n, 1)$ conjugacy classes of regular loxodromic elements. It follows from Lemma 2.1 that the real traces classify a point on C and, up to conjugacy, we can assume that elements of C have the

same projective fixed points. Let $\mathcal{T}$ be the set of real traces $(a_1, \ldots, a_n) \in \mathbb{R}^n$ given by $\Delta^{-1}(0, \infty)$, where $\Delta : C \to (0, \infty)$ is the discriminant function in Lemma 2.5. There is a natural projection map $p : C \to \mathcal{T}$. However, $p^{-1}(t)$ is not unique. The map p has fiber $(\mathbb{CP}^1)^n = \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$. A point on this $(\mathbb{CP}^1)^n$ determines a loxodromic element uniquely up to relabelling of fixed points.

In the case of $SU(n, 1)$ an easier version of the above lemma holds true.

LEMMA 3.2. *Let A and A' be regular loxodromic elements in $SU(n, 1)$. Then $A = A'$ if and only if they have the same projective fixed points and the same characteristic polynomial, where having the same characteristic polynomial is equivalent to the condition of having the same eigenvalues.*

4. Weakly nonsingular pairs

In this section, we mostly work with the group $Sp(n, 1)$. However, the arguments restrict over $SU(n, 1)$ with slight modifications and hence are omitted.

4.1. Gram matrix associated to a pair. Let (A, B) be a weakly nonsingular pair in $Sp(n, 1)$. Condition (3) in Definition 1.5 implies that we may assume, by re-arranging the indices if necessary, that

$$\langle \mathbf{x}_{k,A}, \mathbf{a}_B \rangle \neq 0, \ \langle \mathbf{x}_{k,B}, \mathbf{a}_A \rangle \neq 0, \ \langle \mathbf{r}_A, \mathbf{x}_{k,B} \rangle \neq 0, \ \langle \mathbf{r}_B, \mathbf{x}_{k,A} \rangle \neq 0 \quad \text{for } 1 \le k \le n - 2.$$

We normalize the eigenvectors such that for $1 \le k \le n - 2$,

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{a}_A, \mathbf{a}_B \rangle = \langle \mathbf{a}_A, \mathbf{r}_B \rangle = \langle \mathbf{a}_A, \mathbf{x}_{k,B} \rangle = \langle \mathbf{a}_B, \mathbf{x}_{k,A} \rangle = 1, \quad |\langle \mathbf{a}_B, \mathbf{r}_A \rangle| = 1, \quad (4\text{-}1)$$

and $\langle \mathbf{r}_A, \mathbf{x}_{k,B} \rangle \neq 0 \neq \langle \mathbf{r}_B, \mathbf{x}_{k,A} \rangle$.

For simplicity of notation, we write:

- $p_1 = a_A, \ p_2 = r_A, \ p_3 = a_B, \ p_4 = r_B$;
- for $5 \le j \le n + 2$, $p_j = x_{j-4,A}$;
- for $n + 3 \le j \le 2n$, $p_j = x_{j-(n+2),B}$;
- $p_{2n+1} = x_{n-1,A}, \ p_{2n+2} = x_{n-1,B}$.

Since the eigenvectors of $A \in Sp(n, 1)$ form an orthonormal basis for $\mathbb{H}^{n,1}$, it follows that if $C(p_i) = p'_i$ for $1 \le i \le 2n$, then $C(p_j) = p'_j$ for $j = 2n + 1, 2n + 2$. For this reason, we associate to (A, B) the Gram matrix (g_{ij}), $g_{ij} = \langle \mathbf{p}_i, \mathbf{p}_j \rangle$, of the ordered $(2n)$-tuple $p = (p_1, p_2, \ldots, p_{2n})$. In view of the normalized eigenvectors, the Gram matrix has the form $G(p) = (g_{ij})$, where:

1. $g_{11} = g_{22} = g_{33} = g_{44} = 0$; $g_{12} = g_{13} = g_{14} = 1 = |g_{23}|$;
2. for $5 \le j \le n + 2$, $g_{1j} = 0$, $g_{2j} = 0$; and, for $n + 3 \le k \le 2n$, $g_{1k} = 1$, $g_{2k} \neq 0$;
3. for $5 \le j \le n + 2$, $g_{3j} = 1$, $g_{4j} \neq 0$; and, for $n + 3 \le k \le 2n$, $g_{3k} = 0$, $g_{4k} = 0$;
4. for $5 \le j, k \le n + 2$, $j < k$, $g_{jk} = 0$; and, for $n + 3 \le k, j \le 2n$, $g_{jk} = 0$, $j < k$, $g_{jk} = 0$.

