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TWO RESULTS ON DYNAMIC EXTENSIONS OF DEVIATION MEASURES

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Abstract

We give a dynamic extension result of the (static) notion of a deviation measure. We also study distribution-invariant deviation measures and show that the only dynamic deviation measure which is law invariant and recursive is the variance.

Keywords: Deviation measure; time consistency; law invariance

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1. Introduction

The traditional way of thinking about risk, playing a crucial role in most fields involved with probabilities, is to measure risk as the deviation of the random outcomes from the longtime average, i.e. to measure risk for instance as the variance or the standard deviation involved. This is in particular the case for portfolio choice theory, where most of the finance literature simply describes portfolio selection as the choice between return (mean) and risk (variance). For stock prices in a continuous-time setting risk is also often identified with volatility, i.e. as the local standard deviation on an incremental time unit.

However, variance penalizes positive deviations from the mean in the same way as negative deviations, which in many contexts is not suitable. Furthermore, computing the variance or the standard deviation is mainly justified by its nice analytical, computational, and statistical properties, but is an ad hoc procedure and it is not clear whether better methods could be used. To overcome these shortfalls, Rockafellar *et al.* (2002) developed a general axiomatic framework for static deviation measures; see also, among many others, Rockafellar *et al.* (2006). This work was inspired by the axiomatic construction of coherent and convex risk measures given in Artzner *et al.* (1999), (2000), Föllmer and Schied (2002), and Frittelli and Rosazza Gianin (2002). Coherent or convex risk measures describe the minimal capital reserves a financial institution should hold in order to be 'safe'. As Artzner *et al.* (2000) gave an axiomatic characterization of capital reserves, these works give an axiomatic framework for deviation measures.

This theory of generalized deviation measures can be extended to a dynamic setting using the conditional variance formula (see Pistorius and Stadje (2017)) in the same spirit as convex risk measures have been extended to a dynamic setting using the tower property. For the latter, see, for instance, Artzner *et al.* (2004), Cheridito *et al.* (2006), Klöppel and Schweizer (2007), Delbaen *et al.* (2010), Pelsser and Stadje (2014), and Elliott *et al.* (2015).

For many risk-measure-like operators an important feature is distribution invariance. This is a convenient property as it enables the agent to focus only on the end distribution of the payoff,

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which is often known explicitly or can be simulated through Monte Carlo methods. There are many distribution-invariant static deviation measures, but it is a priori not clear if, apart from variance, there are other dynamic deviation measures belonging to this class. For distribution-invariant convex risk measures, Kupper and Schachermayer (2009) showed, building on results of Gerber (1974), that the so-called entropic risk measures are essentially the only convex risk measures satisfying the tower property; see also Goovaerts and De Vylder (1979) and Kaluszka and Krzeszowiec (2013). The entropic risk measure arises as the negative certainty equivalent of a decision maker with an exponential utility function; see, for instance, Föllmer and Schied (2002). The contribution of this paper is twofold: first we show how a static deviation measure can be extended dynamically, and second we study distribution-invariant deviation measures and show that the only dynamic deviation measure which is law invariant and recursive is the variance. Interestingly, it is also known in other contexts that there is a close relationship between the variance and the entropic risk measure (or equivalently the use of an exponential utility function). For instance, it is well known in the economics literature that the mean-variance principle can be seen as a second-order Taylor approximation to the entropic risk measure. Furthermore, both induce preferences which are invariant under shifts of wealth and lead to the same optimal portfolios under normality assumptions. Moreover, it has been shown, for instance by Pelsser and Stadje (2014), that in a Brownian filtration applying mean-variance recursively over an infinitesimal small time interval is equivalent to applying the entropic risk measure recursively over an infinitesimal small time interval. This paper adds to these results, showing that the entropic risk measure and the variance are the only distribution-invariant risk measures which naturally extend to continuous time under dynamic consistency conditions.

The paper is structured as follows. Section 2 introduces the setting and the basic concepts and definitions. It also shows under which specific conditions a static deviation measure gives rise to a (consistent) dynamic deviation measure. Section 3 analyzes distribution-invariant dynamic deviation measures.

2. Setting

Formally, we consider from now on a filtered, completed, right-continuous probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $T > 0$ and $\mathcal{F}_0$ is the trivial σ-algebra. Throughout the text, equalities and inequalities between random variables are meant to hold P-almost surely (a.s.); two random variables are identified if they are equal P-a.s. For $t \in [0, T]$, we define $L^2(\mathcal{F}_t)$ as the space of $\mathcal{F}_t$-measurable random variables X such that $\mathbb{E}\left[X^2\right] < \infty$. $L^2_+(\mathcal{F}_t)$, $L^\infty(\mathcal{F}_t)$, and $L^\infty_+(\mathcal{F}_t)$ denote the subsets of non-negative, bounded, and non-negative bounded elements in $L^2(\mathcal{F}_t)$, respectively.

