Canad. Math. Bull. Vol. 63 (4), 2020 pp. 738–743
http://dx.doi.org/10.4153/S0008439519000778
© Canadian Mathematical Society 2019

Irregularities in the Distribution of Prime Numbers in a Beatty Sequence

Janyarak Tongsomporn and Jörn Steuding

Abstract. We prove irregularities in the distribution of prime numbers in any Beatty sequence $\mathcal{B}(\alpha,\beta)$, where α is a positive real irrational number of finite type.

1 Introduction and Statement of the Main Result

In the 1980s, Helmut Maier [11] proved with his ingenious matrix method the existence of unexpected irregularities in the distribution of prime numbers. He showed that for every $b > 2$, there exist a constant $\delta_b > 0$ and arbitrarily large x, x' such that

$$\pi(x+y) - \pi(x) > (1+\delta_b)\frac{y}{\log x},$$
$$\pi(x'+y') - \pi(x') < (1-\delta_b)\frac{y'}{\log x'},$$

where $y = y(x) = (\log x)^b$, $y' = y(x')$, and $\pi(x)$ counts the primes up to x. In a later joint work with Adolf Hildebrand [8], even stronger irregularities were established. On the contrary, already in the 1940s, Atle Selberg [12] had shown under assumption of the Riemann hypothesis that

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x} \qquad \text{for almost all integers } x,$$

provided that $\lim_{x\to\infty} \frac{y}{\log x} = \infty$. This result supported the probabilistic model of Harald Cramér [3] from the 1930s. Ever since Maier's discovery of these unexpected exceptions, new applications of the matrix method were found. We refer to Andrew Granville's survey [7] and Frank Thorne's overview [14] for details. In this note we are concerned with the distribution of primes in a Beatty sequence.

Denote by $\lfloor x \rfloor$ the largest integer $\le x$. Given positive real numbers α, β, the set

$$\mathcal{B}(\alpha,\beta) = \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}$$

is called the associated *Beatty sequence* (or *Beatty set*). If α is rational, then $\mathcal{B}(\alpha,\beta)$ is a union of arithmetic progressions. If α is irrational and α' is defined by $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, however, then $\mathcal{B}(\alpha,0) \sqcup \mathcal{B}(\alpha',0)$ yields a partition of $\mathbb{N}$. This is known as Rayleigh's theorem or Beatty's theorem. There is also a more general version for the case of inhomogeneous Beatty sequences due to Aviezri Fraenkel [5] and Suton Tadee and

Received by the editors July 25, 2019; revised December 3, 2019.
Published online on Cambridge Core December 16, 2019.
AMS subject classification: **11A41**, 11B83.
Keywords: prime numbers, irregularities, Maier's matrix method, Beatty sequence.

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press

Vichian Laohakosol [13], respectively. In fact, if additionally $\frac{\beta}{\alpha} + \frac{\beta'}{\alpha'} \equiv 0 \bmod 1$ holds and the intersection of $\alpha'\mathbb{Z}+\beta' := \{\alpha' z+\beta' \,:\, z \in \mathbb{Z}\}$ with $\mathbb{Z}$ is empty, then $\mathcal{B}(\alpha,\beta) \sqcup \mathcal{B}(\alpha',\beta')$ is a disjoint union of all sufficiently large integers. In view of this result, there already exist irregularities in at least one of the appearing Beatty sequences $\mathcal{B}(\alpha,\beta)$ or $\mathcal{B}(\alpha',\beta')$.