We call G a *normalized Gram matrix* associated to (A, B).

LEMMA 4.1. *Suppose that the Gram matrix $G(\mathbf{p})$ is a normalized Gram matrix for p with respect to the lift $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2n})$. Let $G(\mathbf{p}')$ be the normalized Gram matrix with respect to the lift $\mathbf{p}' = (\mathbf{p}_1\lambda_1, \ldots, \mathbf{p}_{2n}\lambda_{2n})$ of p. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_{2n}$ and $\lambda_1 \in Sp(1)$.*

PROOF. We have $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k \rangle = 1$; thus, $\overline{\lambda_k}\lambda_1 = 1$ for $k = 2, 3, 4$ because $\langle \mathbf{p}_1, \mathbf{p}_k \rangle = 1$. Now, from $|\langle \mathbf{p}_2\lambda_2, \mathbf{p}_3\lambda_3 \rangle| = 1$, we have $|\lambda_3||\lambda_2| = 1$ as $|\langle \mathbf{p}_2, \mathbf{p}_3 \rangle| = 1$. Thus, we have $|\lambda_1| = 1$, so $\lambda_1 \in Sp(1)$. Therefore, by $\overline{\lambda_k}\lambda_1 = 1$, for $k = 2, 3, 4$, we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $\lambda_1 \in Sp(1)$.

By $\langle \mathbf{p}_3\lambda_3, \mathbf{p}_j\lambda_j \rangle = 1$, for $j = 5, 6, \ldots, n + 2$, we have $\overline{\lambda_j}\lambda_3 = 1$. Thus, $\lambda_3 = \lambda_j$ for $j = 5, 6, \ldots, n + 2$ satisfies $|\lambda_3| = 1$. Also, from the relations $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k \rangle = 1$, for $k = n + 3, n + 4, \ldots, 2n$, we can see that $\overline{\lambda_k}\lambda_1 = 1$ for $k = n + 3, n + 4, \ldots, 2n$. Now $|\lambda_1| = 1$ gives $\lambda_1 = \lambda_k$ for $k = n + 3, n + 4, \ldots, 2n$. So, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_{2n}$ and $\lambda_1 \in Sp(1)$.   □

The Gram matrix $G(\mathbf{p})$ is well defined up to a scalar action of $Sp(1)$. We denote the $Sp(1)$ orbit of entries of $G(\mathbf{p})$ as $O_{G(p)}$. The following theorem follows using similar arguments as in the proof of [GK2, Lemma 8.9].

LEMMA 4.2. *Let (A, B) and (A', B') be two weakly nonsingular pairs of loxodromic elements in $Sp(n, 1)$. Let $p = (p_1, \ldots, p_{2n})$ and $p' = (p'_1, \ldots, p'_{2n})$ be the associated tuples to the pairs, respectively. Then there exists $C \in Sp(n, 1)$ such that $C(p_i) = p'_i$, $i = 1, \ldots, 2n$, if and only if $O_{G(p)} = O_{G(p')}$.*

REMARK 4.3. We note further that, if we keep the lift of a chosen point p_j from the same hyperplane, for example if we always take $\mathbf{p}_j$ to be standard, then it follows from Lemma 4.1 that there is a unique normalized Gram matrix associated to the tuple p.