2.1. (Conditional) deviation measures

Dynamic deviation measures are given in terms of conditional deviation measures, which are in turn conditional versions of the notion of a (static) deviation measure as in Rockafellar *et al.* (2006), which we describe next. Consider the (risky) positions described by elements in $L^2(\mathcal{F}_t)$.

Definition 1. For any given $t \in [0, T]$, $D_t : L^2(\mathcal{F}_T) \to L^2_+(\mathcal{F}_t)$ is called an *$\mathcal{F}_t$-conditional generalized deviation measure* if it is *normalized* ($D_t(0) = 0$) and the following properties are satisfied:

(D1) *Translation invariance*: $D_t(X + m) = D_t(X)$ for any $m \in L^\infty(\mathcal{F}_t)$ and $X \in L^2(\mathcal{F}_T)$.

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(D2) *Positivity*: $D_t(X) \geq 0$ for any $X \in L^2(\mathcal{F}_T)$, and $D_t(X) = 0$ if and only if X is $\mathcal{F}_t$-measurable.

(D3′) *Subadditivity*: $D_t(X+Y) \leq D_t(X) + D_t(Y)$ for any $X, Y \in L^2(\mathcal{F}_T)$.

(D4′) *Positive homogeneity*: $D_t(\lambda X) = \lambda D_t(X)$ for any $X \in L^2(\mathcal{F}_T)$ and $\lambda \in L^\infty_+(\mathcal{F}_t)$.

D_0 is a deviation measure in the sense of Definition 1 in Rockafellar *et al.* (2006). It is well known that if (D4′) holds, (D3′) is equivalent to

(D3) *Convexity*: For any $X, Y \in L^2(\mathcal{F}_T)$ and any $\lambda \in L^\infty(\mathcal{F}_t)$ that satisfies $0 \leq \lambda \leq 1$,

$$D_t(\lambda X + (1-\lambda)Y) \leq \lambda D_t(X) + (1-\lambda) D_t(Y).$$

Definition 2. For any given $t \in [0, T]$, $D_t : L^2(\mathcal{F}_T) \to L^2_+(\mathcal{F}_t)$ is called an $\mathcal{F}_t$*-conditional convex deviation measure* if it is *normalized* ($D_t(0) = 0$) and satisfies (D1)–(D3) . A deviation measure $D_0 : L^2(\mathcal{F}_T) \to \mathbb{R}^+_0$ is called an *unconditional deviation measure* if it is normalized and satisfies (D1)–(D3) for $t = 0$.

By postulating convexity in the following instead of (D3′) and (D4′) our dynamic theory will be richer and include more examples. If D_0 is an unconditional deviation measure, it is a finite convex functional and hence satisfies the following continuity condition:

- Continuity: *If X^n converges to X in $L^2(\mathcal{F}_T)$ then $D_0(X) = \lim_n D_0(X^n)$.*

A typical example of a conditional deviation measure satisfying (D1)–(D3) would be to identify risk with the conditional variance and to define

$$D_t(X) := \mathrm{Var}_t(X) = \mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{F}_t])^2 \mid \mathcal{F}_t\right].$$

Remark 1. As mentioned in the introduction, the axiomatic development of the theory of deviation measures in Rockafellar *et al*. (2002), (2006) was inspired by the axiomatic development of the theory of convex risk measures. Mappings $\rho_t : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$ are a family of dynamic convex risk measures if the following properties are satisfied:

(R1) *Cash risklessness*: For all $m \in L^\infty(\mathcal{F}_t)$ we have $\rho_t(m) = -m$.

(R2) *Convexity*: For $X, Y \in L^2(\mathcal{F}_T)$, $\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda \rho_t(X) + (1-\lambda)\rho_t(Y)$ for all $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

(R3) *Monotonicity*: If $X, Y \in L^2(\mathcal{F}_T)$ and $X \leq Y$ then $\rho_t(X) \geq \rho_t(Y)$.

(R4) *Strong time consistency*: If $X \in L^2(\mathcal{F}_T)$ and $0 \leq s \leq t \leq T$ then $\rho_s(X) = \rho_s(-\rho_t(X))$.

Monotonicity (R3) is an axiom which does not make sense for deviation measures since, for instance, $D_t(m) = 0$ for all constants m. Obviously (R4) implies (R3) if (R1) holds, but for later discussion it will be useful to list them separately.

Note that (D3) implies that the following two equivalent properties hold for all $X \in L^2(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$ (see Cheridito *et al.* (2006)):

$$D_t(I_A X_1 + I_{A^c} X_2) = I_A D_t(X_1) + I_{A^c} D_t(X_2), \tag{2.1}$$

$$D_t(I_A X) = I_A D_t(X). \tag{2.2}$$

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Equation (2.2) is also called the local property. Now, in a theory of risk in a dynamic setting one needs to specify how the evaluation of risk tomorrow affects the evaluation of risk today. Intuitively it seems appealing to relate the overall deviation to an expectation of the fluctuations we expect after tomorrow plus the fluctuations happening until tomorrow. To be precise, we will postulate the following property:

(D4) *Recursive property*: For $X \in L^2(\mathcal{F}_T)$, $D_t(X) = D_t(\mathbb{E}[X \mid \mathcal{F}_s]) + \mathbb{E}[D_s(X) \mid \mathcal{F}_t]$ for all $t, s \in [0, T]$ with $t \le s$.