In this context it is natural to ask whether there exist values α, β for which the distribution of primes within $\mathcal{B}(\alpha,\beta)$ is without irregularities? To answer this question we first notice that there are rational values α for which $\mathcal{B}(\alpha,\beta)$ does not contain any prime at all; for example,

$$\mathcal{B}\Big(\frac{44}{3},1\Big) = \{15,30,45\} + 3\mathbb{N}_0.$$

For irrational α, however, it follows from a classical result of Ivan Vinogradov [15] (in the context of his research on the Goldbach conjecture) that the number $\pi_{\mathcal{B}(\alpha,\beta)}(x)$ of primes $p \le x$ with $p \in \mathcal{B}(\alpha,\beta)$ is asymptotically given by

$$\pi_{\mathcal{B}(\alpha,\beta)}(x) \sim \tfrac{1}{\alpha}\pi(x) \tag{1.1}$$

(even with an explicit error term); notice that $\frac{1}{\alpha}$ is the natural density of $\mathcal{B}(\alpha,\beta)$ in $\mathbb{N}$. Our main result is the following theorem.

Theorem 1.1 *Let α be a positive real irrational number of finite type and let β be an arbitrary real number. Then for every $b > 2$, there exist a positive constant $\delta_b > 0$ and arbitrarily large x, x' such that*

$$\pi_{\mathcal{B}(\alpha,\beta)}(x+y) - \pi_{\mathcal{B}(\alpha,\beta)}(x) > \frac{1+\delta_b}{\alpha}\frac{y}{\log x},$$
$$\pi_{\mathcal{B}(\alpha,\beta)}(x'+y') - \pi_{\mathcal{B}(\alpha,\beta)}(x') < \frac{1-\delta_b}{\alpha}\frac{y'}{\log x'},$$

where $y = y(x) = (\log x)^b$ and $y' = y(x')$.

Recall that an irrational α is said to be of *type* τ if

$$\tau = \sup\{\rho \in \mathbb{R} : \liminf_{n\to\infty} n^\rho \|n\alpha\| = 0\},$$

where $\|x\|$ denotes the minimal distance of x to the nearest integer; moreover, α is of *finite type* if $\tau < \infty$. For example, $e = \exp(1)$ is of type $\tau = 1$; on the contrary, Liouville numbers are not.

Our reasoning follows Maier's original reasoning closely. The new ingredient is a prime number theorem for Beatty sequences due to William Banks and Igor Shparlinski [1] (which improves (1.1) for parameters α of finite type). In the following section, we collect this and some further useful results; the proof of the theorem is given in the final section.

2 Preliminaries

Maier [10] introduced the notion of a *good* modulus as follows. If the Dirichlet L-functions $L(s;\chi)$ to all characters $\chi \bmod q$ do not vanish for all $s = \sigma + it$

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press

satisfying

$$\sigma > 1 - \frac{C}{\log|q(|t|+1)|},$$

where C is a positive constant, then q is said to be *good*. Since this definition depends on C, it follows for sufficiently small C that either q is good or there is a quadratic character χ such that $L(s;\chi)$ has a an exceptional real zero. In this latter case, the exceptional zero and the character are unique as follows from Page's theorem (see [4]).

Lemma 2.1 *There is a positive constant C such that there exist arbitrarily large values of z for which*

$$P(z) := \prod_{p<z} p \tag{2.1}$$

is a good modulus.

This is [10, Lemma 1] and leads to a uniform prime number theorem for all prime residue classes to a good modulus. Let $\pi(x; a \bmod q)$ denote the number of primes $p \le x$ satisfying $p \equiv a \bmod q$.

Lemma 2.2 *If q is a good modulus, then*

$$\pi(x+h; a \bmod q) - \pi(x; a \bmod q) = \frac{1}{\varphi(q)} \int_x^{x+h} \frac{\mathrm{d}u}{\log u} \Big(1 + O\big(\exp(-cD) + \exp(-\sqrt{\log x})\big)\Big), \tag{2.2}$$

provided that $a \bmod q$ *is a prime residue class,* $x \ge q^D$, *and* $\frac{1}{2}x \le h \le x$, *where* $\log q \ge D \ge D_0$ *with* D_0 *and c being positive constants and the implied constant in the Big O-term depends only on the constant C from Lemma 2.1.*

This result is essentially due to Gallagher [6] (see also [10, 11]).