4.2. Conjugacy invariants.
We consider the following invariants associated to the tuple p.

1.   Angular invariant: $\mathbb{A}(p_1, p_2, p_3)$.
2.   Usual cross ratios: $\mathbb{X}_1(A, B) = \mathbb{X}(p_1, p_2, p_3, p_4)$, $\mathbb{X}_2(A, B) = \mathbb{X}(p_1, p_3, p_2, p_4)$.
3.   Generalized cross ratios:

        For $n + 3 \le k \le 2n$, $\mathbb{X}_{2k}(A, B) = \mathbb{X}(p_1, p_2, p_3, p_k)$.

        For $5 \le j \le n + 2$, $\mathbb{X}_{4j}(A, B) = \mathbb{X}(p_3, p_4, p_1, p_j)$.

        For $5 \le j \le n + 2$, $n + 3 \le k \le 2n$, $\mathbb{X}_{jk}(A, B) = \mathbb{X}(p_3, p_k, p_2, p_j)$.

        Note that we have denoted $\mathbb{X}_{2k}(A, B)$ by $\alpha_k(A, B)$ and $\mathbb{X}_{4j}(A, B)$ by $\beta_j(A, B)$ in Section 1.

4.   Goldman's eta invariants:

        For $5 \le j \le n + 2$, $\eta_j(A, B) = \eta(p_3, p_4; p_j) = \langle \mathbf{p}_3, \mathbf{p}_j \rangle \langle \mathbf{p}_3, \mathbf{p}_4 \rangle^{-1} \langle \mathbf{p}_j, \mathbf{p}_4 \rangle \langle \mathbf{p}_j, \mathbf{p}_j \rangle^{-1}$.

        For $n + 3 \le k \le 2n$, $\eta_k(A, B) = \eta(p_1, p_2; p_k) = \langle \mathbf{p}_1, \mathbf{p}_k \rangle \langle \mathbf{p}_1, \mathbf{p}_2 \rangle^{-1} \langle \mathbf{p}_k, \mathbf{p}_2 \rangle \langle \mathbf{p}_k, \mathbf{p}_k \rangle^{-1}$.

We note that using our notation earlier, $\mathbb{X}_{2j}(A, B) = \alpha_j(A, B)$ and $\mathbb{X}_{4k}(A, B) = \beta_k(A, B)$. However, we slightly change the notation here in order to have uniformity in the symbols.

LEMMA 4.4. *Let (A, B) be a weakly nonsingular pair in $\mathrm{Sp}(n, 1)$. Suppose that the Gram matrix $G(\mathbf{p}) = (g_{ij})$ is a normalized Gram matrix associated to (A, B) with respect to the lift $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2n})$. Then the Gram matrix is determined by the invariants listed above.*

PROOF. The proof is obtained by computing the invariants in view of the normalized Gram matrix and

$$\mathbb{A} = \arg(-g_{23}), \text{ that is, } g_{23} = -e^{i\mathbb{A}},$$
$$\mathbb{X}_1 = \bar{g}_{23}^{-1}\bar{g}_{24}, \ \mathbb{X}_2 = g_{23}^{-1}\bar{g}_{34},$$
$$\mathbb{X}_{2k} = \bar{g}_{23}^{-1}\bar{g}_{2k}, \ \mathbb{X}_{4j} = \bar{g}_{4j},$$
$$\mathbb{X}_{jk} = g_{23}g_{2k}^{-1}g_{jk}, \ 5 \le j \le n + 2, \ n + 3 \le k \le 2n,$$
$$\eta_j = g_{34}^{-1}\bar{g}_{4j}g_{jj}^{-1}, \ \eta_k = \bar{g}_{2k}g_{kk}^{-1}.$$

This clearly shows the result.   □

4.3. Classification of weakly nonsingular pairs.

THEOREM 4.5. *Let (A, B) be a weakly nonsingular pair of loxodromic elements in $\mathrm{Sp}(n, 1)$. Then (A, B) is determined uniquely up to conjugacy in $\mathrm{Sp}(n, 1)$ by the real traces, the angular invariant $\mathbb{A}(a_A, r_A, a_B)$, the $\mathrm{Sp}(1)$ conjugation orbit of the (unordered) tuple of the above conjugacy invariants (2)–(4) and the projective points.*