Obviously, the recursive property corresponds to the conditional variance formula. Axiom (D4) was used in Pistorius and Stadje (2017).

Definition 3. A family $(D_t)_{t\in[0,T]}$ is called a *dynamic deviation measure* if D_t, $t \in [0, T]$, are $\mathcal{F}_t$-conditional deviation measures satisfying (D4) .

Remark 2. Given a conditional risk measure ρ, one can define a conditional deviation measure $D_t(X) = R_t(X - \mathbb{E}[X \mid \mathcal{F}_t])$. Then D_t satisfies (D1)–(D3) . On the other hand, given a dynamic deviation measure D_t we can define a risk measure $\tilde{\rho}_t$ by $\tilde{\rho}_t(X) = D_t(X) - \mathbb{E}[X \mid \mathcal{F}_t]$. Note that $\tilde{\rho}$ satisfies (R1) and (R2) , but it does not necessarily satisfy (R3) (monotonicity). Denote $\operatorname{ess\,inf}_t X := \operatorname{ess\,sup}\{Y \text{ is } \mathcal{F}_t\text{-measurable } \mid Y \le X\}$. One can show, as in Theorem 1 in Rockafellar *et al.* (2002), that if, additionally, the condition $D_t(X) \le \mathbb{E}[X \mid \mathcal{F}_t] - \operatorname{ess\,inf}_t X$ holds for any payoff X then $\tilde{\rho}$ satisfies monotonicity. Therefore, there is a one-to-one relationship between conditional deviation measures satisfying the above condition and conditional convex risk measures. It should be noted that mean-variance does not satisfy the monotonicity property (R3) . Therefore, in general dynamic deviation measures satisfying (D1)–(D4) do not satisfy monotonicity as well. In particular, they do not satisfy the inequality above and thus do not induce convex or coherent risk measures. Finally, we remark that $\tilde{\rho}$ satisfying strong time consistency (R4) does not correspond to D satisfying (D4) . In fact, these are fundamentally different recursions which are mutually exclusive, see also Pistorius and Stadje (2017).

In the following we start with an unconditional deviation measure D_0 which induces a dynamic deviation measure $(D_t)_{t\in[0,T]}$. We make the following two assumptions:

(A1) $D_0(X) = D_0(\mathbb{E}[X \mid \mathcal{F}_t]) + D_0(X - \mathbb{E}[X \mid \mathcal{F}_t])$ for all $X \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$.

(A2) For any $X \in L^2(\mathcal{F}_T)$ and any $t \in [0, T]$ there exists an $\mathcal{F}_t$-measurable, square integrable random variable, say $D_t(X)$, such that, for all $A \in \mathcal{F}_t$,

$$D_0(I_A X) - D_0(I_A \mathbb{E}[X \mid \mathcal{F}_t]) = \mathbb{E}[I_A D_t(X)] . \tag{2.3}$$

In particular, for any $X \in L^2(\mathcal{F}_T)$,

$$D_0(X) = D_0(\mathbb{E}[X \mid \mathcal{F}_t]) + \mathbb{E}[D_t(X)] . \tag{2.4}$$

Theorem 1. *$D_t(X)$ is unique (up to an a.s. modification) if (A2) holds. Furthermore D_0 satisfies (D1)–(D3) (for $t = 0$) and (A1) and (A2) if and only if the corresponding family $(D_t(X))_{t\in[0,T]}$ is a dynamic deviation measure.*

Proof. That $D_t(X)$ is uniquely defined is seen as follows. Suppose that besides $D_t(X)$ there exists another square integrable $\mathcal{F}_t$-measurable random variable $D'_t(X)$ satisfying (A2) for all $A \in \mathcal{F}_t$. Fix X and denote the $\mathcal{F}_t$-measurable set A' by $A' := \{D'_t(X) > D_t(X)\}$. If we assume that A' has non-zero measure then, by (A2) ,

$$\mathbb{E}\left[I_{A'} D_t(X)\right] = D_0(I_{A'} X) - D_0(\mathbb{E}\left[I_{A'} X \mid \mathcal{F}_t\right]) = \mathbb{E}\left[I_{A'} D'{}_t(X)\right],$$