Next we define the quantities

$$W(z) := \prod_{p<z}\Big(1 - \frac{1}{p}\Big), \quad \Phi(x,y) := \sharp\{n \le x : \gcd(n, P(y)) = 1\}. \tag{2.3}$$

Moreover, let the so-called *Buchstab-function* $\omega(u)$ be defined by $\omega(u) = 0$ for $0 < u < 1$ and

$$u\omega(u) = 1 + \int_1^{u-1} \omega(v)\,\mathrm{d}v$$

for $u \ge 1$. The following result is due to Buchstab [2].

Lemma 2.3 *For* $\lambda > 1$,

$$\lim_{z\to\infty} z^{-\lambda} W(z)^{-1} \Phi(z^\lambda, z) = \exp(\gamma)\omega(\lambda).$$

Notice that Mertens' classical theorem states that

$$\prod_{p<z}\Big(1 - \frac{1}{p}\Big) \sim \frac{\exp(-\gamma)}{\log z}$$

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press

with the Euler–Mascheroni constant $\gamma := \lim_{N\to\infty}(\sum_{n\le N}\frac{1}{n} - \log N) = 0.577\ldots$. We conclude with another result due to Iwaniec [9] (see also [11]).

Lemma 2.4 *The function $u \mapsto \omega(u) - \exp(-\gamma)$ changes sign in any interval $[a-1, a]$, where $a \ge 2$ is arbitrary.*

Finally, we quote from Banks and Shparlinski [1] the following prime number theorem for Beatty sequences.

Lemma 2.5 *Given real numbers α and β, where α is a positive irrational of finite type, there exists a positive constant κ such that, for all integers $0 \le a < q \le M^{\kappa}$ with* $\gcd(a,q) = 1$,

$$\sum_{\substack{n\le M\\ \lfloor n\alpha+\beta\rfloor\equiv a \bmod q}} \Lambda(\lfloor n\alpha+\beta\rfloor) = \frac{1}{\alpha}\sum_{\substack{m\le\lfloor M\alpha+\beta\rfloor\\ m\equiv a \bmod q}} \Lambda(m) + O(M^{1-\kappa}).$$

Here, Λ denotes the von Mangoldt–Lambda function. By partial summation, this yields for the number $\pi_{\mathcal{B}(\alpha,\beta)}(x; a \bmod q)$ of primes $p \le x$ in the intersection of $\mathcal{B}(\alpha,\beta)$ and $a + q\mathbb{Z} := \{a + qz : z \in \mathbb{Z}\}$ the asymptotic formula

$$\pi_{\mathcal{B}(\alpha,\beta)}(x; a \bmod q) = \frac{1}{\alpha}\pi(x; a \bmod q) + O\Big(\frac{x^{1-\kappa}}{\log x} + x^{\frac{1}{2}}\log x\Big). \tag{2.4}$$

With an explicit κ, of course, the error term could be simplified.

3 Proof of the Theorem

We begin by recalling the matrix method. For a sufficiently large integer D, let $z \ge \exp(cD)$ be a real number such that $P := P(z) \ge 2$ is a good modulus in the sense of Lemma 2.1 and the definition (2.1). We consider a progression of intervals $(rP, rP + U]$, where $R := R(z)$ and $U := U(z)$ are integers satisfying $R \le r < 2R$ and $U \le P$, and write its integers in form of a matrix:

$$\begin{matrix} RP+1 & RP+2 & \ldots & RP+U \\ (R+1)P+1 & (R+1)P+2 & \ldots & (R+1)P+U \\ \vdots & \vdots & & \vdots \\ (2R-1)P+1 & (2R-1)P+2 & \ldots & (2R-1)P+U. \end{matrix}$$