PROOF. Let (A, B) and (A', B') be loxodromic elements in $\mathrm{Sp}(n, 1)$. Suppose that $p = (p_1, \ldots, p_{2n})$ and $p' = (p'_1, \ldots, p'_{2n})$ are the associated tuples to the pairs, respectively. Assume that $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p_1', p_2', p_3')$ and the $\mathrm{Sp}(1)$ conjugation orbits of the (unordered) tuple of the above conjugacy invariants (2)–(4) with respect to (A, B) and (A', B') are equal. So, there exists $\mu \in \mathrm{Sp}(1)$ such that

$$\mu\mathbb{X}_1(A, B)\bar{\mu} = \mathbb{X}_1(A', B'), \ \mu\mathbb{X}_2(A, B)\bar{\mu} = \mathbb{X}_2(A', B'), \ \mu\mathbb{X}_{2k}(A, B)\bar{\mu} = \mathbb{X}_{2k}(A', B'),$$

$$\mu\mathbb{X}_{4j}(A, B)\bar{\mu} = \mathbb{X}_{4j}(A', B'), \ \mu\mathbb{X}_{jk}(A, B)\bar{\mu} = \mathbb{X}_{jk}(A', B'), \ \mu\eta_j(A, B)\bar{\mu} = \eta_j(A', B'),$$

$$\mu\eta_k(A, B)\bar{\mu} = \eta_k(A', B').$$

By Lemma 4.4, we have $DG(\mathbf{p})D^{-1} = G(\mathbf{p'})$, where $D = \mathrm{diag}(\mu, \mu, \ldots, \mu)$. That is, $O_{G(\mathbf{p})} = O_{G(\mathbf{p'})}$. Then, by Lemma 4.2, there exists $C \in \mathrm{Sp}(n, 1)$ such that $C(p_i) = p'_i$ for $1 \le i \le 2n + 2$. In particular, CAC^{-1} and A' have the same projective fixed points. Since they have the same real traces, they belong to the same conjugacy class. By Lemma 3.1, $CAC^{-1} = A'$ if and only if they have the same projective points. Similarly, $CBC^{-1} = B'$.   □

REMARK 4.6. Let $\mathcal{I}$ denote the tuple of real numbers given by the above invariants, and let $\mathcal{T}$ denote the set of real traces of regular loxodromics. Let $\mathcal{W}$ denote the set of weakly nonsingular representations in $\mathfrak{X}(\mathrm{F}_2, \mathrm{Sp}(n, 1))$. Clearly, by Lemma 4.4, there is a well-defined map $p : \mathcal{W} \to \mathcal{T} \times \mathcal{T} \times \mathcal{I}$. However, given a point t in the image $p(\mathcal{W})$, $p^{-1}(t)$ is not a unique point, but a product of $2n$ copies of $\mathbb{CP}^1$ corresponding to the projective points.

4.3.1. Proof of Theorem 1.7. This is a restatement of the above theorem where $\rho(x) = A$, $\rho(y) = B$.

4.3.2. Proof of Theorem 1.8. This follows from the above by restricting everything over $\mathbb{C}$.

5. The nonsingular pairs

LEMMA 5.1. *Let A, B be loxodromic elements in $\mathrm{SU}(n, 1)$ such that (A, B) is nonsingular. Denote $\mathbb{A}(A, B) = \mathbb{A}(a_A, r_A, a_B)$. Let (A', B') be a nonsingular and loxodromic pair such that the following holds.*

(i) *For $k = 1, 2$, $\mathbb{X}_k(A, B) = \mathbb{X}_k(A', B')$ and $\mathbb{A}(A, B) = \mathbb{A}(A', B')$.*
(ii) *For $1 \le j \le n - 2$, $\alpha_j(A', B') = \alpha_j(A, B)$ and $\beta_j(A', B') = \beta_j(A, B)$.*

Then there exists an element C in $\mathrm{SU}(n, 1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(x_{k,A}) = x_{k,A'}$ and $C(a_B) = a_{B'}$, $C(r_B) = r_{B'}$, $C(x_{k,B}) = x_{k,B'}$.