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which is a contradiction to the definition of the set A'. That the set $\{D'_t(X) < D_t(X)\}$ must have measure zero as well is seen similarly. That D_0 satisfies (D1) –(D3) and (A2) if $(D_t(X))_{t\in[0,T]}$ is a dynamic deviation measure is straightforward to see. To see that (A1) also holds, assume first that $X \in L^\infty(\mathcal{F}_T)$. Then, by (D4) and (D1) , $D_0(X - \mathbb{E}\,[X \mid \mathcal{F}_t]) = \mathbb{E}\,[D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t])] = \mathbb{E}\,[D_t(X)]$. From (D4) we can then conclude that (A1) holds for bounded random variables. For general $X \in L^2(\mathcal{F}_T)$ we can find bounded X_n converging to X in L^2 and obtain

$$
\begin{aligned}
D_0(X) &= \lim_n D_0(X_n) \\
&= \lim_n \{D_0(\mathbb{E}\,[X_n \mid \mathcal{F}_t]) + D_0(X_n - \mathbb{E}\,[X_n \mid \mathcal{F}_t])\} \\
&= D_0(\mathbb{E}\,[X \mid \mathcal{F}_t]) + D_0(X - \mathbb{E}\,[X \mid \mathcal{F}_t]),
\end{aligned}
$$

where we used the continuity of D_0 in the first and last equations. In the second equation we used that we have already shown that (A1) holds for bounded random variables.

Next, let us show that if D_0 satisfies (D1)–(D3) (for $t = 0$) and (A1) and (A2) , the corresponding family $(D_t(X))_{t\in[0,T]}$ is a dynamic deviation measure. First, we will show the local property $D_t(I_B X) = I_B D_t(X)$ for all $B \in \mathcal{F}_t$. To see this, note that, by (2.3), for all $A \in \mathcal{F}_t$,

$$
\begin{aligned}
D_0(I_A(I_B X)) - D_0(I_A \mathbb{E}\,[I_B X \mid \mathcal{F}_t]) &= D_0(I_{A\cap B} X) - D_0(I_{A\cap B}\mathbb{E}\,[X \mid \mathcal{F}_t]) \\
&= \mathbb{E}\,[I_{A\cap B} D_t(X)] = \mathbb{E}\,[I_A I_B D_t(X)]\,.
\end{aligned}
$$

Hence, by the uniqueness of $D_t(I_B X)$ shown before, we must have that $D_t(I_B X) = I_B D_t(X)$.

Let us next show (D1) . Comparing (2.4) to property (A1) (replacing X with $I_A X$) we obtain that $D_0(I_A X - \mathbb{E}\,[I_A X \mid \mathcal{F}_t]) = \mathbb{E}\,[D_t(I_A X)]$. Consequently, for $A \in \mathcal{F}_t$,

$$
\begin{aligned}
\mathbb{E}\,[I_A D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t])] &= \mathbb{E}\,[D_t(I_A X - \mathbb{E}\,[I_A X \mid \mathcal{F}_t])] \\
&= D_0(I_A X - \mathbb{E}\,[I_A X \mid \mathcal{F}_t]) = \mathbb{E}\,[D_t(I_A X)] = \mathbb{E}\,[I_A D_t(X)]\,,
\end{aligned}
$$

where the second equation follows by (2.4) applied to $I_A X$. Since the above equation holds for all $A \in \mathcal{F}_t$, we must have that $D_t(X) = D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t])$ a.s. for $X \in L^2(\mathcal{F}_T)$. This yields, for arbitrary $m \in L^2(\mathcal{F}_t)$, that

$$
D_t(X + m) = D_t(X + m - \mathbb{E}\,[X + m \mid \mathcal{F}_t]) = D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t]) = D_t(X), \tag{2.5}
$$

showing in particular that D_t satisfies (D1) .

Next, let us show that D_t satisfies (D2) . For any $X \in L^2(\mathcal{F}_T)$ with $\mathbb{E}\,[X \mid \mathcal{F}_t] = 0$ and $A \in \mathcal{F}_t$ we have

$$
0 \le D_0(I_A X) = \mathbb{E}\,[D_t(I_A X)] = \mathbb{E}\,[I_A D_t(X)]\,,
$$

where we used positivity of D_0 in the first inequality and (2.4) in the first equality. From this inequality it follows that the set $A = \{D_t(X) < 0\}$ must have measure zero, showing that $D_t(X) \ge 0$ if $\mathbb{E}\,[X \mid \mathcal{F}_t] = 0$. Now, for general $X \in L^2(\mathcal{F}_T)$ we then have, by (D1) (shown before), that $D_t(X) = D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t]) \ge 0$, showing the first part of (D2) for D_t. To prove the second part of positivity for D_t, note that clearly (2.4) directly implies that $\mathbb{E}\,[D_t(X)] = 0$ if X is $\mathcal{F}_t$-measurable. As $D_t(X) \ge 0$, this entails that $D_t(X) = 0$. To see that $D_t(X) = 0$ implies that X is $\mathcal{F}_t$-measurable, note that

$$
D_0(X - \mathbb{E}\,[X \mid \mathcal{F}_t]) = \mathbb{E}\,[D_t(X - \mathbb{E}\,[X \mid \mathcal{F}_t])] = \mathbb{E}\,[D_t(X)] = 0,
$$

where we used (2.4) in the first equality, (2.5) in the second equality, and the assumption that $D_t(X) = 0$ in the third. Using axiom (D2) for D_0 we can infer that $X - \mathbb{E}[X \mid \mathcal{F}_t]$ is constant. But this entails that $X = \mathbb{E}[X \mid \mathcal{F}_t]$, and thus that X is $\mathcal{F}_t$-measurable.