Obviously, the primes are contained in those columns $rP + j$ with $\gcd(j,P) = 1$. Let $R = P^{D-1}$. It then follows from Lemma 2.2 that the number of all primes $p \equiv j \bmod P$ in such a column in Maier's matrix is

$$\begin{aligned} &\pi(2P^D; j \bmod P) - \pi(P^D; j \bmod P)\\ &\quad= \frac{1}{\varphi(P)}\cdot\frac{P^D}{\log(P^D)}\big(1 + O(\exp(-cD))\big)\\ &\quad= \frac{1}{W}\cdot\frac{P^{D-1}}{\log(P^D)}\cdot\big(1 + O(\exp(-cD))\big), \end{aligned} \tag{3.1}$$

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press

where we have used the well-known asymptotics for the logarithmic integral in (2.2) and

$$\varphi(P) = P\prod_{p|P}\Big(1-\frac{1}{p}\Big) = PW \qquad \text{with} \quad W := W(z)$$

as defined in (2.3). For the number of primes in the Beatty sequence $\mathcal{B}(\alpha,\beta)$ in such a column, however, we get via the prime number theorem of Banks and Shparlinski in the form (2.4) that

$$\pi_{\mathcal{B}(\alpha,\beta)}(2P^D; j \bmod P) - \pi_{\mathcal{B}(\alpha,\beta)}(P^D; j \bmod P) = \frac{1}{\alpha}\cdot\frac{1}{W}\cdot\frac{P^{D-1}}{\log(P^D)}\cdot\Big(1+O(\exp(-cD)+(P^D)^{-\kappa})\Big),$$

which is, up to a factor $\frac{1}{\alpha}$, the same as in Maier's case (3.1); the additional term $(P^D)^{-\kappa}$ in the Big O-term results from the error term in (2.4) (under the assumption that $\kappa < \frac{1}{2}$; otherwise, one would have to replace $(P^D)^{-\kappa}$ by $(P^D)^{\epsilon-\frac{1}{2}}$, where $\epsilon > 0$ can be chosen arbitrarily small). It should be noticed that the application of Lemma 2.5 (resp. (2.4)) relies on the inequality $P \le (P^D)^{\kappa} = P^{D\kappa}$, which is obviously fulfilled for sufficiently large D.

The number of columns $rP + j$ with $\gcd(j, P) = 1$ is given by $\Phi(U, z)$, where Φ is given by (2.3). In view of Lemma 2.3, we have

$$\Phi(U,z) \sim \exp(\gamma)\omega(\lambda)\cdot WU \qquad \text{for} \quad U = \lfloor z^{\lambda}\rfloor. \tag{3.2}$$

Writing $\Phi(U, z) = cUW$, the number of all "Beatty primes" in Maier's matrix thus equals

$$\sum_{\substack{1\le j\le U\\ \gcd(j,P)=1}}\Big(\pi_{\mathcal{B}(\alpha,\beta)}(2RP; j \bmod P) - \pi_{\mathcal{B}(\alpha,\beta)}(RP; j \bmod P)\Big) = \frac{1}{\alpha}\cdot\frac{P^{D-1}}{\log(P^D)}\cdot U\cdot c(U,z)\cdot\Big(1+O(\exp(-cD)+P^{-D\kappa})\Big).$$

Next, by Lemma 2.4, we can choose some $\lambda^+ > b$ such that $\omega(\lambda^+) > \exp(-\gamma)$, where $b > 2$ is arbitrary but fixed. There are P^{D-1} rows in Maier's matrix. Hence, it follows from (3.2) that there is at least one row of Maier's matrix with at least

$$\frac{1}{\alpha}\cdot\frac{U}{\log(P^D)}\cdot\exp(\gamma)\omega(\lambda^+)\cdot\Big(1+O(\exp(-cD)+P^{-D\kappa})\Big)$$

primes in $\mathcal{B}(\alpha,\beta)$, which is more than the expected number, since $\exp(\gamma)\omega(\lambda^+) > 1$. To make this more explicit, let $\ell = (\log(P^D))^{\lambda^+}$ and divide this row into $k = \lfloor U/\ell\rfloor + 1$ disjoint subintervals of equal length $\ell + (1 + o(1))$. Then at least one of these intervals $(a, b]$ contains at least

$$n := \frac{1}{\alpha}\cdot\frac{U}{k\log(P^D)}\cdot\exp(\gamma)\omega(\lambda^+)\cdot\Big(1+O(\exp(-cD)+P^{-D\kappa})\Big)$$