PROOF. We follow similar arguments as in the proof of [GP18A, Lemma 5.1].

Since $\mathbb{X}_k(A, B) = \mathbb{X}_k(A', B')$, $\mathbb{A}(A, B) = \mathbb{A}(A', B')$ $k = 1, 2$, by Theorem 2.8, it follows that there exists $C \in \mathrm{SU}(n, 1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(a_B) = a_{B'}$ and $C(r_B) = r_{B'}$. Let $1 \le k \le n - 2$. Since $\alpha_k(A, B) = \alpha_k(A', B')$,

$$\langle \mathbf{x}_{k,B}, \mathbf{r}_A \rangle \langle \mathbf{x}_{k,B}, \mathbf{a}_A \rangle^{-1} \langle \mathbf{a}_B, \mathbf{a}_A \rangle \langle \mathbf{a}_B, \mathbf{r}_A \rangle^{-1} = \langle \mathbf{x}_{k,B'}, \mathbf{r}_{A'} \rangle \langle \mathbf{x}_{k,B'}, \mathbf{a}_{A'} \rangle^{-1} \langle \mathbf{a}_{B'}, \mathbf{a}_{A'} \rangle \langle \mathbf{a}_{B'}, \mathbf{r}_{A'} \rangle^{-1}.$$

Let

$$\langle C^{-1}(\mathbf{x}_{k,B'}), \mathbf{r}_A \rangle^{-1} \langle \mathbf{x}_{k,B}, \mathbf{r}_A \rangle = \langle C^{-1}(\mathbf{x}_{k,B'}), \mathbf{a}_A \rangle^{-1} \langle \mathbf{x}_{k,B}, \mathbf{a}_A \rangle = \lambda.$$

This implies that

$$\langle \mathbf{x}_{k,B} - C^{-1}(\mathbf{x}_{k,B'})\lambda, \mathbf{r}_A \rangle = 0, \tag{5-1}$$

$$\langle \mathbf{x}_{k,B} - C^{-1}(\mathbf{x}_{k,B'})\lambda, \mathbf{a}_A \rangle = 0. \tag{5-2}$$

On the other hand, note that

$$\langle \mathbf{x}_{k,B} - C^{-1}(\mathbf{x}_{k,B'})\lambda, \mathbf{r}_B \rangle = \langle \mathbf{x}_{k,B}, \mathbf{r}_B \rangle - \langle C^{-1}(\mathbf{x}_{k,B'}), \mathbf{r}_B \rangle \lambda = 0 - \langle \mathbf{x}_{k,B'}, \mathbf{r}_{B'} \rangle \lambda = 0. \tag{5-3}$$

Similarly,

$$\langle \mathbf{x}_{k,B} - C^{-1}(\mathbf{x}_{k,B'})\lambda, \mathbf{a}_B \rangle = 0. \tag{5-4}$$

Let L_A and L_B denote the two-dimensional time-like subspaces of $\mathbb{C}^{n,1}$ with $\{\mathbf{a}_A, \mathbf{r}_A\}$ and $\{\mathbf{a}_B, \mathbf{r}_B\}$ the respective bases of L_A and L_B that represent the complex lines.

Thus, it follows from (5-1)–(5-4) that $\mathbf{v} = \mathbf{x}_{k,B} - C^{-1}(\mathbf{x}_{k,B'})\lambda$ is orthogonal to both L_A and L_B. We must have $\langle v, v \rangle > 0$. Thus, v is polar to the $(n-1)$-dimensional totally geodesic complex subspace that is represented by $V = v^\perp$. Since $\mathbb{C}^{n,1} = V \oplus \mathbb{C}v$, L_A and L_B must be subsets in V. Thus, the fixed points of A and B belong to the boundary of the totally geodesic subspace $\mathbb{P}(V)$. This is a contradiction to the nonsingularity of (A, B). Hence, we must have $v = 0$, that is, $C(\mathbf{x}_{k,B}) = \mathbf{x}_{k,B'}\lambda$. Thus, $C(x_{k,B}) = x_{k,B'}$. Consequently, $C(x_{n-1,B}) = x_{n-1,B'}$.