Next, let us show that D_t satisfies (D3) . Let $X, Y \in L^2(\mathcal{F}_T)$, and let us first assume that X and Y both have conditional expectation equal to zero. Then, for any $A \in \mathcal{F}_t$,

$$\begin{aligned}\mathbb{E}[I_A D_t(\lambda X + (1-\lambda)Y)] &= \mathbb{E}[D_t(\lambda I_A X + (1-\lambda) I_A Y)] \\ &= D_0(\lambda I_A X + (1-\lambda) I_A Y) \\ &\leq \lambda D_0(I_A X) + (1-\lambda) D_0(I_A Y) \\ &= \lambda \mathbb{E}[D_t(I_A X)] + (1-\lambda)\mathbb{E}[D_t(I_A Y)] \\ &= \mathbb{E}[I_A(\lambda D_t(X) + (1-\lambda) D_t(Y))], \end{aligned} \tag{2.6}$$

where we used the convexity of D_0 in the inequality. Set $A := \{D_t(\lambda X + (1-\lambda)Y) > \lambda D_t(X) + (1-\lambda) D_t(Y)\}$, and note that (2.6) implies that A must have measure zero. Hence, D_t is convex for random variables with conditional expectation zero. To see that D_t is convex for general $X, Y \in L^2(\mathcal{F}_T)$, note that

$$\begin{aligned}D_t(\lambda X + (1-\lambda)Y) &= D_t(\lambda(X - \mathbb{E}[X \mid \mathcal{F}_t]) + (1-\lambda)(Y - \mathbb{E}[Y \mid \mathcal{F}_t])) \\ &\leq \lambda D_t(X - \mathbb{E}[X \mid \mathcal{F}_t]) + (1-\lambda) D_t(Y - \mathbb{E}[Y \mid \mathcal{F}_t]) \\ &= \lambda D_t(X) + (1-\lambda) D_t(Y),\end{aligned}$$

where the first and the last equations hold by (D1) . This shows that (D3) holds.

Finally, let us show that $(D_t(X))_{t\in[0,T]}$ satisfies (D4) . Specifically, we want to show that

$$D_t(X) = D_t(\mathbb{E}[X \mid \mathcal{F}_s]) + \mathbb{E}[D_s(X) \mid \mathcal{F}_t] \tag{2.7}$$

with $s \in [t, T]$. Equation (2.7) would follow by uniqueness if we could show that the right-hand side of (2.7) satisfies (2.3) for any $A \in \mathcal{F}_t$ when plugged in for $D_t(X)$. We have

$$\begin{aligned}\mathbb{E}[I_A\{D_t(\mathbb{E}[X \mid \mathcal{F}_s]) + \mathbb{E}[D_s(X) \mid \mathcal{F}_t]\}] &= \mathbb{E}[I_A D_t(\mathbb{E}[X \mid \mathcal{F}_s])] + \mathbb{E}[I_A D_s(X)] \\ &= D_0(I_A \mathbb{E}[X \mid \mathcal{F}_s]) - D_0(I_A \mathbb{E}[X \mid \mathcal{F}_t]) \\ &\quad + D_0(I_A X) - D_0(I_A \mathbb{E}[X \mid \mathcal{F}_s]) \\ &= D_0(I_A X) - D_0(I_A \mathbb{E}[X \mid \mathcal{F}_t]),\end{aligned}$$

where the second equation holds by (2.3) applied to the random variable $\mathbb{E}[X \mid \mathcal{F}_s]$ at time t and to the random variable X at time s. Hence, (2.7) holds by the uniqueness of property (A2) for $\mathcal{F}_t$-measurable random variables, and the proof is complete. □

The theorem above and (2.5) yield the following corollary:

Corollary 1. *For any dynamic deviation measure* $(D_t(X))$ *we have*

$$D_t(X + m) = D_t(X) \text{ for } m \in L^2(\mathcal{F}_t) \text{ and } X \in L^2(\mathcal{F}_T).$$

The proof of the following proposition is analogous to Proposition 2.7 in Pistorius and Stadje (2017).

Proposition 1. *Let $I := \{t_0, t_1, \dots, t_n\} \subset [0, T]$ be strictly ordered. $D = (D_t)_{t\in I}$ satisfies (D1)–(D3) and (D4) if and only if, for some collection $\tilde{D} = (\tilde{D}_t)_{t\in I}$ of conditional deviation measures, we have*

$$D_t(X) = \mathbb{E}\left[\sum_{t_i\in I: t_i\geq t} \tilde{D}_{t_i}\left(\mathbb{E}\left[X \mid \mathcal{F}_{t_{i+1}}\right] - \mathbb{E}\left[X \mid \mathcal{F}_{t_i}\right]\right) \,\Big|\, \mathcal{F}_t\right], \quad t \in I,\ X \in L^2(\mathcal{F}_T). \tag{2.8}$$

In particular, a dynamic deviation measure D satisfies (2.8) with $\tilde{D}_{t_i} = D_{t_i}$, $t_i \in I$.