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press

primes in $\mathcal{B}(\alpha, \beta)$. Setting $x = a$ it follows that $(a, b] \subset (x, x + y(x)]$ with $y = y(x) = (\log x)^b$ and the interval $(x, x + y(x)]$ contains at least

$$n = \frac{1 + \delta_b}{\alpha} \cdot \frac{y}{\log x} \cdot \big(1 + O(\exp(-cD) + P^{-D\kappa})\big)$$

primes in $\mathcal{B}(\alpha, \beta)$, where $\exp(\gamma)\omega(\lambda^+) > 1 + \delta_b$ for some positive constant δ_b. This proves the first inequality; the second one follows by the same reasoning with the choice λ' such that $\omega(\lambda') < \exp(-\gamma)$ according to Lemma 2.4.

Acknowledgments The authors are grateful to the anonymous referee for her or his valuable remarks.

References

[1] W. D. Banks and I. E. Shparlinski, *Prime numbers with Beatty sequences*. Colloq. Math. 115(2009), 147–157. https://doi.org/10.4064/cm115-2-1

[2] A. A. Buchstab, *Asymptotic estimates of a general number-theoretic function*. Mat.-Sb. (N.S.) 2(1937), 1239–1246.

[3] H. Cramér, *On the order of magnitude of the difference between consecutive prime numbers*. Acta Arith. 2(1936), 23–46.

[4] H. Davenport, *Multiplicative number theory*, Third ed, Graduate Texts in Mathematics, 74, Springer-Verlag, New York, 2000.

[5] A. S. Fraenkel, *The bracket function and complementary sets of integers*. Canad. J. Math. 21(1969), 6–27. https://doi.org/10.4153/CJM-1969-002-7

[6] P. X. Gallagher, *A large sieve density estimate near $\sigma = 1$*. Invent. Math. 11(1970), 329–339. https://doi.org/10.1007/BF01403187

[7] A. Granville, *Unexpected irregularities in the distribution of prime numbers*. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, 1995, pp. 388–399.

[8] A. Hildebrand and H. Maier, *Irregularities in the distribution of primes in short intervals*. J. Reine Angew. Math. 397(1989), 162–193.

[9] H. Iwaniec, *The sieve of Eratosthenes-Legendre*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4(1977), 257–268.

[10] H. Maier, *Chains of large gaps between consecutive primes*. Adv. in Math. 39(1981), 257–269. https://doi.org/10.1016/0001-8708(81)90003-7

[11] H. Maier], *Primes in short intervals*. Michigan Math. J. 32(1985), 221–225. https://doi.org/10.1307/mmj/1029003189

[12] A. Selberg, *On the normal density of primes in small intervalls and the difference between conscutive primes*. Arch. Math. Naturvid. 47(1943), 87–105.

[13] S. Tadee and V. Laohakosol, *Complementary sets and Beatty functions*. Thai J. Math. 3(2005), 27–33. https://doi.org/10.1155/2005/165785

[14] F. Thorne, *Maier matrices beyond Z.*. In: *Combinatorial number theory*. Walter de Gruyter, Berlin, 2009, pp. 185–192.

[15] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*. Dover Publications, Mineola, NY, 2004.

Walailak University, School of Science, Nakhon Si Thammarat, 80 160, Thailand
e-mail: tjanyarak@gmail.com

Department of Mathematics, Würzburg University, Am Hubland, 97 218 Würzburg, Germany
e-mail: steuding@mathematik.uni-wuerzburg.de

https://doi.org/10.4153/S0008439519000778 Published online by Cambridge University Press