Similarly, $\beta_j(A, B) = \beta_j(A', B')$ implies that $C(x_{j,A}) = x_{j,A'}$ for $1 \le j \le n-1$. This proves the lemma.   □

5.1. Proof of Theorem 1.9.   If (A, B) and (A', B') are conjugate, then it is clear that they have the same invariants.

Conversely, suppose that (A, B) and (A', B') are nonsingular pairs of loxodromics such that $\alpha_k(A, B) = \alpha_k(A', B')$, $\beta_k(A, B) = \beta_k(A', B')$, $1 \le k \le n-2$, $\mathbb{X}_i(A, B) = \mathbb{X}_i(A', B')$, $i = 1, 2$, $\mathbb{A}(A, B) = \mathbb{A}(A', B')$. By Lemma 5.1, it follows that there exists $C \in \mathrm{SU}(n, 1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(x_{k,A}) = x_{k,A'}$ and $C(a_B) = a_{B'}$, $C(r_B) = r_{B'}$, $C(x_{k,B}) = x_{k,B'}$, $1 \le k \le n-1$. Therefore, A', respectively B', and CAC^{-1}, respectively CBC^{-1}, have the same fixed points. Since they also have the same family of traces, $CAC^{-1} = A'$. Similarly, $CBC^{-1} = B'$. This completes the proof.

5.2. Proof of Theorem 1.12.   The following lemma follows by mimicking the proof of Theorem 5.1; the only difference is that instead of Theorem 2.8, one has to apply Theorem 2.7 in the proof.

LEMMA 5.2. *Let A, B be loxodromic elements in $\mathrm{Sp}(n, 1)$ such that (A, B) is nonsingular. Suppose that the lifts of the attracting fixed points of a loxodromic element are always assumed to be standard. Let (A', B') be a nonsingular pair such that the following holds.*

(i)   *For $k = 1, 2, 3$, $\mathbb{X}_k(A, B) = \mathbb{X}_k(A', B')$ and $\mathbb{A}_k(A, B) = \mathbb{A}_k(A', B')$.*
(ii)   *For $1 \le j \le n-2$, $\alpha_j(A', B') = \alpha_j(A, B)$ and $\beta_j(A', B') = \beta_j(A, B)$.*

Then there exists an element C in $\mathrm{Sp}(n, 1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(x_{k,A}) = x_{k,A'}$ and $C(a_B) = a_{B'}$, $C(r_B) = r_{B'}$, $C(x_{k,B}) = x_{k,B'}$.

Now Theorem 1.12 follows using the same arguments as above or in the proof of Theorem 1.7.

6. The twist-bend parameters and surface group representations

6.1. The twist-bend parameters.   Suppose that $\langle A, B \rangle$ is a nonsingular $(0, 3)$ group in $\mathrm{Sp}(3, 1)$, that is, A, B and $B^{-1}A^{-1}$ are loxodromic and $\langle A, B \rangle$ is free. We also assume that (A, B) is nonsingular. We want to attach two such nonsingular subgroups to get a group that is freely generated by three generators. Now two cases are possible. The first case corresponds to the case when two different three-holed spheres (or pairs of pants) are attached along their boundary components. This gives a $(0, 4)$ group generated by

three elements. The second case corresponds to the case when two of the boundary components of the same three-holed sphere are glued. In this case gluing two $(0, 3)$ groups gives a $(1, 1)$ group that is a group generated by two loxodromic elements and their commutator. This process is called 'closing a handle'. To get more details of these terminologies and the gluing process, we refer to [PP08].

Let $\langle A, B \rangle$ and $\langle C, D \rangle$ be two nonsingular $(0, 3)$ groups in $\mathrm{Sp}(3, 1)$ such that the boundary components associated to A and D are compatible. Here compatibility means $A = D^{-1}$. A three-dimensional *quaternionic hyperbolic twist-bend* corresponds to an element K in $\mathrm{Sp}(3, 1)$ that commutes with A and conjugates $\langle C, D \rangle$; see [PP08, Section 8.1]. We assume that up to conjugacy, A fixes 0, ∞ and it is of the form $E(r, \theta, \phi_1, \phi_2)$. Since K commutes with A, it is also of the form $K = E(t, \psi, \xi_1, \xi_2)$; see [Gon13]. Thus, K is either a boundary elliptic or a hyperbolic element.