3. Distribution-invariant deviation measures

The next result investigates the question of what happens if we impose, in addition to axioms (D1)–(D4) , the property of *distribution invariance*. A dynamic deviation measure D is distribution invariant if $D_0(X_1) = D_0(X_2)$ whenever X_1 and X_2 have the same distribution. Distribution invariance is a property which is often not satisfied in a finance context when it comes to evaluation and risk analysis. The reason is that the value of a payoff may not only depend on the nominal discounted value of the payoff itself but also on the whole state of the economy or the performance of the entire financial market. For instance, in no-arbitrage pricing scenarios are additionally weighted with a (risk neutral) density so that the value of a certain payoff in a certain scenario depends not only on the frequency with which the corresponding scenario occurs but also on the state of the whole economy. Also, in most asset pricing models in finance, not only the distribution of an asset matters but also its correlation to the whole market portfolio. However, for deviation measures, distribution invariance is a convenient property as it enables the agent to focus only on the end distribution of the payoff (which is often known explicitly or can be simulated through Monte Carlo methods). There are many distribution-invariant static deviation measures, but it is a priori not clear if, apart from variance, there are other dynamic deviation measures belonging to this class. The next theorem shows that this is actually not the case, and that variance is the only dynamic distribution-invariant deviation measure. This result can also serve as justification for using variance as a dynamic deviation measure. Namely, a decision maker who believes for a static deviation measure in axioms (D1)–(D3) , (A1) and (A2) , and distribution invariance, or for a dynamic deviation measure in axioms (D1)–(D4) and distribution invariance, necessarily has to use variance as a deviation measure. For these results we will assume that the probability space is rich enough to support a one-dimensional Brownian motion.

Theorem 2. *A deviation measure D_0 satisfying (D1)–(D3) and (A1) and (A2) is distribution invariant if and only if D_0 is a positive multiple of the variance, i.e. there exists an $\alpha > 0$ such that*

$$D_0(X) = \alpha \operatorname{Var}(X) \quad \text{for all } X \in L^2(\mathcal{F}_T).$$

For the proof of Theorem 2 we will need the following lemma:

Lemma 1. *Suppose that D_t is a family of dynamic distribution-invariant deviation measures and that Y is independent of $\mathcal{F}_t$. Then $D_t(Y)$ is constant and*

$$D_t(Y) = D_0(Y).$$

Proof. The case $t = 0$ is trivial, so let us assume that $t > 0$. Suppose then that $D_t(Y)$ is not P-a.s. constant. Choose sets $A, A' \in \mathcal{F}_t$ with $\mathbb{P}(A) = \mathbb{P}(A') > 0$ such that $D_t(Y)(\omega) > D_t(Y)(\omega')$

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for all $\omega \in A$, $\omega' \in A'$. Next, note that, by independence, $I_A Y \overset{\mathrm{D}}{\sim} I_{A'} Y$. However,

$$\begin{aligned} D_0(I_A Y) &= D_0(\mathbb{E}\,[I_A Y \mid \mathcal{F}_t]) + \mathbb{E}\,[D_t(I_A Y)] \\ &= D_0(I_A \mathbb{E}\,[Y \mid \mathcal{F}_t]) + \mathbb{E}\,[I_A D_t(Y)] \\ &= D_0(I_A \mathbb{E}\,[Y]) + \mathbb{E}\,[I_A D_t(Y)] \\ &< D_0(I_A \mathbb{E}\,[Y]) + \mathbb{E}\left[I_{A'} D_t(Y)\right] \\ &= D_0(I_{A'} \mathbb{E}\,[Y]) + \mathbb{E}\left[I_{A'} D_t(Y)\right] \\ &= D_0(\mathbb{E}\left[I_{A'} Y \mid \mathcal{F}_t\right]) + \mathbb{E}\left[D_t(I_{A'} Y)\right] = D_0(I_{A'} Y), \end{aligned}$$

which is a contradiction to the distribution invariance of D_0. So, $D_t(Y)$ is indeed constant. Finally, by the first part of the proof,

$$\begin{aligned} D_0(Y) &= \mathbb{E}\,[D_t(Y)] + D_0(\mathbb{E}\,[Y \mid \mathcal{F}_t]) \\ &= \mathbb{E}\,[D_t(Y)] + D_0(\mathbb{E}\,[Y]) = \mathbb{E}\,[D_t(Y)] = D_t(Y). \end{aligned}$$