It follows that there is a total of 10 real parameters associated to K, the real trace (t, ψ, ξ_1, ξ_2), along with six real parameters associated to the projective points. If $t = 1$, then K is a boundary elliptic and the eigenvalue $[e^{i\psi}]$ has multiplicity two. The projective points for these eigenvalues can be defined as before. There are exactly one negative-type and two positive-type eigenvalues of K. Since K commutes with A, the projective points of K are determined by the projective points of A. Hence, there are three projective points of K to determine it. Consequently, we have 10 real parameters associated to a twist-bend K. We denote these parameters by $\kappa = (t, \psi, \xi_1, \xi_2, k_1, k_2, k_3)$, where $k_1 = p_1(K)$, $k_2 = p_2(K)$, $k_3 = p_3(K)$ are the projective points of the similarity classes of eigenvalues of K.

The parameters $\kappa = (t, \psi, \xi_1, \xi_2, k_1, k_2, k_3)$ obtained in this way are called the *twist-bend parameters*. Note that the twist-bend is a relative invariant as it always has to be chosen with respect to some fixed group $\langle A, B, C \rangle$ that one has to specify before applying the twist-bend. When we write $A = QE(r, \theta, \phi_1, \phi_2)Q^{-1}$, if the matrix $K = QE(t, \psi, \xi_1, \xi_2)Q^{-1}$, then we say that the twist-bend parameters κ are *oriented consistently* with A.

To obtain conjugacy invariants to quantify the twist-bend parameters, we define the following numerical objects corresponding to κ:

$$\tilde{\mathbb{X}}_1(\kappa) = \mathbb{X}(a_A, r_A, a_B, K(r_C)), \ \ \tilde{\mathbb{X}}_2(\kappa) = \mathbb{X}(a_A, K(r_C), a_B, r_A), \ \ \tilde{\mathbb{X}}_3(\kappa) = \mathbb{X}(r_A, K(r_C), a_B, a_A);$$

$$\tilde{\mathbb{A}}_1(\kappa) = \mathbb{A}(a_A, r_A, K(r_C)), \ \ \tilde{\mathbb{A}}_3(\kappa) = \mathbb{A}(r_A, K(r_C), a_B).$$

LEMMA 6.1. *Let A, B, C be loxodromic transformations of $\mathbf{H}^3_{\mathbb{H}}$ such that $\langle A, B \rangle$ and $\langle A^{-1}, C \rangle$ are nonsingular $(0, 3)$ subgroups of $\mathrm{Sp}(3, 1)$. We further assume that a_B, r_C do not lie on a proper totally geodesic subspace joining a_A and r_A. Let $K = E_K(t, \psi, \xi_1, \xi_2, k_1, k_2, k_3)$ and $K' = E_{K'}(t', \psi', \xi'_1, \xi'_2, k'_1, k'_2, k'_3)$ represent twist-bend parameters that are oriented consistently with A. If*

$$[\tilde{\mathbb{X}}_1(\kappa)] = [\tilde{\mathbb{X}}_1(\kappa')], \ \ [\tilde{\mathbb{X}}_2(\kappa)] = [\tilde{\mathbb{X}}_2(\kappa')], \ \ [\tilde{\mathbb{X}}_3(\kappa)] = [\tilde{\mathbb{X}}_3(\kappa')];$$

$$\tilde{\mathbb{A}}_1(\kappa) = \tilde{\mathbb{A}}_1(\kappa'), \ \ \tilde{\mathbb{A}}_3(\kappa) = \tilde{\mathbb{A}}_3(\kappa');$$

and $k_1 = k'_1$, $k_2 = k'_2$, $k_3 = k'_3$, then $K = K'$.

PROOF. Without loss of generality, we assume that $a_A = o$ and $r_A = \infty$. In view of the conditions

$$[\tilde{\mathbb{X}}_1(\kappa)] = [\tilde{\mathbb{X}}_1(\kappa')], \quad [\tilde{\mathbb{X}}_2(\kappa)] = [\tilde{\mathbb{X}}_2(\kappa')], \quad [\tilde{\mathbb{X}}_3(\kappa)] = [\tilde{\mathbb{X}}_3(\kappa')] \text{ and}$$

$$\tilde{\mathbb{A}}_1(\kappa) = \tilde{\mathbb{A}}_1(\kappa'), \ \tilde{\mathbb{A}}_3(\kappa) = \tilde{\mathbb{A}}_3(\kappa')$$

and noting that $\tilde{\mathbb{A}}_2(\kappa)$ and $\tilde{\mathbb{A}}_2(\kappa')$ are trivially equal, following similar arguments as in the proof of [Cao16, Theorem 5.2], we have f in $\mathrm{Sp}(3,1)$ such that $f(a_A) = a_A$, $f(r_A) = r_A$, $f(a_B) = a_B$ and $f(E_K(r_C)) = E_{K'}(r_C)$. Since f fixes three points on the boundary, it must be of the form

$$f = \begin{bmatrix} \mu_o & 0 & 0 & 0 \\ 0 & \mu_o & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{bmatrix}.$$

The boundary fixed point set of such a transformation always bounds a proper totally geodesic subspace of $\mathbf{H}_{\mathbb{H}}^3$. Since a_B does not lie on a proper totally geodesic subspace joining a_A and r_A, we must have $f = \pm I$. Thus, it follows that $E_K(r_C) = E_{K'}(r_C)$. Now, by using the fact that $E_K E_{K'}^{-1}$ has the three fixed points $a_A = o, r_A = \infty$ and r_C together with the condition that r_C does not lie on a totally geodesic subspace joining a_A and r_A, we have $E_K = E_{K'}$.

Hence, K and K' are conjugate with the same attracting and the same repelling points. So, by Lemma 3.1, $K = K'$ if and only if they have the same projective points and the same fixed points. This completes the proof.   □

6.2. Proof of Theorem 1.15. After we have Theorem 1.12 and Lemma 6.1, the proof of Theorem 1.15 follows by mimicking the arguments in [PP08, GK1]. We sketch it here.

Let $\Sigma_g \setminus C$ be the complement of the curve system C in Σ_g. This is a disjoint union of $2g - 2$ three-holed spheres. Each of the three-holed spheres corresponds to a nonsingular $(0,3)$ subgroup of $\mathrm{Sp}(3,1)$. By Corollary 1.13, a $(0,3)$ subgroup $\langle A, B \rangle$ is determined up to conjugacy by the 36 real parameters. While attaching two three-holed spheres, we attach two $(0,3)$ groups subject to the compatibility condition that a peripheral element in one group is conjugate to the inverse of a peripheral element in the other group. This gives a $(0,4)$ group that can be seen to be determined by 72 real parameters. Proceeding in this way, attaching $2g - 2$ of the above $(0,3)$ groups, we get a surface with $2g$ handles, and it is determined by $36(2g - 2) = 72g - 72$ real parameters obtained from the attaching process. The handles correspond to the g curves that in turn correspond to the two boundary components of the three-holed spheres. Now there are g quaternionic constraints that are imposed to close these handles: one of the peripheral elements of each of these $(0,3)$ groups must be conjugate to the inverse of the other peripheral element. Note that, corresponding to each peripheral element, there are 10 natural real parameters: the real trace and

two projective points. So, the number of real parameters reduces to $72g - 72 - 10g = 62g - 72$. But there are g twist-bend parameters $\kappa_i = (s_i, \psi_i, \xi_{i_1}, \xi_{i_2}, k_{1i}, k_{2i}, k_{3i})$, one for each handle, and each contributes 10 real parameters. Thus, we need $62g - 72 + 10g = 72g - 72$ real parameters to determine ρ up to conjugacy.

This proves the theorem.

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