Proof of Theorem 2. Clearly, variance is distribution invariant. To see the other direction, assume without loss of generality that $T = 1$. Let us first show that the theorem holds for X having a normal distribution. Let Z be a standard normally distributed random variable. Define $f(\sigma) = D_0(\sigma Z)$ with $\sigma \in \mathbb{R}$. By assumption, there exists an adapted Brownian motion, say $(B_t)_{0 \le t \le 1}$. We thus have, for $0 \le t \le 1$,

$$\begin{aligned} f(\sigma\sqrt{t}) &= D_0(\sqrt{t}\sigma Z) \\ &= D_0(\sigma B_t) \\ &= \sum_{i=0}^{n-1} \mathbb{E}\left[D_{ti/n}(\sigma\,\Delta B_{t(i+1)/n})\right] = \sum_{i=0}^{n-1} D_0(\sigma\,\Delta B_{t(i+1)/n}) \\ &= nD_0\left(\frac{\sqrt{t}\sigma Z}{\sqrt{n}}\right) = nf\left(\frac{\sqrt{t}\sigma}{\sqrt{n}}\right), \end{aligned}$$

where we set $\Delta B_{t(i+1)/n} := B_{t(i+1)/n} - B_{ti/n}$ and used Proposition 1. It follows that $f\left(\frac{\sqrt{t}\sigma}{\sqrt{n}}\right) = \frac{f(\sqrt{t}\sigma)}{n}$. Arguing as before, we also get, for $k \in \mathbb{N}$ with $\frac{k}{n} \le \frac{1}{t}$,

$$\begin{aligned} f\left(\sqrt{\frac{k}{n}t\sigma}\right) &= D_0(\sigma B_{kt/n}) \\ &= \sum_{i=0}^{k-1} \mathbb{E}\left[D_{ti/n}(\sigma\,\Delta B_{t(i+1)/n})\right] \\ &= kD_0(\sigma B_{t/n}) = kD_0\left(\frac{\sigma B_t}{\sqrt{n}}\right) = \frac{k}{n} D_0(\sigma B_t) = \frac{k}{n} f(\sigma\sqrt{t}). \end{aligned}$$

By continuity of D_0 we have that f is continuous. Therefore, for all $0 \le \lambda \le \frac{1}{t}$, $f(\lambda\sigma\sqrt{t}) = \lambda^2 f(\sigma\sqrt{t})$ for any $\sigma \in \mathbb{R}$. Setting, for arbitrary $x \in \mathbb{R}$, $\sigma = x/\sqrt{t}$, we get that $f(\lambda x) = \lambda^2 f(x)$ for

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all $0 \le \lambda \le \frac{1}{t}$ with $t \in [0, 1]$. Choosing t arbitrary small, we may conclude that $f(\lambda x) = \lambda^2 f(x)$ for all $\lambda \in \mathbb{R}_+$. Hence, if we define $\alpha := f(1) > 0$ we have that

$$D_0(\sigma Z) = D_0(|\sigma|Z) = f(|\sigma|) = \sigma^2\alpha = \alpha\,\mathrm{Var}(Z),$$

where the first equality follows by the distribution invariance of D_0.

Next, let us show that, for simple functions of the form $X = \int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s$ with $h_i = \sum_{j=1}^m c_j I_{A_j}$, $c_j \in \mathbb{R}^d$, and disjoint sets $A_j \in \mathcal{F}_{t_i}$ for $j = 1, \ldots, m$, we have

$$D_0(X) = \alpha\,\mathrm{Var}(X).$$

Now,

$$\begin{aligned} D_0(X) &= D_0(h_{t_i}\Delta B_{t_{i+1}}) \\ &= \mathbb{E}\left[\sum_{j=1}^m I_{A_j} D_{t_i}\big(c_j \Delta B_{t_{i+1}}\big)\right] = \alpha\,\mathbb{E}\left[\sum_{j=1}^m I_{A_j} c_j^2 (t_{i+1} - t_i)\right] = \alpha\,\mathrm{Var}(X), \end{aligned}$$

where we used (2.1) in the second equation, and Lemma 1 in the third equation to argue that $D_{t_i}(c_j\Delta B_{t_{i+1}}) = D_0(c_j\Delta B_{t_{i+1}}) = c_j^2(t_{i+1} - t_i)$. For $X = \int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s$ with general $h_i \in L_d^2(\mathcal{F}_{t_i}, P)$, choose simple functions h_i^n converging to h_i in L^2 and define $X^n = \big(\int_0^1 h_i^n I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s\big)_{t_i,t_{i+1}}$. Using the L^2-continuity of D_0, we may conclude that

$$D_0(X) = \lim_n D_0(X^n) = \lim_n \alpha\,\mathrm{Var}(X^n) = \alpha\,\mathrm{Var}(X).$$

Next, note that, for simple functions of the form $X = \sum_{i=1}^l \int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\mathrm{d}B_s$ for $l \in \mathbb{N}$, h_i being $\mathcal{F}_{t_i}$-measurable and square integrable, we have

$$\begin{aligned} D_0(X) &= \sum_{i=1}^l \mathbb{E}\left[D_{t_i}\left(\int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s\right)\right] \\ &= \sum_{i=1}^l D_0\left(\int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s\right) \\ &= \sum_{i=1}^l \alpha\,\mathrm{Var}\left(\int_0^1 h_i I(s)_{(t_i,t_{i+1}]}\,\mathrm{d}B_s\right) = \alpha\,\mathrm{Var}(X), \end{aligned}$$

where we used Proposition 1. Therefore, $D_0(X) = \alpha\,\mathrm{Var}(X)$ for all simple functions X. Using the L^2-continuity of D_0 and $\alpha\,\mathrm{Var}(X)$ as before, we get that equality actually holds for all $X \in L^2(\mathcal{F}_1^B)$, with $\mathcal{F}_1^B$ being the completion of the σ-algebra generated by $(B_t)_{0\le t\le 1}$. Next, take a general $X \in L^2(\mathcal{F}_1)$. Define the uniform [0, 1] distributed random variable $U = F_{B_1}(B_1)$, where F_{B_1} is the cumulative distribution function (cdf) of B_1. Set $X' = q_X(U) \overset{\mathrm{D}}{=} X$, where q_X is the right-continuous inverse of the cdf of X. Then clearly X' is $\mathcal{F}_1^B$-measurable. Therefore,

$$D_0(X) = D_0(X') = \alpha\,\mathrm{Var}(X') = \alpha\,\mathrm{Var}(X).$$

This proves the theorem. □

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Kupper and Schachermayer (2009) showed that a dynamic convex risk measure is law invariant if and only if there exists $\gamma \in [0, \infty]$ such that

$$\rho_t(X) = \frac{1}{\gamma}\mathbb{E}\left[\exp\left(-\gamma X\right) \mid \mathcal{F}_t\right]. \tag{3.1}$$

The limiting cases $\gamma = 0$ and $\gamma = \infty$ are identified with the conditional expectation and the essential supremum, respectively. In the workshop “Stochastic Analysis in Finance and Insurance” (2008) in Oberwolfach, Delbaen presented this result in a continuous time setting using duality representations from Delbaen *et al.* (2010). Related results are also known for insurance premiums: see Gerber (1974), Goovaerts and De Vylder (1979), and Kaluszka and Krzeszowiec (2013).

By Remark 2, the operators studied in this paper are neither monotone themselves (i.e. they do not satisfy (R3)) nor do they give rise to convex risk measures by adding a conditional expectation. Furthermore, our recursiveness condition (D4) does not also give rise to condition (R4) . Hence, the proofs by Delbaen or by Kupper and Schachermayer (2009) cannot be adapted to the results in this paper in a straightforward manner since the underlying properties of the dynamic risk measures are either different or do not hold at all. For instance, monotonicity is used in the last part of the proof of Theorem 1.10 in Kupper and Schachermayer (2009) and in the dual representations in Delbaen *et al.* (2010).

References

Artzner, Ph., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Math. Finance* **9**, 203–228.

Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D. and Ku, H. (2004). Coherent multiperiod risk adjusted values and Bellmans principle. *Ann. Operat. Res.* **152**, 5–22.

Cheridito, P., Delbaen, F. and Kupper, M. (2006). Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Prob.* **11**, 57–106.

Delbaen, F., Peng, S. and Rosazza Gianin, E. (2010). Representation of the penalty term of dynamic concave utilities. *Finance Stoch.* **14**, 449–472.

Elliott, R. J., Siu., T. K. and Cohen, S. N. (2015). Backward stochastic difference equations for dynamic convex risk measures on a binomial tree. *J. Appl. Prob.* **52**, 771–785.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance Stoch.* **6**, 429–447.

Frittelli, M. and Rosazza Gianin, E. (2002). Putting order in risk measures. *J. Bank. Finance* **26**, 1473–1486.

Gerber, H. U. (1974). On iterative premium calculation principles. *Sonderabdruck aus den Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* **74**, 2.

Goovaerts, M. J. De Vylder, F. (1979). A note on iterative premium calculation principles. *Astin Bull.* **10**, 325–329.

Kaluszka, M. and Krzeszowiec, M. (2013). On iterative premium calculation principles under Cumulative Prospect Theory. *Insurance Math. Econom.* **52**, 435–440.

Klöppel, S. and Schweizer, M. (2007). Dynamic indifference valuation via convex risk measures. *Math. Finance* **17**, 599–627.

Kupper, M. and Schachermayer, W. (2009). Representation results for law invariant time consistent functions. *Math. Finan. Econom.* **2**, 189–210.

Pelsser, A. and Stadje, M. (2014). Time-consistent and market-consistent evaluation. *Math. Finance* **24**, 25–62.

Pistorius, M. and Stadje, M. (2017). On dynamic deviation measures and continuous-time portfolio optimization. *Ann. Appl. Prob.* **27**, 3342–3384.

Rockafellar, R. T., Uryasev, S. P. and Zabarankin, M. (2002). Deviation measures in risk analysis and optimization. Working Paper 2002-7, Department of Industrial & Systems Engineering, University of Florida.

Rockafellar, R. T., Uryasev, S. P. and Zabarankin, M. (2006). Master funds in portfolio analysis with general deviation measures. *J. Banking* **30**, 743–778.